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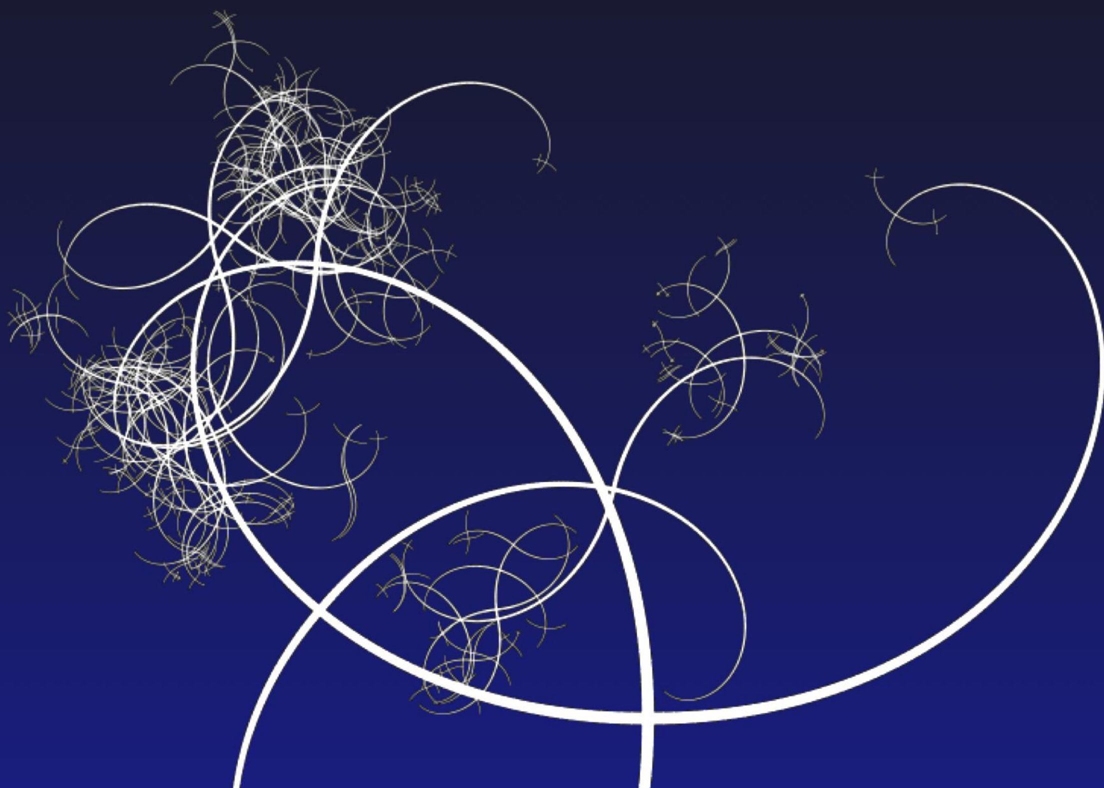

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EFFECTIVE COMPUTATION OF SO(3) AND O(3) LINEAR REPRESENTATION SYMMETRY CLASSES

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We propose a general algorithm to compute all the symmetry classes of any SO(3) or O(3) linear representation. This method relies on a binary operator between sets of conjugacy classes of closed subgroups, called the *clips*. We compute explicit tables for this operation, which allows us to definitively solve the problem.

1. Introduction

The problem of finding the *symmetry classes* (also called *isotropy classes*) of a given Lie group linear representation is a difficult task in general, even for a compact group, where their number is known to be finite [Mostow 1957; Mann 1962].

It is only in 1996 that Forte and Vianello [1996] were able to clearly define the *symmetry classes* of tensor spaces. Such tensor spaces, with natural O(3) and SO(3) representations, appear in continuum mechanics via linear constitutive laws. Thanks to this clarification, Forte and Vianello obtained for the first time the eight symmetry classes of the SO(3) reducible representation on the space of *elasticity tensors*. One goal was to clarify and correct all the attempts already made to model the notion of *symmetry* in continuum mechanics, as initiated by Curie [1894], but strongly influenced by crystallography. The results were contradictory — some authors announced nine different elasticity anisotropies [Love 1944] while others

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announced ten of them [Schouten 1951; Jagodzinski 1955; Fedorov 1968; Gurtin 1973]. Following Forte and Vianello, similar results were obtained in piezoelectricity [Weller 2004], photoelasticity [Forte and Vianello 1997], and flexoelasticity [Le Quang and He 2011].

Besides these results on symmetry classes in continuum mechanics, the subject has also been active in the mathematical community, especially due to its importance in bifurcation theory. For instance, Michel [1980] obtained the isotropy classes for *irreducible* SO(3) representations. These results were confirmed by Ihrig and Golubitsky [1984] and completed by the symmetry classes for O(3). Later, they were corrected by Chossat et al. [1990]. Thereafter, Chossat and Guyard [1994] calculated the symmetry classes of a direct sum of *two irreducible representations of* SO(3).

In this paper, we propose an algorithm — already mentioned in [Olive and Auffray 2013; 2014; Olive 2014] — to obtain the finite set of symmetry classes for any O(3) or SO(3) representation. Such an algorithm uses a binary operation defined over the set of conjugacy classes of a given group G and that we decided to call the *clips operation*. This operation was almost formulated in [Chossat and Guyard 1994], but with no specific name, and only computed for SO(3) closed subgroups. As mentioned in [Chossat and Guyard 1996], the clips operation allows one to compute the set of *symmetry classes* $\mathfrak{I}(V)$ of a direct sum $V = V_1 \oplus V_2$ of linear representations of a group G, if we know the symmetry classes for each individual representation $\mathfrak{I}(V_1)$ and $\mathfrak{I}(V_2)$.

We compute clips tables for all conjugacy classes of closed subgroups of O(3) and SO(3). The clips table for SO(3) was obtained in [Chossat and Guyard 1994], but we give here more detail on the proof and extend the calculation to conjugacy classes of closed O(3) subgroups. These results allow us to obtain, in a finite-step process, the set of symmetry classes for any *reducible* O(3) or SO(3) representation, so that for instance we directly obtain the sixteen symmetry classes of *piezoelectric tensors*.

The subject being particularly important for applications to tensorial properties of any order, the necessity of convincing correctness of the clips tables requires a complete and correct full article with sound proofs. The present article is therefore intended to be a final solution to the theoretical problem and provide the effective calculation of symmetry classes. Of course, we try to show the direct interest in the mechanics of continuous media (for this, references to [Olive and Auffray 2013; 2014] are important), but we have no other choice than to insist on the mathematical formulation of the problem.

The paper is organized as follow. In Section 2, which is close to [Chossat and Guyard 1996, §2.1], the theory of *clips* is introduced for a general group G and applied in the context of symmetry classes where it is shown that isotropy classes of a direct sum correspond to the clips of their respective isotropy classes. In Section 3, we recall classical results on the classification of closed subgroups of SO(3) and O(3) up to conjugacy. Models for irreducible representations of O(3) and SO(3) and their symmetry classes are recalled in Section 4. We then provide in the subsection beginning on page 214 some applications to tensorial mechanical properties, as the nonclassical example of Cosserat elasticity. The clips tables for SO(3) and O(3) are presented in Section 5. The details and proofs of how to obtain these tables are provided in Appendices A and B.

2. A general theory of clips

Given a group G and a subgroup H of G, the conjugacy class of H

$$[H] := \{gHg^{-1} : g \in G\}$$

is a subset of $\mathcal{P}(G)$. We define $\mathrm{Conj}(G)$ to be the set of all conjugacy classes of a given group G:

$$\mathrm{Conj}(G) := \{[H] : H \subset G\}.$$

Recall that, on $\mathrm{Conj}(G)$, there is a preorder relation induced by inclusion. It is defined as

$$[H_1] \preceq [H_2] \quad \text{if } H_1 \text{ is conjugate to a subgroup of } H_2.$$

When restricted to the *closed subgroups* of a topological compact group, this preorder relation becomes a *partial order* [Bredon 1972, Proposition 1.9] and defines the *poset* (partially ordered set) of conjugacy classes of *closed subgroups* of G.

We now define a binary operation called the *clips operation* on the set $\mathrm{Conj}(G)$.

Definition 2.1. Given two conjugacy classes $[H_1]$ and $[H_2]$ of a group G, we define their *clips* as the subset of conjugacy classes

$$[H_1] \circledcirc [H_2] := \{[H_1 \cap gH_2g^{-1}] : g \in G\}.$$

This definition immediately extends to two families (finite or infinite) $\mathcal{F}_1$ and $\mathcal{F}_2$ of conjugacy classes:

$$\mathcal{F}_1 \circledcirc \mathcal{F}_2 := \bigcup_{[H_i] \in \mathcal{F}_i} [H_1] \circledcirc [H_2].$$

Remark 2.2. The clips operation was already introduced, with no specific name, in [Chossat and Guyard 1994], the notation being $\boldsymbol{P}(H_1, H_2)$. In this article, the author only focus on the SO(3) case, with no meaning to deal with a general theory.

This clips operation thus defined a binary operation on the set $\mathcal{P}(\mathrm{Conj}(G))$, which is *associative* and *commutative*. We have moreover

$$[\mathbb{1}] \circledcirc [H] = \{[\mathbb{1}]\} \quad \text{and} \quad [G] \circledcirc [H] = \{[H]\},$$

for every conjugacy class $[H]$, where $\mathbb{1} := \{e\}$ and e is the identity element of G.

Consider now a linear representation (V, ρ) of the group G. Given $\boldsymbol{v} \in V$, its *isotropy group* (or *symmetry group*) is defined as

$$G_{\boldsymbol{v}} := \{g \in G : g \cdot \boldsymbol{v} = \boldsymbol{v}\}$$

and its *isotropy class* is the conjugacy class $[G_{\boldsymbol{v}}]$ of its isotropy group. The *isotropy classes* (or *orbit types*) of the representation V is the family of all isotropy classes of vectors $\boldsymbol{v}$ in V:

$$\mathfrak{I}(V) := \{[G_{\boldsymbol{v}}] : \boldsymbol{v} \in V\}.$$

The central observation is that the isotropy classes of a direct sum of representations is obtained by the clips of their respective isotropy classes.

Lemma 2.3. *Let V_1 and V_2 be two linear representations of G. Then*

$$\mathfrak{I}(V_1 \oplus V_2) = \mathfrak{I}(V_1) \circledcirc \mathfrak{I}(V_2).$$

Proof. Let $[G_{\boldsymbol{v}}]$ be some isotropy class for $\mathfrak{I}(V_1 \oplus V_2)$ and write $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ where $\boldsymbol{v}_i \in V_i$. Note first that $G_{\boldsymbol{v}_1} \cap G_{\boldsymbol{v}_2} \subset G_{\boldsymbol{v}}$. Conversely given $g \in G_{\boldsymbol{v}}$, we get

$$g \cdot \boldsymbol{v} = g \cdot \boldsymbol{v}_1 + g \cdot \boldsymbol{v}_2 = \boldsymbol{v}, \quad g \cdot \boldsymbol{v}_i \in V_i,$$

and thus $g \cdot \boldsymbol{v}_i = \boldsymbol{v}_i$. This shows that $G_{\boldsymbol{v}} = G_{\boldsymbol{v}_1} \cap G_{\boldsymbol{v}_2}$ and therefore that

$$\mathfrak{I}(V_1 \oplus V_2) \subset \mathfrak{I}(V_1) \circledcirc \mathfrak{I}(V_2).$$

Conversely, let $[H] = [H_1 \cap g H_2 g^{-1}]$ in $\mathfrak{I}(V_1) \circledcirc \mathfrak{I}(V_2)$ where $H_i = G_{\boldsymbol{v}_i}$ for some vectors $\boldsymbol{v}_i \in V_i$. Then, if we set

$$\boldsymbol{v} = \boldsymbol{v}_1 + g \cdot \boldsymbol{v}_2,$$

we have $G_{\boldsymbol{v}} = H_1 \cap g H_2 g^{-1}$, as before, which shows that

$$[H_1 \cap g H_2 g^{-1}] \in \mathfrak{I}(V_1 \oplus V_2)$$

and achieves the proof. □

Using this lemma, we deduce a general algorithm to obtain the isotropy classes $\mathfrak{I}(V)$ of a finite-dimensional representation of a reductive algebraic group G, provided we know

(1) a decomposition $V = \bigoplus_i W_i$ into irreducible representations W_i,

(2) the isotropy classes $\mathfrak{I}(W_i)$ for the *irreducible representations* W_i, and

(3) the tables of *clips operations* $[H_1] \circledcirc [H_2]$ between conjugacy classes of closed subgroups $[H_i]$ of G.

In the sequel of this paper, we will successfully apply this program to the linear representations of SO(3) and O(3).

3. Closed subgroups of O(3)

Every closed subgroup of SO(3) is conjugate to one of [Golubitsky et al. 1988]

$$\mathrm{SO}(3), \quad \mathrm{O}(2), \quad \mathrm{SO}(2), \quad \mathbb{D}_n\ (n \geq 2), \quad \mathbb{Z}_n\ (n \geq 2), \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{I}, \quad \text{or} \quad \mathbb{1}$$

where:

- O(2) is the subgroup generated by all the rotations around the z-axis and the order 2 rotation $r : (x, y, z) \mapsto (x, -y, -z)$ around the x-axis.
- SO(2) is the subgroup of all the rotations around the z-axis.
- $\mathbb{Z}_n$ is the unique cyclic subgroup of order n of SO(2) ($\mathbb{Z}_1 = \{I\}$).
- $\mathbb{D}_n$ is the *dihedral* group. It is generated by $\mathbb{Z}_n$ and $r : (x, y, z) \mapsto (x, -y, -z)$ ($\mathbb{D}_1 = \{I\}$).
- $\mathbb{T}$ is the *tetrahedral* group, the (orientation-preserving) symmetry group of the tetrahedron $\mathcal{T}_0$ defined in Figure 10. It has order 12.
- $\mathbb{O}$ is the *octahedral* group, the (orientation-preserving) symmetry group of the cube $\mathcal{C}_0$ defined in Figure 10. It has order 24.
- $\mathbb{I}$ is the *icosahedral* group, the (orientation-preserving) symmetry group of the dodecahedron $\mathcal{D}_0$ in Figure 11. It has order 60.
- $\mathbb{1}$ is the trivial subgroup, containing only the unit element.

The *poset* of conjugacy classes of closed subgroups of SO(3) is completely described by the inclusion of subgroups [Golubitsky et al. 1988]

$$\begin{aligned} &\mathbb{Z}_n \subset \mathbb{D}_n \subset \mathrm{O}(2) && (n \geq 2), \\ &\mathbb{Z}_n \subset \mathbb{Z}_m \quad \text{and} \quad \mathbb{D}_n \subset \mathbb{D}_m && (\text{if } n \text{ divides } m), \\ &\mathbb{Z}_n \subset \mathrm{SO}(2) \subset \mathrm{O}(2) && (n \geq 2), \end{aligned}$$

completed by $[\mathbb{Z}_2] \preceq [\mathbb{D}_n]$ ($n \geq 2$) and by the arrows in Figure 1 (note that an arrow between the classes $[H_1]$ and $[H_2]$ means that $[H_1] \preceq [H_2]$), taking account of the exceptional subgroups $\mathbb{O}$, $\mathbb{T}$, and $\mathbb{I}$.

Classification of O(3) closed subgroups is more involved [Ihrig and Golubitsky 1984; Sternberg 1994] and has been described using *three types of subgroups*. Given a closed subgroup Γ of O(3) this classification runs as follows:

Type I: A subgroup Γ is of type I if it is a subgroup of SO(3).

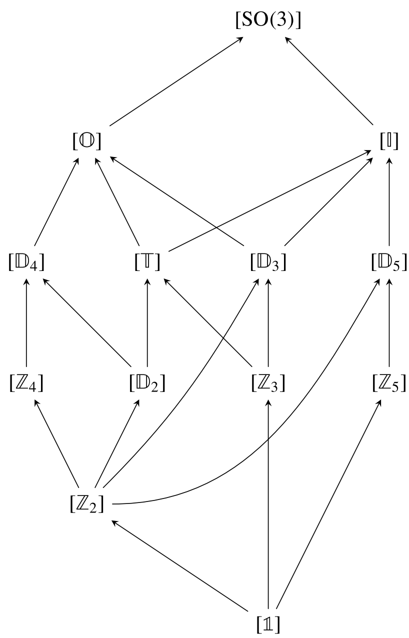

Figure 1. Exceptional conjugacy classes of closed SO(3) subgroups.

Type II: A subgroup Γ is of type II if $-I \in \Gamma$. In that case, Γ is generated by some subgroup K of SO(3) and $-I$.

Type III: A subgroup Γ is of type III if $-I \notin \Gamma$ and Γ is not a subgroup of SO(3).

The description of type III subgroups requires more details. We will denote by $\boldsymbol{Q}(\boldsymbol{v};\theta) \in \mathrm{SO}(3)$ the rotation around $\boldsymbol{v} \in \mathbb{R}^3$ with angle $\theta \in [0, 2\pi[$ and by $\sigma_{\boldsymbol{v}} \in \mathrm{O}(3)$ the reflection through the plane normal to $\boldsymbol{v}$. Finally, we fix an arbitrary orthonormal frame $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, and we introduce the following definitions:

- $\mathbb{Z}_2^-$ is the order-2 reflection group generated by $\sigma_{\boldsymbol{i}}$ (where $\mathbb{Z}_1^- = \{\mathbb{1}\}$).
- $\mathbb{Z}_{2n}^-$ ($n \geq 2$) is the group of order $2n$, generated by $\mathbb{Z}_n$ and $-\boldsymbol{Q}(\boldsymbol{k};\pi/n)$ (see (B-1)).
- $\mathbb{D}_{2n}^h$ ($n \geq 2$) is the group of order $4n$ generated by $\mathbb{D}_n$ and $-\boldsymbol{Q}(\boldsymbol{k};\pi/n)$ (see (B-3)).
- $\mathbb{D}_n^v$ ($n \geq 2$) is the group of order $2n$ generated by $\mathbb{Z}_n$ and $\sigma_{\boldsymbol{i}}$ (where $\mathbb{D}_1^v = \{\mathbb{1}\}$).
- $\mathrm{O}(2)^-$ is generated by SO(2) and $\sigma_{\boldsymbol{i}}$.

These planar subgroups are completed by the subgroup $\mathbb{O}^-$ which is of order 24 (see the subsection starting on page 233 and (B-5) for details).

The *poset* of conjugacy classes of closed subgroups of O(3) is given in Figure 2 for $\mathbb{O}^-$ subgroups and in Figure 3 for O(3) subgroups.

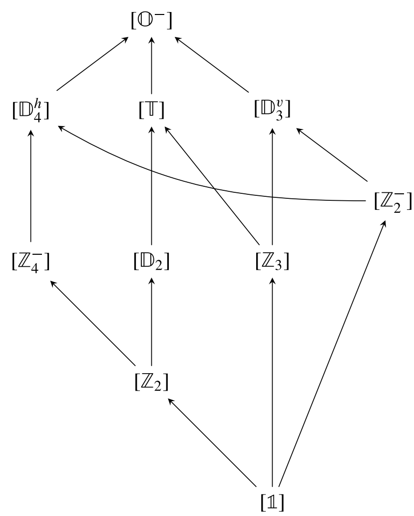

Figure 2. Poset of closed $\mathbb{O}^-$ subgroups.

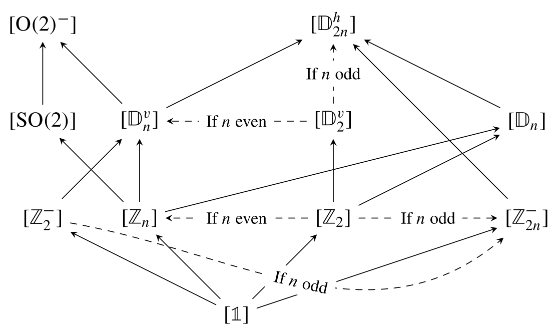

Figure 3. Poset of closed O(3) subgroups.

4. Symmetry classes for irreducible representations

Let $\mathcal{P}_n(\mathbb{R}^3)$ be the space of *homogeneous polynomials* of degree n on $\mathbb{R}^3$. We have two natural representations of O(3) on $\mathcal{P}_n(\mathbb{R}^3)$. The first one, denoted by ρ_n, is given by

$$[\rho_n(\mathrm{p})](\boldsymbol{x}) := \mathrm{p}(g^{-1}\boldsymbol{x}), \quad g \in G,\ \boldsymbol{x} \in \mathbb{R}^3,$$

whereas the second one, denoted by ρ_n^*, is given by

$$[\rho_n^*(\mathrm{p})](\boldsymbol{x}) := \det(g)\mathrm{p}(g^{-1}\boldsymbol{x}), \quad g \in G,\ \boldsymbol{x} \in \mathbb{R}^3.$$

Note that both of them induce the same representation ρ_n of SO(3).

Let $\mathcal{H}_n(\mathbb{R}^3) \subset \mathcal{P}_n(\mathbb{R}^3)$ be the subspace of *homogeneous harmonic polynomials* of degree n (polynomials with null Laplacian). It is a classical fact [Golubitsky et al. 1988] that $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ and $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$ $(n \geq 0)$ are irreducible O(3) representations, and each irreducible O(3) representation is isomorphic to one of them. Models for irreducible representations of SO(3) reduce to $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ $(n \geq 0)$.

Remark 4.1. Other classical models for O(3) and SO(3) irreducible representations, used in mechanics [Forte and Vianello 1996], are given by spaces of *harmonic tensors* (i.e., totally symmetric traceless tensors).

The isotropy classes for irreducible representations of SO(3) was first obtained by Michel [1980]. The same results were then obtained and completed in the O(3) case by Ihrig and Golubitsky [1984] and then by Chossat et al. [1990].

Theorem 4.2. *The isotropy classes for the* SO(3) *representation* $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ *are*

(1) $[\mathbb{1}]$ *for* $n \geq 3$,

(2) $[\mathbb{Z}_k]$ *for* $2 \leq k \leq n$ *if n is odd and* $2 \leq k \leq n/2$ *if n is even*,

(3) $[\mathbb{D}_k]$ *for* $2 \leq k \leq n$,

(4) $[\mathbb{T}]$ *for* $n = 3, 6, 7$ *or* $n \geq 9$,

(5) $[\mathbb{O}]$ *for* $n \neq 1, 2, 3, 5, 7, 11$,

(6) $[\mathbb{I}]$ *for* $n = 6, 10, 12, 15, 16, 18$ *or* $n \geq 20$ *and* $n \neq 23, 29$,

(7) $[\mathrm{SO}(2)]$ *for n odd*,

(8) $[\mathrm{O}(2)]$ *for n even*, *and*

(9) $[\mathrm{SO}(3)]$ *for any n*.

Remark 4.3. The list in Theorem 4.2 is similar to the list in [Michel 1980, Table A1; Chossat et al. 1990, Table A4]. In [Ihrig and Golubitsky 1984, Theorem 6.6] (for SO(3) irreducible representations)

- $[\mathbb{T}]$ is an isotropy class for $n = 6, 7$ and $n \geq 9$ and
- $[\mathbb{O}]$ is an isotropy class for $n \neq 1, 2, 5, 7, 11$.

Such lists are different from (4) and (5) in our Theorem 4.2. But according to [Ihrig and Golubitsky 1984, Proposition 3.7], $[\mathbb{T}]$ is a maximum isotropy class for $n = 3$. We have thus corrected this error in Theorem 4.2.

Theorem 4.4. *The isotropy classes for the* O(3) *representations* $(\mathcal{H}_n(\mathbb{R}^3), \rho_n)$ (*for n odd*) *and* $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$ (*for n even*) *are*

(1) $[\mathbb{1}]$ *for* $n \geq 3$,

(2) $[\mathbb{Z}_k]$ *for* $2 \leq k \leq n/2$,

(3) $[\mathbb{Z}_{2k}^-]$ *for* $k \leq n/3$,

(4) $[\mathbb{D}_k]$ *for* $2 \leq k \leq n/2$ *if n is odd and for* $2 \leq k \leq n$ *if n is even*,

(5) $[\mathbb{D}_k^v]$ *for* $2 \leq k \leq n$ *if n is odd and* $2 \leq k \leq n/2$ *if n is even*,

(6) $[\mathbb{D}_{2k}^h]$ *for* $2 \leq k \leq n$, *except* $\mathbb{D}_4^h$ *for* $n = 3$,

(7) $[\mathbb{T}]$ *for* $n \neq 1, 2, 3, 5, 7, 8, 11$,

(8) $[\mathbb{O}]$ *for* $n \neq 1, 2, 3, 5, 7, 11$,

(9) $[\mathbb{O}^-]$ *for* $n \neq 1, 2, 4, 5, 8$,

(10) $[\mathbb{I}]$ *for* $n = 6, 10, 12, 15, 16, 18$ *or* $n \geq 20$ *and* $n \neq 23, 29$,

(11) $[\mathrm{O}(2)]$ *when n is even*, *and*

(12) $[\mathrm{O}(2)^-]$ *when n is odd.*

Remark 4.5. The list in Theorem 4.4 for n odd is similar to the list in [Chossat et al. 1990, Table A4]. In [Ihrig and Golubitsky 1984, Theorem 6.8] (for O(3) irreducible representations), $[\mathbb{T}]$ is an isotropy class for $n \neq 1, 2, 5, 7, 8, 11$ (which is different from the list (7) in our Theorem 4.4 above). But according to [Geymonat and Weller 2002; Weller 2004; Chossat et al. 1990, Table A4], $[\mathbb{T}]$ is not an isotropy class in the case $n = 3$, and we corrected this error in the list (7) of Theorem 4.4.

5. Clips tables

SO(3) ***closed subgroups.*** The resulting conjugacy classes for the clips operation of closed SO(3) subgroups are given in Table 1.

The following notations have been used:

$$d := \gcd(m, n), \qquad d_2 := \gcd(n, 2), \qquad k_2 := 3 - d_2,$$

$$d_3 := \gcd(n, 3), \qquad d_5 := \gcd(n, 5),$$

$$dz := \begin{cases} 2 & \text{if } m \text{ and } n \text{ even,} \\ 1 & \text{otherwise,} \end{cases} \qquad d_4 := \begin{cases} 4 & \text{if 4 divides } n, \\ 1 & \text{otherwise,} \end{cases} \qquad \mathbb{Z}_1 = \mathbb{D}_1 := \mathbb{1}.$$

Remark 5.1. The clips operations $[\mathbb{T}] \circledcirc [\mathbb{T}]$ and $[\mathbb{T}] \circledcirc [\mathbb{O}]$ were wrong in [Olive and Auffray 2013; Olive 2014], since for instance the isotropy class $[\mathbb{D}_2]$ was omitted.

Example 5.2 (isotropy classes for a family of n vectors). For one vector, we get

$$\mathfrak{I}(\mathcal{H}_1(\mathbb{R}^3)) = \{[\mathrm{SO}(2)], [\mathrm{SO}(3)]\}.$$

From Table 1, we deduce that the isotropy classes for a family of n vectors ($n \geq 2$) is

$$\mathfrak{I}\left(\bigoplus_{k=1}^{n} \mathcal{H}_1(\mathbb{R}^3)\right) = \{[\mathbb{1}], [\mathrm{SO}(2)], [\mathrm{SO}(3)]\}.$$

Example 5.3 (isotropy classes for a family of n quadratic forms). The space of quadratic forms on $\mathbb{R}^3$, $\mathrm{S}_2(\mathbb{R}^3)$, decomposes into two irreducible components (*deviatoric* and *spherical* tensors for the mechanicians):

$$\mathrm{S}_2(\mathbb{R}^3) = \mathcal{H}_2(\mathbb{R}^3) \oplus \mathcal{H}_0(\mathbb{R}^3).$$

$\circledcirc$	$[\mathbb{Z}_n]$	$[\mathbb{D}_n]$	$[\mathbb{T}]$	$[\mathbb{O}]$	$[\mathbb{I}]$	$[\mathrm{SO}(2)]$	$[\mathrm{O}(2)]$
$[\mathbb{Z}_m]$	$[\mathbb{1}]$ $[\mathbb{Z}_d]$						
$[\mathbb{D}_m]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2}]$ $[\mathbb{Z}_d]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$, $[\mathbb{D}_{dz}]$ $[\mathbb{Z}_d]$, $[\mathbb{D}_d]$					
$[\mathbb{T}]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2}]$ $[\mathbb{Z}_{d_3}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_{d_3}]$, $[\mathbb{D}_{d_2}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_2]$, $[\mathbb{Z}_3]$ $[\mathbb{T}]$				
$[\mathbb{O}]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2}]$ $[\mathbb{Z}_{d_3}]$ $[\mathbb{Z}_{d_4}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_{d_3}]$, $[\mathbb{Z}_{d_4}]$ $[\mathbb{D}_{d_2}]$, $[\mathbb{D}_{d_3}]$ $[\mathbb{D}_{d_4}]$	$[\mathbb{1}]$ $[\mathbb{Z}_2]$, $[\mathbb{D}_2]$ $[\mathbb{Z}_3]$ $[\mathbb{T}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_2]$, $[\mathbb{Z}_3]$ $[\mathbb{D}_3]$, $[\mathbb{Z}_4]$ $[\mathbb{D}_4]$, $[\mathbb{O}]$			
$[\mathbb{I}]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2}]$ $[\mathbb{Z}_{d_3}]$ $[\mathbb{Z}_{d_5}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_{d_3}]$, $[\mathbb{Z}_{d_5}]$ $[\mathbb{D}_{d_2}]$ $[\mathbb{D}_{d_3}]$, $[\mathbb{D}_{d_5}]$	$[\mathbb{1}]$ $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$ $[\mathbb{T}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$, $[\mathbb{D}_3]$ $[\mathbb{T}]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$, $[\mathbb{D}_3]$ $[\mathbb{Z}_5]$, $[\mathbb{D}_5]$ $[\mathbb{I}]$		
$[\mathrm{SO}(2)]$	$[\mathbb{1}]$ $[\mathbb{Z}_n]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_n]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$, $[\mathbb{Z}_4]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{Z}_3]$, $[\mathbb{Z}_5]$	$[\mathbb{1}]$ $[\mathrm{SO}(2)]$	
$[\mathrm{O}(2)]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2}]$ $[\mathbb{Z}_n]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_{k_2}]$, $[\mathbb{D}_n]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_2]$, $[\mathbb{Z}_3]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_2]$, $[\mathbb{D}_3]$ $[\mathbb{D}_4]$	$[\mathbb{1}]$, $[\mathbb{Z}_2]$ $[\mathbb{D}_2]$, $[\mathbb{D}_3]$ $[\mathbb{D}_5]$	$[\mathbb{1}]$ $[\mathbb{Z}_2]$ $[\mathrm{SO}(2)]$	$[\mathbb{Z}_2]$ $[\mathbb{D}_2]$ $[\mathrm{O}(2)]$

Table 1. Clips operations for SO(3).

We thus get

$$\mathfrak{I}(\mathrm{S}_2(\mathbb{R}^3)) = \mathfrak{I}(\mathscr{H}_2(\mathbb{R}^3)) = \{[\mathbb{D}_2], [\mathrm{O}(2)], [\mathrm{SO}(3)]\}.$$

The useful part of Table 1, for our purpose, reads:

$\circledcirc$	$[\mathbb{Z}_2]$	$[\mathbb{D}_2]$	$[\mathrm{O}(2)]$
$[\mathbb{Z}_2]$	$\{[\mathbb{1}], [\mathbb{Z}_2]\}$	$\{[\mathbb{1}], [\mathbb{Z}_2]\}$	$\{[\mathbb{1}], [\mathbb{Z}_2]\}$
$[\mathbb{D}_2]$		$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$	$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$
$[\mathrm{O}(2)]$			$\{[\mathbb{Z}_2], [\mathbb{D}_2], [\mathrm{O}(2)]\}$

We deduce therefore that the set of isotropy classes for a family of n quadratic forms ($n \geq 2$) is

$$\mathfrak{I}\left(\bigoplus_{k=1}^{n} \mathrm{S}_2(\mathbb{R}^3)\right) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathrm{O}(2)], [\mathrm{SO}(3)]\}.$$

$\circledcirc$	$[\mathbb{Z}_2^-]$	$[\mathbb{Z}_{2m}^-]$	$[\mathbb{D}_m^v]$	$[\mathbb{D}_{2m}^h]$	$[\mathbb{O}^-]$	$[\mathrm{O}(2)^-]$
$[\mathbb{Z}_2^-]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$					
$[\mathbb{Z}_{2n}^-]$	$[\mathbb{1}]$, $[\mathbb{Z}_{i(n)}^-]$	Figure 4, left				
$[\mathbb{D}_n^v]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$	Figure 4, right	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{D}_d^v]$, $[\mathbb{Z}_d]$			
$[\mathbb{D}_{2n}^h]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$	Figure 5	Figure 6	Figure 7		
$[\mathbb{O}^-]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$	Figure 8, left	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{Z}_{d_3(m)}]$ $[\mathbb{D}_{d_3(m)}^v]$ $[\mathbb{Z}_{d_2(m)}]$ $[\mathbb{D}_{d_2(m)}^v]$	Figure 8, right	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{Z}_4^-]$, $[\mathbb{Z}_3]$	
$[\mathrm{O}(2)^-]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$	$[\mathbb{1}]$, $[\mathbb{Z}_{i(m)}^-]$ $[\mathbb{Z}_m]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{D}_m^v]$	$[\mathbb{1}]$ $[\mathbb{Z}_{d_2(m)}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{D}_{i(m)}^v]$, $[\mathbb{D}_m^v]$	$[\mathbb{1}]$, $[\mathbb{Z}_2^-]$ $[\mathbb{D}_3^v]$, $[\mathbb{D}_2^v]$	$[\mathbb{Z}_2^-]$ $[\mathrm{O}(2)^-]$

Table 2. Clips operations on type III O(3) subgroups.

O(3) *closed subgroups.* Let us first consider an O(3) representation where $-I$ acts as $-\mathrm{Id}$ (meaning that this representation doesn't reduce to some SO(3) representation). In such a case, only the null vector can be fixed by $-\mathrm{Id}$, and so type II subgroups never appear as isotropy subgroups. In that case, we only need to focus on clips operations between type I and type III subgroups, and then between type III subgroups, since clips operations between type I subgroups have already been considered in Table 1. For type III subgroups as detailed in Section B we have:

Lemma 5.4. *Let H_1 be some type III closed subgroup of* O(3) *and H_2 be some type I closed subgroup of* O(3). *Then we have*

$$H_1 \cap H_2 = (H_1 \cap \mathrm{SO}(3)) \cap H_2,$$

and for every closed subgroup H of SO(3), *we get*

$$[\mathbb{Z}_2^-] \circledcirc [H] = \{[\mathbb{1}]\}, \qquad [\mathbb{Z}_{2n}^-] \circledcirc [H] = [\mathbb{Z}_n] \circledcirc [H],$$
$$[\mathbb{D}_n^v] \circledcirc [H] = [\mathbb{Z}_n] \circledcirc [H], \qquad [\mathbb{D}_{2n}^h] \circledcirc [H] = [\mathbb{D}_n] \circledcirc [H],$$
$$[\mathbb{O}^-] \circledcirc [H] = [\mathbb{T}] \circledcirc [H], \qquad [\mathrm{O}(2)^-] \circledcirc [H] = [\mathrm{SO}(2)] \circledcirc [H].$$

The resulting conjugacy classes for the clips operation for type III subgroups are given in Table 2, where the following notations have been used:

$$d := \gcd(n, m), \qquad d_2(n) := \gcd(n, 2),$$
$$d_3(n) := \gcd(n, 3), \qquad i(n) := 3 - \gcd(2, n), \qquad \mathbb{Z}_1^- = \mathbb{D}_1^v = \mathbb{D}_2^h = \mathbb{1}.$$

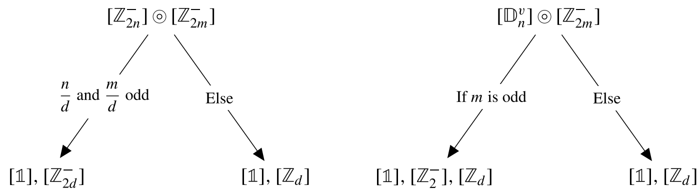

Figure 4. Left: $\mathbb{Z}_{2n}^-$ case. Right: $\mathbb{D}_n^v$ and $\mathbb{Z}_{2n}^-$.

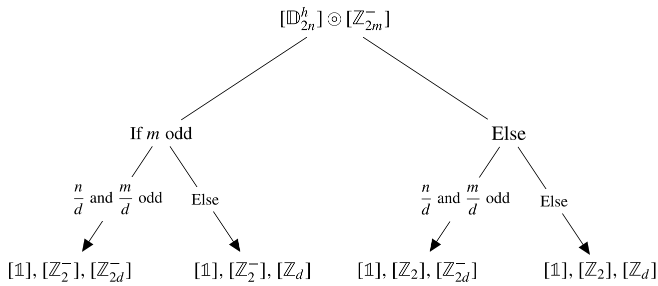

Figure 5. $\mathbb{D}_{2n}^h$ and $\mathbb{Z}_{2n}^-$.

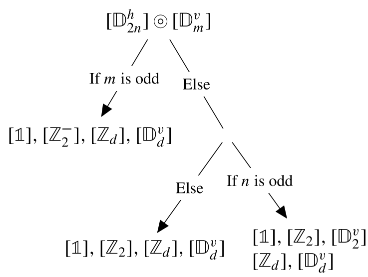

Figure 6. $\mathbb{D}_{2n}^h$ and $\mathbb{D}_n^v$.

Remark 5.5. One misprint in [Olive 2014; Olive and Auffray 2014] for clips operation $[\mathrm{O}(2)^-] \circledcirc [\mathbb{D}_{2m}^h]$ has been corrected: the conjugacy class $[\mathbb{Z}_2]$ appears for m even (and not for m odd).

Application to tensorial mechanical properties. We propose here some direct applications of our results for many different tensorial spaces, each one being endowed with the natural $\mathrm{O}(3)$ representation. Such tensorial spaces occur for instance in the modeling of mechanical properties. The main idea is to use the *irreducible decomposition*, also known as the *harmonic decomposition* [Backus 1970; Forte and Vianello 1996]. We then use clips operations given in Tables 1 and 2. Note that we don't need to know explicit irreducible decomposition of those tensorial representations.

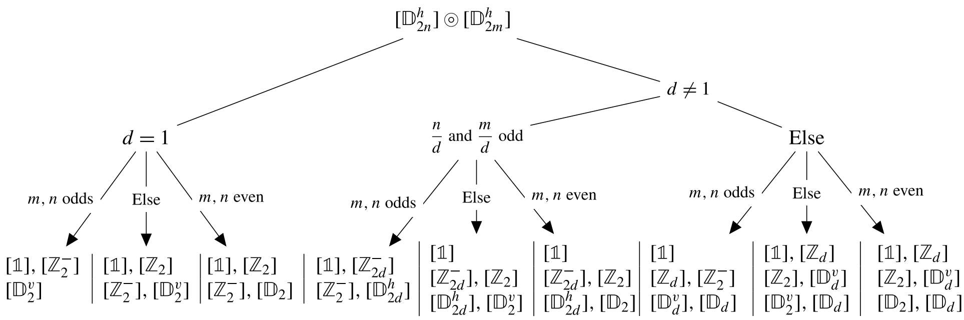

Figure 7. $\mathbb{D}_{2n}^h$ and $\mathbb{D}_{2m}^h$.

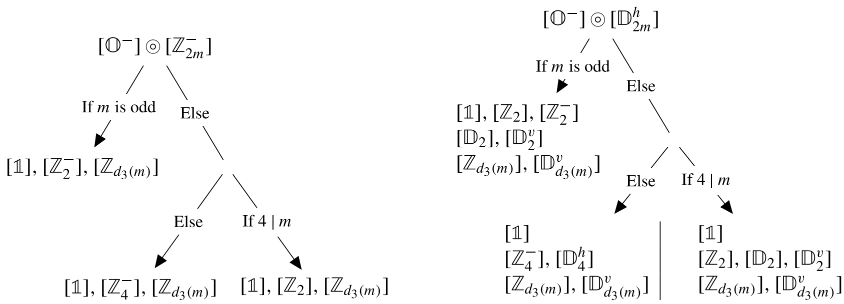

Figure 8. Left: O^- and $\mathbb{Z}_{2m}^-$. Right: O^- and $\mathbb{D}_{2m}^h$.

From now on, we define $\mathcal{H}_n(\mathbb{R}^3)^*$ to be the O(3) representation $(\mathcal{H}_n(\mathbb{R}^3), \rho_n^*)$. Furthermore, we define

$$\mathcal{H}_n(\mathbb{R}^3)^{\oplus k} := \bigoplus_{i=1}^{k} \mathcal{H}_n(\mathbb{R}^3), \qquad \mathcal{H}_n(\mathbb{R}^3)^{\oplus k *} := \bigoplus_{i=1}^{k} \mathcal{H}_n(\mathbb{R}^3)^*.$$

We propose here some specific irreducible decompositions, related to mechanical theory, without any further details. We explain the use of the clips operation in the classical case of elasticity, all other examples being done in the same way.

Classical results. We first give some classical results we obtain here directly.

(1) Elasticity [Forte and Vianello 1996]: SO(3) tensor space

$$\mathbb{E}\mathrm{la} \simeq \mathcal{H}_4(\mathbb{R}^3) \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 2} \oplus \mathcal{H}_0(\mathbb{R}^3)^{\oplus 2}.$$

We have from Theorem 4.2

$$\mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)) = \{[\mathbb{D}_2], [\mathrm{O}(2)], [\mathrm{SO}(3)]\},$$

$$\mathfrak{I}(\mathcal{H}_4(\mathbb{R}^3)) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{O}], [\mathrm{O}(2)], [\mathrm{SO}(3)]\},$$

so to obtain $\mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)^{\oplus 2})$ we use the clips table

$\circledcirc$	$[\mathbb{D}_2]$	$[\mathrm{O}(2)]$
$[\mathbb{D}_2]$	$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$	$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2]\}$
$[\mathrm{O}(2)]$		$\{[\mathbb{Z}_2], [\mathbb{D}_2], [\mathrm{O}(2)]\}$

so that

$$\begin{aligned}\mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)^{\oplus 2}) &= \mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)) \circledcirc \mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)) \\ &= ([\mathbb{D}_2] \circledcirc [\mathbb{D}_2]) \cup ([\mathbb{D}_2] \circledcirc [\mathrm{O}_2]) \cup ([\mathrm{O}_2] \circledcirc [\mathrm{O}_2]) \\ &= \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathrm{O}(2)]\}\end{aligned}$$

and now we conclude using the clips table coming from the clips operation

$$\mathfrak{I}(\mathbb{E}\mathrm{la}) = \mathfrak{I}(\mathcal{H}_4(\mathbb{R}^3)) \circledcirc \mathfrak{I}(\mathcal{H}_2(\mathbb{R}^3)^{\oplus 2}).$$

Finally we obtain eight symmetry classes

$$\mathfrak{I}(\mathbb{E}\mathrm{la}) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{O}], [\mathrm{O}(2)], [\mathrm{SO}(3)]\}.$$

(2) Photoelasticity [Forte and Vianello 1997]: SO(3) tensor space

$$\mathbb{P}\mathrm{la} \simeq \mathcal{H}_4(\mathbb{R}^3) \oplus \mathcal{H}_3(\mathbb{R}^3) \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 3} \oplus \mathcal{H}_1(\mathbb{R}^3) \oplus \mathcal{H}_0(\mathbb{R}^3)^{\oplus 2}$$

and twelve symmetry classes

$$\mathfrak{I}(\mathbb{P}\mathrm{la}) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{T}], [\mathbb{O}], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{SO}(3)]\}.$$

(3) Piezoelectricity [Geymonat and Weller 2002]: O(3) tensor space

$$\mathbb{P}\mathrm{iez} \simeq \mathcal{H}_3(\mathbb{R}^3) \oplus \mathcal{H}_2(\mathbb{R}^3)^* \oplus \mathcal{H}_1(\mathbb{R}^3)^{\oplus 2}$$

and sixteen symmetry classes

$$\begin{aligned}\mathfrak{I}(\mathbb{P}\mathrm{iez}) = \{&[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4^h], [\mathbb{D}_6^h], \\ &[\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathbb{O}^-], [\mathrm{O}(3)]\}.\end{aligned}$$

Remark 5.6. Note that the symmetry classes of the space ($\mathbb{P}\mathrm{iez}$, O(3)) appear in many different works: in the PhD thesis of Weller [2004, Theorem 3.19, p. 84], where fourteen symmetry classes are announced, but fifteen appeared in the poset, in [Geymonat and Weller 2002] where they establish fourteen symmetry classes, in [Olive and Auffray 2014] (with a typo), in [Zou et al. 2013] (without the [O(3)] symmetry class), and finally in [Olive 2014], with sixteen symmetry classes.

Nonclassical results. We now present some nonclassical tensor spaces, coming from Cosserat elasticity [Cosserat and Cosserat 1909; Eringen 1966; Forest 2005; Forest and Sievert 2006] and Strain gradient elasticity [Mindlin 1964; Mindlin and Eshel 1968; Portigal and Burstein 1968; Auffray et al. 2015], with their harmonic decomposition and symmetry classes.

(1) Classical Cosserat elasticity: SO(3) tensor space

$$\mathbb{C}\text{os} \simeq \mathcal{H}_4(\mathbb{R}^3) \oplus \mathcal{H}_3(\mathbb{R}^3) \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 4} \oplus \mathcal{H}_1(\mathbb{R}^3)^{\oplus 2} \oplus \mathcal{H}_0(\mathbb{R}^3)^{\oplus 3}$$

and twelve symmetry classes

$$\mathfrak{I}(\mathbb{C}\text{os}) = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{T}], [\mathbb{O}], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{SO}(3)]\}.$$

(2) Rotational Cosserat elasticity: O(3) tensor space

$$\mathbb{C}\text{hi} \simeq \mathcal{H}_4(\mathbb{R}^3)^* \oplus \mathcal{H}_3(\mathbb{R}^3)^{\oplus 3} \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 6*} \oplus \mathcal{H}_1(\mathbb{R}^3)^{\oplus 6} \oplus \mathcal{H}_0(\mathbb{R}^3)^{\oplus 3*}$$

and twenty-four symmetry classes

$$\begin{aligned}\mathfrak{I}(\mathbb{C}\text{hi}) = \{&[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_6^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4],\\ &[\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{D}_4^v], [\mathbb{D}_4^h], [\mathbb{D}_6^h], [\mathbb{D}_8^h], [\mathbb{T}], [\mathbb{O}], [\mathbb{O}^-], [\mathrm{SO}(2)],\\ &[\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathrm{SO}(3)], [\mathrm{O}(3)]\}.\end{aligned}$$

(3) Fifth-order O(3) tensor space in strain gradient elasticity, given by

$$\mathbb{S}\text{ge} \simeq \mathcal{H}_5(\mathbb{R}^3) \oplus \mathcal{H}_4(\mathbb{R}^3)^{\oplus 2*} \oplus \mathcal{H}_3(\mathbb{R}^3)^{\oplus 5} \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 5*} \oplus \mathcal{H}_1(\mathbb{R}^3)^{\oplus 6} \oplus \mathcal{H}_0(\mathbb{R}^3)^*$$

and twenty-nine symmetry classes

$$\begin{aligned}\mathfrak{I}(\mathbb{S}\text{ge}) = \{&[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_5], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_6^-], [\mathbb{Z}_8^-], [\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{D}_5],\\ &[\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{D}_4^v], [\mathbb{D}_5^v], [\mathbb{D}_4^h], [\mathbb{D}_6^h], [\mathbb{D}_8^h], [\mathbb{D}_{10}^h], [\mathbb{T}], [\mathbb{O}], [\mathbb{O}^-],\\ &[\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathrm{SO}(3)], [\mathrm{O}(3)]\}.\end{aligned}$$

(4) Fifth-order O(3) tensor space of acoustical gyrotropic tensor [Portigal and Burstein 1968] (reducing to fourth-order tensor space), given by

$$\mathbb{A}\text{gy} \simeq \mathcal{H}_4(\mathbb{R}^3)^* \oplus \mathcal{H}_3(\mathbb{R}^3)^{\oplus 2} \oplus \mathcal{H}_2(\mathbb{R}^3)^{\oplus 3*} \oplus \mathcal{H}_1(\mathbb{R}^3)^{\oplus 2} \oplus \mathcal{H}_0(\mathbb{R}^3)^*$$

and twenty-four symmetry classes

$$\begin{aligned}\mathfrak{I}(\mathbb{A}\text{gy}) = \{&[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{Z}_4], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_6^-],\\ &[\mathbb{D}_2], [\mathbb{D}_3], [\mathbb{D}_4], [\mathbb{D}_2^v], [\mathbb{D}_3^v], [\mathbb{D}_4^v], [\mathbb{D}_4^h], [\mathbb{D}_6^h], [\mathbb{D}_8^h],\\ &[\mathbb{T}], [\mathbb{O}], [\mathbb{O}^-], [\mathrm{SO}(2)], [\mathrm{O}(2)], [\mathrm{O}(2)^-], [\mathrm{SO}(3)], [\mathrm{O}(3)]\}.\end{aligned}$$

Appendix A: Proofs for SO(3)

In this section, we provide all the details required to obtain the results in Table 1. We will start with the following definition, which was introduced in [Golubitsky et al. 1988] and happens to be quite useful for this task.

Definition A.1. Let $K_1, K_2, \dots, K_s$ be subgroups of a given group G. We say that G is the *direct union* of the K_i and we write $G = \biguplus_{i=1}^{s} K_i$ if

$$G = \bigcup_{i=1}^{s} K_i \quad \text{and} \quad K_i \cap K_j = \{e\} \quad \text{for all } i \neq j.$$

In the following, we will have to repeatedly identify the conjugacy class of intersections such as

$$H_1 \cap (gH_2g^{-1}), \tag{A-1}$$

where H_1 and H_2 are two closed subgroups of SO(3) and $g \in \mathrm{SO}(3)$. A useful observation is that all closed SO(3) subgroups have some *characteristic axes* and that intersection (A-1) depends only on the relative positions of these characteristic axes.

As detailed below, for any subgroup conjugate to $\mathbb{Z}_n$ or $\mathbb{D}_n$ ($n \geq 3$), the axis of an n-th order rotation (in this subgroup) is called its *primary axis*. For subgroups conjugate to $\mathbb{D}_n$ ($n \geq 3$), axes of order-2 rotations are said to be *secondary axes*. In the special case $n = 2$, the z-axis is the primary axis of $\mathbb{Z}_2$, while any of the x-, y-, or z-axes can be considered as a primary axis of $\mathbb{D}_2$.

Cyclic subgroup. For any axis a of $\mathbb{R}^3$ (throughout the origin), we denote by $\mathbb{Z}_n^a$ the unique cyclic subgroup of order n around the a-axis, which is its primary axis. We have then:

Lemma A.2. *Let $m, n \geq 2$ be two integers and $d = \gcd(n, m)$. Then*

$$[\mathbb{Z}_n] \circledcirc [\mathbb{Z}_m] = \{[\mathbb{1}], [\mathbb{Z}_d]\}.$$

Proof. We have to consider intersections, such as

$$\mathbb{Z}_n \cap (g\mathbb{Z}_m g^{-1}) = \mathbb{Z}_n \cap \mathbb{Z}_m^a,$$

for some axis a, and only two cases occur:

(1) If $a \neq (Oz)$, then necessarily the intersection reduces to $\mathbb{1}$.

(2) If $a = (Oz)$, then the order r of a rotation in $\mathbb{Z}_n \cap \mathbb{Z}_m$ divides both n and m and thus divides $d = \gcd(m, n)$. We therefore get $\mathbb{Z}_n \cap \mathbb{Z}_m \subset \mathbb{Z}_d$. But obviously, $\mathbb{Z}_d \subset \mathbb{Z}_n \cap \mathbb{Z}_m$ and thus $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$. □

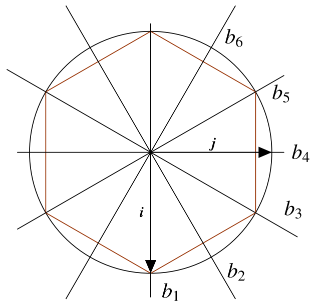

Figure 9. Secondary axis of the dihedral group $\mathbb{D}_6$.

Dihedral subgroup. Let b_1 be the x-axis and b_k ($k = 2, \dots, n$) be the axis recursively defined by

$$b_k := \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{n}\right) b_{k-1}.$$

Then, we have

$$\mathbb{D}_n = \mathbb{Z}_n \uplus \mathbb{Z}_2^{b_1} \uplus \cdots \uplus \mathbb{Z}_2^{b_n}, \tag{A-2}$$

where the z-axis (corresponding to an n-th order rotation) is the primary axis and the b_k-axes (corresponding to order-2 rotations) are the secondary axes of this dihedral group (see Figure 9).

Lemma A.3. *Let $m, n \geq 2$ be two integers, and set $d := \gcd(n, m)$ and $d_2(m) := \gcd(m, 2)$. Then we have*

$$[\mathbb{D}_n] \circledcirc [\mathbb{Z}_m] = \{[\mathbb{1}], [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_d]\}.$$

Proof. Let $\Gamma = \mathbb{D}_n \cap g\mathbb{Z}_m g^{-1}$ for $g \in \mathrm{SO}(3)$. From decomposition (A-2), we have to consider intersections

$$\mathbb{Z}_n \cap g\mathbb{Z}_m g^{-1}, \qquad \mathbb{Z}_2^{b_j} \cap g\mathbb{Z}_m g^{-1}.$$

which thus reduce to Lemma A.2. □

Lemma A.4. *Let $m, n \geq 2$ be two integers. Set $d := \gcd(n, m)$ and*

$$dz := \begin{cases} 2 & \text{if } m \text{ and } n \text{ even}, \\ 1 & \text{otherwise}. \end{cases}$$

Then, we have

$$[\mathbb{D}_n] \circledcirc [\mathbb{D}_m] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_{dz}], [\mathbb{Z}_d], [\mathbb{D}_d]\}.$$

Proof. Let $\Gamma = \mathbb{D}_n \cap (g\mathbb{D}_m g^{-1})$.

(1) If both primary axes and one secondary axis match, $\Gamma = \mathbb{D}_d$ if $d \neq 1$ and $\Gamma \in [\mathbb{Z}_2]$ otherwise.

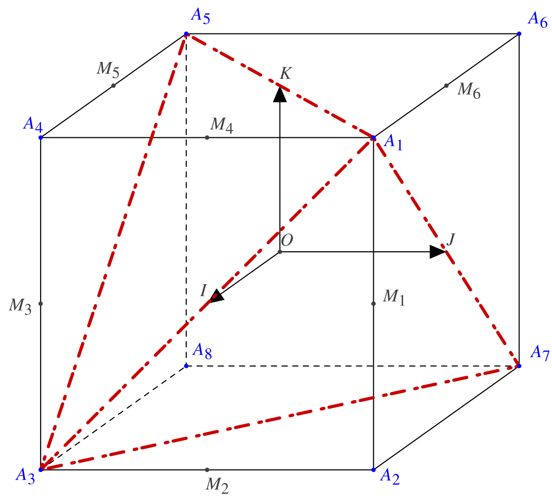

Figure 10. Cube $\mathscr{C}_0$ and tetrahedron $\mathscr{T}_0 := A_1A_3A_7A_5$.

(2) If only the primary axes match, $\Gamma = \mathbb{Z}_d$.

(3) If the angle of primary axes is $\pi/4$ and a secondary axis match, then $\Gamma \in [\mathbb{Z}_2]$

(4) If the primary axis of $g\mathbb{D}_m g^{-1}$ matches with the secondary axis (Ox) of $\mathbb{D}_n$ (or the converse), we obtain $\Gamma \in [\mathbb{D}_2]$ for n and m even and a secondary axis of $g\mathbb{D}_m g^{-1}$ is (Oz); otherwise, we obtain $\Gamma \in [\mathbb{Z}_2]$.

(5) In all other cases, we have $\Gamma = \mathbb{1}$. □

Tetrahedral subgroup. The (orientation-preserving) symmetry group $\mathbb{T}$ of the tetrahedron $\mathscr{T}_0 := A_1A_3A_7A_5$ (see Figure 10) decomposes as [Ihrig and Golubitsky 1984]

$$\mathbb{T} = \biguplus_{i=1}^{4} \mathbb{Z}_3^{\boldsymbol{vt}_i} \uplus \biguplus_{j=1}^{3} \mathbb{Z}_2^{\boldsymbol{et}_j} \tag{A-3}$$

where $\boldsymbol{vt}_i$ and $\boldsymbol{et}_j$ are the *vertices axes* and *edges axes* of the tetrahedron (see Figure 10), respectively:

$$\boldsymbol{vt}_1 := (OA_1), \quad \boldsymbol{vt}_2 := (OA_3), \quad \boldsymbol{vt}_3 := (OA_5), \quad \boldsymbol{vt}_4 := (OA_7),$$
$$\boldsymbol{et}_1 := (Ox), \quad \boldsymbol{et}_2 := (Oy), \quad \boldsymbol{et}_3 := (Oz).$$

Corollary A.5. *Let $n \geq 2$ be an integer. Set $d_2(n) := \gcd(n, 2)$ and $d_3(n) := \gcd(3, n)$. Then we have*

$$[\mathbb{Z}_n] \circledcirc [\mathbb{T}] = \{[\mathbb{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}]\}.$$

Proof. Consider $\mathbb{T}\cap\mathbb{Z}_n^a$ for some axis a. As a consequence of Lemma A.2, we only need to consider the case where a is an edge axis or a face axis of the tetrahedron, reducing to the clips operations

$$[\mathbb{Z}_2]\circledcirc[\mathbb{Z}_n],\qquad [\mathbb{Z}_3]\circledcirc[\mathbb{Z}_n],$$

which directly leads to the lemma. □

Corollary A.6. *Let $n\geq 2$ be some integer. Set $d_2(n):=\gcd(n,2)$ and $d_3(n):=$ $\gcd(3,n)$. Then we have*

$$[\mathbb{D}_n]\circledcirc[\mathbb{T}]=\{[\mathbb{1}],[\mathbb{Z}_2],[\mathbb{Z}_{d_3(n)}],[\mathbb{D}_{d_2(n)}]\}.$$

Proof. Let $\Gamma=\mathbb{T}\cap(g\mathbb{D}_n g^{-1})$. From decomposition (A-3), we only need to consider intersections

$$\mathbb{Z}_3^{\boldsymbol{vt}_i}\cap(g\mathbb{D}_n g^{-1})\quad\text{and}\quad \mathbb{Z}_2^{\boldsymbol{et}_j}\cap(g\mathbb{D}_n g^{-1}),$$

which have already been studied (see Lemma A.3). □

Lemma A.7. *We have*

$$[\mathbb{T}]\circledcirc[\mathbb{T}]=\{[\mathbb{1}],[\mathbb{Z}_2],[\mathbb{D}_2],[\mathbb{Z}_3],[\mathbb{T}]\}.$$

Proof. Let $\Gamma=\mathbb{T}\cap(g\mathbb{T}g^{-1})$.

(1) If no axes match, then $\Gamma=\mathbb{1}$.

(2) If only one edge axis or one face axis from both configurations match, then $\Gamma\in[\mathbb{Z}_2]$ or $[\mathbb{Z}_3]$, respectively.

(3) If $g=\boldsymbol{Q}(\boldsymbol{k},\pi/2)$, then $\Gamma=\mathbb{D}_2$. □

Octahedral subgroup. The group $\mathbb{O}$ is the (orientation-preserving) symmetry group of the cube $\mathscr{C}_0$ (see Figure 10) with vertices

$$\{A_i\}_{i=1,\dots,8}=(\pm1,\pm1,\pm1).$$

We have the decomposition [Ihrig and Golubitsky 1984]

$$\mathbb{O}=\biguplus_{i=1}^{3}\mathbb{Z}_4^{\boldsymbol{fc}_i}\uplus\biguplus_{j=1}^{4}\mathbb{Z}_3^{\boldsymbol{vc}_j}\uplus\biguplus_{l=1}^{6}\mathbb{Z}_2^{\boldsymbol{ec}_l}\tag{A-4}$$

with *vertices*, *edges*, and *faces axes* respectively denoted $\boldsymbol{vc}_i$, $\boldsymbol{ec}_j$, and $\boldsymbol{fc}_j$. For instance we have

$$\boldsymbol{vc}_1:=(OA_1),\qquad \boldsymbol{ec}_1:=(OM_1),\qquad \boldsymbol{fc}_1:=(OI).$$

As an application of decomposition (A-4) and Lemma A.2, we obtain:

Corollary A.8. *Let $n \geq 2$ be some integer. Set*

$$d_2(n) = \gcd(n, 2), \qquad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \text{ divide } n, \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

$$[\mathbb{Z}_n] \circledcirc [\mathbb{O}] = \{[\mathbb{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}]\}.$$

Corollary A.9. *Let $n \geq 2$ be some integer. Set*

$$d_2(n) = \gcd(n, 2), \qquad d_3(n) = \gcd(n, 3),$$

and

$$d_4(n) = \begin{cases} 4 & \text{if } 4 \text{ divides } n, \\ 1 & \text{otherwise.} \end{cases}$$

Then we have

$$[\mathbb{D}_n] \circledcirc [\mathbb{O}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}], [\mathbb{Z}_{d_4(n)}], [\mathbb{D}_{d_2(n)}], [\mathbb{D}_{d_3(n)}], [\mathbb{D}_{d_4(n)}]\}.$$

Proof. Using decomposition (A-4), we have to consider intersections

$$\mathbb{D}_n \cap (g\mathbb{Z}_4^{\boldsymbol{fc}_i} g^{-1}), \qquad \mathbb{D}_n \cap (g\mathbb{Z}_3^{\boldsymbol{vc}_j} g^{-1}), \qquad \mathbb{D}_n \cap (g\mathbb{Z}_2^{\boldsymbol{ec}_l} g^{-1}),$$

which have already been studied in Lemma A.3. □

Lemma A.10. *We have*

$$[\mathbb{T}] \circledcirc [\mathbb{O}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$$

Proof. Let $\Gamma = \mathbb{O} \cap (g\mathbb{T}g^{-1})$. From decompositions (A-3)–(A-4) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_4^{\boldsymbol{fc}_i} \cap (g\mathbb{Z}_2^{\boldsymbol{et}_j} g^{-1}), \qquad \mathbb{Z}_3^{\boldsymbol{vc}_j} \cap (g\mathbb{Z}_3^{\boldsymbol{vt}_i} g^{-1}), \qquad \mathbb{Z}_2^{\boldsymbol{ec}_l} \cap (g\mathbb{Z}_2^{\boldsymbol{et}_j} g^{-1}).$$

Now, we always can find g such that the intersection Γ reduces to some subgroup conjugate to $\mathbb{1}$, $\mathbb{Z}_2$, or $\mathbb{Z}_3$, and taking $g = \boldsymbol{Q}(\boldsymbol{k}, \pi/4)$, we get that Γ is conjugate to $\mathbb{D}_2$, which achieves the proof. □

Lemma A.11. *We have*

$$[\mathbb{O}] \circledcirc [\mathbb{O}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{O}]\}.$$

Proof. Consider the subgroup $\Gamma = \mathbb{O} \cap (g\mathbb{O}g^{-1}) \subset \mathbb{O}$. From the poset in Figure 1, we deduce that the conjugacy class $[\Gamma]$ belong to the list

$$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{T}], [\mathbb{O}]\}.$$

Now:

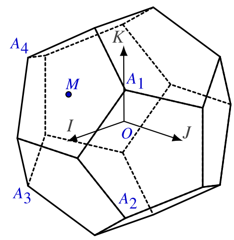

Figure 11. Dodecahedron $\mathscr{D}_0$.

(1) If g fixes only one edge axis or one vertex axis, then $\Gamma \in [\mathbb{Z}_2]$ or $\Gamma \in [\mathbb{Z}_3]$, respectively.

(2) If $g = \boldsymbol{Q}(\boldsymbol{i}; \pi/6)$, only one face axis is fixed by g and $\Gamma \in [\mathbb{Z}_4]$.

(3) If $g = \boldsymbol{Q}(\boldsymbol{i}; \pi/4)$, $\Gamma \supset \mathbb{Z}_4^{\boldsymbol{i}} \uplus \mathbb{Z}_2^{\boldsymbol{k}}$ and thus $\Gamma \in [\mathbb{D}_4]$.

(4) If $g = \boldsymbol{Q}(\boldsymbol{k}; \pi/4) \circ \boldsymbol{Q}(\boldsymbol{i}; \pi/4)$, $\Gamma \in [\mathbb{D}_2]$ with characteristic axes $g\boldsymbol{fc}_3 = \boldsymbol{ec}_6$, $g\boldsymbol{ec}_1 = \boldsymbol{fc}_1$, and $g\boldsymbol{ec}_2 = \boldsymbol{ec}_5$.

(5) If $g = \boldsymbol{Q}(\boldsymbol{vc}_1, \pi)$, $\Gamma \in [\mathbb{D}_3]$ with $\boldsymbol{vc}_1$ as the primary axis and $\boldsymbol{ec}_5$ as the secondary axis of Γ.

(6) If $\Gamma \supset \mathbb{T}$, then g fixes the three edge axes of the tetrahedron, and g fixes the cube. In that case, $\Gamma = \mathbb{O}$. □

Icosahedral subgroup. The group $\mathbb{I}$ is the (orientation-preserving) symmetry group of the dodecahedron $\mathscr{D}_0$ (Figure 11), where we have

- twelve vertices $(\pm\phi, \pm\phi^{-1}, 0)$, $(\pm\phi^{-1}, 0, \pm\phi)$, $(0, \pm\phi, \pm\phi^{-1})$, ϕ being the gold number, or
- eight vertices $(\pm 1, \pm 1, \pm 1)$ of a cube.

We thus have the decomposition

$$\mathbb{I} = \biguplus_{i=1}^{6} \mathbb{Z}_5^{\boldsymbol{fd}_i} \uplus \biguplus_{j=1}^{10} \mathbb{Z}_3^{\boldsymbol{vd}_j} \uplus \biguplus_{l=1}^{15} \mathbb{Z}_2^{\boldsymbol{ed}_l} \tag{A-5}$$

with *vertices*, *edges*, and *faces axes* respectively denoted $\boldsymbol{vd}_i$, $\boldsymbol{ed}_j$, and $\boldsymbol{fd}_j$. For instance we have

$$\boldsymbol{vd}_1 := (OA_1), \qquad \boldsymbol{ed}_1 := (OI), \qquad \boldsymbol{fd}_1 := (OM)$$

where M is the center of some face.

From decomposition (A-5) and Lemma A.2 we obtain the following corollary.

Corollary A.12. *Let $n \geq 2$ be some integer. Set*

$$d_2 := \gcd(n, 2), \qquad d_3 := \gcd(n, 3), \qquad d_5 := \gcd(n, 5).$$

Then

$$[\mathbb{Z}_n] \circledcirc [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_{d_2}], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}]\}.$$

Once again using decomposition (A-5) and the clips operation $[\mathbb{D}_n] \circledcirc [\mathbb{Z}_m]$ in Lemma A.3 we get the following corollary.

Corollary A.13. *Let $n \geq 2$ be some integer. Set*

$$d_2 := \gcd(n, 2) \qquad d_3 := \gcd(n, 3), \qquad d_5 := \gcd(n, 5).$$

Then

$$[\mathbb{D}_n] \circledcirc [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3}], [\mathbb{Z}_{d_5}], [\mathbb{D}_{d_2}], [\mathbb{D}_{d_3}], [\mathbb{D}_{d_5}]\}.$$

Lemma A.14. $\quad [\mathbb{I}] \circledcirc [\mathbb{T}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{T}]\}.$

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{T}g^{-1})$. From decompositions (A-3)–(A-5) and Lemma A.2, we only have to consider intersections

$$\mathbb{Z}_3^{\boldsymbol{vd}_j} \cap (g\mathbb{Z}_3^{\boldsymbol{vt}_i} g^{-1}), \qquad \mathbb{Z}_2^{\boldsymbol{ed}_l} \cap (g\mathbb{Z}_2^{\boldsymbol{et}_j} g^{-1}).$$

First, note that there always exists g such that Γ reduces to a subgroup conjugate to $\mathbb{1}$, $\mathbb{Z}_2$, or $\mathbb{Z}_3$. Now, if Γ contains a subgroup conjugate to $\mathbb{D}_2$, then its three characteristic axes are edge axes of the dodecahedron: say Ox, Oy, and Oz. In that case, g fixes these three axes, and also the eight vertices of the cube $\mathscr{C}_0$. The subgroup $g\mathbb{T}g^{-1}$ is thus the (orientation-preserving) symmetry group of a tetrahedron included in the dodecahedron $\mathscr{D}_0$. Then, $\Gamma \in [\mathbb{T}]$. □

The next two lemmas are more involved.

Lemma A.15. $\quad [\mathbb{O}] \circledcirc [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{O}g^{-1})$. From the poset in Figure 1, we deduce that the conjugacy class $[\Gamma]$ belongs to the list

$$\{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{T}]\}.$$

First, we can always find $g \in \mathrm{SO}(3)$ such that $\Gamma \in [\mathbb{Z}_3]$, $\Gamma \in [\mathbb{Z}_2]$, or $\Gamma = \mathbb{1}$. Moreover, as in the proof of Lemma A.14, if Γ contains a subgroup conjugate to $\mathbb{D}_2$, then $\Gamma \in [\mathbb{T}]$. Finally, we will exhibit some $g \in \mathrm{SO}(3)$ such that $\Gamma \in [\mathbb{D}_3]$. First, recall that

$$A_1(1, 1, 1), \qquad A_2(1, 1, -1), \qquad A_4(1, -1, 1), \qquad A_5(-1, -1, 1)$$

are common vertices of the cube $\mathscr{C}_0$ and dodecahedron $\mathscr{D}_0$. Now let $B_2(\phi^{-1}, 0, -\phi)$ be a vertex of the dodecahedron and I_2 or I_4 be the middle-point of $[B_2A_2]$ or $[A_4A_5]$, respectively (see Figure 12).

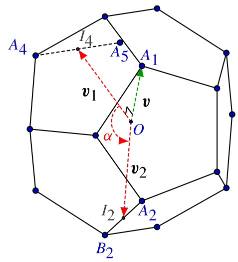

Figure 12. Rotation g to obtain $[\mathbb{D}_3]$ in $[\mathbb{O}] \circledcirc [\mathbb{I}]$.

Then, $a_1 = (OI_4)$ and $a_2 = (OI_2)$ are perpendicular axes to $a = (OA_1)$. Choose α such that $\boldsymbol{Q}(\boldsymbol{v}, \alpha)\boldsymbol{v}_1 = \boldsymbol{v}_2$, with (see Figure 12)

$$\boldsymbol{v} = \overrightarrow{OA_1}, \qquad \boldsymbol{v}_1 = \overrightarrow{OI_4}, \qquad \boldsymbol{v}_2 = \overrightarrow{OI_2},$$

and set $g = \boldsymbol{Q}(\boldsymbol{v}, \alpha)$. From decompositions (A-4) and (A-5), we then deduce that $\mathbb{I} \cap (g\mathbb{O}g^{-1})$ contains the subgroups

$$\mathbb{Z}_2^{a_2} \cap (g\mathbb{Z}_2^{a_1}g^{-1}) = \mathbb{Z}_2^{a_2}, \qquad \mathbb{Z}_3^{a} \cap, (g\mathbb{Z}_3^{a}g^{-1}) = \mathbb{Z}_3^{a}.$$

Therefore, Γ contains a subgroup conjugate to $\mathbb{D}_3$. Using a maximality argument (see poset in Figure 1), we must have $\Gamma \in [\mathbb{D}_3]$, and this concludes the proof. □

Lemma A.16. $[\mathbb{I}] \circledcirc [\mathbb{I}] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_5], [\mathbb{D}_5], [\mathbb{I}]\}$.

Proof. Let $\Gamma = \mathbb{I} \cap (g\mathbb{I}g^{-1})$. Considering the subclasses of $[\mathbb{I}]$, we have to check the classes

$$[\mathbb{T}], \qquad [\mathbb{D}_3], \qquad [\mathbb{D}_5], \qquad [\mathbb{D}_2], \qquad [\mathbb{Z}_3], \qquad [\mathbb{Z}_5], \qquad [\mathbb{Z}_2].$$

First note that there exist rotations g such that $\Gamma \in [\mathbb{Z}_2]$, $\Gamma \in [\mathbb{Z}_3]$, $\Gamma \in [\mathbb{Z}_5]$, or $\Gamma = \mathbb{1}$.

When Γ contains a subgroup conjugate to $\mathbb{T}$ or $\mathbb{D}_2$, using the same argument as in the proof of Lemma A.14, g fixes all the dodecahedron vertices. In that case, $\Gamma = \mathbb{I}$.

We will now exhibit some $g \in \mathrm{SO}(3)$ such that $\Gamma \in [\mathbb{D}_3]$. Consider the dodecahedron $\mathscr{D}_0$ in Figure 11 and the points $A_3(1, -1, -1)$ and $B_3(\phi, -\phi^{-1}, 0)$. Let I_3 be the midpoint of $[A_3B_3]$ and g be the order-2 rotation around $a_1 := (OA_1)$ (see Figure 13). Let

$$b_1 := (OI_3), \qquad b_2 := \boldsymbol{Q}\left(a_1, \frac{2\pi}{3}\right)b_1, \qquad b_3 := \boldsymbol{Q}\left(a_1, \frac{2\pi}{3}\right)b_2.$$

We check directly that a_1 and b_i $(i = 1, \ldots, 3)$ are the only g-invariant characteristic axes of the dodecahedron. We then deduce from decomposition (A-5) that Γ

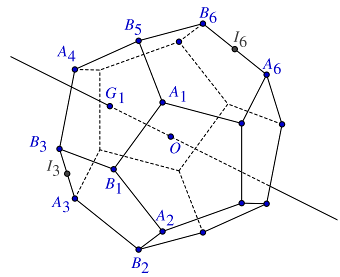

Figure 13. Rotation g to obtain $[\mathbb{D}_3]$ or $[\mathbb{D}_5]$ in $[\mathbb{I}] \circledcirc [\mathbb{I}]$.

reduces to

$$\mathbb{Z}_3^{a_1} \uplus \biguplus_{i=1}^{3} \mathbb{Z}_2^{b_i} \in [\mathbb{D}_3].$$

In the same way, we can find $g \in \mathrm{SO}(3)$, such that $\Gamma \in [\mathbb{D}_5]$. Let

$$B_1(\phi, \phi^{-1}, 0), \qquad B_5(\phi^{-1}, 0, \phi), \qquad B_6(-\phi^{-1}, 0, \phi), \qquad A_6(-1, 1, 1)$$

be vertices of the dodecahedron $\mathscr{D}_0$. Let G_1 be the center of pentagon $A_1B_1B_3A_4B_5$ (see Figure 13) and I_6 be the midpoint of $[B_6A_6]$. Let g be the order-2 rotation around $f_1 := (OG_1)$ and set

$$c_1 := (OI_6), \qquad c_{k+1} := \boldsymbol{Q}\left(\overrightarrow{OA_1}, \frac{2\pi}{5}\right)c_k, \quad 1 \leq k \leq 4.$$

Then we can check that f_1 and c_k ($k = 1, \ldots, 5$) are the only g-invariant characteristic axes of the dodecahedron. Using decomposition (A-5), we then deduce that $\Gamma \in [\mathbb{D}_5]$, which concludes the proof. $\square$

Infinite subgroups. The primary axis of both SO(2) and O(2) is defined as the z-axis, while any perpendicular axis to (Oz) is a secondary axis for O(2).

Clips operation between SO(2) or O(2) and finite subgroups are obtained using simple arguments on characteristic axes. The same holds for the clips $[\mathrm{SO}(2)] \circledcirc [\mathrm{SO}(2)]$. To compute $[\mathrm{O}(2)] \circledcirc [\mathrm{O}(2)]$, consider the subgroup $\Gamma = \mathrm{O}(2) \cap (g\mathrm{O}(2)g^{-1})$ for some $g \in \mathrm{SO}(3)$.

(1) If both primary axes are the same, then $\Gamma = \mathrm{O}(2)$.

(2) If the primary axis of $g\mathrm{O}(2)g^{-1}$ is in the xy-plane, then $\Gamma \in [\mathbb{D}_2]$.

(3) In all other cases, $\Gamma \in [\mathbb{Z}_2]$, where the primary axis of Γ is perpendicular to the primary axes of O(2) and $g\mathrm{O}(2)g^{-1}$.

Γ	H	L
$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbb{1}$
$\mathbb{Z}_{2n}^-$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_n$
$\mathbb{D}_n^v$	$\mathbb{D}_n$	$\mathbb{Z}_n$
$\mathbb{D}_{2n}^h$	$\mathbb{D}_{2n}$	$\mathbb{D}_n$
$\mathbb{O}^-$	$\mathbb{O}$	$\mathbb{T}$
$\mathrm{O}(2)^-$	$\mathrm{O}(2)$	$\mathrm{SO}(2)$

Table 3. Characteristic couples for type III subgroups.

Appendix B: Proofs for O(3)

In this appendix, we provide the details about clips operations between type III closed O(3) subgroups. The proofs follow the same ideas that have been used for SO(3) closed subgroups (decomposition into simpler subgroups and discussion about their characteristic axes), but most of them are unfortunately more involved.

We first recall the general structure of type III subgroups Γ of O(3) (see [Ihrig and Golubitsky 1984] for details). For each such subgroup Γ, there exist a couple $L \subset H$ of SO(3) subgroups such that $H = \pi(\Gamma)$, where

$$\pi : g \in \mathrm{O}(3) \mapsto \det(g) g \in \mathrm{SO}(3)$$

and $L = \mathrm{SO}(3) \cap \Gamma$ is an indexed 2 subgroup of H. These characteristic couples are detailed in Table 3. Note that, for a given couple (L, H), Γ can be recovered as $\Gamma = L \cap gL$, where $-g \in H \setminus L$.

In the following, we shall use the convention

$$\mathbb{Z}_1^\sigma = \mathbb{Z}_1^- = \mathbb{D}_1^v = \mathbb{1}.$$

$\mathbb{Z}_{2n}^-$ subgroups. Consider the couple $\mathbb{Z}_n \subset \mathbb{Z}_{2n}$ $(n > 1)$ in Table 3, where

$$\mathbb{Z}_{2n} = \left\{ I,\ \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{n}\right),\ \boldsymbol{Q}\left(\boldsymbol{k}; \frac{2\pi}{n}\right), \dots \right\}$$

and let $\boldsymbol{r}_n := \boldsymbol{Q}(\boldsymbol{k}; \pi/n) \in \mathbb{Z}_{2n} \setminus \mathbb{Z}_n$. Set

$$\mathbb{Z}_{2n}^- := \mathbb{Z}_n \cup (-\boldsymbol{r}_n \mathbb{Z}_n). \tag{B-1}$$

The *primary axis* of the subgroup $\mathbb{Z}_{2n}^-$ is defined as the z-axis.

Remark B.1. The subgroup $\mathbb{Z}_2^-$ is generated by $-\boldsymbol{Q}(\boldsymbol{k}, \pi)$, which is the reflection through the xy-plane. If σ_b is the reflection through the plane with normal axis b, then $\mathbb{Z}_2^{\sigma_b} := \{e, \sigma_b\}$, which is conjugate to $\mathbb{Z}_2^-$.

We have the following lemma.

Lemma B.2. *Let $m, n \geq 2$ be two integers. Set $d := \gcd(n, m)$ and*

$$\boldsymbol{r}_n := \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{n}\right), \qquad \boldsymbol{r}_m := \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{m}\right).$$

The intersection $(-\boldsymbol{r}_n\mathbb{Z}_n) \cap (-\boldsymbol{r}_m\mathbb{Z}_m)$ does not reduce to $\varnothing$ if and only if m/d and n/d are odd. In such a case, we have

$$(-\boldsymbol{r}_n\mathbb{Z}_n) \cap (-\boldsymbol{r}_m\mathbb{Z}_m) = -\boldsymbol{r}_d\mathbb{Z}_d, \quad \boldsymbol{r}_d = \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{d}\right).$$

Proof. The intersection $(-\boldsymbol{r}_n\mathbb{Z}_n) \cap (-\boldsymbol{r}_m\mathbb{Z}_m)$ differs from $\varnothing$, if and only if there exist integers i, j such that

$$\frac{2i+1}{n}\pi = \frac{2j+1}{m}\pi, \qquad 2i+1 \leq 2n, \qquad 2j+1 \leq 2m.$$

Let $n = dn_1$ and $m = dm_1$. The above equation can then be recast as $(2i+1)m_1 = (2j+1)n_1$, so that

$$2i+1 = pn_1 \quad \text{and} \quad 2j+1 = pm_1.$$

Thus, m_1 and n_1 are necessarily odd, in which case (recall that $\mathbb{Z}_1 = \mathbb{1}$)

$$(-\boldsymbol{r}_n\mathbb{Z}_n) \cap (-\boldsymbol{r}_n\mathbb{Z}_m) = -\boldsymbol{r}_d\mathbb{Z}_d, \quad \boldsymbol{r}_d = \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{d}\right). \qquad \square$$

Corollary B.3. *Let $m, n \geq 1$ be two integers. Set $d := \gcd(n, m)$. Then we have*

$$[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{Z}_{2m}^-] = \begin{cases} \{[\mathbb{1}], [\mathbb{Z}_{2d}^-]\} & \text{if } n/d \text{ and } m/d \text{ are odd}, \\ \{[\mathbb{1}], [\mathbb{Z}_d]\} & \text{otherwise}. \end{cases}$$

Proof. Note first that all intersections reduce to $\mathbb{1}$ when the characteristic axes don't match, so we only have to consider the situation where they match. Now, by (B-1), we only have to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^- = (\mathbb{Z}_n \cap \mathbb{Z}_m) \cup ((-\boldsymbol{r}_n\mathbb{Z}_n) \cap (-\boldsymbol{r}_m\mathbb{Z}_m)).$$

By Lemma A.2, $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ and we directly conclude with Lemma B.2. $\square$

$\mathbb{D}_n^v$ subgroups. Consider the couple $\mathbb{Z}_n \subset \mathbb{D}_n$ in Table 3. Recall that $\mathbb{D}_n$ contains $\mathbb{Z}_n$ and all the second-order rotations about the b_j axes (see (A-2) and Figure 9). Set

$$\mathbb{D}_n^v := \mathbb{Z}_n \uplus \biguplus_{j=1}^{n} \mathbb{Z}_2^{\sigma_{b_j}}. \tag{B-2}$$

Given $g \in \mathrm{O}(3)$, the primary axis of $g\mathbb{D}_n^v g^{-1}$ is $g(Oz)$, and its secondary axes are gb_j.

Lemma B.4. *Let $n \geq 2$ and $m \geq 1$ be two integers. Set $d = \gcd(n, m)$ and*

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ 2 & \text{if } m \text{ is odd}. \end{cases}$$

Then, we have

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{Z}_{2m}^-] = \{\mathbb{1}, [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d]\}.$$

Proof. Let $\Gamma := \mathbb{D}_n^v \cap (g\mathbb{Z}_{2m}^- g^{-1})$ and

$$\mathbb{Z}_{2m}^- = \mathbb{Z}_m \cup (-\boldsymbol{r}_m \mathbb{Z}_m), \qquad \boldsymbol{r}_m = \boldsymbol{Q}\left(\boldsymbol{k}; \frac{\pi}{m}\right).$$

(1) If both primary axes of $\mathbb{D}_n^v$ and $g\mathbb{Z}_{2m}^- g^{-1}$ (generated by $g\boldsymbol{k}$) match, then by decomposition (B-2) and Lemma A.2, Γ reduces to $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$.

(2) If the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$ matches with a secondary axis of $\mathbb{D}_n^v$, say (Ox), then Γ reduces to $\mathbb{Z}_2^{\sigma_{b_0}} \cap (g\mathbb{Z}_{2m}^- g^{-1})$. Such an intersection has already been studied in the clips operation $[\mathbb{Z}_2^-] \circledcirc [\mathbb{Z}_{2m}^-]$ (see Corollary B.3).

(3) Otherwise, $\Gamma = \mathbb{1}$, which concludes the proof. □

Lemma B.5. *Let $m, n \geq 2$ be two integers and $d = \gcd(n, m)$. Then we have*

$$[\mathbb{D}_n^v] \circledcirc [\mathbb{D}_m^v] = \{\mathbb{1}, [\mathbb{Z}_2^-], [\mathbb{D}_d^v], [\mathbb{Z}_d]\}.$$

Proof. Only two cases need to be considered.

(1) If the primary axes of $\mathbb{D}_n^v$ and $g\mathbb{D}_m^v g^{-1}$ do not match, then we get $\mathbb{1}$.

(2) If they have the same primary axis, by decomposition (B-2), we have to consider the intersections

$$\mathbb{Z}_n \cap \mathbb{Z}_m, \qquad \mathbb{Z}_2^{\sigma_{b_j}} \cap \mathbb{Z}_2^{\sigma_{b'_k}},$$

which reduce to

$$\mathbb{Z}_d \uplus \mathbb{Z}_2^{\sigma_{c_l}},$$

where c_l are the common secondary axes of the two subgroups. Then we get either $\mathbb{Z}_d$, $\mathbb{D}_d^v$, or a subgroup conjugate to $\mathbb{Z}_2^-$ (when $d = 1$ and $b_0 = b'_0$), which concludes the proof. □

$\mathbb{D}_{2n}^h$ ***subgroups.*** Consider the couple $\mathbb{D}_n \subset \mathbb{D}_{2n}$ in Table 3. For $j = 0, \ldots, n-1$, let p_j be the axis generated by

$$\boldsymbol{v}_j := \boldsymbol{Q}\left(\boldsymbol{k}; \frac{j\pi}{n}\right) \cdot \boldsymbol{i},$$

(see Figure 14) and q_j the axis generated by

$$\boldsymbol{w}_j := \boldsymbol{Q}\left(\boldsymbol{k}; \frac{(2j+1)\pi}{2n}\right) \cdot \boldsymbol{i}.$$

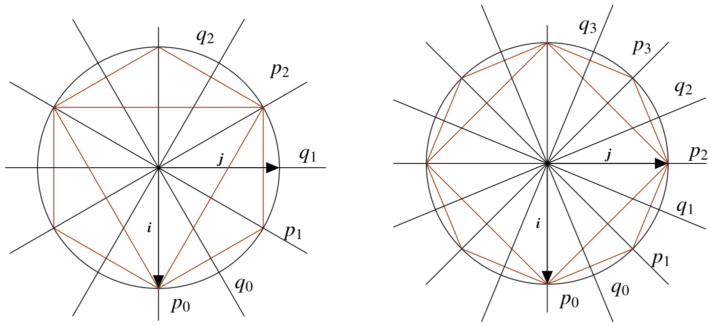

Figure 14. Left: characteristic axes for $\mathbb{D}_6 \supset \mathbb{D}_3$. Right: characteristic axes for $\mathbb{D}_8 \supset \mathbb{D}_4$.

Set

$$\mathbb{D}_n = \left\{\mathbb{1},\ \boldsymbol{Q}\left(\boldsymbol{k};\ \frac{2\pi}{n}\right),\ \boldsymbol{Q}\left(\boldsymbol{k};\ \frac{4\pi}{n}\right), \dots,\ \boldsymbol{Q}(\boldsymbol{v}_0;\pi),\ \boldsymbol{Q}(\boldsymbol{v}_1;\pi), \dots\right\},$$

and

$$-\boldsymbol{r}_n\mathbb{D}_n = \left\{-\boldsymbol{Q}\left(\boldsymbol{k};\ \frac{\pi}{n}\right), -\boldsymbol{Q}\left(\boldsymbol{k};\ \frac{3\pi}{n}\right), \dots, -\boldsymbol{Q}(\boldsymbol{w}_0;\pi), -\boldsymbol{Q}(\boldsymbol{w}_1;\pi), \dots\right\},$$

where $\boldsymbol{r}_n = \boldsymbol{Q}(\boldsymbol{k};\pi/n)$. We define

$$\mathbb{D}_{2n}^h := \mathbb{D}_n \cup (-\boldsymbol{r}_n\mathbb{D}_n), \tag{B-3}$$

which decomposes as

$$\mathbb{D}_{2n}^h = \mathbb{Z}_{2n}^- \uplus \biguplus_{j=0}^{n-1} \mathbb{Z}_2^{p_j} \uplus \biguplus_{j=0}^{n-1} \mathbb{Z}_2^{\sigma_{q_j}}. \tag{B-4}$$

Note that in this decomposition, there are n subgroups conjugate to $\mathbb{Z}_2$ and n others conjugate to $\mathbb{Z}_2^-$. The z-axis and x-axis are said to be the primary and secondary axes of $\mathbb{D}_{2n}^h$, respectively. For each $g \in \mathrm{O}(3)$, the primary and secondary axes of the subgroup $g\mathbb{D}_{2n}^h g^{-1}$ are generated by $g\boldsymbol{k}$ and by $g\boldsymbol{i}$, respectively.

Lemma B.6. *Let $m, n \geq 2$ be two integers. Set $d = \gcd(n, m)$, $d_2(m) = \gcd(m, 2)$, and*

$$i(m) = \begin{cases} 1 & \textit{if } m \textit{ is even}, \\ 2 & \textit{otherwise}. \end{cases}$$

Then:

- *If n/d or m/d is even, we have*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{Z}_{2m}^-] = \{\mathbb{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_d]\}.$$

- *If n/d and m/d are odd, we have*

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{Z}_{2m}^-] = \{\mathbb{1}, [\mathbb{Z}_{d_2(m)}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{2d}^-]\}.$$

Proof. First of all, if no characteristic axes of $\mathbb{D}_{2n}^h$ and $g\mathbb{Z}_{2m}^- g^{-1}$ match, then their intersection reduces to $\mathbb{1}$.

We now have to consider three cases:

(1) The first case is when $\mathbb{D}_{2n}^h$ and $g\mathbb{Z}_{2m}^- g^{-1}$ have the same primary axis. Then, using decompositions (B-4) and (B-1), we only have to consider the intersection

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_{2m}^-.$$

This has already been studied in the clips operation $[\mathbb{Z}_{2n}^-] \circledcirc [\mathbb{Z}_{2m}^-]$ in Corollary B.3, leading to the conjugacy class $[\mathbb{Z}_d]$ or $[\mathbb{Z}_{2d}^-]$.

(2) The second one is when some secondary axis p_j (say p_0) matches with the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$; then we only have to consider intersection

$$\mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_{2m}^- g^{-1}) = \mathbb{Z}_2^{p_0} \cap (g\mathbb{Z}_m g^{-1})$$

leading to $\mathbb{Z}_2^{p_0}$ if m is even.

(3) Finally, we have to consider the case when the primary axis of $g\mathbb{Z}_{2m}^- g^{-1}$ is q_0. In that case the problem reduces to the intersection

$$\mathbb{Z}_2^{\sigma_{q_0}} \cap (g\mathbb{Z}_{2m}^- g^{-1})$$

leading to the conjugacy class $[\mathbb{Z}_2^-]$ for odd m (see Corollary B.3). □

The cases $[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{D}_n^v]$ and $[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{D}_{2m}^h]$ are more involved. We start by formulating the following lemma, without proof (see Figure 9 for an example).

Lemma B.7. *If n is even, then there exist $p_k \perp p_l$ and $q_r \perp q_s$, where no axes p_i, q_j are perpendicular. If n is odd, there exist $p_i \perp q_j$ and no axes p_k, p_l nor q_r, q_s are perpendicular.*

Lemma B.8. *Let $m, n \geq 2$ be two integers. Set $d_2(m) := \gcd(m, 2)$,*

$$i(m,n) := \begin{cases} 2 & \text{if } m \text{ is even and } n \text{ is odd,} \\ 1 & \text{otherwise,} \end{cases}$$

and

$$i(m) := 3 - \gcd(2, m) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 2 & \text{if } m \text{ is odd.} \end{cases}$$

Then we have

$$[\mathbb{D}_{2n}^h] \circledcirc [\mathbb{D}_m^v] = \{[\mathbb{1}], [\mathbb{Z}_{i(m)}^-], [\mathbb{Z}_{d_2(m)}], [\mathbb{D}_{i(m,n)}^v], [\mathbb{Z}_d], [\mathbb{D}_d^v]\}.$$

Proof. The only nontrivial cases are when $g\mathbb{D}_m^v g^{-1}$ and $\mathbb{D}_{2n}^h$ have no matching characteristic axes. Now we have to distinguish whether the principal axis a of $g\mathbb{D}_m^v g^{-1}$ is (Oz) or not:

(1) Let's first suppose that $a = (Oz)$. In that case, we need to compute the intersection $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$. From (B-2) and (B-4), this reduces to studying the three intersections

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m, \qquad \mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}, \qquad \mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}.$$

Now:

(a) The first intersection, $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_m$, reduces to $\mathbb{Z}_n \cap \mathbb{Z}_m = \mathbb{Z}_d$ (from (B-1) and Lemma A.2).

(b) The second one, $\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b_j}}$, reduces to $\mathbb{1}$, since primary axes of $\mathbb{Z}_{2n}^-$ and $\mathbb{Z}_2^{\sigma_{b_j}}$ (conjugate to $\mathbb{Z}_2^-$) do not match.

(c) The last one, $\mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b_j}}$, can reduce to some $\mathbb{Z}_2^{\sigma_{q_k}}$ if $b_0 = q_0$. In that case, $\mathbb{D}_{2n}^h \cap \mathbb{D}_m^v$ contains $\mathbb{Z}_d$ and some $\mathbb{Z}_2^{\sigma_{q_k}}$, which generate $\mathbb{D}_d^v$ (see (B-2)).

This first case thus leads to $\mathbb{Z}_d$ or $\mathbb{D}_d^v$.

(2) Consider now the case when $a \neq (Oz)$. Thus, the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \qquad \mathbb{Z}_2^{p_j} \cap \mathbb{Z}_m^a, \qquad \mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}},$$

where the b'_j axes are the secondary axes of $g\mathbb{D}_m^v g^{-1}$. Now:

(a) First suppose that $a = p_0$ (for instance) and all the other axes are different. Then, $\mathbb{Z}_2^{p_j} \cap \mathbb{Z}_m^a = \mathbb{Z}_{d_2(m)}^a$.

(b) Suppose now that $a = p_0$ and $b'j = q_k$ for some couple (k, j), so that $\mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}} = \mathbb{Z}_2^{\sigma_{q_k}}$. As $q_k \perp p_0$, we deduce from Lemma B.7 that n is odd. All depend on the parity of m: if m is even, then Γ contains $\mathbb{Z}_2^{p_0}$ and $\mathbb{Z}_2^{\sigma_{q_k}}$, and we obtain some subgroup conjugate to $\mathbb{D}_2^v$. If m is odd, then Γ reduces to $\mathbb{Z}_2^{\sigma_{q_k}}$, which is conjugate to $\mathbb{Z}_2^-$.

(c) Finally, suppose that $a \neq p_j$ for all j (and recall that $a \neq (Oz)$), so that the intersections to be considered are

$$\mathbb{Z}_{2n}^- \cap \mathbb{Z}_2^{\sigma_{b'_0}}, \qquad \mathbb{Z}_2^{\sigma_{q_k}} \cap \mathbb{Z}_2^{\sigma_{b'_j}}.$$

We thus only obtain some subgroups already considered in the previous cases, which concludes the proof. □

Lemma B.9. *Let $m, n \geq 2$ be two integers. Set $d = \gcd(n, m)$ and*

$$\Delta = [\mathbb{D}_{2n}^h] \circledcirc [\mathbb{D}_{2m}^h].$$

Then $[\mathbb{1}] \subset \Delta$ *and*:

- *For every integer d:*
  - *If m and n are even, then* $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2]\}$.
  - *If m and n are odd, then* $\Delta \supset \{[\mathbb{Z}_2^-]\}$.
  - *Otherwise,* $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{D}_2^v]\}$.
- *If $d = 1$, then:*
  - *If m and n are odd, then* $\Delta \supset \{[\mathbb{D}_2^v]\}$.
  - *Otherwise, m or n is even and* $\Delta \supset \{[\mathbb{Z}_2], [\mathbb{Z}_2^-]\}$.
- *If $d \neq 1$, then:*
  - *If m/d and n/d are odd, then* $\Delta \supset \{[\mathbb{Z}_{2d}^-], [\mathbb{D}_{2d}^h]\}$.
  - *Otherwise, m/d or n/d is even and* $\Delta \supset \{[\mathbb{Z}_d], [\mathbb{D}_d], [\mathbb{D}_d^v]\}$.

Sketch of the proof. We consider decomposition (B-3). If no characteristic axes $\mathbb{D}_{2n}^h$ and $g\mathbb{D}_{2m}^h g^{-1}$ match, then their intersection reduces to $\mathbb{1}$. Otherwise, from (B-3) it reduces to

$$\mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_{2m}^- g^{-1}), \qquad \mathbb{Z}_{2n}^- \cap (g\mathbb{Z}_2^{\sigma_{q_k'}} g^{-1}), \qquad \mathbb{Z}_2^{p_j} \cap (g\mathbb{Z}_2^{p_k'} g^{-1}),$$
$$\mathbb{Z}_2^{\sigma_{q_j}} \cap (g\mathbb{Z}_{2m}^- g^{-1}), \qquad \mathbb{Z}_2^{\sigma_{q_j}} \cap (g\mathbb{Z}_2^{\sigma_{q_k'}} g^{-1}),$$

where all $\mathbb{Z}_2^{\sigma_{q_j}}, \mathbb{Z}_2^{\sigma_{q_k'}}$ are subgroups conjugate to $\mathbb{Z}_2^-$.

Now all these intersections have already been studied in the clips operation $[\mathbb{Z}_{2r}^-] \circledcirc [\mathbb{Z}_{2s}^-]$. We can thus use Corollary B.3, argue on the characteristic axes, and use Lemma B.7 to conclude the proof in each case. □

$\mathbb{O}^-$ ***subgroup.*** Consider the couple $\mathbb{T} \subset \mathbb{O}$ in Table 3 and the decompositions

$$\mathbb{O} = \biguplus_{i=1}^{3} \mathbb{Z}_4^{\boldsymbol{fc}_i} \uplus \biguplus_{j=1}^{4} \mathbb{Z}_3^{\boldsymbol{vc}_j} \uplus \biguplus_{l=1}^{6} \mathbb{Z}_2^{\boldsymbol{ec}_l},$$

and

$$\mathbb{T} = \biguplus_{j=1}^{4} \mathbb{Z}_3^{\boldsymbol{vt}_j} \uplus \mathbb{Z}_2^{\boldsymbol{et}_1} \uplus \mathbb{Z}_2^{\boldsymbol{et}_2} \uplus \mathbb{Z}_2^{\boldsymbol{et}_3}, \qquad \mathbb{Z}_2^{\boldsymbol{et}_i} \subset \mathbb{Z}_4^{\boldsymbol{fc}_i}, \ i = 1, 2, 3.$$

This leads (see [Ihrig and Golubitsky 1984] for details) to the decomposition

$$\mathbb{O}^- := \biguplus_{i=1}^{3} (\mathbb{Z}_4^{\boldsymbol{fc}_i})^- \uplus \biguplus_{j=1}^{4} \mathbb{Z}_3^{\boldsymbol{vc}_j} \uplus \biguplus_{l=1}^{6} \mathbb{Z}_2^{\sigma_{\boldsymbol{ec}_l}}, \tag{B-5}$$

where $(\mathbb{Z}_4^{\boldsymbol{fc}_i})^-$ is the subgroup conjugate to $\mathbb{Z}_4^-$ with $\boldsymbol{fc}_i$ as primary axis. Note also that $\mathbb{Z}_2^{\sigma_{e_l}}$ are subgroups conjugate to $\mathbb{Z}_2^-$ with $\boldsymbol{ec}_l$ as primary axis.

Using this decomposition (B-5) and those of type III closed O(3) subgroups previously mentioned directly lead to the following corollaries.

Corollary B.10. *Let $n \geq 2$ be an integer, and set $d_2(n) = \gcd(n, 2)$ and $d_3(n) = \gcd(3, n)$. Then:*

- *If n is odd, we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{Z}_{2n}^-] = \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}]\}.$$

- *If $n = 2 + 4k$ for $k \in \mathbb{N}$, we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{Z}_{2n}^-] = \{[\mathbb{1}], [\mathbb{Z}_4^-], [\mathbb{Z}_{d_3(n)}]\}.$$

- *If n is even and 4 does not divide n, we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{Z}_{2n}^-] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_{d_3(n)}]\}.$$

Moreover, in all cases, we have

$$[\mathbb{O}^-] \circledcirc [\mathbb{D}_n^v] = \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v], [\mathbb{Z}_{d_2(n)}], [\mathbb{D}_{d_2(n)}^v]\}.$$

Corollary B.11. *Let $n \geq 2$ be an integer and $d_3(n) := \gcd(n, 3)$.*

- *If n is even and $n = 2 + 4k$ for $k \in \mathbb{N}$, then we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{D}_{2n}^h] = \{[\mathbb{1}], [\mathbb{Z}_4^-], [\mathbb{D}_4^h], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

- *If n is even and 4 divides n, then we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{D}_{2n}^h] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

- *If n is odd, then we have*

$$[\mathbb{O}^-] \circledcirc [\mathbb{D}_{2n}^h] = \{[\mathbb{1}], [\mathbb{Z}_2], [\mathbb{Z}_2^-], [\mathbb{D}_2], [\mathbb{D}_2^v], [\mathbb{Z}_{d_3(n)}], [\mathbb{D}_{d_3(n)}^v]\}.$$

Corollary B.12. $[\mathbb{O}^-] \circledcirc [\mathbb{O}^-] = \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{Z}_4^-], [\mathbb{Z}_3]\}.$

$\mathrm{O}(2)^-$ *subgroup.* Consider the couple $\mathrm{SO}(2) \subset \mathrm{O}(2)$ in Table 3 and set

$$\mathrm{O}(2)^- := \mathrm{SO}(2) \uplus \biguplus_{v \subset xy\text{-plane}} \mathbb{Z}_2^{\sigma_v}. \tag{B-6}$$

As $\mathbb{Z}_2^{\sigma_v}$ are subgroups conjugate to $\mathbb{Z}_2^-$, previous results on the clips operation of $[\mathbb{Z}_2^-]$ and type III subgroups except $[\mathrm{O}(2)^-]$ leads to the following lemma.

Lemma B.13. *Let $n \geq 2$ be some integer. Set $d_2(n) := \gcd(2, n)$ and*

$$i(n) := 3 - \gcd(2, n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Then, we have

$$[\mathrm{O}(2)^-] \circledcirc [\mathbb{Z}_{2n}^-] = \{[\mathbb{1}], [\mathbb{Z}_{i(n)}^-], [\mathbb{Z}_n]\},$$
$$[\mathrm{O}(2)^-] \circledcirc [\mathbb{D}_n^v] = \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{D}_n^v]\},$$
$$[\mathrm{O}(2)^-] \circledcirc [\mathbb{D}_{2n}^h] = \{[\mathbb{1}], [\mathbb{Z}_{d_2(n)}], [\mathbb{Z}_2^-], [\mathbb{D}_{i(n)}^v], [\mathbb{D}_n^v]\},$$
$$[\mathrm{O}(2)^-] \circledcirc [\mathbb{O}^-] = \{[\mathbb{1}], [\mathbb{Z}_2^-], [\mathbb{D}_3^v], [\mathbb{D}_2^v]\},$$
$$[\mathrm{O}(2)^-] \circledcirc [\mathrm{O}(2)^-] = \{[\mathbb{Z}_2^-], [\mathrm{O}(2)^-]\}.$$

Sketch of proof. We will only focus on the clips operation $[\mathrm{O}(2)^-] \circledcirc [\mathbb{D}_{2n}^h]$ and consider thus intersections

$$\mathrm{O}(2)^- \cap (g\mathbb{D}_{2n}^h g^{-1}), \quad g \in \mathrm{O}(3).$$

There are only two nontrivial cases to work on, whether characteristic axes match or not.

(1) If primary axes match, then by (B-4) and (B-6) we must consider intersections

$$\mathbb{Z}^{\sigma_v} \cap \mathbb{Z}_{2n}^-, \qquad \mathrm{SO}(2) \cap \mathbb{Z}_2^{p_j}, \qquad \mathbb{Z}^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}, \tag{B-7}$$

which reduce to $\mathbb{D}_n^v$ (see decomposition (B-2)).

(2) Suppose, moreover, that $p_0 = (Oz)$, in which case $\mathrm{SO}(2) \cap \mathbb{Z}_2^{p_j} = \mathbb{Z}_2$.

(a) For n odd, there exists some secondary axes q_k in the xy-plane (see Lemma B.7) and thus $\mathbb{Z}^{\sigma_b} \cap \mathbb{Z}_2^{\sigma_{q_j}}$ reduces to $\mathbb{Z}_2^{\sigma_{q_k}}$. Moreover, $\mathbb{Z}^{\sigma_v} \cap \mathbb{Z}_{2n}^-$ reduces to some $\mathbb{Z}_2^{\sigma_v}$ with v perpendicular to p_0 and q_j and the final result is a subgroup conjugate to $\mathbb{D}_2^v$.

(b) For n even, we obtain $\mathbb{Z}_2$.

(3) If now the primary axis a of $g\mathbb{D}_{2n}^h g^{-1}$ is (Ox), and no other characteristic axes correspond to (Oz) nor (Oy), then intersections (B-7) reduce to $\mathbb{Z}_2^{\sigma_a} \cap (g\mathbb{Z}_{2n}^- g^{-1})$, which is conjugate to $\mathbb{Z}_2^-$. □

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