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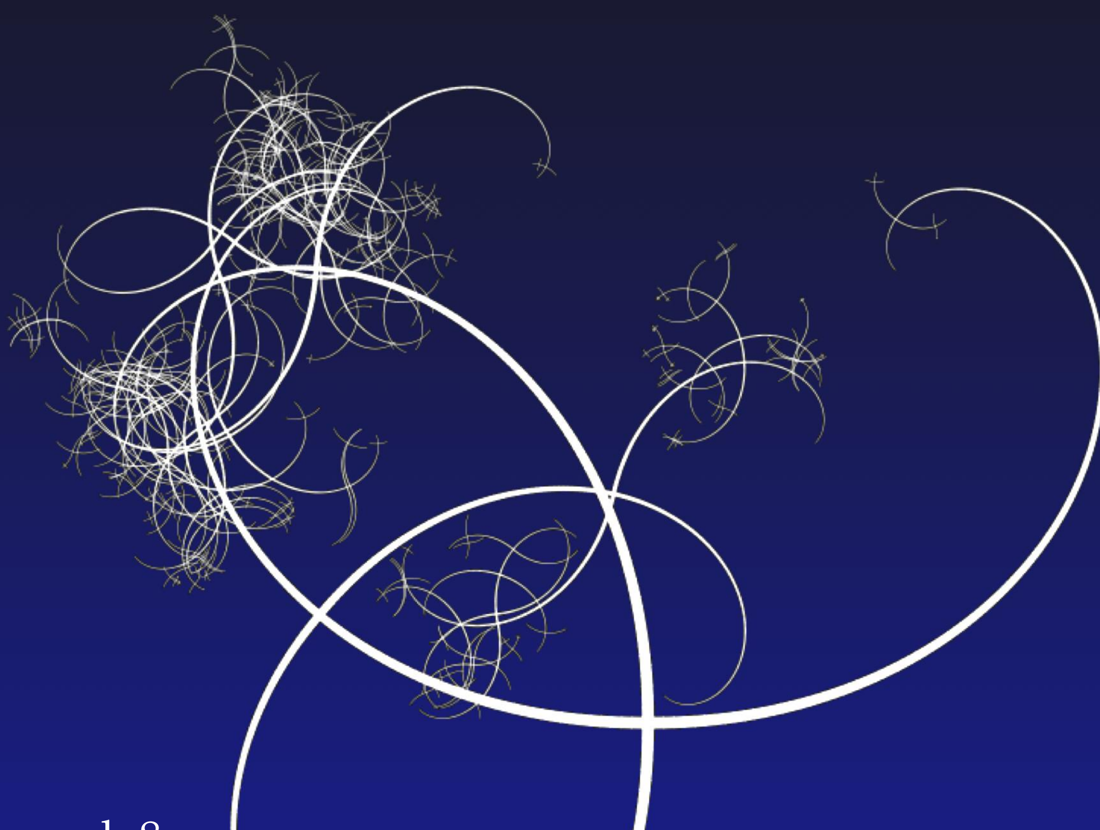

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IBVP FOR ELECTROMAGNETO-ELASTIC MATERIALS: VARIATIONAL APPROACH

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This paper aims to establish a variational framework for materials having coupling interactions between electromagnetic and mechanical fields. Based on coupled constitutive equations and the alternative field equations, a general variational form, imposing no restriction on the fields involved, is given. Subsequently, the result is derived for the case when satisfaction of the strain-displacement equation is presumed as a restriction. Next, the variational forms for kinematically admissible processes and, in turn, for kinematically admissible displacement-potential processes are found. Finally, the principles characterizing the stress field instead of the displacement field are formulated. The results of the present work provide a framework in which the satisfaction of initial boundary conditions is inherently considered. The proposed framework furnishes an alternative path for the implementation of numerical approaches for PDEs governing the motion of electromagneto-elastic materials.

1. Introduction

Electromagneto-elastic materials, a category of materials that contains both piezoelectric and piezomagnetic phases, are being widely used in several devices including ultrasonic transducers and microactuators, thermal-imaging devices, health-monitoring devices, biomedical devices, biomimetics, and energy harvesting [Li 2003; Miehe et al. 2011]. Also, these materials have found applications in microwave electronic and optoelectronic instruments because of their flat frequency response as well as the capability of energy conversion [Li and Kardomateas 2006]. Consequently, to mathematically understand the physics of such materials, several studies have been carried out by employing a continuum approach in which classical laws are generalized to account for the coupling between mechanical, electric, and magnetic fields. Some of the most prominent contributions in this regard can be found in [Guggenheim 1936a; 1936b; Penfield et al. 1963; Brown 1966; Coleman

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and Dill 1971; Tiersten and Tsai 1972; Mindlin 1972; Nelson 1979; Maugin 1988; Eringen and Maugin 1990; Landau et al. 1984].

In the linearized classical isothermal continuum mechanics, the governing equations to model a physical phenomenon are the balance of mass and balances of linear and angular momenta along with desired constitutive equations for the phenomenon at hand. These, accompanied by the initial and boundary conditions, often lead to a mixed initial-boundary value problem (IBVP) in terms of the displacement field. However, [Ignaczak 1959; 1963] proposed a robust alternative approach in which the governing equations are formulated in terms of stress. This formulation motivated several researchers to assess the potential of this strategy, which offers a much more straightforward framework when the boundary conditions are of Neumann type. For a comprehensive review of those works, the readers are referred to the review paper [Ostoja-Starzewski 2019]. Among them, the most pioneering works are convolutional variational principles, i.e., variational principles containing convolution products with respect to time, developed in [Gurtin 1963; 1964]. In these works, the framework has been rationally developed to be applicable for mixed initial-boundary value problems, leading to integro-partial-differential equations and the corresponding convolutional variational principles. From there on, [Nickell and Sackman 1968] generalized Gurtin's work to thermoelasticity and, subsequently, a specific form of such formulation has been obtained for piezoelectric materials in [Oden and Reddy 1983]. Recently, one can note the results given in [El-Karamany and Ezzat 2011] for two-temperature thermoelasticity.

Owing to the fact that the analytical methods are only sufficient tools for problems with simple geometry and rather strict assumptions, variational principles are of great importance in engineering sciences as they pave the way for developing numerical approaches to solve PDEs with either more relaxed assumptions or arbitrary/complicated boundary conditions. The finite element, mesh-free particle, and Ritz methods are examples stemming from variational principles. As an alternative application of such principles, we note the homogenization theory which supplies bounds for properties of materials (e.g., [Hashin and Shtrikman 1962]). In the case of solid mechanics, however, the classical variants of seminal work of [Washizu 1957] are not applicable to the case of a mixed problem of elastodynamics since the prescribed initial velocity is not realized and the knowledge of displacement field at a later time is only presupposed [Gurtin 1964]. Therefore, starting with [Gurtin 1964], as an alternative approach appropriate to elastodynamics, convolutional variational principles have been developed. The elegance of the approach consists of imposing the initial conditions implicitly in the form of a body force and thus assuring appropriate satisfaction of them.

Concerning the variational principles for electromagneto-elastic materials, several studies have been carried out dating back to [Toupin 1956].Variational principles for various problems including piezoelectric ceramics have been proposed in [He 2000; 2001a; 2001b; 2001c]. Convolutional regionwise variational principles for thermopiezoelectric media can be found in [Bo 2003]. Also, for the nonlinear case, some studies have been done, e.g., by making use of the first law of thermodynamics in [Kuang 2008]. Convolutional variational principles have then been proposed for the case of nonlinear electromagneto-elastic solids in [Wang et al. 2010]. Recently, on the basis of incremental variational principles, a general framework for functional dissipative materials has been obtained in [Miehe et al. 2011] and employed in [Miehe and Rosato 2011] to analyze piezoelectric ceramics.

To the best of the authors' knowledge, a comprehensive and systematic generalization of the results initially developed in [Gurtin 1964] for the case of linear electromagneto-elastic materials is not available in the literature and this challenge defines the focus of the present study. Indeed, it is of interest to enrich the numerical framework relevant to the analysis of electromagneto-elastic materials because of the progressive increase in the application of such materials in the realm of structural mechanics; see, for example, a recent review [Irschik et al. 2010]. As an example of the recent development in the use of smart materials, one can mention the recent paper [Schoeftner and Irschik 2016] in which the design of piezoelectric devices controlling the level of stress in thin bars has been discussed. The methodology to form and prove the results obtained in this study, similar to the presentation given in [Nickell and Sackman 1968], is based on [Gurtin 1964]. For the sake of completeness of the presentation, we collect in Appendix A the mathematical concepts and lemmas originally proved in [Gurtin 1964] along with a corollary obtained in [Nickell and Sackman 1968]. Alternative field equations, the main ground for establishing the corresponding convolutional variational principles, for electromagneto-elastic materials are described in Appendix B. Field equations, along with some continuity conditions, are given in detail in Section 2. Subsequently, analogously to [Gurtin 1964], the convolutional variational forms of the alternative integro-partial-differential field equations are obtained and proved comprehensively in Section 3. As mentioned earlier, the derivations in the main body of the present study can be useful in the sense of analysis and design of electromagneto-elastic materials in both practice and research.

2. Problem statement

In this section, the field equations for an electromagneto-elastic material are listed. Throughout the paper we indicate the position vector and time parameter, respectively, by $\boldsymbol{x}$ and t. Also, the standard index notation for Cartesian tensors is used. The mathematical notation used herein along with some lemmas and theorems that are the primary tools to obtain the results of this paper can be found in Appendix A.

Let $\overline{V}$ denote a closed and bounded subset of 3D Euclidean space, with interior V and boundary ∂V occupied by a deformable electromagneto-elastic material. Furthermore, assume that V is a regular region in the sense of [Gurtin 1964]. Let $u_i(\boldsymbol{x}, t)$, $\sigma_{ij}(\boldsymbol{x}, t)$, $\varepsilon_{ij}(\boldsymbol{x}, t)$, $f_i(\boldsymbol{x}, t)$, $E_i(\boldsymbol{x}, t)$, $D_i(\boldsymbol{x}, t)$, $B_i(\boldsymbol{x}, t)$, and $H_i(\boldsymbol{x}, t)$ with $(\boldsymbol{x}, t) \in \overline{V} \times [0, \infty)$, respectively, denote the components of the displacement vector, stress tensor, strain tensor, body force, electric field, electric displacement field, magnetic field, and magnetic field strength. In addition, scalar fields $\rho(\boldsymbol{x}, t)$, $q_e(\boldsymbol{x}, t)$, $\varphi(\boldsymbol{x}, t)$, and $\psi(\boldsymbol{x}, t)$ denote the mass density, charge density, electric potential, and magnetic potential, respectively.

Let ∂V_i denote a subset of ∂V over which $i = \boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \psi, \boldsymbol{B}$ is prescribed with the condition

$$\begin{aligned} &\partial V_{\boldsymbol{u}} \cup \partial V_{\boldsymbol{\sigma}} = \partial V, \qquad && \partial V_{\boldsymbol{u}} \cap \partial V_{\boldsymbol{\sigma}} = \varnothing, \\ &\partial V_{\varphi} \cup \partial V_{\boldsymbol{D}} = \partial V, && \partial V_{\varphi} \cap \partial V_{\boldsymbol{D}} = \varnothing, \\ &\partial V_{\psi} \cup \partial V_{\boldsymbol{B}} = \partial V, && \partial V_{\psi} \cap \partial V_{\boldsymbol{B}} = \varnothing. \end{aligned} \tag{2-1}$$

Moreover, the symbol $\overline{\partial V_i}$ with $i = \boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \psi, \boldsymbol{B}$ stands for the closure of the aforementioned sets. Furthermore, the quasistatic electromagnetic condition is presumed; that is, it is assumed that the electric and magnetic fields are both curl free. This approximation leads to accurate results for instance, as a particular case, in analysis of nonmagnetizable elastic dielectrics when the wavelengths of mechanical waves are negligible if compared to wavelengths of electromagnetic waves of the same frequency [Eringen and Maugin 1990]. Accordingly, the governing equations read [Li 2003]

$$\begin{aligned} &\rho \ddot{u}_i = \sigma_{ij,j} + F_i \quad && \text{on } V \times (0, \infty), \\ &D_{i,i} = q_e && \text{on } V \times (0, \infty), \\ &B_{i,i} = 0 && \text{on } V \times (0, \infty), \end{aligned} \tag{2-2}$$

in which $\sigma_{ij} = \sigma_{ji}$. Also, kinematic equations are

$$\begin{aligned} &\epsilon_{ij} = u_{(i,j)} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad && \text{on } V \times (0, \infty), \\ &E_i = -\varphi_{,i} && \text{on } V \times (0, \infty), \\ &H_i = -\psi_{,i} && \text{on } V \times (0, \infty), \end{aligned} \tag{2-3}$$

in which $u_{(i,j)}$ denotes the symmetric part of the second-order tensor $u_{i,j}$.

Next, the constitutive equations need to be set. To that end, one needs to define which of the physical quantities are dependent variables and which are independent ones. Thus, one can define various forms of constitutive equations based on different independent variables. In general, in a nonlinear theory, it is a difficult task to obtain one form of the constitutive equations from the other. Nevertheless, in linear theory, the necessary Legendre transformations are easy to manipulate and readily obtain the desired variants of constitutive equations [Pérez-Fernández et al. 2009]. Assuming the isothermal condition and hyperelasticity, one can obtain the relations

$$\begin{aligned} \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e^{E}_{kij}E_k - e^{H}_{kij}H_k && \text{on } V \times (0, \infty), \\ D_i &= e^{E}_{ikl}\varepsilon_{kl} + \kappa^{E}_{ij}E_j + \kappa^{EH}_{ij}H_j && \text{on } V \times (0, \infty), \\ B_i &= e^{H}_{ikl}\varepsilon_{kl} + \kappa^{EH}_{ji}E_j + \kappa^{H}_{ij}H_j && \text{on } V \times (0, \infty), \end{aligned} \tag{2-4}$$

with symmetry conditions

$$\begin{aligned} C_{ijkl} = C_{klij} = C_{jikl} &= C_{ijlk} && \text{on } V, \\ e^{E}_{kij} = e^{E}_{kji}, \qquad e^{H}_{kij} &= e^{H}_{kji} && \text{on } V, \\ \kappa^{E}_{ij} = \kappa^{E}_{ji}, \qquad \kappa^{H}_{ij} &= \kappa^{H}_{ji} && \text{on } V. \end{aligned} \tag{2-5}$$

Applying the Legendre transformation, one can find

$$\begin{aligned} \varepsilon_{ij} &= S_{ijkl}\sigma_{kl} + d^{E}_{kij}E_k + d^{H}_{kij}H_k && \text{on } V \times (0, \infty), \\ D_i &= d^{E}_{ikl}\sigma_{kl} + \chi^{E}_{ij}E_j + \chi^{EH}_{ij}H_j && \text{on } V \times (0, \infty), \\ B_i &= d^{H}_{ikl}\sigma_{kl} + \chi^{EH}_{ji}E_j + \chi^{H}_{ij}H_j && \text{on } V \times (0, \infty), \end{aligned} \tag{2-6}$$

with symmetry conditions

$$\begin{aligned} S_{ijkl} = S_{klij} = S_{jikl} &= S_{ijlk} && \text{on } V, \\ d^{E}_{kij} = d^{E}_{kji}, \qquad d^{H}_{kij} &= d^{H}_{kji} && \text{on } V, \\ \chi^{E}_{ij} = \chi^{E}_{ji}, \qquad \chi^{H}_{ij} &= \chi^{H}_{ji} && \text{on } V. \end{aligned} \tag{2-7}$$

In (2-4) and (2-6) the coefficients S_{ijkl}, C_{ijkl}, d^{E}_{kij}, d^{H}_{kij}, e^{E}_{kij}, e^{H}_{kij}, χ^{E}_{ij}, χ^{H}_{ij}, κ^{E}_{ij}, κ^{H}_{ij}, χ^{EH}_{ij}, and κ^{EH}_{ij}, all functions of position, represent, respectively, components of the compliance tensor, stiffness tensor, direct piezoelectric tensor, direct piezomagnetic tensor, reverse piezoelectric tensor, reverse piezomagnetic tensor, permittivity under constant stress, permeability under constant stress, permittivity under constant strain, permeability under constant strain, magnetoelectric tensor under constant stress, and magnetoelectric tensor under constant strain. The aforementioned variables are related as

$$\begin{aligned} S_{ijkl}C_{ijpq} &= \delta_{kp}\delta_{lq} && \text{on } V, \\ d^{E}_{kij} &= S_{pqij}e^{E}_{kpq} && \text{on } V, \\ d^{H}_{kij} &= S_{pqij}e^{H}_{kpq} && \text{on } V, \\ \chi^{E}_{ij} &= S_{pqrs}e^{E}_{ipq}e^{E}_{jrs} + \kappa^{E}_{ij} && \text{on } V, \\ \chi^{H}_{ij} &= S_{pqrs}e^{H}_{ipq}e^{H}_{jrs} + \kappa^{H}_{ij} && \text{on } V, \\ \chi^{EH}_{ij} &= S_{pqrs}e^{H}_{jpq}e^{E}_{irs} + \kappa^{EH}_{ij} && \text{on } V. \end{aligned} \tag{2-8}$$

To form the IBVP, define the boundary conditions

$$
\begin{aligned}
&\text{mechanical boundary conditions} \begin{cases} u_i = \hat{u}_i(\boldsymbol{x}, t) & \text{on } \partial V_{\boldsymbol{u}} \times [0, \infty), \\ t_i = \hat{t}_i(\boldsymbol{x}, t) & \text{on } \partial V_{\boldsymbol{\sigma}} \times [0, \infty), \end{cases} \\
&\text{electric boundary conditions} \begin{cases} \varphi = \hat{\varphi}(\boldsymbol{x}, t) & \text{on } \partial V_{\varphi} \times [0, \infty), \\ d = \hat{d}(\boldsymbol{x}, t) & \text{on } \partial V_{\boldsymbol{D}} \times [0, \infty), \end{cases} \\
&\text{magnetic boundary conditions} \begin{cases} \psi = \hat{\psi}(\boldsymbol{x}, t) & \text{on } \partial V_{\psi} \times [0, \infty), \\ b = \hat{b}(\boldsymbol{x}, t) & \text{on } \partial V_{\boldsymbol{B}} \times [0, \infty), \end{cases}
\end{aligned}
\tag{2-9}
$$

where $t_i = \sigma_{ij} n_j$, $d = D_i n_i$, $b = B_i n_i$, and the initial conditions are

$$
\begin{aligned}
u_i(\boldsymbol{x}, 0) &= u_i^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{V}, \\
\dot{u}_i(\boldsymbol{x}, 0) &= v_i^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{V}.
\end{aligned}
\tag{2-10}
$$

In (2-9), $\hat{u}_i(\boldsymbol{x}, t)$, $\hat{t}_i(\boldsymbol{x}, t)$, $\hat{\varphi}(\boldsymbol{x}, t)$, $\hat{d}(\boldsymbol{x}, t)$, $\hat{\psi}(\boldsymbol{x}, t)$, and $\hat{b}(\boldsymbol{x}, t)$ are, respectively, the prescribed displacement components, traction components, electric potential, electric displacement, magnetic potential, and magnetic field over the boundary. In addition, by the displacement-potential boundary conditions we mean

$$
\begin{aligned}
u_i &= \hat{u}_i(\boldsymbol{x}, t) \quad \text{on } \partial V_{\boldsymbol{u}} \times [0, \infty), \\
\varphi &= \hat{\varphi}(\boldsymbol{x}, t) \quad \text{on } \partial V_{\varphi} \times [0, \infty), \\
\psi &= \hat{\psi}(\boldsymbol{x}, t) \quad \text{on } \partial V_{\psi} \times [0, \infty).
\end{aligned}
\tag{2-11}
$$

Similar to [Gurtin 1964], for reference in the remainder of the paper, the regularity assumptions are listed here:

(i) $\rho > 0$ is continuously differentiable on $\overline{V}$,

(ii) $\boldsymbol{C}$, $\boldsymbol{e}$, $\boldsymbol{\kappa}$ and $\boldsymbol{S}$, $\boldsymbol{d}$, $\boldsymbol{\chi}$ are continuously differentiable on $\overline{V}$ and meet (2-5), (2-7), and (2-8).

(iii) $\boldsymbol{u}^0(\boldsymbol{x})$ is continuously differentiable on $\overline{V}$,

(iv) $\boldsymbol{v}^0(\boldsymbol{x})$ is continuously differentiable on $\overline{V}$,

(v) $\boldsymbol{f}$ and q_e are continuously differentiable on $\overline{V}$,

(vi) $\hat{\boldsymbol{u}}$, $\hat{\varphi}$, and $\hat{\psi}$ are continuous on $\overline{\partial V_{\boldsymbol{u}}} \times [0, \infty)$, $\overline{\partial V_{\boldsymbol{D}}} \times [0, \infty)$, and $\overline{\partial V_{\boldsymbol{B}}} \times [0, \infty)$, respectively, and

(vii) $\hat{\boldsymbol{t}}$, $\hat{d}$, and $\hat{b}$ are piecewise continuous on $\overline{\partial V_{\boldsymbol{\sigma}}} \times [0, \infty)$, $\overline{\partial V_{\boldsymbol{D}}} \times [0, \infty)$, and $\overline{\partial V_{\boldsymbol{B}}} \times [0, \infty)$, respectively.

Since our goal is to obtain variational principles for electromagneto-elastic materials, analogous to [Gurtin 1964], we define what we mean by an admissible process:

Definition. An ordered array $S = [\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \varphi, \psi]$ is called an admissible process on $\overline{V} \times [0, \infty)$ provided that $u_i \in C^{1,2}$, $\varepsilon_{ij} \in C^{0,0}$, $\sigma_{ij} \in C^{1,0}$, $D_i \in C^{1,0}$, $B_i \in C^{1,0}$, $\varphi \in C^{1,0}$, $\psi \in C^{1,0}$, $E_i \in C^{0,0}$, $H_i \in C^{0,0}$, $\varepsilon_{ij} = \varepsilon_{ji}$, and $\sigma_{ij} = \sigma_{ji}$.

In addition, a solution of the mixed initial-boundary value problem (i.e., IBVP) is an admissible process which meets (2-2), (2-3), (2-4), (2-9), and (2-10).

3. Variational principles

Now, for an electromagneto-elastic material, convolutional variational forms, originally developed for the case of pure elasticity in [Gurtin 1964], will be derived. The first part of the results, i.e., Theorems 1, 2, and 3, is devoted to the characterization of $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \varphi, \psi]$. In the second part, as a corollary of Theorem 3, a variational form characterizing $[\boldsymbol{u}, \varphi, \psi]$ is set. Finally, the results of the third part are applied to the characterization of $[\boldsymbol{\sigma}, \varphi, \psi]$. In the results, t_i, $\tilde{t}_i$, d, $\tilde{d}$, b, and $\tilde{b}$ will be consistently used in place of $\sigma_{ij}n_j$, $\tilde{\sigma}_{ij}n_j$, $D_i n_i$, $\widetilde{D}_i n_i$, $B_i n_i$, and $\tilde{B}_i n_i$, respectively. Additionally, the definitions of $h(t)$ and $f_i^b(\boldsymbol{x}, t)$, which shall be used in the sequel, have been given in (B-1).

3.1. *Variational principles characterizing* $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$.

First let us derive a general form which imposes no restriction on the fields:

Theorem 1. *Let Ω denote the set of all admissible processes. Let $S = [\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \varphi, \psi]$ be an element of Ω and define the functional ϑ_t on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$
\begin{aligned}
\vartheta_t(S) = {} & \frac{1}{2}\int_V C_{ijkl}(\boldsymbol{x})[h * \varepsilon_{ij} * \varepsilon_{kl}](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V [h * \sigma_{ij} * \varepsilon_{ij}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \int_V e^E_{kij}(\boldsymbol{x})[h * \varepsilon_{ij} * E_k](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V e^H_{kij}(\boldsymbol{x})[h * \varepsilon_{ij} * H_k](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \frac{1}{2}\int_V \kappa^E_{ij}(\boldsymbol{x})[h * E_i * E_j](\boldsymbol{x}, t)\, d\boldsymbol{x} - \frac{1}{2}\int_V \kappa^H_{ij}(\boldsymbol{x})[h * H_i * H_j](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \int_V \kappa^{EH}_{ij}(\boldsymbol{x})[h * E_i * H_j](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_V [h * D_i * E_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \int_V [h * H_i * B_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V [h * (D_{i,i} - q_e) * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \int_V [h * B_{i,i} * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x} + \frac{1}{2}\int_V \rho(\boldsymbol{x})[u_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \int_V [(h * \sigma_{ij,j} + f_i^b) * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{\sigma}}} [h * (t_i - \hat{t}_i) * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \int_{\partial V_{\boldsymbol{u}}} [h * t_i * \hat{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\varphi}} [h * d * \hat{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \int_{\partial V_{\psi}} [h * b * \hat{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{D}}} [h * (d - \hat{d}) * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \int_{\partial V_{\boldsymbol{B}}} [h * (b - \hat{b}) * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x}. \qquad (3\text{-}1)
\end{aligned}
$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta\vartheta_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{D}}, \widetilde{\boldsymbol{B}}, \tilde{\varphi}, \tilde{\psi}] \in \Omega$ be an admissible process and suppose that $S + \lambda\tilde{S} \in \Omega$ for all real values of λ. Using (3-1), (A-5), the symmetry condition (2-5), the divergence theorem, and properties of convolution product given in Appendix A, we obtain

$$\begin{aligned}
\delta_{\tilde{S}}\vartheta_t(S) = &\int_V [h * (C_{ijkl}\varepsilon_{kl} - e^E_{kij}E_k - e^H_{kij}H_k - \sigma_{ij}) * \tilde{\varepsilon}_{ij}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+ \int_V [h*(-e^E_{kij}\varepsilon_{ij} - \kappa^E_{ik}E_i - \kappa^{EH}_{ki}H_i + D_k)*\tilde{E}_k](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+ \int_V [h*(-e^H_{kij}\varepsilon_{ij} - \kappa^{EH}_{ik}E_i - \kappa^H_{ik}H_i + B_k)*\tilde{H}_k](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+ \int_V [h*(E_i + \varphi_{,i})*\widetilde{D}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_V [h*(H_i + \psi_{,i})*\tilde{B}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&- \int_V [h*(D_{i,i} - q_e)*\tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V [h*B_{i,i}*\tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+ \int_V [h*(u_{(i,j)} - \varepsilon_{ij})*\tilde{\sigma}_{ij}](\boldsymbol{x}, t) d\boldsymbol{x} - \int_V [(h*\sigma_{ij,j} + f^b_i - \rho u_i)*\tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&- \int_{\partial V_{\boldsymbol{u}}} [h*(u_i - \hat{u}_i)*\tilde{t}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_\sigma} [h*(t_i - \hat{t}_i)*\tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&- \int_{\partial V_\varphi} [h*(\varphi - \hat{\varphi})*\tilde{d}](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_\psi} [h*(\psi - \hat{\psi})*\tilde{b}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+ \int_{\partial V_{\boldsymbol{D}}} [h*(d - \hat{d})*\tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{B}}} [h*(b - \hat{b})*\tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x}. \qquad \text{(3-2)}
\end{aligned}$$

First, based on Theorem B.2, for every $\tilde{S} \in \Omega$ $(0 \le t < \infty)$ we immediately find $\delta_{\tilde{S}}\vartheta_t(S) = 0$ when S is a solution of the IBVP, implying $\delta\vartheta_t(S) = 0$ over Ω. Conversely, suppose $\delta\vartheta_t(S) = 0$ over Ω. Let $\tilde{S} = [\tilde{\boldsymbol{u}}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0, 0] \in \Omega$ where $\tilde{\boldsymbol{u}}$ and all its spatial derivatives are identical to zero on $\partial V \times [0, \infty)$. Then, based on $\delta\vartheta_t(S) = 0$, (3-2), and Lemma A.1, we obtain $h * \sigma_{ij,j} + f^b_i - u_i = 0$ on $V \times [0, \infty)$. Next, suppose $\tilde{\boldsymbol{u}}$ and all its spatial derivatives are identical to zero on $\partial V_{\boldsymbol{u}} \times [0, \infty)$. Based on Lemma A.2, $h * \sigma_{ij,j} + f^b_i - u_i = 0$ on $V \times [0, \infty)$, $\delta\vartheta_t(S) = 0$, and (3-2), we have $h * (t_i - \hat{t}_i) = 0$ on $\partial V_\sigma \times [0, \infty)$. Since $h \neq 0$, the property of convolution reads $(t_i - \hat{t}_i) = 0$ on $\partial V_\sigma \times [0, \infty)$. Considering (2-5), by the same token, $-e^E_{kij}\varepsilon_{ij} - \kappa^E_{ki}E_i - \kappa^{EH}_{ki}H_i + D_k = 0$ on $V \times (0, \infty)$ and $-e^H_{kij}\varepsilon_{ij} - \kappa^{EH}_{ik}E_i - \kappa^H_{ki}H_i + B_k = 0$ on $V \times (0, \infty)$ can be obtained. With the same logic mentioned so far, one can readily deduce $(2\text{-}2)_{2\text{–}3}$, $(2\text{-}3)_{2\text{–}3}$, and $(2\text{-}9)_{4,6}$. Next, let $\tilde{S} = [\boldsymbol{0}, \tilde{\boldsymbol{\varepsilon}}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}, 0, 0] \in \Omega$ in which $\tilde{\boldsymbol{\varepsilon}}$ is a symmetric second-order tensor and zero-valued on the whole boundary at all times. Thus, the symmetry of the constitutive equations and symmetry of $\boldsymbol{\sigma}$,

$\delta\vartheta_t(S)=0$, (3-2), and Lemma A.1 imply $h*(C_{ijkl}\varepsilon_{kl}-e^E_{kij}E_k-e^H_{kij}H_k-\sigma_{ij})=0$ on $V\times[0,\infty)$, leading to $(2\text{-}4)_1$. Similarly, by taking $\tilde{S}=[\mathbf{0},\mathbf{0},\tilde{\boldsymbol{\sigma}},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},0,0]$ in which $\tilde{\boldsymbol{\sigma}}$ is a symmetric second-order tensor and zero-valued on the whole boundary at all times, considering symmetry of ϵ_{ij}, we conclude $(2\text{-}3)_1$. Moreover, let us define $\tilde{S}=[\mathbf{0},\mathbf{0},\tilde{\boldsymbol{\sigma}},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},0,0]\in\Omega$ in which $\tilde{\boldsymbol{\sigma}}$ is a symmetric second-order tensor and zero-valued on the boundary $\partial V_{\boldsymbol{\sigma}}$ at all times. By taking into account $(2\text{-}3)_1$, $\delta\vartheta_t(S)=0$, (3-2), and Lemma A.3, we immediately find $u_i-\hat{u}_i=0$ on $\partial V_{\boldsymbol{u}}\times[0,\infty)$. Also, by having $\tilde{S}=[\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\widetilde{\boldsymbol{D}},\mathbf{0},0,0]$ in which $\widetilde{\boldsymbol{D}}$ is zero on $\partial V_{\boldsymbol{D}}$ at all times, using $(2\text{-}3)_2$, $\delta\vartheta_t(S)=0$, (3-2), and the Corollary A.4, we conclude $h*(\varphi-\hat{\varphi})=0$ on $\partial V_\varphi\times[0,\infty)$, which implies $(2\text{-}9)_3$. In a similar fashion, we have $(2\text{-}9)_5$. Hence, based on Theorem B.2, $\delta\vartheta_t(S)=0$ over Ω yields a solution of the mixed initial-boundary value problem, and the proof is complete. □

Next, as the first example in which there is a restriction on fields, analogous to [Gurtin 1964], we obtain a variational form of the mixed initial-boundary value problem of the admissible process for which the kinematic equation $(2\text{-}3)_1$ is identically satisfied.

Theorem 2. *Let Ω denote the set of all admissible processes which satisfy* $(2\text{-}3)_1$. *Let $S=[\boldsymbol{u},\boldsymbol{\varepsilon},\boldsymbol{\sigma},\boldsymbol{E},\boldsymbol{H},\boldsymbol{D},\boldsymbol{B},\varphi,\psi]$ be an element of Ω and define the functional Ξ_t on Ω at each time*, *say $t\in[0,\infty)$*, *in the form of*

$$
\begin{aligned}
\Xi_t(S)={}&\int_V[h*\sigma_{ij}*\varepsilon_{ij}](\boldsymbol{x},t)\,d\boldsymbol{x}-\frac{1}{2}\int_V S_{ijkl}(\boldsymbol{x})[h*\sigma_{ij}*\sigma_{kl}](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\int_V d^E_{kij}(\boldsymbol{x})[h*\sigma_{ij}*E_k](\boldsymbol{x},t)\,d\boldsymbol{x}-\int_V d^H_{kij}(\boldsymbol{x})[h*\sigma_{ij}*H_k](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\frac{1}{2}\int_V \chi^E_{ij}(\boldsymbol{x})[h*E_i*E_j](\boldsymbol{x},t)\,d\boldsymbol{x}-\frac{1}{2}\int_V \chi^H_{ij}(\boldsymbol{x})[h*H_i*H_j](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\int_V \chi^{EH}_{ij}(\boldsymbol{x})[h*E_i*H_j](\boldsymbol{x},t)\,d\boldsymbol{x}+\int_V[h*D_i*E_i](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&+\int_V[h*B_i*H_i](\boldsymbol{x},t)\,d\boldsymbol{x}-\int_V[h*(D_{i,i}-q_e)*\varphi](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\int_V[h*B_{i,i}*\psi](\boldsymbol{x},t)\,d\boldsymbol{x}+\frac{1}{2}\int_V\rho(\boldsymbol{x})[u_i*u_i](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\int_V[f^b_i*u_i](\boldsymbol{x},t)\,d\boldsymbol{x}-\int_{\partial V_{\boldsymbol{u}}}[h*t_i*(u_i-\hat{u}_i)](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&-\int_{\partial V_{\boldsymbol{\sigma}}}[h*\hat{t}_i*u_i](\boldsymbol{x},t)\,d\boldsymbol{x}+\int_{\partial V_\varphi}[h*d*\hat{\varphi}](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&+\int_{\partial V_{\boldsymbol{D}}}[h*(d-\hat{d})*\varphi](\boldsymbol{x},t)\,d\boldsymbol{x}+\int_{\partial V_\psi}[h*b*\hat{\psi}](\boldsymbol{x},t)\,d\boldsymbol{x}\\
&+\int_{\partial V_{\boldsymbol{B}}}[h*(b-\hat{b})*\psi](\boldsymbol{x},t)\,d\boldsymbol{x}.
\end{aligned}\tag{3-3}
$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta\Xi_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{D}}, \widetilde{\boldsymbol{B}}, \tilde{\varphi}, \tilde{\psi}] \in \Omega$ be an admissible process and suppose that $S + \lambda\tilde{S} \in \Omega$ for all real values of λ. By employing (3-3) and using (A-5), the compatibility equation $(2\text{-}3)_1$, the symmetry condition (2-7), the divergence theorem, and properties of convolution product given in Appendix A, one can find

$$
\begin{aligned}
\delta_{\tilde{S}}\Xi_t(S) = &-\int_V [h * (S_{ijkl}\sigma_{kl} + d^E_{kij}E_k + d^H_{kij}H_k - \varepsilon_{ij}) * \tilde{\sigma}_{ij}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+\int_V [h*(D_k - d^E_{kij}\sigma_{ij} - \chi^E_{ik}E_i - \chi^{EH}_{ki}H_i) * \tilde{E}_k](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+\int_V [h*(B_k - d^H_{kij}\sigma_{ij} - \chi^{EH}_{ik}E_i - \chi^H_{ik}H_i) * \tilde{H}_k](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&-\int_V [(h*\sigma_{ij,j} + f^b_i - \rho u_i) * \tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_V [h*(E_i + \varphi_{,i}) * \widetilde{D}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+\int_V [h*(H_i + \psi_{,i}) * \tilde{B}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V [h*(D_{i,i} - q_e) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&-\int_V [h*B_{i,i} * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_{\boldsymbol{u}}} [h*(u_i - \hat{u}_i) * \tilde{t}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+\int_{\partial V_{\boldsymbol{\sigma}}} [h*(t_i - \hat{t}_i) * \tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_\varphi} [h*(\varphi - \hat{\varphi}) * \tilde{d}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&-\int_{\partial V_\psi} [h*(\psi - \hat{\psi}) * \tilde{b}](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{D}}} [h*(d - \hat{d}) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
&+\int_{\partial V_{\boldsymbol{B}}} [h*(b - \hat{b}) * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x}.
\end{aligned} \tag{3-4}
$$

Due to Theorem B.2, if S is a solution to the IBVP, then we conclude $\delta_{\tilde{S}}\Xi_t(S) = 0$ for every $\tilde{S} \in \Omega$ $(0 \le t < \infty)$, leading us to $\delta\Xi_t(S) = 0$ over Ω. Also, with the same path given in Theorem 1, by using Lemmas A.1, A.2, and A.3, Corollary A.4, (3-4), $\delta\Xi_t(S) = 0$ over Ω, properties of convolution product, considering (2-6), (2-7), (2-8), and Theorem B.2, the implication in the other direction is proved. □

Theorem 1 is the most general variational form giving a solution of the elastodynamics IBVP for electromagneto-elastic materials. The displacement-strain kinematic equation in Theorem 2 is employed as the only restriction. Hence, one can further restrict the admissible process by which it automatically satisfies some of the field equations and boundary conditions. In doing so, define a kinematically admissible process and consequently obtain a relevant variational form.

Definition. An admissible process is called a kinematically admissible process if it satisfies the kinematic equations (2-3), the constitutive equations (2-4), and the displacement-potential boundary conditions.

Theorem 3. *Let Ω denote the set of all kinematically admissible processes. Let $S = [\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \varphi, \psi]$ be an element of Ω and define the functional $\Sigma_t(S)$ on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}\Sigma_t(S) = \frac{1}{2}\int_V [h * \sigma_{ij} * \varepsilon_{ij}](\boldsymbol{x}, t)\, d\boldsymbol{x} + \frac{1}{2}\int_V \rho(\boldsymbol{x})[u_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_V [f_i^b * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \frac{1}{2}\int_V [h * D_i * E_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \frac{1}{2}\int_V [h * B_i * H_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_V [h * q_e * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_{\partial V_\sigma} [h * \hat{t}_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_{\boldsymbol{D}}} [h * \hat{d} * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_{\partial V_{\boldsymbol{B}}} [h * \hat{b} * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x}.\end{aligned} \tag{3-5}$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta\Sigma_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{H}}, \widetilde{\boldsymbol{D}}, \widetilde{\boldsymbol{B}}, \tilde{\varphi}, \tilde{\psi}]$ be an admissible process and suppose that $S+\lambda\tilde{S} \in \Omega$ for all real values of λ. Obviously, it implies $\tilde{u}_i = 0$ on $\partial V_{\boldsymbol{u}} \times [0, \infty)$, $\tilde{\varphi} = 0$ on $\partial V_\varphi \times [0, \infty)$, and $\tilde{\psi} = 0$ on $\partial V_\psi \times [0, \infty)$. By making use of (3-5), (A-5), kinematic equation (2-3), the constitutive equation (2-4), the symmetry condition (2-5), and the divergence theorem, we obtain

$$\begin{aligned}\delta_{\tilde{S}}\Sigma_t(S) = -\int_V [(h * \sigma_{ij,j} + f_i^b - \rho u_i) * \tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_V [h * (D_{i,i} - q_e) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V [h * B_{i,i} * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_\sigma} [h * (t_i - \hat{t}_i) * \tilde{u}_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{D}}} [h * (d - \hat{d}) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_{\boldsymbol{B}}} [h * (b - \hat{b}) * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x}.\end{aligned} \tag{3-6}$$

As is clear when S is a solution of the mixed initial-boundary value problem, then $\delta_{\tilde{S}}\Sigma_t(S) = 0$ for every admissible $\tilde{S}$ ($0 \le t < \infty$), leading to $\delta\Sigma_t(S) = 0$ over Ω. Conversely, similar to Theorem 1, since the array $[\tilde{\boldsymbol{u}}, \tilde{\varphi}, \tilde{\psi}]$ can be defined arbitrarily, for every $t \in [0, \infty)$, on the domain and the boundary ∂V, then by employing $\delta\Sigma_t(S) = 0$ over Ω, (3-6), Lemmas A.1 and A.2, properties of convolution product, and Theorem B.2, we obtain the desired result. □

3.2. ***Variational principles characterizing [$\boldsymbol{u}, \boldsymbol{\varphi}, \boldsymbol{\psi}$].*** With the aid of Theorem 3, it is straightforward to obtain a variational form in terms of the displacement field, electric potential, and magnetic potential. In this regard, define an admissible and a kinematically admissible displacement-potential process as follows:

Definition. An array $S = [\boldsymbol{u}, \varphi, \psi]$ is called an admissible displacement-potential process if $\boldsymbol{u} \in C^{1,2}$, $\varphi \in C^{1,0}$, and $\psi \in C^{1,0}$.

Definition. An array $S = [\boldsymbol{u}, \varphi, \psi]$ is called a kinematically admissible displacement-potential process if it is an admissible displacement-potential process and meets the displacement-potential boundary conditions.

Now, to obtain the desired variational form as a corollary of Theorem 3, the constitutive equations (2-4) and kinematic relations (2-3) need to be employed in (3-5). Doing so, one can easily obtain the variational form corresponding to a kinematically admissible displacement-potential process:

Theorem 4. *Let Ω denote the set of all kinematically admissible displacement-potential processes. Let $S = [\boldsymbol{u}, \varphi, \psi]$ be an element of Ω and define the functional Θ_t on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}\Theta_t(S) = \frac{1}{2}\int_V &[h * (C_{ijkl}u_{k,l} + e^E_{kij}\varphi_{,k} + e^H_{kij}\psi_{,k}) * u_{i,j}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &+ \frac{1}{2}\int_V [h * (e^E_{ikl}u_{k,l} - \kappa^E_{ij}\varphi_{,j} - \kappa^{EH}_{ij}\psi_{,j}) * \varphi_{,i}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &+ \frac{1}{2}\int_V [h * (e^H_{ikl}u_{k,l} - \kappa^{EH}_{ji}\varphi_{,j} - \kappa^H_{ij}\psi_{,j}) * \psi_{,i}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &- \int_V [f^b_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_V [h * q_e * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &+ \frac{1}{2}\int_V \rho(\boldsymbol{x})[u_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_{\boldsymbol{\sigma}}} [h * \hat{t}_i * u_i](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &- \int_{\partial V_{\boldsymbol{D}}} [h * \hat{d} * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_{\partial V_{\boldsymbol{B}}} [h * \hat{b} * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x}. \end{aligned} \tag{3-7}$$

Then, $S = [\boldsymbol{u}, \varphi, \psi]$ is a solution of the mixed initial-boundary value problem if and only if $\delta\Theta_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

3.3. *Variational principles characterizing* $[\boldsymbol{\sigma}, \boldsymbol{\varphi}, \boldsymbol{\psi}]$. Theorem B.4 motivates us to develop variational forms in terms of the stress field rather than the displacement field, which is more desirable when the mechanical boundary conditions are traction-type. In other words, it is of interest to obtain conditions by which the array $S = [\boldsymbol{\sigma}, \varphi, \psi]$ is a solution to the mixed initial-boundary value problems. To this end, let first define what we mean by a kinematically admissible electromagneto-stress process:

Definition. An array $[\boldsymbol{\sigma}, \varphi, \psi]$ in which $\boldsymbol{\sigma}$ is a second-order symmetric tensor and $\boldsymbol{\sigma} \in C^{2,0}$, $\varphi \in C^{2,0}$, and $\psi \in C^{2,0}$ is called a kinematically admissible electromagneto-stress process if $\varphi = \hat{\varphi}(\boldsymbol{x}, t)$ on $\partial V_\varphi \times [0, \infty)$ and $\psi = \hat{\psi}(\boldsymbol{x}, t)$ on $\partial V_\psi \times [0, \infty)$.

Now, the following statement for the kinematically admissible electromagneto-stress processes holds true.

Theorem 5. *Let Ω denote the set of all kinematically admissible electromagneto-stress processes. Let $S = [\boldsymbol{\sigma}, \varphi, \psi]$ be an element of Ω and define the functional Υ_t on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}\Upsilon_t(S) = \frac{1}{2}\int_V \Big[\frac{h}{\rho} * \sigma_{ij,j} * \sigma_{ik,k}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V \Big[\Big(\frac{1}{\rho} f^b_{(i}\Big)_{,j)} * \sigma_{ij}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \frac{1}{2}\int_V [S_{ijkl}\sigma_{ij} * \sigma_{kl} + \chi^E_{ij}\varphi_{,i} * \varphi_{,j} + \chi^H_{ij}\psi_{,i} * \psi_{,j}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_V [-d^E_{ikl}\sigma_{kl} * \varphi_{,i} - d^H_{ikl}\sigma_{kl} * \psi_{,i} + \chi^{EH}_{ij}\psi_{,j} * \varphi_{,i}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_V [q_e * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{u}}} \Big[\Big(\frac{f^b_i}{\rho} - \hat{u}_i\Big) * t_i\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_{\boldsymbol{\sigma}}} \Big[\frac{h}{\rho} * (\hat{t}_i - t_i) * \sigma_{ij,j}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{D}}} [\hat{d} * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_{\boldsymbol{B}}} [\hat{b} * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x}. \qquad (3\text{-}8)\end{aligned}$$

Then, S is a solution of the mixed initial-boundary value problem if and only if $\delta\Upsilon_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

Proof. Let $\tilde{S} = [\tilde{\boldsymbol{\sigma}}, \tilde{\varphi}, \tilde{\psi}]$ be an ordered array in which $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$, $\tilde{\boldsymbol{\sigma}} \in C^{2,0}$, $\tilde{\varphi} \in C^{2,0}$, and $\tilde{\psi} \in C^{2,0}$ such that $S + \lambda\tilde{S} \in \Omega$ for all real values of λ — which implies $\tilde{\varphi} = 0$ on $\partial V_\varphi \times [0, \infty)$ and $\tilde{\psi} = 0$ on $\partial V_\psi \times [0, \infty)$. By making use of (3-8), (A-5), and symmetry condition (2-7), applying the divergence theorem, properties of convolution, and the above-mentioned restriction, we find

$$\begin{aligned}\delta_{\tilde{S}}\Upsilon_t(S) = \int_V \Big[\Big(-\Big(\frac{h}{\rho} * \sigma_{(ik,k}\Big)_{,j)} + S_{ijkl}\sigma_{kl} - d^E_{kij}\varphi_{,k} - d^H_{kij}\psi_{,k} - \Big(\frac{1}{\rho} f^b_{(i}\Big)_{,j)}\Big) * \tilde{\sigma}_{ij}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_V [((d^E_{ikl}\sigma_{kl} - \chi^E_{ij}\varphi_{,j} - \chi^{EH}_{ij}\psi_{,j})_{,i} - q_e) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_V [(d^H_{ikl}\sigma_{kl} - \chi^{EH}_{ji}\varphi_{,j} - \chi^H_{ij}\psi_{,j})_{,i} * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_{\boldsymbol{u}}} \Big[\Big(\frac{h}{\rho} * \sigma_{ik,k} + \frac{f^b_i}{\rho} - \hat{u}_i\Big) * \tilde{t}_i\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ + \int_{\partial V_{\boldsymbol{\sigma}}} \Big[\frac{h}{\rho} * (\hat{t}_i - t_i) * \tilde{\sigma}_{ij,j}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_{\partial V_{\boldsymbol{D}}} [((d^E_{ikl}\sigma_{kl} - \chi^E_{ij}\varphi_{,j} - \chi^{EH}_{ij}\psi_{,j})n_i - \hat{d}) * \tilde{\varphi}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\ - \int_{\partial V_{\boldsymbol{B}}} [((d^H_{ikl}\sigma_{kl} - \chi^{EH}_{ji}\varphi_{,j} - \chi^H_{ij}\psi_{,j})n_i - \hat{b}) * \tilde{\psi}](\boldsymbol{x}, t)\, d\boldsymbol{x}. \qquad (3\text{-}9)\end{aligned}$$

Obviously, if S is a solution of the mixed initial-boundary value problem, then $\delta_{\tilde{S}}\Upsilon_t(S) = 0$ for every above-defined $\tilde{S}$ ($0 \le t < \infty$) is implied from Theorem B.4, resulting in $\delta\Upsilon_t(S) = 0$ over Ω. Conversely, if $\delta\Upsilon_t(S) = 0$ over Ω, then, by utilizing Lemmas A.1, A.2, A.3, and A.5, and Theorem B.4, we obtain the desired result. □

When the mechanical boundary condition is entirely traction-type, say traction problems, one can establish a more convenient variational form in terms of the ordered array $S = [\boldsymbol{\sigma}, \varphi, \psi]$.

Definition. A kinematically admissible electromagneto-stress process is called a dynamically admissible electromagneto-stress process if $\sigma_{ij} n_j = \hat{t}_i(\boldsymbol{x}, t)$ on $\partial V_{\boldsymbol{\sigma}} \times [0, \infty)$.

Now, based on Theorem 5, it is straightforward to obtain a variational framework for the dynamically admissible electromagneto-stress processes:

Theorem 6. *Let Ω denote the set of all dynamically admissible electromagneto-stress processes. Let $S = [\boldsymbol{\sigma}, \varphi, \psi]$ be an element of Ω and define the functional $\Im_t$ on Ω at each time, say $t \in [0, \infty)$, in the form of*

$$\begin{aligned}
\Im_t(S) = {} & \frac{1}{2}\int_V \Big[\frac{h}{\rho} * \sigma_{ij,j} * \sigma_{ik,k}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} - \int_V \Big[\Big(\frac{1}{\rho} f^b_{(i}\Big)_{,j)} * \sigma_{ij}\Big](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \frac{1}{2}\int_V [S_{ijkl}\sigma_{ij} * \sigma_{kl} + \chi^E_{ij}\varphi_{,i} * \varphi_{,j} + \chi^H_{ij}\psi_{,i} * \psi_{,j}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& + \int_V [-d^E_{ikl}\sigma_{kl} * \varphi_{,i} - d^H_{ikl}\sigma_{kl} * \psi_{,i} + \chi^{EH}_{ij}\psi_{,j} * \varphi_{,i}](\boldsymbol{x}, t)\, d\boldsymbol{x} \\
& - \int_V [q_e * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{D}}} [\hat{d} * \varphi](\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial V_{\boldsymbol{B}}} [\hat{b} * \psi](\boldsymbol{x}, t)\, d\boldsymbol{x}. \qquad (3\text{-}10)
\end{aligned}$$

Then, S is a solution of the traction problem (i.e., $\partial V_{\boldsymbol{u}} = \varnothing$) if and only if $\delta\Im_t(S) = 0$ over Ω, within the time interval $t \in [0, \infty)$.

Proof. The proof is analogous to that of Theorem 5. □

4. Conclusion

In parallel to [Gurtin 1964], on the basis of alternative field equations for electro-magneto-elastic materials, which are comprehensively given in Appendix B, the convolutional variational principles have been derived and proved rigorously. In Theorem 1, a general convolutional variational form, in which the admissible process is not required to meet any field equations and/or boundary/initial conditions, has been derived. The convolutional variational principle corresponding to an admissible process that meets only the strain-displacement relation has been formulated in Theorem 2. Next, the result for a more restricted process — namely, a kinematically admissible process — has been formulated in Theorem 3. As a corollary of Theorem 3, the convolutional variational principle corresponding to a kinematically admissible displacement-potential has been set in Theorem 4. Lastly, through Theorems 5 and 6, variational principles in terms of stress rather than displacement have been established, respectively, for general problems and traction problems. On the application side, the results of the present work provide a robust basis for numerical analysis of electromagneto-elastic materials with general material domain geometry and boundary/initial conditions.

Appendix A: Mathematical background

Here, for the sake of completeness, we summarize the basic concepts, originally developed in [Gurtin 1964; Nickell and Sackman 1968], that are employed in the main body of the paper. For a comprehensive discussion, the readers are referred to those references.

Smoothness of a vector (or scalar) function $\boldsymbol{f}$ (or f) is expressed mathematically by $C^{M,N}$, where M and N are nonnegative integers, with the following definition: $\boldsymbol{f}$ (or f) $\in C^{M,N}$, in which $\boldsymbol{f}$ (or f) is a function of position and time defined on $\overline{V} \times (0, \infty)$, if and only if the functions $f^{(n)}_{\underbrace{,ij\cdots k}_{m \text{ indices}}}$ ($m = 0, 1, \ldots, M$ and $n = 0, 1, \ldots, N$) exist and are continuous.

The pair $(\boldsymbol{x}, t) \in \overline{\partial V} \times [0, \infty)$ is called a regular point if the unit outward normal $\boldsymbol{n}$ at $\boldsymbol{x}$, and at any time, is continuous. Moreover, the function f is called a piecewise regular function on boundary $\overline{\partial V_i} \times [0, \infty)$ with $i = \boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \boldsymbol{D}, \psi, \boldsymbol{B}$ if and only if it is piecewise continuous on $\overline{\partial V_i} \times [0, \infty)$ and continuous on every regular point of that region. Additionally, for piecewise regular functions f and $\hat{f}$ on $\partial \overline{V}_i \times [0, \infty)$, we say $f = \hat{f}$ if and only if the equality holds true for any regular point $(\boldsymbol{x}, t) \in \overline{\partial V_i} \times [0, \infty)$.

The symbol $f * g$, in which f and g are functions of the position and continuous functions of time defined on $\Re \times [0, \infty)$, with $\Re$ a subset of the Euclidean space, indicates the convolution of two functions in the sense of

$$[f * g](\boldsymbol{x}, t) = \int_0^t f(\boldsymbol{x}, t-\lambda) g(\boldsymbol{x}, \lambda)\, d\lambda, \quad (\boldsymbol{x}, t) \in \Re \times [0, \infty). \tag{A-1}$$

In this regard, one can show that the following properties hold true:

$$f * g = g * f, \tag{A-2}$$

$$f * g = 0 \iff f = 0 \vee g = 0, \tag{A-3}$$

$$f * (g * h) = (f * g) * h = f * g * h. \tag{A-4}$$

A functional is a real-valued function on a subset of a linear space. Denoting a linear space by L and a subset of L by K, and defining $\Phi(S)$ as a functional on K, we define

$$\delta_{\tilde{S}} \Phi(S) = \frac{d}{d\lambda} \Phi(S + \lambda \tilde{S}) \bigg|_{\lambda=0} \tag{A-5}$$

for all real numbers λ, where S, $\tilde{S} \in L$, and $S + \lambda\tilde{S} \in K$. And we say the variation of $\Phi(S)$ is zero and write $\delta\Phi(S) = 0$ over K if and only if $\delta_{\tilde{S}}\Phi(S)$ exists and equals zero for all $\tilde{S}$ such that S, $\tilde{S} \in L$, and $S + \lambda\tilde{S} \in K$.

Now, we list four lemmas and one corollary proved in [Gurtin 1964; Nickell and Sackman 1968]. However, we write them in such a way that they are applicable to the present study.

Lemma A.1 [Gurtin 1964]. *Let ϑ be a continuous function on $\overline{V} \times [0, \infty)$ and suppose*

$$\int_V \vartheta * \omega(\boldsymbol{x}, t)\, d\boldsymbol{x} = 0, \quad 0 \le t < \infty, \tag{A-6}$$

for every $\omega \in C^{\infty,\infty}$ which, together with its spatial derivatives, vanishes on $\partial V \times [0, \infty)$. Then

$$\vartheta = 0 \quad \text{on } \overline{V} \times [0, \infty). \tag{A-7}$$

Lemma A.2 [Gurtin 1964]. *Let ϑ be a piecewise regular function on $\overline{\partial V_i} \times [0, \infty)$ with $i = \boldsymbol{\sigma}, \boldsymbol{D}, \boldsymbol{B}$, and suppose*

$$\int_{\partial V_i} \vartheta * \omega(\boldsymbol{x}, t)\, d\boldsymbol{x} = 0, \quad 0 \le t < \infty, \tag{A-8}$$

for every $\omega \in C^{\infty,\infty}$ that vanishes on $\partial V_j \times [0, \infty)$ with, respectively, $j = \boldsymbol{u}, \varphi, \psi$. Then

$$\vartheta = 0 \quad \text{on } \overline{\partial V_i} \times [0, \infty). \tag{A-9}$$

Lemma A.3 [Gurtin 1964]. *Let ϑ_i be continuous on $\overline{\partial V_{\boldsymbol{u}}} \times [0, \infty)$, and suppose we have*

$$\int_{\partial V_{\boldsymbol{u}}} \vartheta_i * (\omega_{ij} n_j)(\boldsymbol{x}, t)\, d\boldsymbol{x} = 0, \quad 0 \le t < \infty, \tag{A-10}$$

for every $\omega_{ij} \in C^{\infty,\infty}$ which, together with all of its spatial derivatives, vanishes on $\partial V_{\boldsymbol{\sigma}} \times [0, \infty)$ and has the property $\omega_{ij} = \omega_{ji}$. Then

$$\vartheta_i = 0 \quad \text{on } \overline{\partial V_{\boldsymbol{u}}} \times [0, \infty). \tag{A-11}$$

The following statement is a corollary of Lemma A.3.

Corollary A.4 [Nickell and Sackman 1968]. *Let ϑ be continuous on $\overline{\partial V_\varphi}$ (or $\overline{\partial V_\psi}$)$\times$ $[0, \infty)$ and suppose*

$$\int_{\partial V_\varphi \ (or\ \partial V_\psi)} [\vartheta * (\omega_i n_i)](\boldsymbol{x}, t)\, d\boldsymbol{x} = 0, \quad 0 \le t < \infty, \tag{A-12}$$

for every $\omega_i \in C^{\infty,\infty}$ which, together with its spatial derivatives, vanishes on $\partial V_{\boldsymbol{D}}$ (or $\partial V_{\boldsymbol{B}}$) $\times [0, \infty)$. Then

$$\vartheta = 0 \quad \text{on } \overline{\partial V_\varphi} \ (or\ \overline{\partial V_\psi}) \times [0, \infty). \tag{A-13}$$

Lemma A.5 [Gurtin 1964]. *Let ϑ_i be a piecewise regular function on $\overline{\partial V_{\boldsymbol{\sigma}}} \times [0, \infty)$, and suppose*

$$\int_{\partial V_\sigma} \vartheta_i * (\omega_{ij,j})(\boldsymbol{x}, t)\, d\boldsymbol{x} = 0, \quad 0 \le t < \infty, \tag{A-14}$$

for all $\omega_{ij} \in C^{\infty,\infty}$ with $\omega_{ij} = \omega_{ji}$. Then

$$\vartheta_i = 0 \quad on\ \overline{\partial V_{\boldsymbol{\sigma}}} \times [0, \infty). \tag{A-15}$$

Appendix B: Integro-partial-differential field equations

The alternative integro-partial-differential field equations of motion of an electromagneto-elastic body are derived in this part. To start with, define the functions [Gurtin 1964]

$$\begin{aligned} h(t) &= t, \quad 0 \le t < \infty, \\ f_i^b(\boldsymbol{x}, t) &= h * f(\boldsymbol{x}, t) + \rho(\boldsymbol{x})(t v_i^0(\boldsymbol{x}) + u_i^0(\boldsymbol{x})) \end{aligned} \tag{B-1}$$

in which f_i^b is a vector field obtained from the prescribed data (2-10) and the body force. We now have the following alternative formulation of $(2\text{-}2)_1$.

Theorem B.1. *Let $u_i \in C^{0,2}$ and $\sigma_{ij} \in C^{1,0}$ be a vector field and a second-order symmetric tensor field, respectively. Then $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ meet* $(2\text{-}2)_1$ *and the associated initial boundary conditions* (2-10) *if and only if*

$$\rho \boldsymbol{u} = h * \nabla.\boldsymbol{\sigma} + \boldsymbol{f}^b \quad on\ V \times [0, \infty). \tag{B-2}$$

Proof. See [Gurtin 1964]. □

Now, with the help of the following theorem, which is the direct result of Theorem B.1, one can define alternative field equations of the mixed initial-boundary value problem.

Theorem B.2. *The admissible process $S = [\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}, \boldsymbol{B}, \varphi, \psi]$ is a solution of the mixed initial-boundary value problem if and only if it satisfies* (B-2), $(2\text{-}2)_{2\text{–}3}$, (2-3), (2-4), *and* (2-9).

Now, through the next two theorems, we obtain two variants of field equations for electromagneto-elastic materials.

Theorem B.3. *Let $u_i \in C^{2,2}$, $\varphi \in C^{2,0}$, and $\psi \in C^{2,0}$. Then the ordered array $[\boldsymbol{u}, \varphi, \psi]$ corresponds to a solution of the mixed initial-boundary value problem if* *and only if the following equations hold true*:

$$
\begin{array}{ll}
u_i = h * (C_{ijkl}u_{k,l} + e^E_{kij}\varphi_{,k} + e^H_{kij}\psi_{,k})_{,j} + f^b_i & \textit{on } V \times [0,\infty), \\
(e^E_{ikl}u_{k,l} - \kappa^E_{ij}\varphi_{,j} - \kappa^{EH}_{ij}\psi_{,j})_{,i} = q_e & \textit{on } V \times [0,\infty), \\
(e^H_{ikl}u_{k,l} - \kappa^{EH}_{ji}\varphi_{,j} - \kappa^H_{ij}\psi_{,j})_{,i} = 0 & \textit{on } V \times [0,\infty), \\
u_i = \hat{u}_i(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{u}} \times [0,\infty), \\
(C_{ijkl}u_{k,l} + e^E_{kij}\varphi_{,k} + e^H_{kij}\psi_{,k})n_j = \hat{t}_i(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{\sigma}} \times [0,\infty), \\
\varphi = \hat{\varphi}(\boldsymbol{x},t) & \textit{on } \partial V_{\varphi} \times [0,\infty), \\
(e^E_{ikl}u_{k,l} - \kappa^E_{ij}\varphi_{,j} - \kappa^{EH}_{ij}\psi_{,j})n_i = \hat{d}(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{D}} \times [0,\infty), \\
\psi = \hat{\psi}(\boldsymbol{x},t) & \textit{on } \partial V_{\psi} \times [0,\infty), \\
(e^H_{ikl}u_{k,l} - \kappa^{EH}_{ji}\varphi_{,j} - \kappa^H_{ij}\psi_{,j})n_i = \hat{b}(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{B}} \times [0,\infty).
\end{array}
\tag{B-3}
$$

Proof. First, suppose that relations (B-3) hold true. Thus, $(2\text{-}9)_{1,3,5}$ are automatically satisfied. Define $\boldsymbol{\varepsilon}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ through (2-3). Also, define $\boldsymbol{\sigma}$, $\boldsymbol{D}$, and $\boldsymbol{B}$ via (2-4). Then, $(2\text{-}9)_{2,4,6}$ are identically satisfied due to the symmetry (2-5); the above-defined $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, and (2-5) together with $(\text{B-3})_1$ give (B-2); $(\text{B-3})_{2\text{–}3}$ together with the above-defined $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{E}$, $\boldsymbol{H}$, and symmetry (2-5) give $(2\text{-}2)_{2\text{–}3}$. Hence, by Theorem B.2, (B-3) is a solution to the mixed initial-boundary value problem. On the other hand, (B-2), $(2\text{-}2)_{2\text{–}3}$, (2-3), (2-4), (2-5), and (2-9) imply (B-3) and the proof is complete. □

Theorem B.4. *Let $\sigma_{ij} \in C^{2,0}$, $\varphi \in C^{2,0}$, and $\psi \in C^{2,0}$ with $\sigma_{ij} = \sigma_{ji}$. Then the ordered array $[\boldsymbol{\sigma}, \varphi, \psi]$ is a solution to the mixed initial-boundary value problem if and only if the following equations hold true*:

$$
\begin{array}{ll}
S_{ijkl}\sigma_{kl} = \Big(\dfrac{h}{\rho} * \sigma_{(ik,k}\Big)_{,j)} + \Big(\dfrac{1}{\rho}f^b_{(i}\Big)_{,j)} + d^E_{kij}\varphi_{,k} + d^H_{kij}\psi_{,k} & \textit{on } V \times [0,\infty), \\
(d^E_{ikl}\sigma_{kl} - \chi^E_{ij}\varphi_{,j} - \chi^{EH}_{ij}\psi_{,j})_{,i} = q_e & \textit{on } V \times [0,\infty), \\
(d^H_{ikl}\sigma_{kl} - \chi^{EH}_{ji}\varphi_{,j} - \chi^H_{ij}\psi_{,j})_{,i} = 0 & \textit{on } V \times [0,\infty), \\
\dfrac{h}{\rho} * \sigma_{ik,k} + \dfrac{1}{\rho}f^b_i = \hat{u}_i(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{u}} \times [0,\infty), \\
\sigma_{ij}n_j = \hat{t}_i(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{\sigma}} \times [0,\infty), \\
\varphi = \hat{\varphi}(\boldsymbol{x},t) & \textit{on } \partial V_{\varphi} \times [0,\infty), \\
(d^E_{ikl}\sigma_{kl} - \chi^E_{ij}\varphi_{,j} - \chi^{EH}_{ij}\psi_{,j})n_i = \hat{d}(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{D}} \times [0,\infty), \\
\psi = \hat{\psi}(\boldsymbol{x},t) & \textit{on } \partial V_{\psi} \times [0,\infty), \\
(d^H_{ikl}\sigma_{kl} - \chi^{EH}_{ji}\varphi_{,j} - \chi^H_{ij}\psi_{,j})n_i = \hat{b}(\boldsymbol{x},t) & \textit{on } \partial V_{\boldsymbol{B}} \times [0,\infty).
\end{array}
\tag{B-4}
$$

Proof. First, suppose that (B-4) holds. Hence, $(2\text{-}9)_{2,3,5}$ are automatically satisfied. Define $\boldsymbol{u}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ through (B-2), $(2\text{-}3)_2$, and $(2\text{-}3)_3$, respectively. Also, define $\boldsymbol{\varepsilon}$, $\boldsymbol{D}$, and $\boldsymbol{B}$ via (2-6). Then, $(2\text{-}9)_{1,4,6}$ are identically satisfied; (2-6) and (2-8) imply (2-4); $(2\text{-}3)_1$ holds because of $(\text{B-4})_1$, $(\text{B-4})_4$, and the above-defined $\boldsymbol{u}$ and $\boldsymbol{\varepsilon}$; $(\text{B-4})_{2\text{–}3}$ together with the above-defined $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{H}$ give $(2\text{-}2)_{2\text{–}3}$. Hence, by Theorem B.2, (B-4) is a solution to the mixed initial-boundary value problem. On the other hand, (B-2), $(2\text{-}2)_{2\text{–}3}$, (2-3), (2-4), (2-8), and (2-9) imply (B-4), and the proof is complete. □

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References

[Bo 2003] H. Bo, "Gurtin-type region-wise variational principles for thermopiezoelectric elastodynamics", *Appl. Math. Mech.* **24**:5 (2003), 576–584.

[Brown 1966] W. F. Brown, Jr., *Magnetoelastic interactions*, Springer Tracts in Natural Philosophy **9**, Springer, 1966.

[Coleman and Dill 1971] B. D. Coleman and E. H. Dill, "Thermodynamic restrictions on the constitutive equations of electromagnetic theory", *Z. Angew. Math. Phys.* **22**:4 (1971), 691–702.

[El-Karamany and Ezzat 2011] A. S. El-Karamany and M. A. Ezzat, "Convolutional variational principle, reciprocal and uniqueness theorems in linear fractional two-temperature thermoelasticity", *J. Thermal Stresses* **34**:3 (2011), 264–284.

[Eringen and Maugin 1990] A. C. Eringen and G. A. Maugin, *Electrodynamics of continua, I: Foundations and solid media*, Springer, 1990.

[Guggenheim 1936a] E. A. Guggenheim, "On magnetic and electrostatic energy", *Proc. Roy. Soc. Lond. A* **155**:884 (1936), 49–70.

[Guggenheim 1936b] E. A. Guggenheim, "The thermodynamics of magnetization", *Proc. Roy. Soc. Lond. A* **155**:884 (1936), 70–101.

[Gurtin 1963] M. E. Gurtin, "Variational principles in the linear theory of viscoelasticity", *Arch. Rational Mech. Anal.* **13** (1963), 179–191.

[Gurtin 1964] M. E. Gurtin, "Variational principles for linear elastodynamics", *Arch. Rational Mech. Anal.* **16** (1964), 34–50.

[Hashin and Shtrikman 1962] Z. Hashin and S. Shtrikman, "On some variational principles in anisotropic and non-homogeneous elasticity", *J. Mech. Phys. Solids* **10** (1962), 335–342.

[He 2000] J.-H. He, "A variational approach to electroelastic analysis of piezoelectric ceramics with surface electrodes", *Mech. Res. Comm.* **27**:4 (2000), 445–450.

[He 2001a] J.-H. He, "Coupled variational principles of piezoelectricity", *Internat. J. Engrg. Sci.* **39**:3 (2001), 323–341.

[He 2001b] J.-H. He, "Hamilton principle and generalized variational principles of linear thermopiezoelectricity", *J. Appl. Mech.* **68**:4 (2001), 666–667.

[He 2001c] J.-H. He, “Variational theory for linear magneto-electro-elasticity”, *Int. J. Nonlinear Sci. Numer. Simul.* **2**:4 (2001), 309–316.

[Ignaczak 1959] J. Ignaczak, “Direct determination of stresses from the stress equations of motion in elasticity”, *Arch. Mech. Stos.* **11** (1959), 671–678.

[Ignaczak 1963] J. Ignaczak, “A completeness problem for stess equations of motion in the linear elasticity theory”, *Arch. Mech. Stos.* **15** (1963), 225–234.

[Irschik et al. 2010] H. Irschik, M. Krommer, and Y. Vetyukov, “On the use of piezoelectric sensors in structural mechanics: some novel strategies”, *Sensors* **10**:6 (2010), 5626–5641.

[Kuang 2008] Z.-B. Kuang, “Some variational principles in elastic dielectric and elastic magnetic materials”, *Eur. J. Mech. A Solids* **27**:3 (2008), 504–514.

[Landau et al. 1984] L. D. Landau, E. M. Lifshitz, and Pitaevskiĭ, *Electrodynamics of continuous media*, 2nd ed., Course of Theoretical Physics **8**, Pergamon, Oxford, 1984.

[Li 2003] J. Y. Li, “Uniqueness and reciprocity theorems for linear thermo-electro-magneto-elasticity”, *Quart. J. Mech. Appl. Math.* **56**:1 (2003), 35–43.

[Li and Kardomateas 2006] R. Li and G. A. Kardomateas, “The mode III interface crack in piezo-electro-magneto-elastic dissimilar bimaterials”, *J. Appl. Mech.* **73**:2 (2006), 220–227.

[Maugin 1988] G. A. Maugin, *Continuum mechanics of electromagnetic solids*, North-Holland Series in Applied Mathematics and Mechanics **33**, North-Holland, Amsterdam, 1988.

[Miehe and Rosato 2011] C. Miehe and D. Rosato, “A rate-dependent incremental variational formulation of ferroelectricity”, *Internat. J. Engrg. Sci.* **49**:6 (2011), 466–496.

[Miehe et al. 2011] C. Miehe, D. Rosato, and B. Kiefer, “Variational principles in dissipative electro-magneto-mechanics: a framework for the macro-modeling of functional materials”, *Internat. J. Numer. Methods Engrg.* **86**:10 (2011), 1225–1276.

[Mindlin 1972] R. D. Mindlin, “Elasticity, piezoelectricity and crystal lattice dynamics”, *J. Elast.* **2**:4 (1972), 217–282.

[Nelson 1979] D. F. Nelson, *Electric, optic, and acoustic interactions in dielectrics*, Wiley, New York, 1979.

[Nickell and Sackman 1968] R. E. Nickell and J. L. Sackman, “Variational principles for linear coupled thermoelasticity”, *Quart. Appl. Math.* **26** (1968), 11–26.

[Oden and Reddy 1983] J. T. Oden and J. N. Reddy, *Variational methods in theoretical mechanics*, 2nd ed., Springer, 1983.

[Ostoja-Starzewski 2019] M. Ostoja-Starzewski, “Ignaczak equation of elastodynamics”, *Math. Mech. Solids* **24**:11 (2019), 3674–3713.

[Penfield et al. 1963] P. L. Penfield, Jr., L. J. Chu, and H. A. Haus, “Electrodynamics of moving media”, quarterly progress report 70, Massachusetts Institute of Technology Research Laboratory of Electronics, 1963, Available at http://hdl.handle.net/1721.1/53884.

[Pérez-Fernández et al. 2009] L. D. Pérez-Fernández, J. Bravo-Castillero, R. Rodríguez-Ramos, and F. J. Sabina, “On the constitutive relations and energy potentials of linear thermo-magneto-electro-elasticity”, *Mech. Res. Comm.* **36**:3 (2009), 343–350.

[Schoeftner and Irschik 2016] J. Schoeftner and H. Irschik, “Stress tracking in thin bars by eigen-strain actuation”, *J. Sound Vibration* **383** (2016), 35–45.

[Tiersten and Tsai 1972] H. F. Tiersten and C. F. Tsai, “On the interaction of the electromagnetic field with heat conducting deformable insulators”, *J. Math. Phys.* **13**:3 (1972), 361–378.

[Toupin 1956] R. A. Toupin, “The elastic dielectric”, *J. Rational Mech. Anal.* **5** (1956), 849–915.

[Wang et al. 2010] Z.-J. Wang, D.-Z. Zheng, and C.-B. Zheng, "Simplified Gurtin-type generalized variational principles for fully dynamic magneto-electro-elasticity with geometrical nonlinearity", *Int. J. Solid. Struct.* **47**:22–23 (2010), 3115–3120.

[Washizu 1957] K. Washizu, "On the variational principles applied to dynamics problems of elastic bodies", tech. report 25-23, Massachusetts Institute of Technology Aeroelastic and Structures Research Laboratory, 1957.

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