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HOMOGENIZATION OF NONLINEAR INEXTENSIBLE PANTOGRAPHIC STRUCTURES BY Γ-CONVERGENCE

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We prove the Γ-convergence of a pantographic microstructured sheet with inextensible fibers to a 2D generalized continuum model. Large deformations considered as geometrical nonlinearities are taken into account, and the Γ-convergence argument is developed in terms of convergence of measure functionals. We also prove a relative compactness property for the sequence of discrete energy functionals.

1. Introduction

Pantographic structures can be basically described as microstructured artifacts in which two families of parallel fibers are mechanically connected in such a way that changing the angle between two fibers, belonging to the two different families, costs deformation energy. Here is an image of a possible physical realization of a pantographic structure:

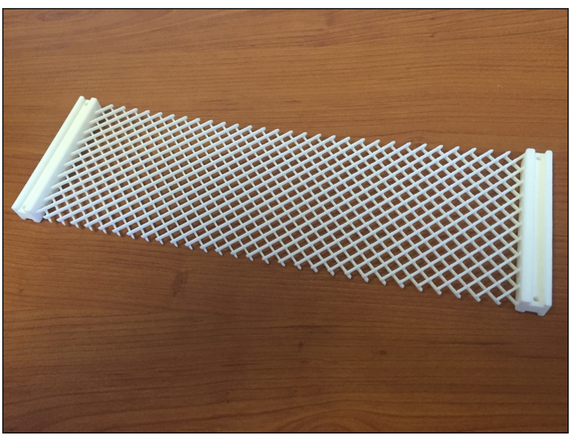

Figure 1. A 3D-printed pantographic sheet made of polyamide (courtesy of Professor Tomasz Lekszycki, Warsaw University of Technology).

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The problem of the equilibrium of lattice structures has been studied for a long time (see for instance [Rivlin 1964; Pipkin 1984; 1986; Steigmann and Pipkin 1991]), and pantographic structures in particular are currently of definite interest both as a structural element, due to their interesting mechanical properties (see for instance [dell'Isola et al. 2016b; Turco et al. 2016b; Battista et al. 2015; Barchiesi et al. 2018b]), and as an experimental and theoretical model case for the onset of behaviors that cannot be described by means of the theory of classical Cauchy continua. In particular it has been shown that generalized continua, in which the energy density depends explicitly on the second gradient of the placement function (see [Mindlin 1964; 1965; Mindlin and Eshel 1968; Germain 1973] for historically important references), are suitable for the description of the deformation of the homogenized version of truss-like [Seppecher et al. 2011; Alibert and Della Corte 2015; Alibert et al. 2003; Turco et al. 2017a] and pantographic structures [Turco et al. 2016a; Rahali et al. 2015].

Here we prove a rigorous homogenization result, namely that a discretized model of pantographic structures (introduced in [dell'Isola et al. 2016a]) Γ-converges to a homogenized 2D continuum model described by an energy functional in which second partial derivatives of the placement appear. The mathematical study of

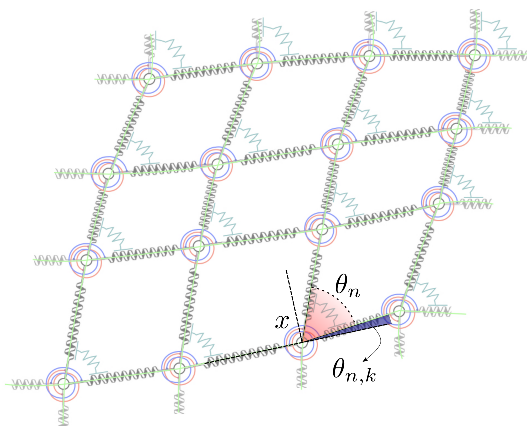

Figure 2. A schematic representation of a pantographic structure in an arbitrary deformed configuration. At every node x of the square lattice there are rotational springs acting between adjacent orthogonal segments (in this case the energy depends, using the notation of Section 3, on the angle θ_n) and between adjacent parallel segments (in this case the energy depends, using the notation of Section 3, on the angle $\theta_{n,k}$). The nodes are connected by means of extensional springs that in the present paper are particularized to be rigid bars.

linear pantographic structures is already an active research field (see, e.g., [Boutin et al. 2017; Eremeyev et al. 2018]). In the present paper, the main result will be proven in the large-deformation regime, that is, taking into account geometrical nonlinearities and in particular the actual curvature of the fibers and not only its linearized form.

The microstructure considered herein consists of a square lattice, at each node of which are positioned two types of rotational springs, one acting between adjacent orthogonal segments and the other one acting between adjacent parallel segments (a schematic representation of the structure, in an arbitrary deformed configuration, is shown in Figure 2). In the general case, between the nodes are positioned extensional springs allowing changes in the distances separating adjacent nodes. In the present paper, however, we consider the inextensible case, i.e., we assume that the nodes of the lattice are connected by rigid bars.

The Γ-convergence argument is developed in terms of convergence of measure functionals. This is, in our opinion, the most sensible approach, since in the real object (at least in planar deformations) most of the deformation energy is actually concentrated in the nodes, stored as torsional deformation energy of the cylindrical pivots interconnecting the two layers of parallel fibers (see, e.g., [Giorgio 2016; dell'Isola et al. 2015]). Therefore, it is quite natural to take this into account in the mathematical modeling introducing a set of vector-valued measures concentrated in the nodes of the lattice. Then we circumscribe the admissible measures by identifying them with functions belonging to suitable Sobolev spaces. This approach allows us to avoid the use of (arbitrary, to some degree) interpolating functions between the nodes. Along with the Γ-convergence result, we prove a relative compactness property, which ensures that controlling the total deformation energy is enough to control the norm of the measure used for the description of the current configuration of the discrete model.

The paper is organized as follows. In Section 2 the general concept of Γ-convergence of measure functionals is introduced; in Section 3 the admissible measures are introduced and the energy of the discrete micromodel as well as the boundary conditions are formally described; in Section 4 the same is done for the continuous macromodel and the main result is stated; in Section 5 the main result is proven, including the relative compactness property for the sequence of discrete energy functionals; finally, in Section 6 some conclusions are stated and some possible directions for future studies are indicated.

2. Γ-convergence of measure functionals

We start by recalling the definition of Γ-convergence for a sequence of measure functionals.

Let $K := [0, 1]^2$ and $(C(K))^2$ be the space of vector-valued continuous functions on K endowed with the uniform norm $\|\varphi\|_\infty := \sup\{\|\varphi(x)\| : x \in K\}$ where $\|\cdot\|$ denotes the euclidean norm of $\mathbb{R}^2$. Let $(\mathcal{M}(K))^2$ be the set of vector-valued bounded measures on K endowed with the norm

$$\|\mu\|_{\mathcal{M}} := \sup\{\langle \mu, \varphi\rangle : \varphi \in (C(K))^2,\ \|\varphi\|_\infty = 1\}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality bracket between $(\mathcal{M}(K))^2$ and $(C(K)^2)$. We simply write $\mu_n \rightharpoonup \mu$ to specify that the sequence (μ_n) of vector bounded measures converges to μ with respect to the weak* topology, i.e., $\lim_{n\to\infty}\langle \mu_n, \varphi\rangle = \langle \mu, \varphi\rangle$ for every $\varphi \in (C(K))^2$. Recall (see for instance [Evans and Gariepy 2015]) that, if a sequence of vector-valued bounded measures (μ_n) satisfies

$$\sup_n \|\mu_n\|_{\mathcal{M}} < +\infty,$$

then there exists $\mu \in (\mathcal{M}(K))^2$ and a subsequence (n_k) such that $\mu_{n_k} \rightharpoonup \mu$. Let (F_n) be a sequence of functionals on $(\mathcal{M}(K))^2$ with values in $\mathbb{R}\cup\{+\infty\}$. We say that the *relative compactness property* holds for the sequence (F_n) if for all sequences (μ_n) in $(\mathcal{M}(K))^2$

$$\sup_n F_n(\mu_n) < +\infty \quad\Longrightarrow\quad \sup_n \|\mu_n\|_{\mathcal{M}} < +\infty.$$

We say that the sequence (F_n) Γ- converges to F if the following two properties are satisfied.

Lower-bound inequality. For all $\mu \in (\mathcal{M}(K))^2$ and all sequences (μ_n) in $(\mathcal{M}(K))^2$

$$\mu_n \rightharpoonup \mu \quad\Longrightarrow\quad \liminf_{n\to\infty} F_n(\mu_n) \geq F(\mu).$$

Upper-bound inequality. For each $\mu \in (\mathcal{M}(K))^2$, there exists a sequence (μ_n) in $(\mathcal{M}(K))^2$ such that

$$\mu_n \rightharpoonup \mu \quad \text{and} \quad \limsup_{n\to\infty} F_n(\mu_n) \leq F(\mu).$$

For a general introduction to Γ-convergence the reader is referred to [Braides 2002].

3. Micromodel for nonlinear pantographic lattices

3.1. ***Reference configuration and basic operators.*** Let δ_t be the Dirac measure concentrated at the point $t \in [0, 1]$. We define four Radon measures on $[0, 1]$ by setting

$$\nu_n := \frac{1}{n}\sum_{i=0}^{n} \delta_{i/n}, \qquad \nu_n^+ := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{i/n}, \qquad \nu_n^- := \frac{1}{n}\sum_{i=1}^{n} \delta_{i/n}, \qquad \nu_n^2 := \frac{1}{n}\sum_{i=1}^{n-1} \delta_{i/n}.$$

The reference configuration of the microsystem is described by the measure $\sigma_n := \nu_n \otimes \nu_n$ on $K := [0, 1] \times [0, 1]$. The support of the measure σ_n is the finite set

$$\text{Support}(\sigma_n) = \left\{ \left(\frac{i}{n}, \frac{j}{n} \right) : 0 \leq i, j \leq n \right\}.$$

Each point of the support of σ_n is called *a node*. Two nodes x and y are called adjacent when $\|y - x\| = 1/n$ and diagonally adjacent when $\|y - x\| = \sqrt{2}/n$. We define six measures on K and six discrete partial derivative operators by setting

$$\begin{aligned}
\sigma^+_{n,1} &:= \nu^+_n \otimes \nu_n, & \partial^+_{n,1} u(x) &:= n\Big(u\Big(x + \frac{1}{n} e_1\Big) - u(x)\Big),\\
\sigma^-_{n,1} &:= \nu^-_n \otimes \nu_n, & \partial^-_{n,1} u(x) &:= n\Big(u(x) - u\Big(x - \frac{1}{n} e_1\Big)\Big),\\
\sigma^2_{n,1} &:= \nu^2_n \otimes \nu_n, & \partial^2_{n,1} u(x) &:= n(\partial^+_{n,1} u(x) - \partial^-_{n,1} u(x)),\\
\sigma^+_{n,2} &:= \nu_n \otimes \nu^+_n, & \partial^+_{n,2} u(x) &:= n\Big(u\Big(x + \frac{1}{n} e_2\Big) - u(x)\Big),\\
\sigma^-_{n,2} &:= \nu_n \otimes \nu^-_n, & \partial^-_{n,2} u(x) &:= n\Big(u(x) - u\Big(x - \frac{1}{n} e_2\Big)\Big),\\
\sigma^2_{n,2} &:= \nu_n \otimes \nu^2_n, & \partial^2_{n,2} u(x) &:= n(\partial^+_{n,2} u(x) - \partial^-_{n,2} u(x))
\end{aligned}$$

with $e_1 := (1, 0)$ and $e_2 := (0, 1)$. Note that, if $u : K \to \mathbb{R}^2$ is defined at every point in the support of σ_n, then for $k \in \{1, 2\}$ and $s \in \{+, -, 2\}$ the function $\partial^s_{n,k} u : K \to \mathbb{R}^2$ is defined at every point in the support of $\sigma^s_{n,k}$. For $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$, we set

$$a \wedge b := a_1 b_2 - a_2 b_1.$$

We define four measures on K and four discrete Jacobian determinant operators by setting, for $s, s' \in \{+, -\}$

$$\sigma^{(s,s')}_n := \nu^s_n \otimes \nu^{s'}_n, \qquad J^{(s,s')}_n(u)(x) := \partial^s_{n,1} u(x) \wedge \partial^{s'}_{n,2} u(x).$$

Note that, if $u : K \to \mathbb{R}^2$ is defined at every point in the support of σ_n, then the function $J^{(s,s')}_n(u) : K \to \mathbb{R}$ is defined at every point in the support of $\sigma^{(s,s')}_n$.

3.2. ***Current configuration and deformation energy of the n-micromodel.*** The current configuration of the object is described by a vector-valued bounded measure of the special set $\mathcal{M}_n$ defined below.

Definition (admissible measures of the n-micromodel). The set of admissible measures of the n-micromodel is denoted by $\mathcal{M}_n$ and consists of those vector bounded measures $\mu \in (\mathcal{M}(K))^2$ of the form

$$\mu(dx) = u(x)\sigma_n(dx)$$

where the function $u : K \to \mathbb{R}^2$ is defined at any point x in the support of σ_n and such that $\|\partial^s_{n,k} u(x)\| > 0$ for every $(k, s) \in \{1, 2\} \times \{+, -\}$. For any admissible measure μ of the n-micromodel, the following notation is used:

$$\rho^s_{n,k}(\mu)(x) := \|\partial^s_{n,k} u(x)\| \quad \text{and} \quad v^s_{n,k}(\mu)(x) := \frac{\partial^s_{n,k} u(x)}{\|\partial^s_{n,k} u(x)\|}. \tag{1}$$

Let $\mu(dx) = u(x)\sigma_n(dx) \in \mathcal{M}_n$. The function u is called the placement function. The point $u(x)$ is the current position of the node x. The above definition of admissible measures imposes the natural requirement that two adjacent nodes should not be mapped by the deformation on the same point (of course this does not exclude the possibility that two generic nodes are mapped on the same point).

At each node x are placed extensional springs which connect x to the adjacent nodes. The deformation energy associated with these extensional springs depends on the distance between the current positions of adjacent nodes and is equal to zero when the distance is equal to $1/n$. So we introduce the following definition.

Definition (extensional deformation energy of the n-micromodel). The extensional deformation energy $E^{(\text{ext})}_n$ is defined on $(\mathcal{M}(K))^2$ by setting $E^{(\text{ext})}_n(\mu) = +\infty$ if $\mu \notin \mathcal{M}_n$ and

$$E^{(\text{ext})}_n(\mu) := \sum_{k=1}^{2} \int f_k(\rho^+_{n,k}(\mu))\, d\sigma^+_{n,k} \quad \text{otherwise,}$$

where the functions $f_k : (0, +\infty) \to [0, +\infty]$ are assumed to be such that $f_k(1) = 0$ and $f_k(\rho) > 0$ if $\rho \neq 1$.

Remark 1. Our main result is obtained in the particular case when the springs between the nodes are just rigid bars, i.e., when f_1 and f_2 are the indicator function of the set $\{1\}$:

$$f_k(\rho) := \begin{cases} 0 & \text{if } \rho = 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

At each node x are placed four rotational springs (to provide shear stiffness) which connect a pair of segments $([x, x + (s/n)e_1], [x, x + (s'/n)e_2])$ with $s, s' \in \{+1, -1\}$. Its energy at the node x depends on the angle $\theta^{(s,s')}_n(\mu)(x)$ formed by the vectors $\partial^s_{n,1} u(x)$ and $\partial^{s'}_{n,2} u(x)$. This energy is equal to zero if and only if the angle is equal to $\pi/2$. We also assume that angles with finite energy must be in the interval $(0, \pi)$, so as to ensure that nodes diagonally adjacent are not mapped by the deformation on the same point (again, this does not exclude the possibility that two generic nodes are mapped on the same point). One has

$$\sin(\theta^{(s,s')}_n(\mu)(x)) = v^s_{n,1}(\mu)(x) \wedge v^{s'}_{n,2}(\mu)(x).$$

So we introduce the following definition.

Definition (first rotational deformation energy of the n-micromodel). The first rotational deformation energy $E_n^{(\mathrm{shear})}$ is defined on $(\mathcal{M}(K))^2$ by setting $E_n^{(\mathrm{shear})}(\mu) = +\infty$ if $\mu \notin \mathcal{M}_n$ and

$$E_n^{(\mathrm{shear})}(\mu) := \sum_{s,s' \in \{+,-\}} \int g^{(s,s')}(v_{n,1}^{s}(\mu) \wedge v_{n,2}^{s'}(\mu))\, d\sigma_n^{(s,s')} \quad \text{otherwise,}$$

where the four functions $g^{(s,s')} : [-1, 1] \to [0, +\infty]$ are assumed to be such that $g^{(s,s')}(1) = 0$.

Remark 2. Our main result is obtained in the particular case when the four functions $g^{(s,s')}$ are assumed to be lower semicontinuous, convex, and such that $\{g^{(s,s')} < +\infty\}$ is compact and $g^{(s,s')}(\delta) = +\infty$ if $\delta \leq 0$.

The second type of rotational springs (providing bending rigidity along each coordinate line) is those which connect a pair of segments $([x, x + (1/n)e_k], [x, x - (1/n)e_k])$ with $k \in \{1, 2\}$. Their energy at the node x depends on the angle $\theta_{n,k}(\mu)(x)$ formed by the vectors $v_{n,k}^{+}(\mu)(x)$ and $v_{n,k}^{-}(\mu)(x)$. This energy is equal to zero if and only if the angle is equal to 0 and one has

$$1 - \cos(\theta_{n,k}(\mu)(x)) = \tfrac{1}{2}\|v_{n,k}^{+}(\mu)(x) - v_{n,k}^{-}(\mu)(x)\|^2 = \frac{1}{2n^2}\|\partial_{n,k}^{-} v_{n,k}^{+}(\mu)(x)\|^2.$$

So, we introduce the following definition.

Definition (second rotational deformation energy of the n-micromodel). Second rotational deformation energy $E_n^{(\mathrm{bend})}$ is defined on $(\mathcal{M}(K))^2$ by setting $E_n^{(\mathrm{bend})}(\mu) = +\infty$ if $\mu \notin \mathcal{M}_n$ and

$$E_n^{(\mathrm{bend})}(\mu) := \sum_{k \in \{1,2\}} \int \frac{\kappa_k}{2} \|\partial_{n,k}^{-} v_{n,k}^{+}(\mu)\|^2 \, d\sigma_{n,k}^2 \quad \text{otherwise.}$$

Remark 3. Our main result is obtained in the case when the two real numbers κ_k are assumed to be positive (which is quite natural since they represent material coefficients accounting for the bending stiffness of the fibers).

Definition (Dirichlet boundary condition for the n-micromodel). Let ∂K denote the boundary of K. Let Σ be a subset of ∂K and $\mathcal{M}_n^{\Sigma}$ be the set of those measures $\mu(dx) = u(x)\sigma_n(dx) \in \mathcal{M}_n$ such that $u(x) = x$ for every $x \in \Sigma \cap \mathrm{Support}(\sigma_n)$. We denote by $E_n^{(\Sigma)}$ the indicator functional of the set $\mathcal{M}_n^{\Sigma}$, i.e.,

$$E_n^{(\Sigma)}(\mu) := \begin{cases} 0 & \text{if } \mu \in \mathcal{M}_n^{\Sigma}, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 4. Our main result is obtained in the particular case when $\Sigma := (a, b) \times \{0\}$ or $\Sigma := \{0\} \times (a, b)$ with $0 \leq a < b \leq 1$.

4. Macromodel and main result

4.1. ***Deformation energy of the macromodel.*** Let $\Omega := (0,1) \times (0,1)$. For each $p \in [1, +\infty]$ we denote by $L^p(\Omega)$ the usual Lebesgue space. Distributional partial derivative operators are denoted by ∂_k. Distributional second partial derivative operators are denoted by $\partial_k \partial_{k'}$. If $k = k'$, we also write ∂_k^2 in place of $\partial_k \partial_k$.

Definition (admissible measures of the macromodel). The set of admissible measures is denoted by $\mathcal{M}_\infty$ and consists of those measures $\mu \in (\mathcal{M}(K))^2$ of the form

$$\mu(dx) = u(x)\, dx$$

where the function $u : \Omega \to \mathbb{R}^2$ is such that $u \in (L^1(\Omega))^2$, $\partial_k u \in (L^1(\Omega))^2$, $\partial_k^2 u \in (L^2(\Omega))^2$, and $\|\partial_k u\| > 0$ a.e. in Ω for every $k \in \{1, 2\}$. For any admissible measure μ, the following notation will be used:

$$\rho_k(\mu) := \|\partial_k u\| \quad \text{and} \quad v_k(\mu) := \frac{\partial_k u}{\|\partial_k u\|}. \tag{3}$$

Definition (extensional deformation energy of the macromodel). The extensional deformation energy $E^{(\mathrm{ext})}$ is defined on $(\mathcal{M}(K))^2$ by setting $E^{(\mathrm{ext})}(\mu) = +\infty$ if $\mu \notin \mathcal{M}_\infty$ and

$$E^{(\mathrm{ext})}(\mu) := \sum_{k=1}^{2} \int_\Omega f_k(\rho_k(\mu)(x))\, dx \quad \text{otherwise,}$$

where f_k was defined in Remark 1.

Definition (first rotational deformation energy of the macromodel). The first rotational deformation energy $E^{(\mathrm{shear})}$ is defined on $(\mathcal{M}(K))^2$ by setting $E^{(\mathrm{shear})}(\mu) = +\infty$ if $\mu \notin \mathcal{M}_\infty$ and

$$E^{(\mathrm{shear})}(\mu) := \int_\Omega g(v_1(\mu)(x) \wedge v_2(\mu)(x))\, dx \quad \text{otherwise,}$$

where $g := g^{(+,+)} + g^{(-,+)} + g^{(-,-)} + g^{(+,-)}$, $g^{s,s'}$ was defined in Remark 2, and $v_k(\mu)(x)$ was defined in (3).

Definition (second rotational deformation energy of the macromodel). The second rotational deformation energy $E^{(\mathrm{bend})}$ is defined on $(\mathcal{M}(K))^2$ by setting $E^{(\mathrm{bend})}(\mu) = +\infty$ if $\mu \notin \mathcal{M}_\infty$ and

$$E^{(\mathrm{bend})}(\mu) := \sum_{k=1}^{2} \int_\Omega \frac{\kappa_k}{2} \|\partial_k v_k(\mu)(x)\|^2\, dx \quad \text{otherwise.}$$

If $\mu(dx) = u(x)\, dx$ is admissible for the macromodel, the placement function u admits a trace on the boundary of Ω because $u \in (L^1(\Omega))^2$ and $\partial_k u \in (L^1(\Omega))^2$ for

$k = 1, 2$. The trace function associated with u will be also denoted by u and belongs to the Lebesgue space of integrable functions with respect to the 1D Hausdorff measure supported on $\partial\Omega = \partial K$. This last measure will be denoted by $\mathcal{H}^1_{\partial\Omega}$.

Definition (Dirichlet boundary condition for the macromodel). Let Σ be a subset of the boundary of Ω and $\mathcal{M}^{\Sigma}_{\infty}$ be the set of those measures $\mu(dx) = u(x)\,dx \in \mathcal{M}_{\infty}$ such that $u(x) = x$ for $\mathcal{H}^1_{\partial\Omega}$-a.e. $x \in \Sigma$. We denote by $E^{(\Sigma)}$ the indicator functional of the set $\mathcal{M}^{\Sigma}_{\infty}$, i.e.,

$$E^{(\Sigma)}(\mu) := \begin{cases} 0 & \text{if } \mu \in \mathcal{M}^{\Sigma}_{\infty}, \\ +\infty & \text{otherwise.} \end{cases}$$

4.2. ***Main result.*** Our main result states that the total deformation energy of the n-micromodel, namely

$$E_n := E_n^{(\text{ext})} + E_n^{(\text{shear})} + E_n^{(\text{bend})} + E_n^{(\Sigma)},$$

Γ-converges to the total deformation energy of the macromodel

$$E := E^{(\text{ext})} + E^{(\text{shear})} + E^{(\text{bend})} + E^{(\Sigma)}.$$

This result is obtained under the assumption that two adjacent nodes are connected by an inextensible bar. This is taken into account by assuming that f_1 and f_2 are the indicator function of the set $\{1\}$. Moreover, we need that the angles $\theta_n^{(s,s')}(\mu)$ remain in the interval $(0, \pi)$. Such an assumption is taken into account by assuming that the four functions $g^{(s,s')}$ are greater than the indicator function of the set $(0, 1]$. Finally, we need that the part of the boundary on which the displacement is zero is not too small. More precisely, we have:

Theorem. *If we assume that*

(H_1) *the two functions* $f_k : (0, +\infty) \to [0, +\infty]$ *are the indicator function of the set* $\{1\}$,

(H_2) *the four functions* $g^{(s,s')} : [-1, 1] \to [0, +\infty]$ *are lower semicontinuous, convex, and such that* $\{g^{(s,s')} < +\infty\}$ *is compact,* $g^{(s,s')}(1) = 0$, *and* $g^{(s,s')}(\delta) = +\infty$ *if* $\delta \le 0$,

(H_3) *the two real numbers* κ_k *are positive, and*

(H_4) $\Sigma := (a, b) \times \{0\}$ *or* $\Sigma := \{0\} \times (a, b)$ *with* $0 \le a < b \le 1$,

then the sequence (E_n) Γ*-converges to the functional* E *and the relative compactness property holds.*

5. Proof of the main theorem

5.1. ***Consequences of the assumptions (H$_1$) and (H$_2$) on placement functions.*** For the sake of simplicity, in Section 5 we will write $\rho^+_{n,k}$ and $v^+_{n,k}$ instead of $\rho^+_{n,k}(\mu)$ and $v^+_{n,k}(\mu)$.

When property (H$_1$) is assumed, the extensional deformation energy of the n-micromodel is just the indicator functional of the set of those admissible measures $\mu \in \mathcal{M}_n$ such that $\rho^s_{n,k}(\mu)(x) = 1$ for every node $x \in \mathrm{Support}(\sigma^s_{n,k})$ and every $(k, s) \in \{1, 2\} \times \{+, -\}$. Moreover, if (H$_2$) is assumed, then measures with finite deformation energy have a very special form.

Lemma 5. *If (H$_1$)–(H$_2$) are assumed and $E^{(\mathrm{ext})}_n(\mu) + E^{(\mathrm{shear})}_n(\mu) < +\infty$, then the placement function u associated with μ is such that*

$$\|\partial^+_{n,k}u(x)\| = 1 \qquad \textit{for every node } x \in \mathrm{Support}(\sigma^+_{n,k}),$$

$$u(x) + u(0) = \sum_{k=1}^{2} u((x \cdot e_k)e_k) \quad \textit{for every node } x \in \mathrm{Support}(\sigma_n).$$

Proof. Since $E^{(\mathrm{ext})}_n(\mu) < +\infty$ and (H$_1$) is assumed, the first claim of Lemma 5 is clear and for any node $x \in \mathrm{Support}(\sigma^{(+,+)}_n)$ one has $v^+_{n,k}(x) = \partial^+_{n,k}u(x)$. As a consequence,

$$v^+_{n,1}\Big(x + \frac{1}{n}e_2\Big) - v^+_{n,1}(x) = v^+_{n,2}\Big(x + \frac{1}{n}e_1\Big) - v^+_{n,2}(x).$$

Let us denote by $w(x)$ any member of the previous identity. Observing that $\|w(x)+v^+_{n,1}(x)\|^2$ and $\|w(x) + v^+_{n,2}(x)\|^2$ are equal to 1 and $v^-_{n,2}(x + (1/n)e_2) = v^+_{n,2}(x)$, we obtain

$$w(x) \cdot (v^+_{n,2}(x) - v^+_{n,1}(x)) = 0,$$
$$\|w(x)\|^2 + w(x) \cdot (v^+_{n,1}(x) + v^+_{n,2}(x)) = 0,$$
$$(w(x) + v^+_{n,1}(x)) \wedge v^+_{n,2}(x) = v^+_{n,1}\Big(x + \frac{1}{n}e_2\Big) \wedge v^-_{n,2}\Big(x + \frac{1}{n}e_2\Big).$$

As $E^{(\mathrm{shear})}_n(\mu) < +\infty$ and (H$_2$) is assumed, $v^+_{n,1}(x+(1/n)e_2)\wedge v^-_{n,2}(x+(1/n)e_2) > 0$ and $v^+_{n,1}(x) \wedge v^+_{n,2}(x) > 0$. As a consequence $(v^+_{n,1}(x) + v^+_{n,2}(x), v^+_{n,2}(x) - v^+_{n,1}(x))$ is an orthogonal basis of $\mathbb{R}^2$, and for some real number λ, one has

$$w(x) = \lambda(v^+_{n,1}(x) + v^+_{n,2}(x)),$$
$$\lambda^2 + \lambda = 0,$$
$$(\lambda + 1)v^+_{n,1}(x) \wedge v^+_{n,2}(x) = (w(x) + v^+_{n,1}(x)) \wedge v^+_{n,2}(x) > 0.$$

We obtain $w(x) = 0$. Consequently $v^+_{n,k}(x) = v^+_{n,k}((x \cdot e_k)e_k)$ for every $x \in \mathrm{Support}(\sigma^+_{n,k})$ and $\partial^+_{n,1}\partial^+_{n,2}u(y) = 0$ for every node $y \in \mathrm{Support}(\sigma^{(+,+)}_n)$. The last identity is obtained by observing that for all $x=(i/n,\, j/n)\in \mathrm{Support}(\sigma_n)$

$$u(x)+u(0)-\sum_{k=1}^{2} u((x\cdot e_k)e_k)=\frac{1}{n^2}\sum_{q=0}^{i-1}\sum_{p=0}^{j-1}\partial^+_{n,1}\partial^+_{n,2}u\left(\frac{q}{n},\,\frac{p}{n}\right). \qquad \square$$

5.2. ***Relative compactness property.*** The fact that the relative compactness property holds for the sequence (E_n) is a direct consequence of the following result.

Lemma 6. *If* (H$_1$) *and* (H$_4$) *are assumed, then the relative compactness property holds for the sequence* $(E_n^{(\mathrm{ext})}+E_n^{(\Sigma)})$.

Proof. Without loss of generality, we may assume that there exist two real numbers a and b such that $0\le a<b\le 1$ and $\{0\}\times(a,b)\subset\Sigma$. For all n such that $b-a>1/n$ we set

$$\varphi_n(t):=\frac{\delta_t(a,b)}{\nu_n(a,b)}.$$

Let $\mu\in(\mathcal{M}(K))^2$ such that $E_n^{(\mathrm{ext})}(\mu)+E_n^{(\Sigma)}(\mu)<+\infty$. Then the measure μ is such that $\mu(dx)=u(x)\sigma_n(dx)$ with $\|\partial^s_{n,k}u(x)\|=1$ for every node $x\in\mathrm{Support}(\sigma_n)$ and $v(x):=u(x)-x=0$ for every node $x\in\{0\}\times(a,b)$. One has

$$\|\mu\|_{\mathcal{M}}=\int\|u(x)\|\sigma_n(dx)\le\int\|x\|\sigma_n(dx)+\int\|v(x)\|\sigma_n(dx)$$

and $\int\varphi_n(t)\nu_n(dt)=1$; then for any node $x=(x_1,x_2)$,

$$v(x)=\int(v(x_1,x_2)-v(x_1,t))\varphi_n(t)\nu_n(dt)+\int(v(x_1,t)-v(0,t))\varphi_n(t)\nu_n(dt),$$

which implies

$$\begin{aligned}\int\|v(x)\|\sigma_n(dx)&\le 2\int\|\partial^+_{n,2}v\|\,d\sigma^+_{n,2}+\frac{4}{\nu_n(a,b)}\int_{[0,1]\times(a,b)}\|\partial^+_{n,1}v\|\,d\sigma^+_{n,1}\\&\le 2\int\|\partial^+_{n,2}u-e_2\|\,d\sigma^+_{n,2}+\frac{4}{\nu_n(a,b)}\int_{[0,1]\times(a,b)}\|\partial^+_{n,1}u-e_1\|\,d\sigma^+_{n,1}\\&\le 8+2\int\|\partial^+_{n,2}u\|\,d\sigma^+_{n,2}+\frac{4}{\nu_n(a,b)}\int\|\partial^+_{n,1}u\|\,d\sigma^+_{n,1}\\&\le 12+\frac{8}{\nu_n(a,b)}.\end{aligned}$$

Since $(\nu_n(a,b)^{-1})$ and $\left(\int\|x\|\sigma_n(dx)\right)$ are bounded sequences, the proof is complete. $\square$

5.3. ***Lower-bound inequality.*** Throughout this subsection, (μ_n) is a sequence of measures with bounded deformation energy, which implies $\mu_n(dx) = u_n(x)\sigma_n(dx)$ is an admissible measure for the n-micromodel and

$$\sup_n E_n(\mu_n) < +\infty.$$

To gain some regularity properties for the placement functions, which will be exploited in the following, it is convenient to introduce a sequence $\bar{\mu}_n(dx) = \bar{u}_n(x)\,dx$ of admissible measures for the macromodel which will be called the *equivalent sequence* because of Lemma 9 below.

Lemma 7 (equivalent sequence). *We define a sequence $\bar{\mu}_n(dx) = \bar{u}_n(x)\,dx$ in $(\mathcal{M}(K))^2$ by setting, for all $t \in [0, 1]$ and all $x \in K$,*

$$\overline{w}_{n,k}(t) := \sum_{i=1}^{n-1} \delta_t\left(\frac{i}{n} - \frac{1}{2n}, \frac{i}{n} + \frac{1}{2n}\right)\partial^-_{n,k} v^+_{n,k}\left(\frac{i}{n} e_k\right),$$

$$\bar{v}_{n,k}(t) := v^+_{n,k}(0) + \int_0^t \overline{w}_{n,k}(s)\,ds,$$

$$\bar{u}_n(x) := u_n(0) + \sum_{k=1}^{2} \int_0^{x\cdot e_k} \bar{v}_{n,k}(t)\,dt.$$

We assume that (H$_1$), (H$_2$), *and* (H$_3$) *hold. Then $\bar{u}_n$ is $C^1(K)$ regular with distributional second partial derivatives in $(L^2(\Omega))^2$. Moreover,*

$$\sum_{k=1}^{2} \int_\Omega \frac{\kappa_k}{2} \|\partial_k^2 \bar{u}_n(x)\|^2\,dx = E_n^{(\text{bend})}(\mu_n) \quad \textit{and} \quad \partial_1\partial_2\bar{u}_n = 0 \quad \textit{in } \Omega,$$

$$\sum_{k=1}^{2} \frac{\kappa_k}{2} \|\partial_k \bar{u}_n(y) - \partial_k \bar{u}_n(x)\|^2 \le \|y - x\| E_n^{(\text{bend})}(\mu_n) \quad \textit{for every } x, y \in K,$$

$$\partial_k \bar{u}_n\left(x + \frac{1}{2n} e_k\right) = v^+_{n,k}(x) \quad \textit{for every } k \in \{1, 2\} \textit{ and } x \in \text{Support}(\sigma^+_{n,k}).$$

Proof. In the proof of Lemma 5 we proved that $v^+_{n,k}(x) = v^+_{n,k}((x \cdot e_k)e_k)$ for every node $x \in \text{Support}(\sigma^+_{n,k})$. Then a direct computation gives us

$$\int_0^1 \|\overline{w}_{n,k}(t)\|^2\,dt = \frac{n}{n+1} \int \|\partial^-_{n,k} v^+_{n,k}\|^2\,d\sigma^2_{n,k}.$$

As a consequence $\bar{v}_{n,k} : [0, 1] \to \mathbb{R}^2$ is continuous on $[0, 1]$ with distributional derivative in $(L^2(0, 1))^2$, which implies that $\bar{u}_n : K \to \mathbb{R}^2$ is $C^1(K)$ regular with distributional second partial derivatives in $(L^2(\Omega))^2$ and such that $\partial_1\partial_2\bar{u}_n = 0$ in Ω

and

$$\int_\Omega \|\partial_k^2 \bar{u}_n(x)\|^2\, dx = \int_\Omega \|\overline{w}_{n,k}(x\cdot e_k)\|^2\, dx = \int \|\partial_{n,k}^- v_{n,k}^+\|^2\, d\sigma_{n,k}^2.$$

Let $x, y \in K$. One has

$$\begin{aligned}\|\partial_k \bar{u}_n(y) - \partial_k \bar{u}_n(x)\|^2 = \|\bar{v}_{n,k}(y\cdot e_k) - \bar{v}_{n,k}(x\cdot e_k)\|^2 &= \left\| \int_{x\cdot e_k}^{y\cdot e_k} \overline{w}_{n,k}(t)\, dt \right\|^2 \\ &\le \|y-x\| \int_0^1 \|\overline{w}_{n,k}(t)\|^2\, dt \\ &\le \|y-x\| \int \|\partial_{n,k}^- v_{n,k}^+\|^2\, d\sigma_n^2.\end{aligned}$$

Let $x \in \mathrm{Support}(\sigma_{n,k}^+)$. There exists $i \in \{0, \ldots, n-1\}$ such that $i/n = x\cdot e_k$; then

$$\begin{aligned}\partial_k \bar{u}_n\left(x + \frac{1}{2n} e_k\right) &= \bar{v}_{n,k}\left(x\cdot e_k + \frac{1}{2n}\right) \\ &= v_{n,k}^+(0) + \sum_{q=1}^{i} \int_{q/n-1/2n}^{q/n+1/2n} \partial_{n,k}^- v_{n,k}^+\left(\frac{q}{n} e_k\right) dt \\ &= v_{n,k}^+(0) + \sum_{q=1}^{i} \left(v_{n,k}^+\left(\frac{q}{n} e_k\right) - v_{n,k}^+\left(\frac{q-1}{n} e_k\right)\right) \\ &= v_{n,k}^+((x\cdot e_k)e_k) = v_{n,k}^+(x).\end{aligned}$$

□

Lemma 8 (admissibility for the equivalent sequence). *We assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold. Then* $\bar{\mu}_n$ *is admissible for the macromodel. More precisely, for all* $k \in \{1, 2\}$,

$$\partial_1 \bar{u}_n(x) \wedge \partial_2 \bar{u}_n(x) > 0 \quad \textit{and} \quad 1 \ge \|\partial_k \bar{u}_n(x)\| > 0$$

for every $x \in K$.

Proof. When $x = (i/n, j/n) \in \mathrm{Support}(\sigma_n^{(+,+)})$, using Lemma 5 and the fact that μ_n is admissible for the micromodel (we recall that μ_n is a sequence of measures with bounded energy), we obtain $v_{n,1}^+((i/n)e_1) \wedge v_{n,2}^+((j/n)e_2) = v_{n,1}^+(x) \wedge v_{n,2}^+(x) > 0$. As a consequence, for all $q_1, q_2 \in \{1, \ldots, n-1\}$ and all $\theta_1, \theta_2 \in [0, 1]$,

$$\left((1-\theta_1) v_{n,1}^-\left(\frac{q_1}{n} e_1\right) + \theta_1 v_{n,1}^+\left(\frac{q_1}{n} e_1\right)\right) \wedge \left((1-\theta_2) v_{n,2}^-\left(\frac{q_2}{n} e_2\right) + \theta_2 v_{n,2}^+\left(\frac{q_2}{n} e_2\right)\right) > 0.$$

Let $x \in K$. When $x \in [1/(2n), 1 - 1/(2n)]^2$ one has

$$\frac{q_k}{n} - \frac{1}{2n} \le x\cdot e_k \le \frac{q_k}{n} + \frac{1}{2n}$$

for some $q_1, q_2 \in \{1, \dots, n-1\}$. A direct computation gives

$$\partial_k \bar{u}_n(x) = (1-\theta_k) v^-_{n,k}\left(\frac{q_k}{n} e_k\right) + \theta_k v^+_{n,k}\left(\frac{q_k}{n} e_k\right) \text{ with } \theta_k := n\left(x \cdot e_k - \left(\frac{q_k}{n} - \frac{1}{2n}\right)\right).$$

Since $\theta_k \in [0, 1]$, we obtain $\partial_1 \bar{u}_n(x) \wedge \partial_2 \bar{u}_n(x) > 0$ and $1 \geq \|\partial_k \bar{u}_n(x)\| > 0$. The proof is easily completed when $x \in K \setminus [1/(2n), 1 - 1/(2n)]^2$. □

Lemma 9 (asymptotic equivalence). *Assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold and set*

$$\|u_n - \bar{u}_n\|_{L^\infty(\sigma_n)} := \sup\{\|u_n(x) - \bar{u}_n(x)\| : x \in \text{Support}(\sigma_n)\}.$$

Then

$$\lim_n \|u_n - \bar{u}_n\|_{L^\infty(\sigma_n)} = 0.$$

Moreover, if $\sup_n \|\mu_n\|_{\mathcal{M}} < +\infty$, *then*

$$\sup_n \|u_n\|_{L^\infty(\sigma_n)} < +\infty \quad \textit{and} \quad \mu_n - \bar{\mu}_n \rightharpoonup 0$$

where $(\bar{\mu}_n)$ *is the equivalent sequence defined in Lemma 7.*

Proof. Step 1. Let $x = (i_1/n, i_2/n) \in \text{Support}(\sigma_n)$. Using Lemmas 5 and 7 we obtain

$$\begin{aligned}
\|u_n(x) - \bar{u}_n(x)\| &\leq \sum_{k=1}^{2} \left\| u_n((x \cdot e_k)e_k) - u_n(0) - \int_0^{x \cdot e_k} \partial_k \bar{u}_n(t e_k)\, dt \right\| \\
&= \sum_{k=1}^{2} \left\| u_n\left(\frac{i_k}{n} e_k\right) - u_n(0) - \int_0^{i_k/n} \partial_k \bar{u}_n(t e_k)\, dt \right\| \\
&\leq \sum_{k=1}^{2} \sum_{q=0}^{i_k-1} \int_{q/n}^{q/n+1/n} \left\| \partial^+_{n,k} u_n\left(\frac{q}{n} e_k\right) - \partial_k \bar{u}_n(t e_k) \right\| dt \\
&= \sum_{k=1}^{2} \sum_{q=0}^{i_k-1} \int_{q/n}^{q/n+1/n} \left\| \partial_k \bar{u}_n\left(\left(\frac{q}{n} + \frac{1}{2n}\right) e_k\right) - \partial_k \bar{u}_n(t e_k) \right\| dt \\
&\leq \sum_{k=1}^{2} \sum_{q=0}^{i_k-1} \int_{q/n}^{q/n+1/n} \left| \left(\frac{q}{n} + \frac{1}{2n}\right) + t \right|^{1/2} \left(\int \|\partial^-_{n,k} v^+_{n,k}\|^2 \, d\sigma^2_{n,k} \right)^{1/2} dt \\
&\leq \sqrt{\frac{1}{2n}} \sum_{k=1}^{2} \left(\int \|\partial^-_{n,k} v^+_{n,k}\|^2 \, d\sigma^2_{n,k} \right)^{1/2};
\end{aligned}$$

then the first claim of Lemma 9 holds because the sequence $(E^{(\text{bend})}_n(\mu_n))$ is bounded and (H$_3$) is assumed.

Step 2. Let x be any node of Support(σ_n). Using Lemma 5 we obtain

$$\begin{aligned}\|u_n(x)\| &\le \frac{1}{\sigma_n(K)}\int \|u_n(y)\|\sigma_n(dy) + \frac{1}{\sigma_n(K)}\int \|u_n(x)-u_n(y)\|\sigma_n(dy)\\ &\le \|\mu_n\|_{\mathcal{M}} + \frac{2}{\sigma_n(K)}\int \|x-y\|\sigma_n(dy)\\ &\le \|\mu_n\|_{\mathcal{M}} + 2\sqrt{2}.\end{aligned}$$

It is assumed that the sequence $(\|\mu_n\|_{\mathcal{M}})$ is bounded; then the second claim of Lemma 9 holds.

Step 3. Let us set $\Omega_x := \{y \in K : \max_k |(y-x)\cdot e_k| < 1/(2n)\}$, and let $|\Omega_x|$ be the area of Ω_x. Observe that for any node $x \in$ Support(σ_n)

$$|\Omega_x| = \begin{cases} 1/(n^2) & \text{if } x \text{ is an interior point of } K,\\ 1/(4n^2) & \text{if } x \text{ is an extreme point of } K,\\ 1/(2n^2) & \text{otherwise.}\end{cases}$$

Let $\varphi \in C(K)^2$ be a test function. Since $\varphi \cdot \bar{u}_n$ is continuous on K one has

$$|\Omega_x|\varphi(y_x)\cdot\bar{u}_n(y_x) = \int_{\Omega_x} \varphi(y)\cdot\bar{u}_n(y)\,dy$$

for some $y_x \in \Omega_x$. As a consequence $\langle \bar{u}_n - u_n, \varphi\rangle = A_n + B_n + C_n + D_n$ with

$$\begin{aligned} A_n &= \sum_{x\in\text{Support}(\sigma_n)} |\Omega_x|\varphi(y_x)\cdot(\bar{u}_n(y_x) - \bar{u}_n(x)),\\ B_n &= \sum_{x\in\text{Support}(\sigma_n)} |\Omega_x|\varphi(y_x)\cdot(\bar{u}_n(x) - u_n(x)),\\ C_n &= \sum_{x\in\text{Support}(\sigma_n)} |\Omega_x|(\varphi(y_x) - \varphi(x))\cdot u_n(x),\\ D_n &= \sum_{x\in\text{Support}(\sigma_n)} \left(|\Omega_x| - \frac{1}{n^2}\right)\varphi(x)\cdot u_n(x).\end{aligned}$$

By Lemma 8, the sequence $(\bar{u}_n)$ is uniformly equicontinuous on K; therefore, $\lim_n A_n = 0$. By Lemma 9, $\lim_n \|\bar{u}_n - u_n\|_{L^\infty(\sigma_n)} = 0$; thus, $\lim_n B_n = 0$. The test function φ is uniformly continuous on K, and it is assumed that the sequence $\left(\int \|u_n\|\,d\sigma_n\right)$ is bounded; thus, $\lim_n C_n = 0$. Observe that

$$|D_n| \le \frac{2n+1}{n^2}\|\varphi\|_{L^\infty(\sigma_n)}\|u_n\|_{L^\infty(\sigma_n)};$$

then by Lemma 9, $\lim_n D_n = 0$. □

Lemma 10 (convergence of the equivalent sequence). *We assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold and* $\mu_n \rightharpoonup \mu$. *Then the limit measure* μ *is of the form* $\mu(dx) = u(x)\,dx$ *where* u *is* $C^1(K)$ *regular with distributional second partial derivatives in* $(L^2(\Omega))^2$ *and* $\partial_1\partial_2 u = 0$ *in* Ω. *Moreover, for all* $k \in \{1, 2\}$

$$\begin{aligned} \bar{u}_n &\to u && \textit{with respect to the uniform norm on } K, \\ \partial_k \bar{u}_n &\to \partial_k u && \textit{with respect to the uniform norm on } K, \\ \partial_k^2 \bar{u}_n &\rightharpoonup \partial_k^2 u && \textit{with respect to the weak topology of } (L^2(\Omega))^2. \end{aligned}$$

Proof. Since $\mu_n \rightharpoonup \mu$, the Banach–Steinhaus theorem implies $\sup_n \|\mu_n\|_{\mathcal{M}} < +\infty$. Using Lemma 9, we obtain

$$\bar{u}_n(x)dx \rightharpoonup \mu(dx).$$

Lemmas 7 and 8 imply that $(\bar{u}_n)$ and $(\partial_k \bar{u}_n)$ are uniformly equicontinuous on K. By the Ascoli theorem, $\bar{u}_n \to u$ and $\partial_k \bar{u}_n \to \partial_k u$ with respect to the uniform norm on K for some $u \in (C^1(K))^2$. As a consequence $\partial_k^2 \bar{u}_n \rightharpoonup \partial_k^2 u$ in the sense of distributions on Ω. Since by Lemma 7 the sequence $(\partial_k^2 \bar{u}_n)$ is bounded with respect to the $(L^2(\Omega))^2$ norm, the above convergence holds with respect to the weak topology of $(L^2(\Omega))^2$. □

Lemma 11 (lower-bound inequalities). *We assume that* (H$_1$), (H$_2$), (H$_3$), *and* (H$_4$) *hold and* $\mu_n \rightharpoonup \mu$. *Then the measure* $\mu(dx) = u(x)\,dx$ *is such that*

$$\begin{aligned} E^{(\text{ext})}(\mu) &= 0, \\ E^{(\text{shear})}(\mu) &\le \liminf_n E_n^{(\text{shear})}(\mu_n), \\ E^{(\text{bend})}(\mu) &\le \liminf_n E_n^{(\text{bend})}(\mu_n). \end{aligned}$$

If moreover $\sup_n E_n^{(\Sigma)}(\mu_n) < +\infty$, *then* $E^{(\Sigma)}(\mu) = 0$.

Proof. Let $(\bar{\mu}_n)$ be the equivalent sequence associated with (μ_n).

Step 1. Lemma 7 implies that $\|\partial_k \bar{u}_n(x + (1/(2n))e_k)\| = 1$ for every node $x \in \text{Support}(\sigma_{n,k}^+)$, and Lemma 10 asserts that $\partial_k \bar{u}_n \to \partial_k u$ with respect to the uniform norm on K. Hence, $\|\partial_k u(x)\| = 1$ for every $x \in K$, which implies $E^{(\text{ext})}(\mu) = 0$. As a consequence μ is admissible for the macromodel and

$$E^{(\text{shear})}(\mu) = \int_\Omega g(\partial_1 u(x) \wedge \partial_2 u(x))\,dx, \qquad E^{(\text{bend})}(\mu) = \sum_{k=1}^{2} \int_\Omega \frac{\kappa_k}{2} \|\partial_k^2 u(x)\|^2\,dx.$$

Step 2. Since the four functions $g^{(s,s')} : [-1, 1] \to [0, +\infty]$ are lower semicontinuous, there exist four sequences of nonnegative functions $(g_p^{(s,s')})$ in $C[0, 1]$ such that

$$g_{p+1}^{(s,s')} \geq g_p \quad \text{and} \quad \sup_p g_p^{(s,s')} = g^{(s,s')}.$$

Using Lemmas 7 and 10 again, the fact that $\sigma_n^{(s,s')}(dx) \rightharpoonup dx$, and the monotone convergence theorem, we obtain

$$\begin{aligned}
\liminf_n E_n^{(\text{shear})}(\mu_n) &:= \liminf_n \sum_{s,s'} \int g^{(s,s')}(v_{n,1}^+(x) \wedge v_{n,2}^+(x))\sigma_n^{(s,s')}(dx) \\
&\geq \sum_{s,s'} \liminf_n \int g^{(s,s')}\Big(\partial_1 \bar{u}_n\Big(x + \frac{s}{2n}e_1\Big) \wedge \partial_2 \bar{u}_n\Big(x + \frac{s'}{2n}e_2\Big)\Big)\sigma_n^{(s,s')}(dx) \\
&\geq \sup_p \sum_{s,s'} \liminf_n \int g_p^{(s,s')}\Big(\partial_1 \bar{u}_n\Big(x + \frac{1}{2n}e_1\Big) \wedge \partial_2 \bar{u}_n\Big(x + \frac{1}{2n}e_2\Big)\Big)\sigma_n^{(s,s')}(dx) \\
&\geq \sup_p \sum_{s,s'} \int_\Omega g_p^{(s,s')}(\partial_1 u(x) \wedge \partial_2 u(x))\, dx \\
&\geq \int_\Omega g(\partial_1 u(x) \wedge \partial_2 u(x))\, dx = E^{(\text{shear})}(\mu).
\end{aligned}$$

Step 3. Using Lemmas 7 and 10 and remembering that the $(L^2(\Omega))^2$ norm is weak lower semicontinuous we obtain

$$\liminf_n E_n^{(\text{bend})}(\mu_n) = \liminf_n \sum_{k=1}^{2} \int_\Omega \frac{\kappa_k}{2} \|\partial_k^2 \bar{u}_n(x)\|^2\, dx \geq E^{(\text{bend})}(\mu).$$

Step 4. The condition $\sup_n E_n^{(\Sigma)}(\mu_n) < +\infty$ says that $u_n(x) = x$ for every $x \in \Sigma \cap \text{Support}(\sigma_n)$. Using Lemma 9 (i.e., $\lim_n \|u_n - \bar{u}_n\|_{L^\infty(\sigma_n)} = 0$) we obtain

$$\limsup_n \{\|\bar{u}_n(x) - x\| : x \in \Sigma \cap \text{Support}(\sigma_n)\} = 0.$$

Using Lemma 10 we obtain

$$\limsup_n \{\|u(x) - x\| : x \in \Sigma \cap \text{Support}(\sigma_n)\} = 0.$$

Since $\Sigma = O \cap \partial\Omega$ where O is an open subset of $\mathbb{R}^2$ we deduce that $u(x) = x$ at any point of Σ. Hence, $E^{(\Sigma)}(\mu) = 0$. □

5.4. *Upper-bound inequality.* We recall that the upper-bound inequality for the sequence (E_n) means that for each $\mu \in (\mathcal{M}(K))^2$, there exists a sequence (μ_n) in $(\mathcal{M}(K))^2$ such that

$$\mu_n \rightharpoonup \mu \quad \text{and} \quad \limsup_{n\to\infty} E_n(\mu_n) \leq E(\mu).$$

This is easily obtained by means of Lemma 13 below.

Lemma 12. *We assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold and* $\mu(dx) = u(x)\,dx$ *is an admissible measure for the macromodel such that* $E(\mu) < +\infty$. *Then the placement function* u *is* $C^1(K)$ *regular with distributional second partial derivatives in* $(L^2(\Omega))^2$ *and for all* $x \in K$ *and all* $k \in \{1, 2\}$,

$$\|\partial_k u(x)\| = 1 \quad \textit{and} \quad u(x) + u(0) = \sum_{k=1}^{2} u((x \cdot e_k)e_k).$$

As a consequence

$$\begin{aligned} E^{(\mathrm{ext})}(\mu) &= 0, \\ E^{(\mathrm{shear})}(\mu) &= \int_\Omega g(\partial_1 u(x) \wedge \partial_2 u(x))\,dx, \\ E^{(\mathrm{bend})}(\mu) &= \int_\Omega \sum_{k=1}^{2} \frac{\kappa_k}{2} \|\partial_k^2 u(x)\|^2\,dx. \end{aligned}$$

Proof. Step 1 ($H^2(\Omega)$ regularity). Let $H^{-1}(\Omega)$ denote the dual space of the usual Sobolev space $H_0^1(\Omega)$. Since $E^{(\mathrm{ext})}(\mu) < +\infty$ and (H$_1$) is assumed, one has $\|\partial_k u\| = \rho_k(\mu) = 1$ a.e. in Ω and $\partial_k \partial_k u = \partial_k v_k(\mu) \in (L^2(\Omega))^2$ for every $k \in \{1, 2\}$. As a consequence

$$\begin{aligned} \partial_1 \partial_2 u &\in (H^{-1}(\Omega))^2, \\ \partial_1(\partial_1 \partial_2 u) = \partial_2(\partial_1^2 u) &= \partial_2(\partial_1 v_1(\mu)) \in (H^{-1}(\Omega))^2, \\ \partial_2(\partial_1 \partial_2 u) = \partial_1(\partial_2^2 u) &= \partial_1(\partial_2 v_2(\mu)) \in (H^{-1}(\Omega))^2. \end{aligned}$$

A well known result by Necas [Carroll et al. 1966] asserts that, if a distribution and its first distributional derivatives are $H^{-1}(\Omega)$ regular and if Ω is bounded with Lipschitz boundary, then the distribution is $L^2(\Omega)$ regular. Hence, $\partial_1 \partial_2 u \in (L^2(\Omega))^2$.

Step 2 ($\partial_1 \partial_2 u = 0$ a.e. in Ω). Thanks to Step 1 one has

$$\begin{aligned} 2(\partial_1 \partial_2 u) \cdot (\partial_1 u + \partial_2 u) &= \partial_2 \|\partial_1 u\|^2 + \partial_1 \|\partial_2 u\|^2 = \partial_2 \rho_1(\mu)^2 + \partial_1 \rho_2(\mu)^2 = 0, \\ 2(\partial_1 \partial_2 u) \cdot (\partial_2 u - \partial_1 u) &= \partial_1 \|\partial_2 u\|^2 - \partial_2 \|\partial_1 u\|^2 = \partial_1 \rho_2(\mu)^2 - \partial_2 \rho_1(\mu)^2 = 0. \end{aligned}$$

Since $E^{(\mathrm{shear})}(\mu) < +\infty$ and (H$_3$) is assumed, one has $\partial_1 u \wedge \partial_2 u > 0$ a.e. in Ω, which implies that $(\partial_1 u + \partial_2 u, \partial_2 u - \partial_1 u)$ is a direct orthogonal basis of $\mathbb{R}^2$ a.e. in Ω; therefore, $\partial_1 \partial_2 u = 0$ a.e. in Ω.

Step 3 ($C^1(K)$ regularity). Let $k \in \{1, 2\}$. By Step 1, Step 2, and the usual Sobolev embedding theorem one has

$$\partial_k u(x) = \bar{v}_k(x \cdot e_k) \quad \text{for a.e. } x \in \Omega$$

for some $\bar{v}_k \in (C[0, 1])^2$ with distributional derivative in $(L^2(0, 1))^2$. As a consequence $u \in (C^1(K))^2$ and the proof is easily completed. □

Lemma 13 (approximating sequence). *We assume that* (H$_1$), (H$_2$), (H$_3$), *and* (H$_4$) *hold and* $\mu(dx) = u(x)\,dx$ *is an admissible measure for the macromodel such that* $E(\mu) < +\infty$. *Then there exists a sequence* $\mu_n(dx) = u_n(x)\sigma_n(dx)$ *such that*

$$\begin{aligned}
E_n^{(\Sigma)}(\mu_n) &= E^{(\sigma)}(\mu) = 0 && \textit{for every integer } n,\\
E_n^{(\mathrm{ext})}(\mu_n) &= E^{(\mathrm{ext})}(\mu) = 0 && \textit{for every integer } n,\\
\mu_n &\rightharpoonup \mu,\\
\lim_n E_n^{(\mathrm{shear})}(\mu_n) &= E^{(\mathrm{shear})}(\mu),\\
E_n^{(\mathrm{bend})}(\mu_n) &\le E^{(\mathrm{bend})}(\mu) && \textit{for every integer } n.
\end{aligned}$$

Proof. Step 1 (construction of the sequence (μ_n)). It is assumed that n is large enough so that at least two nodes are contained in $\Sigma := (a, b) \times \{0\}$. As a consequence of Lemma 12, one has $\partial_1 u(te_1) = e_1$ for every $t \in [a, b]$. It is then possible to define ν_n-a.e. two functions $u_{n,k} : [0, 1] \to \mathbb{R}^2$ by setting

$$\begin{aligned}
u_{n,1}\left(\frac{i}{n}\right) &= \frac{i}{n}e_1 && \text{if } a < \frac{i}{n} < b,\\
n\left(u_{n,1}\left(\frac{i+1}{n}\right) - u_{n,1}\left(\frac{i}{n}\right)\right) &= \partial_1 u\left(\left(\frac{i}{n} + \frac{1}{2n}\right)e_1\right) && \text{for every } i \in \{0, \dots, n-1\},
\end{aligned}$$

and

$$\begin{aligned}
u_{n,2}(0) &= u_{n,1}(0)\\
n\left(u_{n,2}\left(\frac{j+1}{n}\right) - u_{n,2}\left(\frac{j}{n}\right)\right) &= \partial_2 u\left(\left(\frac{j}{n} + \frac{1}{2n}\right)e_2\right) && \text{for every } j \in \{0, \dots, n-1\}.
\end{aligned}$$

We finally define $\mu_n(dx) := u_n(x)\sigma_n(dx)$ by setting

$$u_n(x) := -u_{n,1}(0) + \sum_{k=1}^{2} u_{n,k}(x \cdot e_k)$$

for every x in the support of σ_n. It follows from the definition of μ_n that $u_n(x) = x$ for every node $x \in \Sigma$, and then $E_n^{(\Sigma)}(\mu_n) = 0$. We have also

$$\partial_{n,k}^+ u_n(x) = n\left(u_{n,k}\left(x \cdot e_k + \frac{1}{n}\right) - u_{n,k}(x \cdot e_k)\right) = \partial_k u\left(x + \frac{1}{2n}e_k\right).$$

Using Lemma 12, we obtain $\|\partial_{n,k}^+ u_n(x)\| = 1$ for every node x in the support of $\sigma_{n,k}^+$; then $E_n^{(\mathrm{ext})}(\mu_n) = 0$.

Step 2 (weak convergence of the sequence (μ_n)). Let us denote

$$\varepsilon_{n,k} := \frac{1}{n}\sum_{q=0}^{n-1}\left\|\partial_k u\left(\left(\frac{q}{n}+\frac{1}{2n}\right)e_1\right) - n\int_{q/n}^{(q+1)/n} \partial_k u(te_1)\,dt\right\|.$$

By Lemma 12, the placement function u is $C^1(K)$ regular and then the sequences $(\varepsilon_{n,k})$ converge to 0 as n tends to ∞. Since $u_{n,1}(x\cdot e_1) - u(x) = 0$ for some node $x \in \Sigma$ and $u_{n,1}(0) = u_{n,2}(0)$, one has

$$\left\|u_{n,1}\left(\frac{i}{n}\right) - u\left(\frac{i}{n}e_1\right)\right\| \le \varepsilon_{n,1} \qquad \text{for every } i \in \{0,\dots,n-1\},$$
$$\left\|u_{n,2}\left(\frac{j}{n}\right) - u\left(\frac{j}{n}e_2\right)\right\| \le \varepsilon_{n,1}+\varepsilon_{n,2} \quad \text{for every } j \in \{0,\dots,n-1\}.$$

Let φ be a test function in $C(K)$. Using Lemma 12 and the definition of u_n, a direct computation gives us

$$\begin{aligned}\langle \mu_n - \mu, \varphi\rangle = {} & \left(\int \varphi(x)\sigma_n(dx)\right)(u(0) - u_{n,1}(0)) \\ & + \sum_{k=1}^{2}\int \varphi(x)(u_{n,k}(x\cdot e_k) - u((x\cdot e_k)e_k))\sigma_n(dx) \\ & + \left(\int_\Omega \varphi(x)\,dx - \int \varphi(x)\sigma_n(dx)\right)u(0) \\ & + \sum_{k=1}^{2}\left(\int \varphi(x)u((x\cdot e_k)e_k)\sigma_n(dx) - \int_\Omega \varphi(x)u((x\cdot e_k)e_k)\,dx\right);\end{aligned}$$

then

$$\begin{aligned}|\langle \mu_n - \mu, \varphi\rangle| \le {} & \left|\int \varphi(x)\sigma_n(dx)\right|(3\varepsilon_{n,1}+\varepsilon_{n,2}) + \left|\int_\Omega \varphi(x)\,dx - \int \varphi(x)\sigma_n(dx)\right|\|u(0)\| \\ & + \sum_{k=1}^{2}\left\|\int \varphi(x)u((x\cdot e_k)e_k)\sigma_n(dx) - \int_\Omega \varphi(x)u((x\cdot e_k)e_k)\,dx\right\|.\end{aligned}$$

Since the measure σ_n weakly converges to the Lebesgue measure on K and the function $x \to \varphi(x)u((x\cdot e_k)e_k)$ is continuous on K, we obtain

$$\lim_n |\langle \mu_n - \mu, \varphi\rangle| = 0.$$

Step 3 (convergence of the sequence $E_n^{(\text{shear})}(\mu_n)$). Let x be a node in the support of $\sigma_n^{s,s'}$. Using the definition of u_n we obtain

$$\partial_{n,1}^{s} u_n(x) \wedge \partial_{n,2}^{s'} u_n(x) = \partial_1 u\left(x + \frac{s}{2n}e_1\right) \wedge \partial_2 u\left(x + \frac{s'}{2n}e_2\right).$$

By Lemma 12, the function u is $C^1(K)$ regular. Assumption (H$_2$) and the fact that $E(\mu) < +\infty$ imply that $u(x) \in \{g^{(s,s')} < +\infty\}$ for every $x \in K$ and the function $g^{(s,s')}$ restricted to the closed set $\{g^{(s,s')} < +\infty\}$ is continuous. As a consequence, the function

$$x \to g^{(s,s')}(\partial_1 u(x) \wedge \partial_2 u(x))$$

is uniformly continuous on K, which implies that

$$\lim_n \int g^{(s,s')}(\partial^s_{n,1} u_n(x) \wedge \partial^{s'}_{n,2} u_n(x)) \sigma^{(s,s')}_n(dx) = \int_\Omega g^{(s,s')}(\partial_1 u(x) \wedge \partial_2 u(x))\, dx$$

so that $\lim_n E^{(\text{shear})}_n(\mu_n) = E^{(\text{shear})}(\mu)$.

Step 4 (upper-bound inequality of the sequence $E^{(\text{bend})}_n(\mu_n)$). Let x be a node in the support of $\sigma^2_{n,k}$. Using Lemma 12 and the definition of u_n we obtain

$$\partial^2_{n,k} u_n(x) = n\left(\partial_k u\left(\left(x \cdot e_k + \frac{1}{2n}\right)e_k\right) - \partial_k u\left(\left(x \cdot e_k - \frac{1}{2n}\right)e_k\right)\right);$$

then Jensen inequality gives us

$$\|\partial^2_{n,k} u_n(x)\|^2 \le n \int_{x \cdot e_k - 1/(2n)}^{x \cdot e_k + 1/(2n)} \|\partial^2_k u(te_k)\|^2\, dt.$$

Integrating with respect to the measure $\sigma^2_{n,k}$ we obtain

$$\int \|\partial^2_{n,k} u_n(x)\|^2 \sigma^2_{n,k}(dx) \le \frac{n-1}{n} \int_\Omega \|\partial^2_k u(x)\|^2\, dx,$$

and therefore, $E^{(\text{bend})}_n(\mu_n) \le E^{(\text{bend})}(\mu)$. □

6. Conclusions

In the present paper we proved the Γ-convergence of a discrete lattice of rigid bars and rotational springs to a 2D generalized continuum model, along with a relative compactness property for the sequence of discrete energy functionals. The result is proven taking into account geometrical nonlinearities.

The main result can be generalized in various ways, the most important of which is probably the extension of the Γ-convergence argument to less restrictive hypotheses on the function f_k, in particular allowing extensional deformation, i.e., changes in the distances of adjacent nodes. The assumptions on the functions $g^{(s,s')}$ can also be relaxed in future investigations.

Of course, future mathematical studies have to take into account also the novelties of mechanical nature coming from experimental and numerical results. For instance, worth mentioning are the recent results on the peculiar 3D (out-of-the-plane) behavior of pantographic structures (see, e.g., [Steigmann and dell'Isola

2015; Misra et al. 2018; Giorgio et al. 2017; Barchiesi et al. 2018a]) and the investigation of generalized pantographic sheets with nonstraight or nonorthogonal fibers [Turco et al. 2017b; Giorgio et al. 2016]. These findings will probably require the development of new techniques in order to obtain rigorous homogenization results. Finally, the possibility of oscillations at the lattice level would require more complex homogenization formulas where the extensional, bending, and shear deformation energies may not be uncoupled anymore.

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