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THE METHOD OF VIRTUAL POWER IN THE MECHANICS OF CONTINUOUS MEDIA I: SECOND-GRADIENT THEORY

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Translated by Marcelo Epstein and Ronald E. Smelser

The systematic application of the definition of internal forces, by means of the virtual power produced in a class of virtual motions, leads to a consistent mathematical representation of stresses and strains in any given mechanical model. It is thus possible to write the statical and dynamical equations and to state well posed boundary value problems. The second-gradient theory, presented here by way of example, can be developed without any ambiguity. An essential distinction is drawn between intrinsic and classical stresses so as to avoid certain issues of interpretation. It is shown that all the results of classical linear elasticity can be immediately extended to the case of second-gradient elastic media. The constitutive equations of nonlinear elasticity are also formulated.

Main notation

(1) *Kinematic quantities*

velocity	$U_i(\underline{x}, t)$
strain-rate tensor	$D_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$
rotation-rate tensor	$\Omega_{ij} = \frac{1}{2}(U_{i,j} - U_{j,i})$
{ rotation-rate vector	$\omega_i = -\frac{1}{2}\varepsilon_{ipq}\Omega_{pq},\ \Omega_{ij} = -\varepsilon_{ijk}\omega_k$
{ tangential component	$\widetilde{\omega}_i$
rotation-gradient tensor	$K_{ij} = \omega_{i,j} = -\frac{1}{2}\varepsilon_{ipq}\Omega_{pq,j} = -\frac{1}{2}\varepsilon_{ipq}U_{p,qj}$
symmetric part of the tensor of second gradient of the velocities	$K_{ijk} = \frac{1}{3}(U_{i,jk} + U_{j,ki} + U_{k,ij})$

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See also the next article in this issue: Marcelo Epstein and Ronald E. Smelser, “An appreciation and analysis of Paul Germain’s ‘The method of virtual power in the mechanics of continuous media, I: Second-gradient theory’ ”, *Math. Mech. Complex Systems*, **8**:1 (2020), pp. 191–199.

(2) *Internal and external forces*

volumetric force	f_i
volumetric couple	$C_{ij}\ (= -C_{ji})$
volumetric symmetric double force	$\Phi_{ij}\ (= \Phi_{ji})$
surface traction (stress vector)	T_i
surface double traction (couple stress vector)	$\widetilde{M}_i$
doubly normal stress (surface density)	N
edge stress (line density)	R_i
surface force on ∂S	t_i
tangential surface couple	$\widetilde{m}_i$
doubly normal force	n
edge force (line density)	r_i
intrinsic stress tensor (1st order)	$\sigma_{ij}\ (= \sigma_{ji})$
intrinsic stress tensors (2nd order)	$\begin{cases} \mu_{ij} & (\mu_{ii} = 0) \\ \mu_{ijk} & \text{(completely symmetric)} \end{cases}$

(3) *Derivative operators on a surface* (with unit vector n_i)

normal gradient	
scalar function φ	$D\varphi = n_l \varphi_{,l}$
vector function V_i	$DV_i = n_l V_{i,l}$
tangential gradient	
scalar function φ	$D_i\varphi = \varphi_{,i} - n_i D\varphi$
vector function V_i	$D_j V_i = V_{i,j} - n_j DV_i$

(4) *Small strains*

displacement	$X_i(\underline{x}, t)$
strain tensor	$\varepsilon_{ij} = \frac{1}{2}(X_{i,j} + X_{j,i})$
rotation tensor	$\varphi_{ij} = \frac{1}{2}(X_{i,j} - X_{j,i})$
rotation vector	$\varphi_i = -\frac{1}{2}\varepsilon_{ipq}\varphi_{pq}$
tangential component	$\widetilde{\varphi}_i$
rotation-gradient tensor	$\eta_{ij} = -\frac{1}{2}\varepsilon_{ipq} X_{p,qj}$
symmetric part of the tensor of second gradient of the displacements	$\eta_{ijk} = \frac{1}{3}(X_{i,jk} + X_{j,ki} + X_{k,ij})$

(5) *Finite strains*

gradient matrix	$F_{i\alpha} = \dfrac{\partial x_i}{\partial a_\alpha}$
Green–Lagrange strain tensor	$L_{\alpha\beta} = \frac{1}{2}(F_{i\alpha}F_{i\beta} - \delta_{\alpha\beta})$
intrinsic Piola–Kirchhoff stress tensor (1st order)	$s_{\alpha\beta}$
intrinsic stress tensors (2nd order)	$\begin{cases} \Pi_{\alpha\beta} & (\Pi_{\alpha\alpha} = 0) \\ \Pi_{\alpha\beta\gamma} & \text{(completely symmetric)} \end{cases}$

1. Introduction

It has long been recognized that in mechanics there are two ways to represent mathematically the forces exerted at a given time t upon a system S. The first, age-old, way consists in representing a force by means of a vector, a mathematical entity that has an origin, a direction, and a magnitude. The completely natural generalization of this idea when attempting to represent the forces exerted on a continuous system leads to a description in terms of a field of vectors associated with a measure, and it is in this way that one speaks of "volumetric forces", "surface forces", "forces per unit mass", etc. While forces are thus represented, it is desirable to utilize as a basic statement of dynamics the fundamental law stipulating that "*there exists at least one reference* (*frame and time*)*, said to be absolute, in which at each time and for each system, the wrench produced by the masses times the accelerations is equal to the wrench of the exterior forces exerted on the system*".

But there exists also, at least since d'Alembert, a second possible avenue, namely, the method of virtual power (or virtual work). Contrary to what is sometimes believed, this second way is as absolutely natural as the first, since it is nothing but the expression of a very common physical experience. If one wants to know if a suitcase is heavy, one tries to lift it slightly; to estimate the tension in a transmission belt, one displaces it a little bit from its stable position; and it is while trying to push a car that one becomes aware of the presence of the internal and external friction forces opposing the motion. From the mathematical point of view, the situation can be described as follows: at a given time t, one considers on S a vector field $\boldsymbol{V}$ that defines at that instant a virtual movement of S — the vectors of the field representing the velocities or the elementary (infinitesimal) displacements during an elementary (infinitesimal) time δt —; the forces that produce this virtual movement $\boldsymbol{V}$ are known if their "virtual power" $\mathcal{P}$ (a real number associated with $\boldsymbol{V}$) is known. More precisely, we consider a set $\mathcal{V}$ of virtual motions $\boldsymbol{V}$, where $\mathcal{V}$ is a normed vector space, and we say that we know the forces exerted on S by the space $\mathcal{V}$ if there exists a continuous linear form $\mathcal{L}(\boldsymbol{V})$, defined on $\mathcal{V}$, whose value for each field $\boldsymbol{V}$ is equal to the virtual power of these forces during the virtual motion defined by $\boldsymbol{V}$.

The essential idea of this second avenue is that of "duality". Moreover, this avenue is not only very close to everyday experience, as we have already remarked, but it is also very versatile; according to the choice of a vector space more, or less, "vast", we will have a description of forces more, or less, fine. Thus, we can consider only those expressions that describe the forces that we need by conveniently choosing the space $\mathcal{V}$. Once $\mathcal{V}$ has been fixed, the set of forces recognized by $\mathcal{V}$ form themselves a vector space, namely, the dual $\mathcal{V}^*$ of $\mathcal{V}$.

We must acknowledge that the notion of a linear map over a vector space is more abstract than that of a vector field, and it is for this reason that the "virtual power approach" has always — at least up until now — appeared to be more difficult. Above all, one has to acknowledge that at the time when the notion of virtual motion was introduced in mechanics the mathematical idea of duality had not yet been sufficiently elaborated so as to completely translate this new notion, which in a certain sense can be considered as the precursor of the notions of measure or of distribution. It is only when the space $\mathcal{V}$ is of finite dimension that no special difficulty arises, and this is why, very early on, we have witnessed the development of the analytical mechanics of systems of a finite number of degrees of freedom, which utilizes in fact, with the notion of "generalized forces", a description of forces by means of the concept of virtual power.

The situation today is different. Functional analysis has been considerably developed and its applications to mechanics, and most particularly to the mechanics of continuous media, are already numerous and of great importance, as it is demonstrated, for example, by the recent work of Duvaut and Lions [1]. The concept of duality is imparted very early on in university curricula. Moreover, the time seems to have arrived to attempt a rather systematic application of the notion of virtual power to continuum mechanics. Such is the objective of this article and of those that will follow under the same general heading.

When utilizing a description of forces by means of virtual power, the most suitable fundamental statement of the laws of dynamics is the *principle of virtual power*. We will limit ourselves in this first article to the case of statics, a case where this principle can be stated as follows:

> "*In an absolute reference, at each time t and for every system, the virtual power of all the forces, internal as well as external, applied to the system vanishes, whatever the virtual motion considered.*"

As is well known, the statement valid for the dynamic case is obtained by adding to the external forces the absolute forces of inertia.

Our procedure is thus very simple and elementary. We want to show that a mechanical theory — and in particular a theory of the mechanics of continuous media — is completely determined once we provide the space $\mathcal{V}$ of virtual motions that we intend to consider and that establishes in some sense the degree of fineness of the theory. The corresponding representation of the forces is deduced by duality, and the collection of all the equations of statics (and more generally, of dynamics) is obtained by application of the principle of virtual power. By way of example, we will start in Section 2 with a short review of the cases of a material point and of a rigid body, classical cases where $\mathcal{V}$ is of finite dimension. After some general remarks about the application of the method to continuous media (Section 3), we will satisfy ourselves in Section 4 with an examination of the so-called first-gradient theory — which is in fact nothing but a slight generalization of the classical theory — and in Section 5 we will present the second-gradient theory. To avoid any error of interpretation, let us note at the outset that our terminology differs from that employed by other authors; the second gradient under consideration here is that of the field of velocities, so that the theory thus named must be compared with the theory of first gradient of the strain (for instance, that developed by Mindlin and Eshel [2]). The results obtained lay the groundwork for the formulation of the thermodynamic properties of the media considered and, consequently, the formulation of the constitutive laws, at least if we resort to the method of local state (see, for example, Germain [3]). By way of illustration we indicate in Section 6 the general features of linear elasticity within the second-gradient theory, and in Section 7, some remarks about nonlinear elasticity that lead in a simple way to the formulation of constitutive laws.

It is not our intention in this article to provide many new results; a large part of the results established below can be found in the literature, in a more or less equivalent form, particularly in the articles listed in the bibliography, at least for the case of elastic media. But, to the best of this author's knowledge, the use of the method of virtual power to define a mechanical model within a given framework of representation has not been the object of a systematic exposition. The second-gradient theory, offered here by way of illustration, permits us to reveal the advantages of this approach. On the one hand, the results remain valid if one wants to take into consideration nonelastic effects. On the other hand, a certain number of difficulties of interpretation that are often present in previous presentations are here automatically removed.

We will not insist here on the strictly mechanical interest of the second-gradient theory, and we refer the reader in this regard to the articles of Mindlin and Eshel [2] and of Toupin [6] and, above all, to the contributions of Casal [13; 14; 15], who has clearly exposed the points of contact between this theory and that of the phenomena of capillarity, thus bringing to light a very interesting physical interpretation, which has not yet received the attention that it deserves.

2. Elementary remarks on the material point and the rigid body

The case of the material point is reviewed here only for reference: at a given time, a virtual motion of the point M is determined by giving the virtual velocity $\boldsymbol{V}_M$ of M; the space $\mathcal{V}$ is, therefore, a Euclidean vector space (of dimension 3). A linear form on $\mathcal{V}$ can be written as an inner product; thus, it determines an element $\boldsymbol{F}_M$ of the dual space, and one can write the virtual power as

$$\mathcal{P} = \mathcal{L}(\boldsymbol{V}_M) = \boldsymbol{F}_M \cdot \boldsymbol{V}_M.$$

In this way, we recover the representation of the forces exerted on a point by the force vector $\boldsymbol{F}_M$, such as provided by the first kind of description of forces recalled at the beginning of the introduction.

Analogously, the case of the rigid body leads to a classical result, although its meaning does not always emerge quite as clearly. Let us consider at a fixed time t a system S, which we will refer to an orthonormal frame — with x_1, x_2, x_3 indicating the coordinates of a point of S, which we will simply denote by $\underline{x}$. It is known that, if S is a rigid body, the velocity field $U_i(\underline{x})$ of the points of S satisfies an identity of the form

$$U_i(\underline{x}) = U_i(\underline{o}) + \Omega_{ij} x_j, \tag{1}$$

where Ω_{ij} is a skew-symmetric matrix, independent of $\underline{x}$, called the rotation-rate matrix, representing, in this frame, the skew-symmetric second-order rotation-rate tensor (or spin tensor). A field that satisfies the identity (1) for every $\underline{x}$ in S is said to be defined by means of a *twist* (or a *distributor*). A twist is thus defined by the six scalars $\Omega_{ij} = -\Omega_{ji}$ and $U_i(\underline{o})$, which are called its *elements of reduction* at the origin. It is also equally well defined by its elements of reduction at any other point of S.

At some fixed time t, let us take as the space of virtual motions $\mathscr{V}$ the (6-dimensional) vector space $\mathscr{C}$ of the twists, a twist being denoted by $\{\mathscr{C}\}$. We say that these virtual motions "*preserve the rigidity of S*" if S is a rigid body or, if S is a deformable medium, that these are "*virtual motions that rigidify S*". The virtual power of the forces applied on S is a linear form over $\mathscr{C}$, namely,

$$\mathscr{P} = \mathscr{L}(\{\mathscr{C}\}). \tag{2}$$

We say that this form defines the screw or *wrench* of the forces, which we denote by $[\mathscr{T}]$. Such a screw is an element of the dual space $\mathscr{T}$ of the space $\mathscr{C}$. If we represent $\{\mathscr{C}\}$ by its elements of reduction at the origin, we can write $\mathscr{P}$ in the form

$$\mathscr{P} = T_i(\underline{o}) U_i(\underline{o}) + M_{ij}(\underline{o}) \Omega_{ij}. \tag{3}$$

The real numbers $T_i(\underline{o})$ and $M_{ij}(\underline{o})$ are the *elements of reduction* of $[\mathscr{T}]$ at the origin. It is clear that, since Ω_{ij} is a skew-symmetric matrix, we can assume without loss of generality that M_{ij} is also skew-symmetric. It is also clear that, since U_i and Ω_{ij} are, respectively, components of a vector and of a second-order skew-symmetric tensor, the same is true for T_i and M_{ij}, respectively.* Naturally, we could also have expressed the linear form (2) while representing $\{\mathscr{C}\}$ by its elements of reduction at another point $\underline{x}$ arbitrarily chosen, and we could have written

$$\mathscr{P} = T_i(\underline{x}) U_i(\underline{x}) + M_{ij}(\underline{x}) \Omega_{ij}. \tag{4}$$

*In other words, these are also components of a vector and of a tensor, respectively. — (TN)

Subtracting (3) from (4), and invoking (1), we obtain

$$(T_i(x) - T_i(\underline{o}))U_i(\underline{o}) + (M_{ij}(\underline{x}) - M_{ij}(\underline{o}) + T_i(\underline{x})x_j)\Omega_{ij} = 0.$$

This equation holds true for arbitrary values of $U_i(\underline{o})$ and of $\Omega_{ij} = -\Omega_{ji}$. This implies that the coefficient of $U_i(\underline{o})$ must vanish identically and that the coefficient of Ω_{ij} must be symmetric in i and j. The vector $\boldsymbol{T}$, called the *vector* or *resultant* of the wrench $[\mathcal{T}]$, is therefore independent of $\underline{x}$, and the skew-symmetric second-order tensor field defined by the matrices M_{ij} is an affine linear function of the coordinates and satisfies the identity

$$M_{ij}(\underline{x}) = M_{ij}(\underline{o}) + x_{[i}T_{j]}, \tag{5}$$

where $A_{[ij]}$ denotes the skew-symmetric part of A_{ij}.*

The preceding treatment is valid regardless of the (finite) dimension of the Euclidean space in which the system S is found. The velocity field (1) and the moment field (5), associated, respectively, to the twist and the wrench, are entities of different mathematical nature. Indeed, the velocity is a vector field while the moment is a second-order skew-symmetric tensor field. It is only in the case of a 3-dimensional space that certain correspondences between these two entities can be made. Let us introduce in this space the alternating tensor with components ε_{ijk}, and let us define

$$\omega_k = -\tfrac{1}{2}\varepsilon_{kij}\Omega_{ij}, \qquad m_k = -\varepsilon_{kij}M_{ij}. \tag{6}$$

Equations (1) and (5) can then be rewritten in the classical form

$$\boldsymbol{U}_M = \boldsymbol{U}_0 + \boldsymbol{\omega} \wedge \boldsymbol{OM}; \qquad \boldsymbol{m}_M = \boldsymbol{m}_0 + \boldsymbol{MO} \wedge \boldsymbol{T}, \tag{7}$$

while $\mathcal{P}$ can be expressed as

$$\mathcal{P} = \boldsymbol{T} \cdot \boldsymbol{U}_M + \boldsymbol{\omega} \cdot \boldsymbol{m}_M = [\mathcal{T}] \cdot \{\mathcal{C}\}. \tag{8}$$

The vector $\boldsymbol{\omega}$ is the *rate of rotation* (or *angular velocity*) vector of the twist $\{\mathcal{C}\}$. The vector field $\boldsymbol{m}_M$ is the *moment field* of the wrench $[\mathcal{T}]$.

Let us underscore once again the significance of the results just obtained: *in the mechanics of rigid bodies, it is futile or superfluous to represent the forces acting on the rigid body other than by means of the wrench that they determine; any other finer representation is redundant.*

*Note, as a matter of detail, that on taking the skew-symmetric part of the tensor product $\boldsymbol{x} \otimes \boldsymbol{T}$, a factor of $\frac{1}{2}$ is introduced, thus explaining the lack of it later in the second equation (6). —(TN)

3. General remarks on the application of the virtual power method in continuum mechanics

From the outset, let us note the essential role played by the following *axiom of the virtual power of the internal forces*:

> *The virtual power of the internal forces of a system S vanishes for any rigidifying virtual motion of S at the time t being considered.*

Let us recall that it is thanks to this axiom that the statement of the principle of virtual power entails the fundamental classical law of mechanics. If, at an arbitrary time t, we consider a rigidifying virtual motion of S, defined by a twist $\{\mathscr{C}\}$, the virtual power of *all* the applied forces is reduced to that of the external forces *alone*, and since S is assumed to be in equilibrium, the virtual power, written as $[\mathscr{T}] \cdot \{\mathscr{C}\}$, where $[\mathscr{T}]$ is the wrench of the *external forces*, must vanish for arbitrary $\{\mathscr{C}\}$. We immediately deduce that $[\mathscr{T}] = \mathbf{0}$, which is precisely the statement of the fundamental law of statics.

The remarks that follow do not have the compulsory and general character of the axiom just formulated; rather, they constitute working hypotheses that could be called into question in theories other than those presented below by way of illustration of the general method.

(a) The systems S to be considered will always be 3-dimensional. We will assume that S is a connected and bounded open domain of the Euclidean space and that its boundary ∂S is piecewise twice continuously differentiable, namely that, except on certain lines which are the edges of ∂S, the surface ∂S has at each of its points a well defined exterior unit normal vector, say $\boldsymbol{n}$, and a curvature tensor which is continuous in a neighborhood of each P belonging to ∂S.

(b) We will apply the principle of virtual power, be it to S or to any subsystem $\mathscr{D}$ of S, for which we will make the same regularity assumptions as for S.

(c) The functions chosen to describe the virtual motion of $\mathscr{D}$, that is, those functions that define an arbitrary element of the normed vector space $\mathscr{V}$, will be assumed to be continuously differentiable over the closure $\mathscr{D} + \partial\mathscr{D}$ of $\mathscr{D}$, as many times as necessary (for example, infinitely differentiable).

(d) The natural language suited to such a theory is that of the theory of distributions. Nevertheless, in order to simplify the exposition and so as to recover directly the classical formulas, we will not make use of it here. This is tantamount to admitting that the distributions that represent the forces are sufficiently regular to be defined in terms of densities, that is, (continuously differentiable) functions defined over certain manifolds. This simplification is more often than not a legitimate one, since we are dealing with notions pertaining to continuous media, themselves a depiction of an essentially discontinuous reality. The linear forms that will define the virtual power can, therefore, be written by means of volume, surface, or line integrals.

In fact, the results obtained under this working hypothesis remain valid in the general case as long as we interpret the various quantities appearing there in the sense of distributions.

(e) The virtual power of the internal forces in the subsystem $\mathcal{D}$ will be denoted as $\mathcal{P}_{(i)}(\mathcal{D})$. We will always assume that it can be expressed in the form of a volume integral over the open set $\mathcal{D}$.

(f) The external forces exerted on the subsystem $\mathcal{D}$, interior to S, will be assumed to be of two types. The first consists of these forces exerted on $\mathcal{D}$ by the systems external to S; these are the *actions at a distance*, which, moreover, will be considered in general as *given*. We will denote their virtual power by $\mathcal{P}_{(d)}$, and we will assume that it is expressed in the form of a volume integral over the open set $\mathcal{D}$. The second kind of external forces consists of those forces exerted on $\mathcal{D}$ by the parts of S exterior to $\mathcal{D}$. We will assume here that, as is customary in the mechanics of continuous media, these are contact forces — thus implying that the actions at a distance originating within S are negligible. The virtual power of the contact forces will be denoted by $\mathcal{P}_{(c)}$, which will be expressible by means of a surface integral[1] over $\partial\mathcal{D}$.

Analogously, we will assume that the external forces exerted on S also comprise actions at a distance and contact forces on ∂S.

(g) Since we limit ourselves to the case of statics, the principle of virtual power is expressed by the equation

$$\mathcal{P}_{(d)} + \mathcal{P}_{(c)} + \mathcal{P}_{(i)} = 0, \tag{9}$$

which must be satisfied for any subdomain $\mathcal{D}$ and any virtual motion considered in $\mathcal{V}$. The relations that express the necessary and sufficient conditions for this to be true constitute the set of equations of statics for the system under consideration.

The meaning of these remarks will become clearer through the two examples that we will presently consider.

4. First-gradient theory

The first-gradient theory is, in fact, a rather simple generalization of the classical formulation of the theory of continuous media. The name given to this first-gradient

[1] As we will see in Section 5, in certain cases it may be appropriate to include an additional term expressed in the form of a line integral (over the edges of $\partial\mathcal{D}$).

theory arises from the fact that for a given subsystem $\mathcal{D}$ the space of virtual motions $\mathcal{V}$ is that of continuous and at least once continuously differentiable velocities over the closure $\mathcal{D} + \partial\mathcal{D}$ of $\mathcal{D}$, where the norm of $\mathcal{V}$ is that of the uniform convergence for the velocities and their first derivatives with respect to the coordinates x_i. We will denote by U_i the velocity components and by $U_{i,j}$ their first derivatives. We will, moreover, introduce the canonical decomposition of the velocity gradient into a symmetric part and a skew-symmetric part, namely,

$$U_{i,j} = D_{ij} + \Omega_{ij}; \qquad D_{ij} = D_{ji}, \quad \Omega_{ij} = -\Omega_{ji}. \tag{10}$$

We know that D_{ij} is the matrix that represents the strain-rate tensor and that Ω_{ij} is the matrix representing the rotation-rate tensor. The continuous linear forms on this space $\mathcal{V}$ are, in all generality, distributions of order 1, that is, measure derivatives. But we have already explained in the preceding section that we will not resort here to this generality and that we will assume that the virtual power can be expressed by means of integrals.

4.1. *We shall always commence by formulating the virtual power of the internal forces*. This will be done for two reasons. In the first place, this is the essentially new notion brought about by continuum mechanics. In the second place, we have at our disposal the axiom stated above, which permits us to simplify its expression. Moreover, we will find that, in writing the virtual work of the external forces, we will be guided by the results already gained for the expression of the virtual power of the internal forces.

We know that $\mathcal{P}_{(i)}$ is a volume integral over $\mathcal{D}$ (in accordance with remark (e) above). We will write, therefore,

$$\mathcal{P}_{(i)} = -\int_{\mathcal{D}} p\, dv. \tag{11}$$

Except for the sign, p is *the virtual power of the internal forces per unit volume*, or *the energy of the internal forces per unit volume*. Furthermore, by hypothesis, p must be a linear form in the arguments U_i, D_{ij}, and Ω_{ij}. But, by virtue of the axiom, we can state:

Proposition 1. *The density p can be written in the form*

$$p = \sigma_{ij} D_{ij}, \tag{12}$$

where σ_{ij} is a symmetric matrix representing a symmetric second-order tensor called the intrinsic stress tensor (*of order* 1), *which is an objective quantity.*

The proof is straightforward. It is clear that, without loss of generality, we can assume that σ_{ij} is symmetric with respect to the indices i and j.* Since D_{ij}

*Assuming, of course, that we're in the case where (12) is valid. — (TN) represents a tensor, the same must be true for σ_{ij}, as can be concluded by a change of frame at the time under consideration. Moreover, by virtue of the axiom, $\mathcal{P}_{(i)}$ and p preserve their values under a change of reference, since evidently the difference of the velocity fields of one and the same virtual motion as observed in two different references is the velocity field of some twist. We still need to show that p can depend neither on U_i nor on Ω_{ij}. If, for instance, there were in (12) a term in U_i having a coefficient not identically zero in the neighborhood of a point M of S, one could find a subsystem Δ of S, containing M, and a virtual motion of translation defined on Δ, for which p would not vanish identically, contrary to the stipulation of the axiom. The impossibility of having a nonvanishing term in Ω_{ij} in (12) can be established by a similar reasoning. The proposition is thus proven.

We assume (remark (d)) that the components σ_{ij} are continuously differentiable in x_i. The divergence theorem permits us to derive, having duly noted that $\sigma_{ij} D_{ij} = \sigma_{ij} U_{i,j}$, the following useful expression for the virtual power of the internal forces:

$$\mathcal{P}_{(i)} = \int_{\mathcal{D}} \sigma_{ij,j} U_i \, dv - \int_{\partial\mathcal{D}} \sigma_{ij} n_j U_i \, da. \tag{13}$$

If we take U_i as the velocity field of a twist,* the left-hand side of (13) vanishes,† so that, incidentally, we obtain the following:

Proposition 2. *The wrench defined by the volumetric density $\sigma_{ij,j}$ in $\mathcal{D}$ is equal to that defined by the surface density $\sigma_{ij} n_j$ on $\partial\mathcal{D}$.*

This result is usually conveyed in more compact notation‡ as

$$[\sigma_{ij,j}]_{\mathcal{D}} = [\sigma_{ij} n_j]_{\partial\mathcal{D}}. \tag{14}$$

4.2. *We will presently formulate the power of the external forces exerted on $\mathcal{D}$*, while adhering to the working hypotheses stated in the preceding section. We will proceed systematically by writing general linear forms over $\mathcal{V}$ and postponing until the end of this section a brief discussion of the physical meaning of the quantities used. As far as $\mathcal{P}_{(d)}$ is concerned, we will write

$$\mathcal{P}_{(d)} = \int_{\mathcal{D}} (f_i U_i + C_{ij} \Omega_{ij} + \Phi_{ij} D_{ij}) \, dv. \tag{15}$$

This definition implies that the external actions at a distance can be represented by

- a field of volumetric forces defined by the density f_i,
- a field of volumetric couples defined by the density C_{ij}, representing a skew-symmetric tensor, namely, $C_{ij} = -C_{ji}$,

*That is, a rigidifying motion. — (TN)
†According to the axiom. — (TN)
‡That is, using screw (or torsor) "notation" as in Equation (8) of [16]. — (TN)

- a field of volumetric "symmetric double forces" defined by the density Φ_{ij}, representing a symmetric tensor, namely, $\Phi_{ij} = \Phi_{ji}$.

It is convenient to transform the equality (15) following the procedure used above. Noting the identities

$$C_{ij}\Omega_{ij} = C_{ij}U_{i,j} = (C_{ij}U_i)_{,j} - C_{ij,j}U_i,$$
$$\Phi_{ij}D_{ij} = \Phi_{ij}U_{i,j} = (\Phi_{ij}U_i)_{,j} - \Phi_{ij,j}U_i,$$

and applying the divergence theorem, we obtain

$$\mathcal{P}_{(d)} = \int_{\mathfrak{D}} (f_i - C_{ij,j} - \Phi_{ij,j})U_i\, dv + \int_{\partial\mathfrak{D}} (C_{ij} + \Phi_{ij})n_j U_i\, da. \tag{16}$$

All that remains is to deal with the external contact forces. Their virtual power is defined by a scalar surface density which is, *a priori*, a linear function of U_i and of the first derivatives of U_i. But, anticipating the formulation of the principle of virtual power, we become aware that these last terms vanish identically, since they could not be possibly balanced by any analogous term in the expressions (13) and (16) of the virtual power of the internal forces and the actions at a distance. Moreover, we can obtain this result in an absolutely explicit fashion from the expression (34), given farther below. We will, therefore, simply write

$$\mathcal{P}_{(c)} = \int_{\partial\mathfrak{D}} T_i U_i\, da, \tag{17}$$

where, by definition, T_i represents the *stress vector* at a point of $\partial\mathfrak{D}$ acting perpendicularly to $\partial\mathfrak{D}$; this is a surface density of contact forces.

4.3. It remains to apply the *principle of virtual power*, that is, (9). Taking (13), (16), and (17) into consideration, we obtain

$$0 = \int_{\mathfrak{D}} (f_i + \sigma_{ij,j} - C_{ij,j} - \Phi_{ij,j})U_i\, dv + \int_{\partial\mathfrak{D}} (T_i - (\sigma_{ij} - C_{ij} - \Phi_{ij})n_j)U_i\, da. \tag{18}$$

We are thus led to define

$$\tau_{ij} = \sigma_{ij} - C_{ij} - \Phi_{ij}. \tag{19}$$

By definition, τ_{ij} represents the *stress tensor*.

Let us apply first the identity (18) taking as U_i an arbitrary field that vanishes outside a compact set contained in $\mathfrak{D}$. In that case, we are just left with the volume integral, and since U_i is otherwise arbitrary, we obtain at each point of this compact set, that is, at each interior point of $\mathfrak{D}$, the equation

$$f_i + \tau_{ij,j} = 0. \tag{20}$$

Consequently, the volume integral in (18) vanishes identically. Introducing now in (18) a field U_i that vanishes outside a compact set with an arbitrarily chosen nonempty intersection Σ with $\partial\mathcal{D}$, we obtain

$$\int_\Sigma (T_i - \tau_{ij} n_j) U_i \, da = 0.$$

Since U_i itself can be chosen arbitrarily in the interior of Σ, we conclude that at each point of $\partial\mathcal{D}$ we must have necessarily that

$$T_i = \tau_{ij} n_j. \tag{21}$$

Equation (21) is the usual relation providing the stress vector in terms of the direction $\boldsymbol{n}$ of the exterior normal. Equation (20) is nothing but the classical equilibrium equation. Splitting in (19) symmetric and skew-symmetric parts, we can write

$$\tau_{[ij]} + C_{ij} = 0, \tag{22}$$

$$\tau_{ij} + \Phi_{ij} = \sigma_{ij}. \tag{23}$$

Up to this point we have always assumed that $\mathcal{D}$ is interior to S. For the sake of completeness, we should apply the principle of virtual power to S itself. To this end, we will assume that the forces external to S comprise, beyond the actions at a distance already mentioned, contact forces (whether known or unknown) defined by surface forces of density t_i. Reasoning just as before to obtain (21), we find the boundary condition that must be satisfied at each point of $\partial\mathcal{S}$, namely,

$$t_i = T_i = \tau_{ij} n_j. \tag{24}$$

In this equation, n_i denotes the exterior unit normal at a point of $\partial\mathcal{S}$, and T_i denotes the stress vector for the direction $\boldsymbol{n}$ obtained by a passage to the limit, the point of $\partial\mathcal{S}$ being an accumulation point of a set of nearby points interior to S. The collection of the results obtained thus far can be summarized in the following statement.

Theorem 1. *The necessary and sufficient conditions ensuring that the system S is in equilibrium establish that the stress tensor satisfies Equations* (20) *and* (22) *at each interior point of S and Equation* (24) *at each point of the boundary ∂S. Moreover, the intrinsic stress tensor and the volumetric energy of the internal forces are given, respectively, by* (23) *and* (12).

Equations (20), (22), and (24) are those provided by the application of the fundamental law of conservation of linear momentum (Germain [3]); in addition, in that case, it is necessary to assume from the start that, at each point of $\partial\mathcal{D}$, T_i is a function of n_j, an assumption which we did not need to invoke in the present treatment. On the other hand, this fundamental law cannot give us any information about the influence of the symmetric double forces that participate in the determination of the volumetric energy of the internal forces. Thus, even in the simple case of the first-gradient theory, it is not without interest to construct the general equations supplying the mechanical description of the system starting from the notion of virtual power. In the classical formulation of continuum mechanics, this advantage disappears, since in that case

$$\Phi_{ij} = C_{ij} = 0,$$

so that

$$\tau_{ij} = \sigma_{ij}.$$

The intrinsic stress tensor, therefore, coincides in this case with the stress tensor proper.

Remark. As we have already stated, we wanted to present the first-gradient theory in a systematic fashion. One can legitimately ask if, except for the classical case, this theory presents any physical interest. This point raises the question as to the physical meaning of the volumetric double forces, that is, the couples C_{ij} and the symmetric double forces Φ_{ij}. These forces can be properly interpreted if we assume that each material point of S is equipped with a microstructure, and it is, in fact, very instructive to draw a correlation between the present theory and the theory of media endowed with microstructure, which we intend to do in a forthcoming article. At first sight, it may seem strange that the microstructure might participate at the level of the modeling of the external actions at a distance or that it might play any role in the modeling of the internal forces. It appears, however (see, for example, Lobdell [4]), that the first-gradient theory may be useful to describe certain electromechanical phenomena in solids.

Be that as it may, it is clear that this first-gradient theory is nothing but a slight extension of the classical theory, an extension that manifests itself in a nutshell in the formula (19). We only developed this theory here so as to show how to apply the virtual power method in a simple context in order to build a mechanical model of continuous media.

5. Second-gradient theory

The theory we are about to construct will be finer than the preceding one. We will consider as the space $\mathcal{V}$ of virtual motions the space of continuous and at least twice differentiable velocity fields defined on the closure $\mathcal{D} + \partial\mathcal{D}$ of $\mathcal{D}$, the norm in $\mathcal{V}$ being that of the uniform convergence for the velocities and their derivatives up to order 2 with respect to x_i. Our calculations will be analogous to those that can be found in [2] (see also [5; 6]), but the interpretation given here is different and more comprehensive, and since our notation is not exactly the same as in those works, we believe that it is a good idea to repeat it here, at least in the Appendix, for the sake of assisting in the reading process. It is appropriate to choose a canonical representation of the (third-order) tensor of the second derivatives of U_i. Mindlin and Eshel [2] propose three different ones. We will content ourselves here with choosing the third one. Defining in the first place the rate of rotation vector ω_i as

$$\omega_i = -\tfrac{1}{2}\varepsilon_{ipq}\Omega_{pq}, \quad \Omega_{ij} = -\varepsilon_{ijk}\omega_k, \tag{25}$$

we introduce the gradient tensor of the rate of rotation

$$K_{ij} = \omega_{i,j} = -\tfrac{1}{2}\varepsilon_{ipq}\Omega_{pq,j} = -\tfrac{1}{2}\varepsilon_{ipq}U_{p,qj}. \tag{26}$$

This tensor K_{ij} is actually a deviator; that is, its trace K_{ii} vanishes. It has, therefore, eight independent components. In the second place, we consider the completely symmetric part of the second gradient of the velocities

$$K_{ijk} = \tfrac{1}{3}(U_{i,jk} + U_{j,ki} + U_{k,ij}). \tag{27}$$

The value of K_{ijk} remains invariant under every permutation of the indices i, j, k. This tensor, therefore, has ten different components. The collection of the K_{ij} and the K_{ijk} determines completely the eighteen second derivatives $U_{i,jk}$ (and vice versa):

$$U_{i,jk} = K_{ijk} - \tfrac{2}{3}\varepsilon_{ijl}K_{lk} - \tfrac{2}{3}\varepsilon_{ikl}K_{lj}. \tag{28}$$

This relation can be easily established noting beforehand that, according to (26), we have

$$\varepsilon_{lmi}K_{ij} = \tfrac{1}{2}(U_{l,mj} - U_{m,lj}).$$

5.1. We will start once again by formulating the *virtual power of the internal forces*, adopting evidently the working hypotheses stated in Section 3. Recalling the considerations developed in Section 4.1, and particularly formula (11), we see that, by virtue of the axiom, we can write the volumetric energy of the internal forces in the form

$$p = \sigma_{ij}D_{ij} + \mu_{ij}K_{ij} + \mu_{ijk}K_{ijk}. \tag{29}$$

The coefficients μ_{ij} are components of a second-order tensor, which is, incidentally, a deviator ($\mu_{ii} = 0$); the μ_{ijk} are components of a totally symmetric third-order tensor (μ_{ijk} remains invariant under every permutation of the indices). *These two tensors constitute a* (*canonical*) *representation of the intrinsic stresses of order 2*. All that is left now is to write $\mathcal{P}_{(i)}$ in the appropriate canonical form necessary to be able to apply the principle of virtual power. This is achieved proceeding, as in the preceding section, to carry out integrations by parts on the expression

$$\mathcal{P}_{(i)} = -\int_{\mathcal{D}}(\sigma_{ij}D_{ij} + \mu_{ij}K_{ij} + \mu_{ijk}K_{ijk})\,dv. \tag{30}$$

For the first term, the integration by parts needs to be performed once; the result is the one obtained on the right-hand side of formula (13). For the remaining two terms, the calculation is slightly more complicated, since it is necessary to integrate twice; this is shown in the Appendix, and the results are those of formulas (A-11) and (A-9), respectively. We see, therefore, that we can write $\mathcal{P}_{(i)}$ in the form

$$\mathcal{P}_{(i)} = \int_{\mathcal{D}} \mathcal{F}_i U_i \, dv + \int_{\partial\mathcal{D}} (\mathcal{T}_i U_i + \widetilde{\mathcal{M}}\widetilde{\omega}_i + \mathcal{N} D_{\underline{\underline{nn}}}) \, da + \int_{\Gamma \curvearrowleft} \mathcal{R}_i U_i \, ds, \tag{31}$$

where we have put

$$\begin{cases} \mathcal{F}_i = \sigma_{ij,j} - \frac{1}{2}\varepsilon_{ipj}\mu_{pq,qj} - \mu_{ijk,jk}, \\ \mathcal{T}_i = -\big(\sigma_{ij} - \frac{1}{2}\varepsilon_{ipj}\mu_{pq,q} - \mu_{ijk,k}\big)n_j + \frac{1}{2}\varepsilon_{ipj} D_j(\mu_{\underline{\underline{nn}}} n_p) \\ \qquad\qquad + (D_j - n_j(D_p n_p))(\mu_{ijk}n_k + \mu_{jlk}n_i n_l n_k), \\ \widetilde{\mathcal{M}}_i = 2\varepsilon_{ikq}\mu_{kjp}n_j n_p n_q - (\mu_{iq}n_q - n_i \mu_{\underline{\underline{nn}}}), \\ \mathcal{N} = -\mu_{ijk}n_i n_j n_k, \\ \mathcal{R}_i = -\big[\!\big[\frac{1}{2}\delta_{im}\mu_{\underline{\underline{nn}}} + \varepsilon_{jmq}n_k n_q(\mu_{ijk} + \mu_{pjk}n_i n_p) \big]\!\big] \tau_m. \end{cases} \tag{32}$$

The meaning of the symbols used in the formulas (32) is better given in the Appendix; $\widetilde{\omega}_i$ is the part of the rate of rotation vector ω_i tangential to $\partial\mathcal{D}$; $D_{\underline{\underline{nn}}}$ and $\mu_{\underline{\underline{nn}}}$ are real numbers representing the doubly normal component of the tensors D_{ij} and μ_{ij}.* The symbol D_i is an operator of derivation tangential to the surface $\partial\mathcal{D}$, whose explicit expression is given in (A-2) (and, incidentally, $D_p n_p$ is nothing other than twice the mean curvature); Γ denotes the edges of the boundary surface $\partial\mathcal{D}$ along which the tangent plane (or the normal vector $\boldsymbol{n}$) is discontinuous; τ_i is the unit vector tangent to Γ, whose orientation can be chosen arbitrarily, but consistently; finally, the symbol $[\![\ ]\!]$ denotes the jump of the bracketed quantity across Γ. It is worthwhile noting that the sense across Γ on which the jump takes place, and the sense of Γ must be related, in agreement with the usual Stokes' formula.

It should be noted that the vector $\widetilde{\mathcal{M}}_i$ is tangential to the surface $\partial\mathcal{D}$, since we have that $\mathcal{R}_i\widetilde{\mathcal{M}}_i = 0$. It is precisely this fact that the tilde is supposed to indicate.

5.2. We must presently formulate the expression of the *virtual power of the external forces*. If we want to proceed systematically, we must write the power of the actions at a distance, taking into consideration the remarks made in Section 3, in the form

$$\mathcal{P}_{(d)} = \int_{\mathcal{D}} (f_i U_i + C_{ij}\Omega_{ij} + \Phi_{ij}D_{ij} + \Xi_{ij}K_{ij} + \Xi_{ijk}K_{ijk}) \, dv. \tag{33}$$

On comparing (33) with (15), we perceive that there appear here additional forces, namely, the volumetric triple forces defined by Ξ_{ij} — which is a deviator — and those defined by Ξ_{ijk}, which is a completely symmetric third-order tensor. In fact, we will assume, for the sake of simplicity, that these triple forces vanish;

*As indicated later, an underlined subscript nullifies the summation convention with respect to that subscript. — (TN) it should not be difficult to indicate how the expressions given below are to be modified if we want to take these forces into account. We will, therefore, accept that $\mathcal{P}_{(d)}$ is still given by (15) or, better, by (16), which is more suitable for the application of the principle of virtual power.

As far as the virtual power of the contact forces acting on the boundary surface $\partial\mathcal{D}$ is concerned, we are guided, as we were in the preceding section, to write the most suitable expression, by anticipating the application of the principle and taking into consideration the formula (31) already found for $\mathcal{P}_{(i)}$. We are thus led to write

$$\mathcal{P}_{(c)} = \int_{\partial\mathcal{D}} (T_i U_i + \widetilde{M}_i \widetilde{\omega}_i + N D_{\underline{nn}})\, da + \int_{\Gamma\frown} R_i U_i \, ds. \tag{34}$$

As before, T_i denotes the stress vector; $\widetilde{M}_i$, a vector tangent to $\partial\mathcal{D}$, is a surface density of a *tangential couple*; N is a scalar surface density of a doubly normal double force; R_i is a vector that defines a line density of a force applied along the edges of Γ.

5.3. It remains only to apply the principle of virtual power, that is, equality (9). We will substitute in it the expressions given in (16), (31), and (34); we thus obtain an equation that must be satisfied for every field U_i twice continuously differentiable in the closure $\mathcal{D} + \partial\mathcal{D}$ of $\mathcal{D}$.

Let us consider first fields U_i that vanish outside a compact set interior to $\mathcal{D}$. The only survivor is the volume integral that can be written as

$$\int_{\mathcal{D}} (f_i + \tau_{ij,j}) U_i \, dv = 0,$$

if we set

$$\tau_{ij} = \sigma_{ij} - \tfrac{1}{2}\varepsilon_{ipj}\mu_{pq,q} - \mu_{ijk,k} - C_{ij} - \Phi_{ij}. \tag{35}$$

It follows that at each interior point of $\mathcal{D}$ we necessarily have

$$f_i + \tau_{ij,j} = 0. \tag{36}$$

In the equation expressing the principle of virtual power, therefore, the only remaining terms in the general case are the surface integral over $\partial\mathcal{D}$ and the line integral over the edges Γ. Let us consider a fixed arbitrary closed connected area Σ which is a subset of $\partial\mathcal{D}$ not having any point in common with an edge, and let us denote by $C_2(\Sigma)$ the collection of twice continuously differentiable scalar-valued functions defined over Σ and vanishing outside a compact set interior to Σ. We state:

Lemma. *Given seven functions in $C_2(\Sigma)$ — $V_1(P)$, $V_2(P)$, $V_3(P)$, $\widetilde{\Omega}_1(P)$, $\widetilde{\Omega}_2(P)$, $\widetilde{\Omega}_3(P)$, $D(P)$ — constrained by the single relation $n_i \widetilde{\Omega}_i = 0$, it is possible to construct a twice continuously differentiable field U_i on the closure $\mathcal{D} + \partial\mathcal{D}$ attaining* *on the boundary* $\partial\mathcal{D}$ *the following values*:

$$\begin{aligned} &\textit{at each point of } \Sigma, && U_i = V_i,\ \widetilde{\omega}_i = \widetilde{\Omega}_i,\ D_{\underline{\underline{nn}}} = D, \\ &\textit{at all other points of } \partial\mathcal{D}, && U_i = \widetilde{\Omega}_i = D_{\underline{\underline{nn}}} = 0. \end{aligned} \tag{37}$$

The proof is straightforward. The velocity field that we are trying to construct has, according to equalitites (37), known values on Σ and a gradient that also has known values on Σ (the tangential derivatives are determined by those of V_i, the normal derivatives of the tangential components are next determined by $\widetilde{\Omega}_i$, and the normal derivative of the normal component by D). Let us consider the subset Δ of $\mathcal{D}$ made up of the points M such that $\boldsymbol{MP} = \zeta \boldsymbol{n}$, with $0 \leq \zeta \leq \zeta_0(P)$, where $\zeta_0(P)$ is a function defined over Σ and infinitely differentiable on Σ vanishing on the boundary and attaining at each point of Σ sufficiently small values so that at each point of Δ there is a unique normal to Σ. It is clear that we can construct in Δ a field $U_i(M) = U_i(P, \zeta)$ such that it and its first derivatives attain the values prescribed on Σ and also such that it and its derivatives up to order 2 vanish over the part of $\partial\Delta - \Sigma$ of the boundary of Δ. Thus, in a trivial fashion, we complete the definition of U_i by assigning to it zero values on the set $\mathcal{D} + \partial\mathcal{D} - \Delta$, and this field satisfies perfectly the conditions of the lemma.

Applying the equation of virtual power to such a field U_i yields

$$\int_\Sigma \big\{(T_i + \mathcal{T}_i + C_{ij}n_j + \Phi_{ij}n_j)V_i + (\widetilde{M}_i + \widetilde{\mathcal{M}}_i)\widetilde{\Omega}_i + (N + \mathcal{N})D\big\}\, da = 0,$$

for arbitrary functions V_i, $\widetilde{\Omega}_i$, D, constrained by the single relation $n_i\widetilde{\Omega}_i = 0$. Furthermore, since the vector $\widetilde{M}_i + \widetilde{\mathcal{M}}_i$ is a vector tangent to the surface Σ, the quantities within the parentheses ( ) under the integral sign must vanish individually at each point of Σ and, consequently, taking into account the latitude with which Σ can be chosen, also at each point of the boundary $\partial\mathcal{D}$ not belonging to an edge. In accordance with (32), we can write

$$\begin{cases} T_i = \tau_{ij}n_j + T_i' = \widehat{T}_i + T_i', \\ T_i' = (n_j(D_p n_p) - D_j)(\mu_{ijk}n_k + \mu_{ljk}n_i n_l n_k) - \frac{1}{2}\varepsilon_{ipj}D_j(n_p\mu_{\underline{\underline{nn}}}), \\ \widetilde{M}_i = \mu_{iq}n_q - n_i\mu_{\underline{\underline{nn}}} - 2\varepsilon_{ikq}\mu_{kjp}n_j n_p n_q, \\ N = \mu_{ijk}n_i n_j n_k. \end{cases} \tag{38}$$

Let us note that the last term appearing in the expression of T_i' also can be written as in (A-12), that is,

$$-\tfrac{1}{2}\varepsilon_{ipj}D_j(n_p\mu_{\underline{\underline{nn}}}) = -\tfrac{1}{2}\varepsilon_{ipj}n_p\mu_{\underline{\underline{nn}},j}.$$

These equations show how the stress vector T_i, the surface tangential couple $\widetilde{M}_i$, and the (surface) normal double force N are expressed in terms of the intrinsic stress tensors of orders 1 and 2. We can still call τ_{ij} the stress tensor, given that the equilibrium equation has the usual form, but it must be noted that in the present case this tensor is no longer sufficient to define the stress vector T_i. Finally, taking into consideration the results already obtained, the equality of virtual powers reduces to

$$\int_\Gamma (R_i + \mathcal{R}_i) U_i \, ds = 0,$$

and therefore, taking (32) into account, we have

$$R_i = [\![\tfrac{1}{2} \delta_{mi} \mu_{\underline{\underline{nn}}} + \varepsilon_{jmq} n_k n_q (\mu_{ijk} + \mu_{pjk} n_i n_p)]\!] \tau_m, \tag{39}$$

an equation that allows us to express the line force along the edges as a function of the intrinsic stress tensors.

In addition, it remains to apply the principle to S itself in order to find the boundary conditions. To this end, we are led to admit that the external forces exerted on S comprise, beyond the volume actions at a distance already considered, surface effects defined by surface forces of density t_i, couples tangential to S of surface density $\widetilde{m}_i$, and doubly normal double forces of surface density n. We can easily show that we must have

$$\begin{cases} T_i = t_i, \;\; \widetilde{M}_i = m_i, \;\; N = n & \text{on } \partial S, \\ R_i = r_i & \text{on } \Gamma. \end{cases} \tag{40}$$

We have thus exhausted the consequences that can be extracted from the principle of virtual power, and we can consequently establish the following:

Theorem 2. *The necessary and sufficient conditions ensuring that the system S is in equilibrium, for the case in which the external triple volume forces are neglected, are expressed by the relations* (35) *and* (36), *that must be satisfied at each interior point of S, and the relations* (40), *that must be satisfied at each point of the boundary ∂S. Moreover, in the interior of S, the surface contact forces are defined by* (38) *and the line forces by* (39). *When the external triple volume forces are taken into consideration, the fundamental equations of statics of the second-gradient theory are obtained by replacing in the previous equations μ_{ij} and μ_{ijk} by $\mu_{ij} - \Xi_{ij}$ and $\mu_{ijk} - \Xi_{ijk}$, respectively.*

5.4. It is instructive to interpret the preceding results in terms of the classical fundamental law. In so doing, we will discover the extra information and precision contributed by the formulation herein advocated, which turns out to be better suited when we are dealing with a medium for which the modeling must be finer than in the classical case.

Let us recall a remark already made above: the statements purveying the fundamental law can be obtained from the equations expressing the principle of virtual power by considering only the rigidifying virtual motions of $\mathcal{D}$, for which the velocity field is determined by a twist, that is, a velocity field of the form

$$U_i = V_i + \varepsilon_{ikj}\bar{\omega}_k x_j, \tag{41}$$

where $\bar{\omega}_i$ and V_i are the elements of reduction of the twist at the origin.

Here is a first application. Let us take up again the expression of the virtual power of the internal forces given by (30), and instead of proceeding to carry out two integrations by parts to end up with (31), let us retain the intermediate result obtained after a single integration. Setting

$$\beta_{ij} = \sigma_{ij} - \tfrac{1}{2}\varepsilon_{ipj}\mu_{pq,q} - \mu_{ijk,k},$$

that is, according to (35),

$$\tau_{ij} = \beta_{ij} - C_{ij} - \Phi_{ij}, \tag{42}$$

we can write

$$0 = \int_{\mathcal{D}} \beta_{ij,j} U_i \, dv - \int_{\partial\mathcal{D}} \beta_{ij} n_j U_i \, da + \int_{\partial\mathcal{D}} \left(\tfrac{1}{2}\varepsilon_{ijp}\mu_{pq}n_q U_{i,j} - \mu_{ijk}n_k U_{i,j}\right) da.$$

Let us apply this equation using the field (41); the term in μ_{ijk} does not contribute at all, by virtue of the symmetry with respect to the first two indices. If we set

$$\gamma_i = \mu_{ij}n_j, \qquad (\gamma_i)_{\partial\mathcal{D}} = \int_{\partial\mathcal{D}} \gamma_i \, da,$$

where $(\gamma_i)_{\partial\mathcal{D}}$ denotes the couple defined by the surface density of the couples γ_i on $\partial\mathcal{D}$, we immediately obtain, using the notations introduced above,

$$[\beta_{ij,j}]_{\mathcal{D}} = [\beta_{ij}n_j]_{\partial\mathcal{D}} + (\gamma_i)_{\partial\mathcal{D}}. \tag{43}$$

With the same notation, an equation of the form

$$\int_{\partial\mathcal{D}} \tau_{ij} n_j U_i \, da = \int_{\mathcal{D}} \tau_{ij,j} U_i \, dv + \int_{\mathcal{D}} \tau_{ij} U_{i,j} \, dv,$$

when applied to the field (41), leads to the wrench equation

$$[\tau_{ij}n_j]_{\partial\mathcal{D}} = \{[\tau_{ij,j}]_{\mathcal{D}} + (\varepsilon_{kij}\tau_{kj})_{\mathcal{D}}\} = 0. \tag{44}$$

We have, therefore,

$$[\Phi_{ij,j}]_{\mathcal{D}} = [\Phi_{ij}n_j]_{\partial\mathcal{D}}, \qquad [C_{ij,j}]_{\mathcal{D}} = [C_{ij}n_j]_{\mathcal{D}} - (c_i)_{\mathcal{D}}, \tag{45}$$

where we have set

$$c_k = -\varepsilon_{kij}C_{ij}. \tag{46}$$

The vector c_i is the couple vector associated with the skew-symmetric tensor C_{ij}. Combining (42), (43), and (45), we can formulate the following:

Proposition 3. *For every part $\mathscr{D}$ of S, we have the following wrench equation*:

$$[\tau_{ij,j}]_{\mathscr{D}} - \{[\tau_{ij}n_j]_{\partial\mathscr{D}} + (\gamma_i)_{\partial\mathscr{D}} + (c_i)_{\mathscr{D}}\} = 0. \tag{47}$$

Recall that $\widehat{T}_i = \tau_{ij}n_j$.

Furthermore, when applied to a twist such as (41), since $\mathscr{P}_{(i)}$ vanishes, $\mathscr{P}_{(d)}$ and $\mathscr{P}_{(c)}$ being given by (15) and (34), respectively, the statement of the principle of virtual power leads to the wrench equation

$$[f_i]_{\mathscr{D}} + (c_i)_{\mathscr{D}} + [\widehat{T}_i]_{\partial\mathscr{D}} + [T_i']_{\partial\mathscr{D}} + (\widetilde{M}_i)_{\partial\mathscr{D}} + [R_i]_\Gamma = 0. \tag{48}$$

Let us add (47) and (48), and let us take (36) into account to obtain

$$(\gamma_i)_{\partial\mathscr{D}} = [T_i']_{\partial\mathscr{D}} + (\widetilde{M}_i)_{\partial\mathscr{D}} + [R_i]_\Gamma. \tag{49}$$

The right-hand side must be, just as the left-hand side, a couple, which implies in particular that the resultant of the wrench that it defines vanishes, namely,

$$\int_{\partial\mathscr{D}} T_i'\,da + \int_\Gamma R_i\,ds = 0.$$

Taking (49) into account, we can write (48) in the form

$$[f_i]_{\mathscr{D}} + (c_i)_{\mathscr{D}} + [\widehat{T}_i]_{\partial\mathscr{D}} + (\gamma_i)_{\partial\mathscr{D}} = 0. \tag{50}$$

The first two terms of the left-hand side represent the wrench of the actions at a distance exerted on $\mathscr{D}$ — the double forces Φ_{ij} must be considered as defining a zero wrench, since their virtual power vanishes in every rigidifying motion. It follows that the last two terms represent the wrench of the contact actions. Consequently, we obtain the following:

Proposition 4. *On every part $\mathscr{D}$ of S the contact actions determine a wrench that can be defined by a surface force $\widehat{T}_i$ and a surface couple γ_i which are linear functions of the unit normal vector $\boldsymbol{n}$. We have indeed*

$$\widehat{T}_i = \tau_{ij}n_j, \qquad \gamma_i = \mu_{ij}n_j. \tag{51}$$

This representation of the contact forces is in appearance simpler than that given in Section 5.2; but this is no more than a globally valid representation. For this reason, in the absence of supplementary details, this representation is in effect insufficient for the study of media endowed with stress couples, such as those that fall under the scope of the second-gradient theory. Let us, moreover, recall that by means of a statement such as that of the classical fundamental law appearing in equality (50) it is not possible to take into account the external double forces. Finally, we should note that, adding (44) and (47), we obtain

$$(\varepsilon_{kij}\tau_{kj})_{\mathscr{D}} + (\gamma_i)_{\partial\mathscr{D}} + (c_i)_{\mathscr{D}} = 0,$$

and using equality (51), transforming the surface integral into a volume integral, and considering (46), we find in the end, since $\mathscr{D}$ is an arbitrary part of S, the equation

$$\tau_{[ij]} = -\tfrac{1}{2}\varepsilon_{ipj}\mu_{pq,q} - C_{ij},$$

expressing the equality of the skew-symmetric parts of the two sides of (35).

5.5. The representation adopted for the second-gradient (26) and (27) allows us to deal immediately with the case where K_{ijk} does not participate in the virtual power. Indeed, it suffices to set $\mu_{ijk} = 0$ in the previous formulas. For example, if we consider (38) and (39), we notice that $N = 0$ and that, more precisely, we have

$$T_i = \tau_{ij,j} + T'_i = \widehat{T_i} + T'_i, \quad T'_i = -\tfrac{1}{2}\varepsilon_{ipj}D_j(n_p\mu_{\underline{nn}}),$$
$$\widetilde{M}_i = \mu_{iq}n_q - n_i\mu_{\underline{nn}} = (\widetilde{\mu_{iq}n_q}) = \tilde{\gamma}_i,$$
$$R_i = \tfrac{1}{2}\mu_{nn}\tau_i.^*$$

The notations of (51) have been used. Naturally, we have

$$\tau_{ij} = (\sigma_{ij} - \Phi_{ij}) + \left(\tfrac{1}{2}\varepsilon_{ijp}\mu_{pq,q} - C_{ij}\right).$$

On the right-hand side, the first term in parentheses is the even part of τ_{ij} — which reduces to σ_{ij} if we neglect the volumetric double forces — while the second term in brackets is the odd part.

The boundary conditions associated naturally to this model are still given by (40), on condition that the equation $N = n$ be omitted, since it is superfluous here. It is precisely these conditions that can suggest the boundary conditions to take into consideration to formulate problems with regular boundaries. We leave to the reader the effort of particularizing the results given in Sections 6 and 7 for this simplified case.

5.6. ***A final remark.*** We have not treated here either the dynamical case or the case of equations with discontinuities. We intend to revisit this topic in a forthcoming article in which we will specify in advance the relations between the second-gradient theory and the theory of media endowed with microstructure. The physical meaning of the first will then be more easily brought to light.

6. Constitutive behavior of a medium under the umbrella of the second-gradient theory within the framework of small perturbations. The case of elastic media

To study the response of a system S under the action of external agents, it is necessary to supplement the general equations obtained above by means of constitutive equations. In this section, we intend to briefly examine how this can be achieved within the framework of problems that can be treated under the hypothesis of small strains.

Under these conditions, indeed, there is no need to distinguish between the Lagrangian and Eulerian representations. The motion of the medium is defined by the displacement field $X_i(x_1, x_2, x_3, t)$ defined over a well determined domain S, which plays the role of a reference configuration. The strains for a second-gradient theory can then be fully characterized by the tensors

$$\varepsilon_{ij} = \tfrac{1}{2}(X_{i,j} + X_{j,i}), \tag{52}$$

$$\eta_{ij} = -\tfrac{1}{2}\varepsilon_{ipq} X_{p,qj} \quad (\eta_{ii} = 0), \tag{53}$$

$$\eta_{ijk} = \tfrac{1}{3}(X_{i,jk} + X_{j,ki} + X_{k,ij}), \tag{54}$$

since, under the hypothesis of small perturbations, the material derivative being identical in this case to the partial derivative with respect to time, the strain-rate tensors of the second-gradient theory are obtained by simple differentiation. We have, therefore (see (26), (27)):

$$D_{ij} = \dot{\varepsilon}_{ij}, \qquad K_{ij} = \dot{\eta}_{ij}, \qquad K_{ijk} = \dot{\eta}_{ijk}. \tag{55}$$

6.1. ***The strain energy.*** Let us suppose now that the medium is elastic, and the evolution is isothermal. The existence of the free-energy density implies that of a volumetric strain energy having the usual convexity properties. More precisely, we will state, in view of the small-perturbation hypothesis, the following:

Definition 1. *There exists a volumetric strain energy* $\mathrm{w}(\varepsilon_{ij}, \eta_{ij}, \eta_{ijk})$, *depending on the variables* $\varepsilon_{ij}, \eta_{ij}, \eta_{ijk}$, *which is a nonnegative quadratic form*, *invariant under any permutation of the two indices of the variable* ε_{ij}, *under any permutation of the three indices of* η_{ijk}, *and under the addition of the same constant to the three variables* $\eta_{11}, \eta_{22}, \eta_{33}$, *which vanishes if*, *and only if*,

$$\varepsilon_{ij} = 0, \qquad \eta_{ijk} = 0, \qquad \eta_{ij} = c\delta_{ij}.$$

Moreover, *for every motion*, *the derivative with respect to time of* w *is equal to the volumetric energy* p *of the internal forces.*

It follows from this definition that the function w satisfies the equation

$$\frac{\partial \mathrm{w}}{\partial \eta_{11}} + \frac{\partial \mathrm{w}}{\partial \eta_{22}} + \frac{\partial \mathrm{w}}{\partial \eta_{33}} = 0, \tag{56}$$

and that, if the strain tensors are defined in terms of a displacement field by the formulas (52), (53), and (54), w vanishes if, and only if, the field X_i is given by a twist, that is, a geometric infinitesimal rigid-body displacement.

Furthermore, in a given state ε_{ij}, η_{ij}, η_{ijk}, it is possible to assign to the strain-rate tensors (55) arbitrary values subject to the single restriction

$$\dot{\eta}_{ii} = 0. \tag{57}$$

According to the definition, if λ denotes a Lagrange multiplier, we have identically that

$$\dot{\mathrm{w}} = \sigma_{ij}\dot{\varepsilon}_{ij} + \mu_{ij}\dot{\eta}_{ij} + \mu_{ijk}\dot{\eta}_{ijk} + \lambda\dot{\eta}_{ii} \tag{58}$$

and, therefore, necessarily that

$$\sigma_{ij} = \frac{\partial \mathrm{w}}{\partial \varepsilon_{ij}}, \qquad \mu_{ij} = \frac{\partial \mathrm{w}}{\partial \eta_{ij}}, \qquad \mu_{ijk} = \frac{\partial \mathrm{w}}{\partial \eta_{ijk}}, \tag{59}$$

since, by virtue of (56) and of the relation $\mu_{ii} = 0$, the Lagrange multiplier λ is necessarily zero.

Equations (59) are the constitutive laws of the elastic medium.

In order to simplify the notation, we will denote by c the collection of the tensors ε_{ij}, η_{ij} ($\eta_{ii} = 0$), η_{ijk}, and by σ the collection of the tensors σ_{ij}, μ_{ij} ($\mu_{ii} = 0$), μ_{ijk}. We will write the strain energy in the form $\mathrm{w}(c)$ and the constitutive laws (59) as

$$\sigma = \frac{\partial \mathrm{w}(c)}{\partial c} = E(c). \tag{60}$$

With these notations, Euler's identity for w attains the form

$$2\mathrm{w} = \sigma \cdot c. \tag{61}$$

The general form of w involves 300 coefficients. But this number is reduced to seven in the case of isotropic media. (See, for example, Mindlin [5], Mindlin and Eshel [2].)

All the results of the classical theory of linear elasticity (see, for example, Germain [7]) are easily extended to the case of the present theory. To this end we introduce the following notations:

- *bilinear form w associated to* $\mathrm{w}(c)$:

$$\mathrm{w}(c + c^*) = \mathrm{w}(c) + 2w(c, c^*) + \mathrm{w}(c^*), \tag{62}$$

- *strain energy of the system S and associated bilinear form*:

$$W(C) = \int_S \mathrm{w}(c)\, dv, \qquad \mathcal{W}(C, C^*) = \int_S w(c, c^*)\, dv. \tag{63}$$

Here, C denotes a strain tensor field c defined over S.

• *Dual notions*:

$$\mathrm{w}^*(\sigma) = \sup_c\{\sigma \cdot c - \mathrm{w}(c)\},$$

$$\mathrm{w}^*(\sigma + \sigma^*) = \mathrm{w}^*(\sigma) + 2w^*(\sigma, \sigma^*) + \mathrm{w}^*(\sigma^*), \tag{64}$$

$$W^*(\Sigma) = \int_S \mathrm{w}(\sigma)\, dv, \qquad \mathcal{W}^*(\Sigma, \Sigma^*) = \int_L w(\sigma, \sigma^*)\, dv_0. \tag{65}$$

Obviously, $\mathrm{w}^*(\sigma)$ is a quadratic form of σ having invariance properties analogous to those of $\mathrm{w}(c)$; Σ denotes a field of stress tensors σ defined on S.

The constitutive equations can be written as

$$c = \frac{\partial \mathrm{w}^*(\sigma)}{\partial \sigma} = E^{-1}(\sigma), \tag{66}$$

where E^{-1} denotes the inverse function of $E(c)$, given in (60).

It should be noted that

$$\mathrm{w}^*(\sigma) = \mathrm{w}(c) \quad \text{if } \sigma = E(c). \tag{67}$$

6.2. *The energy theorems.* Let Σ be a field of intrinsic stresses $\sigma(\sigma_{ij}, \mu_{ij}, \mu_{ijk})$, in equilibrium with given external forces $\mathcal{F}(f_i, C_{ij}, \Phi_{ij}, t_i, \widetilde{m}_i, n, r_i)$ defined as in Section 5 (see (15), (34), and (40)). If $\mathcal{C}^*$ denotes the virtual motion defined by the twice continuously differentiable field X_i^* and the corresponding strain-rate tensors, ε_{ij}^*, η_{ij}^*, η_{ijk}^* according to the formulas (52), (53), (54), as well as the rotation rates

$$\varphi_{ij}^* = \tfrac{1}{2}(X_{i,j}^* - X_{j,i}^*), \qquad \varphi_i^* = -\tfrac{1}{2}\varepsilon_{ijk}\varphi_{jk}^*,$$

the virtual power of the forces $\mathcal{F}$ in the virtual motion $\mathcal{C}^*$ can be written as

$$\begin{aligned}\langle \mathcal{F}, \mathcal{C}^*\rangle &= \int_S (f_i X_i^* + C_{ij}\varphi_{ij}^* + \Phi_{ij}\varepsilon_{ij}^*)\, dv \\ &\qquad + \int_{\partial\mathcal{D}} (t_i X_i^* + \widetilde{m}_i \widetilde{\varphi}_i^* + n\varepsilon_{\underline{\underline{nn}}}^*) da + \int_\Gamma r_i X_i^*\, ds.\end{aligned} \tag{68}$$

We can, therefore, state:

Proposition 5. *If the elastic medium S is in equilibrium under the action of external forces $\mathcal{F}$, the equation expressing the principle of virtual power can be written as*

$$\langle \mathcal{F}, \mathcal{C}^*\rangle = 2\mathcal{W}(C, C^*), \tag{69}$$

where C is the field of strains c of the medium S in elastic equilibrium.

Indeed, if $\sigma = E(c)$, then $2w(c, c^*) = \sigma \cdot c^*$. Analogously:

Proposition 6. *If a stress field $\widetilde{\Sigma}$ is in equilibrium with external forces $\widetilde{\mathcal{F}}$, and if $\mathcal{C}(C)$, Σ denote, respectively, a field of displacements, rotations, and strains, and a field of stresses of an elastic equilibrium state, then*

$$\langle \widetilde{\mathcal{F}}, \mathcal{C}\rangle = 2\mathcal{W}^*(\Sigma, \widetilde{\Sigma}). \tag{70}$$

These two propositions furnish an interesting interpretation of the functions $\mathscr{W}$ and $\mathscr{W}^*$.

It is not necessary to dwell on showing how the work and reciprocity theorems are derived. We leave their formulation as an exercise to the reader. It is a straightforward matter also to obtain the uniqueness theorem for regular problems.

To avoid a complicated notation and following the current usage, we will examine a problem of the following type (type III). Let Σ_1 and Σ_2 be two disjoint parts of the boundary ∂S of S. The problem data are

(a) f_i, C_{ij}, Φ_{ij} in S,

(b) on Σ_1, $X_i = \overline{X}_i, \widetilde{\varphi}_i = \overline{\widetilde{\varphi}}_i, \varepsilon_{\underline{nn}} = \bar{\varepsilon}_{\underline{nn}}$,

(c) on Σ_2, $t_i = \bar{t}_i, \widetilde{m}_i = \overline{\widetilde{m}}_i, n = \bar{n}$, and $r_i = \bar{r}_i$ on $\Gamma \cap \Sigma_2$.

In other words, we are given in S and on Σ_2 the external forces and on Σ_1 the displacements, the tangential rotation, and the unitary elongation in the normal direction. We then have (Mindlin and Eshel [2]):

Theorem 3. *The problem thus posed has at most one nontrivial solution.*

Indeed, applying (68) to the homogeneous problem associated with $\mathscr{C}^* = \mathscr{C}$, we have, by virtue of the data of this homogeneous problem, $\langle \mathscr{F}, \mathscr{C} \rangle = 0$ and, therefore, $W(C) = 0$, which implies that, at every point of S, $\mathrm{w}(c) = 0$, since $\mathrm{w}(c)$ is nonnegative and continuous. Consequently, the displacement field is a twist, and the strain and stress fields vanish identically. These properties are precisely what characterizes a trivial solution. We can also give a variational formulation of the regular problems. We will so do in the case of the problem formulated above. To this end, let us propose the following definitions.

Definition 2. *A field $\mathscr{C}'$, determined by the field $X_i'(x_1, x_2, x_3)$, assumed to be defined and twice differentiable in the closure of S, is said to be kinematically admissible for the problem under consideration if it satisfies the kinematic boundary conditions on Σ_1 (condition* (b)).* *We will denote by C' the strain field defined by $\mathscr{C}$.*

Definition 3. *A field Σ' determined by the tensor fields σ'_{ij}, μ'_{ij} $(\mu'_{ii} = 0)$, μ'_{ijk} defined and twice differentiable in the closure of S is said to be statically admissible for the problem under consideration if it satisfies the equilibrium equations* (35) *and* (36) *in the interior of S and the boundary conditions* (38) *and* (40) *on Σ_2 and on $\Gamma \cap \Sigma_2$.*

Let us remark that if we denote by $\overline{\overline{\mathscr{F}}}$ the data of external forces defined in S and on Σ_2, by $\overline{\mathscr{C}}$ the kinematic data on Σ_1, and by $\mathscr{F}'$ the system of external forces

*The original has Σ_2 instead of Σ_1.—(TN) in equilibrium with Σ', we can write

$$\langle \mathcal{F}', \mathcal{C}' \rangle = \{\bar{\bar{\mathcal{F}}}, \mathcal{C}'\} + (\mathcal{F}', \bar{\mathcal{C}}), \tag{71}$$

where $\{\}$ collects in (68) the integrals taken on S and on Σ_2, while $()$ collects the integrals on Σ_1.

We can still formulate the following definitions.

Definition 4. *We call the potential energy of a kinematically admissible field for the problem under consideration the function*

$$V(C') = W(C') - \{\bar{\bar{\mathcal{F}}}, C'\}, \tag{72}$$

and the potential energy of a statically admissible field for the problem under consideration the function

$$V^*(\Sigma') = -W(\Sigma') + (\mathcal{F}', \bar{\mathcal{C}}). \tag{73}$$

We can then prove the following:

Theorem 4. *Suppose that the problem under consideration admits a solution. If C and Σ denote the strain and stress fields defining this solution, we have that for every field of admissible $\mathcal{C}'$ and σ'*

$$V^*(\Sigma') \leq V^*(\Sigma) = V(C) \leq V(C'). \tag{74}$$

Moreover, it is not possible to have the equality $V(C') = V(C)$ except if C and C' define the same strain tensor fields, and it is not possible to have $V^*(\Sigma') = V(\Sigma)$ except if Σ' and Σ define the same intrinsic stress fields.

The proof is easy and classic. Let us set $\mathcal{C}' = \mathcal{C} + \mathcal{C}^*$. Then, $C' = C + C^*$ and, with obvious notation,

$$V(C') - V(C) = W(C + C^*) = W(C) - \{\bar{\bar{\mathcal{F}}}, \mathcal{C}^*\}. \tag{75}$$

By virtue of (62) and (63),

$$W(C + C^*) - W(C) = 2\mathcal{W}(C, C^*) + W(C^*).$$

On the other hand, according to (71), for every $\mathcal{F}'$, and in particular for $\mathcal{F}$ corresponding to the solution,

$$\langle \mathcal{F}', \mathcal{C}^* \rangle = \langle \mathcal{F}', \mathcal{C}' \rangle - \langle \mathcal{F}', \mathcal{C} \rangle = \{\bar{\bar{\mathcal{F}}}, \mathcal{C}^*\} = \langle \mathcal{F}, \mathcal{C}^* \rangle\,;$$

and finally, according to (69), $\mathcal{C}$ and $\mathcal{F}$ being associated with the solution,

$$\langle \mathcal{F}, \mathcal{C}^* \rangle = 2\mathcal{W}(C, C^*).$$

Plugging these results into (75) yields

$$V(C') - V(C) = W(C^*). \tag{76}$$

The right-hand side of (76) is always nonnegative, and it doesn't vanish except when the strain tensors defined by C and C' are identical. Let us set $\Sigma' = \Sigma + \widetilde{\Sigma}$ and, accordingly, $\mathcal{F}' = \mathcal{F} + \widetilde{\mathcal{F}}$. Then

$$V^*(\Sigma') - V^*(\Sigma) = W^*(\Sigma + \widetilde{\Sigma}) - W^*(\Sigma) - (\widetilde{\mathcal{F}}, \overline{\mathcal{C}}). \tag{77}$$

But

$$W^*(\Sigma + \widetilde{\Sigma}) - W^*(\Sigma) = 2\mathcal{W}^*(\Sigma, \widetilde{\Sigma}) + W^*(\widetilde{\Sigma}).$$

Moreover, according to (71), for every admissible $\mathcal{C}'$ and in particular for the solution $\mathcal{C}$,

$$\langle \widetilde{\mathcal{F}}^*, \mathcal{C}' \rangle = \langle \mathcal{F}', \mathcal{C}' \rangle - \langle \mathcal{F}, \mathcal{C}' \rangle = (\widetilde{\mathcal{F}}, \tilde{\mathcal{C}}) = \langle \mathcal{F}, \mathcal{C} \rangle\,;$$

and finally, according to (70)

$$\langle \widetilde{\mathcal{F}}, \mathcal{C} \rangle = 2\mathcal{W}^*(\Sigma, \widetilde{\Sigma}).$$

It follows then that

$$V^*(\Sigma) - V^*(\Sigma') = W(\widetilde{\Sigma}),$$

which establishes the first inequality (74).

Finally, quite evidently by virtue of (69) or of (70) or, more directly, by the work theorem, we have for the solution $\mathcal{C}$, Σ

$$V(C) = V^*(\Sigma).$$

This completes the proof of the energy theorem.

6.3. *Hints on the existence theorems.* It is not difficult to extend most of the theorems of existence established in the classical theory (Duvaut–Lions [1], Ch. 3) to the problems of linear elasticity formulated within the framework of the second-gradient theory. We will content ourselves here with the formulation of one of these theorems, always in the case of the problem of type III considered above, assuming, moreover, for the sake of simplicity, on the one hand, that the boundary ∂S of S is a twice continuously differentiable manifold (which, in fact, eliminates the edges) and, on the other hand, that the part Σ_1 contains at least three noncollinear points. We need then to specify the functional framework within which the problem must be formulated.

(1) The displacements X_i are assumed[2] to belong to $H^2(S)$. It follows that on ∂S the displacements X_i belong to $H^{3/2}(\partial S)$ and that the tangential rotation $\widetilde{\varphi}_i$

[2]The notation used for the Sobolev spaces is the classical one. See, for example, Lions–Magenes [8] or Duvaut–Lions [1].

and the doubly normal unitary elongation $\varepsilon_{\underline{\underline{nn}}}$ belong to $H^{1/2}(\partial S)$. This defines the functional space in which the data $\overline{\overline{\mathscr{C}}}$ on Σ_1 must be taken.

(2) The given external volume forces f_i, C_{ij}, Φ_{ij} will be taken in $L^2(S)$. The external forces given on Σ_2 will be, as far as T_i is concerned, in the restriction to Σ_2 of $H^{-3/2}(\partial S)$, and as far as $\widetilde{m}_i$ and n are concerned, in the restriction to Σ_2 of $H^{-1/2}(\partial S)$.

(3) It will be assumed that the coefficients of the quadratic form $\mathrm{w}(c)$ belong to $L^\infty(S)$, that is, that they are essentially bounded.

Under these conditions, we can state:

Theorem 5. *If the data $\overline{\mathscr{C}}$ and $\overline{\overline{\mathscr{F}}}$ belong to the functional spaces introduced above, there exists a unique displacement field that minimizes the potential energy $V(C')$ among all the kinematically admissible fields* (*defined by $X_i' \in (H^2(S))$ and satisfying the boundary conditions $\overline{\mathscr{C}}$ on Σ_1*).

The proof, which we will not carry out here, has been developed[3] by Duvaut [12]. It relies on a generalized Korn inequality that can be deduced very simply from the classical inequality according to which the norm defined in $(H^2(\Omega))^3$ by the inner product

$$((\mathscr{C}_i^{(1)}, \mathscr{C}_i^{(2)})) = \int_S (X_i^{(1)} X_i^{(2)} + \varepsilon_{ij}^{(1)} \varepsilon_{ij}^{(2)} + \eta_{ij}^{(1)} \eta_{ij}^{(2)} + \eta_{ijk}^{(1)} \eta_{ijk}^{(2)})\, dv$$

is equivalent to the classical norm on $(H^2(\Omega))^3$, and moreover, from the fact that if $\overline{\mathscr{C}} = 0$, there exists a constant M such that $\mathrm{w}(C) \geq M((\mathscr{C}, \mathscr{C}))$.

6.4. ***Possible generalizations.*** When extending the methods described in [3] to construct in the classical case the constitutive laws of viscoelastic or elastic-perfectly plastic media, it is not difficult to formulate, in the framework of the small perturbation hypothesis, theories of viscoelasticity or plasticity for a description of a continuous medium by means of a second-order theory. It will suffice, for example, to add to the function $\mathrm{w}(c)$ a conveniently chosen dissipation function. We will content ourselves here with this simple remark.

7. Constitutive laws of a hyperelastic medium in a second-gradient theory

In the preceding section, we have shown how the construction of the proposed second-gradient theory leads quite naturally to the formulation of constitutive laws in the case where the medium sustained only small perturbations. It remains for us to show how to proceed in the case of finite strains. We will content ourselves with considering the elastic case.

[3]Only the case of isotropic media is considered in [12]. But by means of appropriate conditions, the extension to the general case is straightforward.

As is well known in the classical theory, the essential point is to obtain a representation of the stresses after convected transport from the configuration under study to the reference configuration. For the sake of simplicity, we will assume this configuration to be defined by the coordinates a_α ($\alpha = 1, 2, 3$) in the orthonormal Cartesian frame used to describe the system S. We will denote by $F_{i\alpha}$ the elements of the gradient matrix and by a superposed dot the material derivative. If U_i denotes the velocity vector, we can write

$$U_{i,\alpha} = U_{i,j} F_{j\alpha} = \dot{F}_{i\alpha},$$

where a Greek index such as α placed after a comma indicates a derivative with respect to a_α, while the function being differentiated is expressed in terms of the Lagrangian variables a_1, a_2, a_3, t.

If we denote by $L_{\alpha\beta}$ the Green–Lagrange strain tensor

$$2L_{\alpha\beta} = F_{i\alpha} F_{i\beta} - \delta_{\alpha\beta}, \tag{78}$$

we easily obtain the classical formula

$$\dot{L}_{\alpha\beta} = D_{ij} F_{i\alpha} F_{j\beta}, \tag{79}$$

whose interpretation would become clearer when using curvilinear coordinates, since it would make manifest that the material derivative of the Green–Lagrange tensor is nothing but the result of the convected transport of the rate of strain tensor to the reference configuration.

We deduce from (79) that

$$\dot{L}_{\alpha\beta,\gamma} = D_{ij}(F_{i\alpha} F_{j\beta})_{,\gamma} + D_{ij,k} F_{i\alpha} F_{j\beta} F_{k\gamma},$$

which shows that the derivatives $D_{ij,k}$ (or, what amounts to the same, the $U_{i,jk}$) can be calculated by means of the gradient of $\dot{L}_{\alpha\beta}$. To utilize the representation of the second gradient that we have chosen, it is convenient to introduce the quantities

$$\Lambda_{\alpha\beta\gamma} = \tfrac{1}{3}(L_{\alpha\beta,\gamma} + L_{\beta\gamma,\alpha} + L_{\gamma\alpha,\beta}), \qquad \Lambda_{\alpha\beta} = -2\varepsilon_{\alpha\rho\sigma} L_{\beta\rho,\sigma}. \tag{80}$$

It should be noted that $\Lambda_{\alpha\beta\gamma}$ is invariant under any permutation of the indices α, β, γ, and that $\Lambda_{\alpha\alpha} = 0$. Indeed,

$$2\varepsilon_{\alpha\rho\sigma} L_{\beta\rho,\sigma} = \varepsilon_{\alpha\rho\sigma}(F_{i\beta} F_{i\rho})_{,\sigma} = \varepsilon_{\alpha\rho\sigma} F_{i\beta,\sigma} F_{i\rho},$$

since $F_{i\rho,\sigma} = \dfrac{\partial^2 x_i}{\partial a_\rho\,\partial a_\sigma}$ is symmetric in ρ and σ. For the same reason, $\varepsilon_{\alpha\rho\sigma} L_{\alpha\rho,\sigma} = 0$. We have, therefore, in the first place

$$\begin{aligned}3\dot{\Lambda}_{\alpha\beta\gamma} &= \dot{L}_{\alpha\beta,\gamma} + \dot{L}_{\beta\gamma,\alpha} + \dot{L}_{\gamma\alpha,\beta} \\ &= D_{ij}[(F_{i\alpha}F_{j\beta})_{,\gamma} + (F_{i\beta}F_{j\gamma})_{,\alpha} + (F_{i\gamma}F_{j\alpha})_{,\beta}] \\ &\qquad + D_{ij,k}[F_{i\alpha}F_{j\beta}F_{k\gamma} + F_{i\beta}F_{j\gamma}F_{k\alpha} + F_{i\gamma}F_{j\alpha}F_{k\beta}],\end{aligned}$$

or

$$\dot{\Lambda}_{\alpha\beta\gamma} = P_{\alpha\beta\gamma ij}D_{ij} + K_{ijk}F_{i\alpha}F_{j\beta}F_{k\gamma}. \tag{81}$$

Here, K_{ijk} denotes the completely skew-symmetric part of $U_{i,jk}$, defined in (27), and $P_{\alpha\beta\gamma ij}$ are coefficients which remain invariant under permutations of the indices α, β, γ, on the one hand, and the indices i, j, on the other. In fact, we may state

$$P_{\alpha\beta\gamma ij} = \tfrac{1}{6}\{(F_{i\alpha}F_{j\beta} + F_{j\alpha}F_{i\beta})_{,\gamma} + (F_{i\beta}F_{j\gamma} + F_{j\beta}F_{i\gamma})_{,\alpha} + (F_{i\gamma}F_{j\alpha} + F_{i\alpha}F_{j\gamma})_{,\beta}\}. \tag{82}$$

Accordingly, we calculate

$$\begin{aligned}\dot{\Lambda}_{\alpha\beta} &= -2\varepsilon_{\alpha\rho\sigma}\dot{L}_{\beta\rho,\sigma} \\ &= -2[\varepsilon_{\alpha\rho\sigma}D_{ij}(F_{i\beta}F_{j\rho})_{,\sigma} + \varepsilon_{\alpha\rho\sigma}D_{ij,k}F_{i\beta}F_{j\rho}F_{k\sigma}].\end{aligned} \tag{83}$$

It proves convenient to introduce the coefficients

$$Q_{\alpha\beta ij} = -\varepsilon_{\alpha\rho\sigma}[(F_{i\beta}F_{j\rho})_{,\sigma} + (F_{j\beta}F_{i\rho})_{,\sigma}] = -\varepsilon_{\alpha\rho\sigma}[F_{i\beta,\sigma}F_{j\rho} + F_{j\beta,\sigma}F_{i\rho}]. \tag{84}$$

The last equality is a result of the symmetry of $F_{j\rho,\sigma}$ with respect to the indices ρ and σ. It should be noted, moreover, that

$$\varepsilon_{\alpha\rho\sigma}[D_{ij,k}F_{j\rho}F_{k\sigma} + D_{ik,j}F_{j\rho}F_{k\sigma}] = \varepsilon_{\alpha\rho\sigma}[F_{j\rho}F_{k\sigma}][D_{ij,k} + D_{ik,j}] = 0,$$

so that in (83) we can replace $D_{ij,k}$ by

$$\tfrac{1}{2}(D_{ij,k} - D_{ik,j}) = \tfrac{1}{2}\Omega_{jk,i} = -\tfrac{1}{2}\varepsilon_{jkp}\omega_{p,i} = -\tfrac{1}{2}\varepsilon_{pjk}K_{pi},$$

where the last equalities stem from (25) and (26).

Thus, (83) can be written as

$$\dot{\Lambda}_{\alpha\beta} = Q_{\alpha\beta ij}D_{ij} + \varepsilon_{pjk}\varepsilon_{\alpha\rho\sigma}F_{i\beta}F_{j\rho}F_{k\sigma}K_{pi}. \tag{85}$$

It should be noted that the right-hand side of (85) is actually a deviator. Indeed, $Q_{\alpha\alpha ij} = 0$, since $\varepsilon_{\alpha\rho\sigma}F_{i\alpha,\sigma}F_{j\rho} = 0$ by virtue of the symmetry of $F_{i\alpha,\sigma}$ in α and σ. Furthermore, if we denote, according to common usage,

$$J = \det(F_{i\alpha}), \tag{86}$$

we obtain

$$\varepsilon_{pjk}\varepsilon_{\alpha\rho\sigma}F_{i\alpha}F_{j\rho}F_{k\sigma}K_{pi} = J\varepsilon_{pjk}\varepsilon_{ijk}K_{pi} = 2JK_{ii} = 0,$$

since K_{ij} is a deviator.

These preliminary computations having been carried out, if we denote by ρ_0 the mass per unit volume in the reference state and by ρ the mass per unit volume in the present state, it is clear that the energy per unit mass of the internal forces will be linear with respect to $L_{\alpha\beta}$, $\Lambda_{\alpha\beta}$, $D_{\alpha\beta\gamma}$. We thus write

$$\frac{1}{\rho}(\sigma_{ij}D_{ij} + \mu_{ij}K_{ij} + \mu_{ijk}K_{ijk}) = \frac{1}{\rho_0}(s_{\alpha\beta}\dot{L}_{\alpha\beta} + \Pi_{\alpha\beta}\dot{\Lambda}_{\alpha\beta} + \Pi_{\alpha\beta\gamma}\dot{\Lambda}_{\alpha\beta\gamma}). \quad (87)$$

In this equation, we can assume, without loss of generality, that $s_{\alpha\beta}$ is symmetric, that $\Pi_{\alpha\beta}$ is a deviator, and that $\Pi_{\alpha\beta\gamma}$ is completely skew-symmetric in α, β, γ. The tensors $s_{\alpha\beta}$, $\Pi_{\alpha\beta}$, $\Pi_{\alpha\beta\gamma}$ provide a representation of the intrinsic stresses in the reference configuration for a medium described in a second-gradient theory. Naturally, $s_{\alpha\beta}$ is the classical symmetric Piola–Kirchhoff tensor.

Using (79), (81), (85), (86), we can write (87) in the form

$$\begin{aligned} D_{ij}[J\sigma_{ij} - F_{i\alpha}F_{j\beta}s_{\alpha\beta} - Q_{\alpha\beta ij}\Pi_{\alpha\beta} - P_{\alpha\beta\gamma ij}\Pi_{\alpha\beta\gamma}] \\ + K_{ij}(J\mu_{ij} - \varepsilon_{ipq}\varepsilon_{\alpha\rho\sigma}F_{j\beta}F_{p\rho}F_{q\sigma}\Pi_{\alpha\beta}) \\ + K_{ijk}(J\mu_{ijk} - F_{i\alpha}F_{j\beta}F_{k\gamma}\Pi_{\alpha\beta\gamma}) = 0. \end{aligned} \quad (88)$$

Since, by hypothesis, we are assuming that the medium is not subjected to any internal constraints, D_{ij}, K_{ij}, and K_{ijk} can take arbitrary values. Moreover, the coefficient of D_{ij} is symmetric in i, j; the coefficient of K_{ij} represents the components of a deviator (since $\varepsilon_{\alpha\rho\sigma}\varepsilon_{\beta\rho\sigma}\Pi_{\alpha\beta} = 0$); and the coefficient of K_{ijk} is completely skew-symmetric in i, j, k. We obtain, therefore, that

$$\begin{cases} J\sigma_{ij} = F_{i\alpha}F_{j\beta}s_{\alpha\beta} + Q_{\alpha\beta ij}\Pi_{\alpha\beta} + P_{\alpha\beta\gamma ij}\Pi_{\alpha\beta\gamma}, \\ J\mu_{ij} = \varepsilon_{ipq}\varepsilon_{\alpha\rho\sigma}F_{j\beta}F_{p\rho}F_{q\sigma}\Pi_{\alpha\beta}, \\ J\mu_{ijk} = F_{i\alpha}F_{j\beta}F_{k\gamma}\Pi_{\alpha\beta\gamma}, \end{cases} \quad (89)$$

identically. These equations express the intrinsic stress tensors in the present configuration as functions of the intrinsic stress tensors in the reference configuration. It is easy, moreover, to solve them in terms of $\Pi_{\alpha\beta\gamma}$, $\Pi_{\alpha\beta}$, $s_{\alpha\beta}$.

Let us assume that the medium is elastic and, more precisely, hyperelastic. The classical theory can be easily generalized (see, for example, Germain [3]) supposing that the specific free energy Ψ is a function of the absolute temperature and of the variables $L_{\alpha\beta}$, $\Lambda_{\alpha\beta}$, $\Lambda_{\alpha\beta\gamma}$, it being understood that $L_{\alpha\beta}$ and $\Lambda_{\alpha\beta\gamma}$ make their contributions symmetrically and that Ψ is invariant under the transformation $\Lambda_{\alpha\beta} \to \Lambda_{\alpha\beta} + C\delta_{\alpha\beta}$ for arbitrary values of the constant C. The differential of Ψ, while the temperature is kept constant, namely,

$$\rho_0^{-1}(s_{\alpha\beta} dL_{\alpha\beta} + \Pi_{\alpha\beta} d\Lambda_{\alpha\beta} + \Pi_{\alpha\beta\gamma} d\Lambda_{\alpha\beta\gamma}),$$

is identical to minus the infinitesimal work of the internal forces, so that we can write

$$s_{\alpha\beta} = \rho_0 \frac{\partial \Psi}{\partial L_{\alpha\beta}}, \qquad \Pi_{\alpha\beta} = \rho_0 \frac{\partial \Psi}{\partial \Lambda_{\alpha\beta}}, \qquad \Pi_{\alpha\beta\gamma} = \rho_0 \frac{\partial \Psi}{\partial \Lambda_{\alpha\beta\gamma}}, \tag{90}$$

which provides a simple first way to write the constitutive laws. Equations (89) lead directly to the expression of the stresses in the present configuration as

$$\begin{cases} \sigma_{ij} = \rho \left[\dfrac{\partial \Psi}{\partial L_{\alpha\beta}} F_{i\alpha} F_{j\beta} + \dfrac{\partial \Psi}{\partial \Lambda_{\alpha\beta}} Q_{\alpha\beta ij} + \dfrac{\partial \Psi}{\partial \Lambda_{\alpha\beta\gamma}} P_{\alpha\beta\gamma ij} \right], \\ \mu_{ij} = \rho \varepsilon_{ipq} \varepsilon_{\alpha\rho\sigma} F_{j\beta} F_{p\rho} F_{q\sigma} \dfrac{\partial \Psi}{\partial \Lambda_{\alpha\beta}}, \\ \mu_{ijk} = \rho \dfrac{\partial \Psi}{\partial \Lambda_{\alpha\beta\gamma}} F_{i\alpha} F_{j\beta} F_{k\gamma}. \end{cases} \tag{91}$$

Recall that the coefficients $P_{\alpha\beta\gamma ij}$ and $Q_{\alpha\beta ij}$ are expressed directly in terms of the gradient matrix and its derivatives according to (82) and (84).

It is to be noted that, under the hypothesis of small strains, we may use in (91)

$$\begin{cases} w = \rho_0 \Psi = \rho \Psi, \\ L_{\alpha\beta} = \varepsilon_{\alpha\beta}, \qquad \Lambda_{\alpha\beta} = \eta_{\alpha\beta}, \qquad \Lambda_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma}, \\ F_{i\alpha} = \delta_{i\alpha}, \qquad P_{\alpha\beta\gamma ij} = 0, \qquad Q_{\alpha\beta ij} = 0, \end{cases} \tag{92}$$

so that we still recover (59).

We have obtained, as we intended, the constitutive laws of a hyperelastic medium under finite strains in the framework of the second-gradient theory.

Conclusion

The aim of this first article — which develops a part of the results announced in a recent note [9] — has been to show how the method of virtual power provides a means at the same time powerful and natural to construct a theory of continuous media. The main results that emerge are the following:

(a) Construction of a first-gradient theory that offers itself as a very simple generalization of the classical theory and that exposes the distinction between the classical stress tensor and the intrinsic stress tensor (Theorem 1).

(b) Construction of a second-gradient-theory that generalizes the preceding one (Theorem 2). The fundamental formulas, which were already known in the case of elastic media, have a general scope. This appears to be a new result.

(c) Statement of the main results of the second-gradient theory in the case of elastic media for infinitesimal strains. These results, without being essentially new, are collected in a systematic presentation.

Incidentally, a new introduction is put forward of the notion of screw (torsor) that perhaps better highlights its mechanical meaning. It was introduced by the author in an unpublished course [11]. See also [3] and [10].

Finally, we would like to remark that, in addition to the advantage of introducing an exact representation of the internal forces suited to the adopted description, this method allows us to obtain as naturally as possible and without new computations the notions of strain energy and the variational formulations that derive from it in the case where the medium is elastic. This notion is, more generally, perfectly adapted to the application of the principles of the thermodynamics of continuous media to obtain the constitutive laws of nonelastic media. We can also say that, without any new effort, this method permits to extend the results obtained in elasticity to the most general media by means of any formulation that takes as its point of departure a variational or Hamiltonian formulation.

Appendix

A.1. ***Preliminary formulas.*** It will be useful to introduce at each point of the surface $\partial\mathcal{D}$, the boundary of a connected domain $\mathcal{D}$, the operators of normal and tangential differentiation. Let $\varphi(x_1, x_2, x_3)$ be a continuous and continuously differentiable scalar-valued function defined on the closure $\mathcal{D} + \partial\mathcal{D}$ of $\mathcal{D}$. Its normal derivative (toward the exterior) is a scalar denoted by $D\varphi$ and its tangential derivative on the surface is the vector $D_j\varphi$ given by

$$D\varphi = n_l\varphi_{,l}, \quad \varphi_{,j} = D_j\varphi + n_j D\varphi. \tag{A-1}$$

We proceed in the same way with a vector-valued function $q_i(x_1, x_2, x_3)$. The normal derivative is the vector Dq_i and the tangential derivative is the tensor $D_j q_i$ given by the following expressions:

$$Dq_i = n_l q_{i,l}, \quad q_{i,j} = D_j q_i + n_j Dq_i. \tag{A-2}$$

Recall the statement of the divergence theorem on a surface. Let Σ be a closed area, with continuous tangent plane and curvature, traced over the surface $\partial\mathcal{D}$, and let τ_i be the unit vector tangent to the boundary $\partial\Sigma$ oriented in the direct sense around the normal n_i to Σ. Finally, let ν_i denote the exterior unit normal to $\partial\Sigma$ lying on the tangent plane to Σ, so that

$$\nu_j = \varepsilon_{jml}\tau_m n_l. \tag{A-3}$$

We can write

$$\int_{\Sigma} D_j q_j \, da = \int_{\Sigma} n_j q_j (D_l n_l) \, da + \int_{\partial\Sigma\curvearrowleft} \nu_j q_j \, ds. \tag{A-4}$$

Note that $D_l n_l$ is twice the mean curvature. We can verify this formula by noticing, for example,[4] that

$$D_j q_j = (D_l n_l) n_j q_j + \varepsilon_{srm} \varepsilon_{mlj} n_s (n_l q_j)_{,r}.$$

The last term is then $\boldsymbol{n} \cdot \mathrm{rot}(\boldsymbol{n} \wedge \boldsymbol{q})$, and the identity results from Stokes' formula.

Suppose now that the boundary $\partial\mathfrak{D}$ is a closed surface with piecewise continuous tangent and curvature, and denote by Γ the "edges" of $\partial\mathfrak{D}$, along which there is a discontinuity of the tangent plane. We can write, according to (A-4),

$$\int_{\partial\mathfrak{D}} D_j q_j \, da = \int_{\partial\mathfrak{D}} (D_p n_p) n_j q_j \, da + \int_{\Gamma\curvearrowleft} [\![\nu_j q_j]\!] \, ds, \tag{A-5}$$

where $[\![\;]\!]$ denotes the jump of the enclosed quantity. Another form is obtained by applying (A-3):

$$\int_{\partial\mathfrak{D}} D_j q_j \, da = \int_{\partial\mathfrak{D}} (D_p n_p) n_j q_j \, da + \int_{\Gamma\curvearrowleft} \varepsilon_{jmp} \tau_m [\![n_p q_j]\!] \, ds. \tag{A-6}$$

We will need, furthermore, a canonical decomposition of the velocity gradient tensor different from that given by (A-2) and involving tangential derivatives of the velocities, the tangential components of the rate of rotation vector, and the doubly normal (to $\partial\mathfrak{D}$) component[5] $D_{\underline{nn}}$ of the rate of strain tensor. To this end it is sufficient to express the normal derivative of the velocity vector. But

$$\begin{aligned} DU_k &= n_p (D_{kp} + \Omega_{kp}) \\ &= n_p \Omega_{kp} + \tfrac{1}{2} n_p U_{k,p} + \tfrac{1}{2} n_p U_{p,k} = n_p \Omega_{kp} + \tfrac{1}{2} DU_k + \tfrac{1}{2} n_p U_{p,k}. \end{aligned}$$

We have, therefore,

$$\begin{aligned} DU_k &= 2 n_p \Omega_{kp} + n_p U_{p,k} = 2 n_p \Omega_{kp} + n_p (D_k U_p + n_k DU_p) \\ &= 2 n_p \Omega_{kp} + n_p D_k U_p + n_k n_p n_q U_{p,q}, \end{aligned}$$

and since $\Omega_{pq} n_p n_q = 0$, we obtain

$$DU_k = 2 n_p \Omega_{kp} + n_p D_k U_p + n_k D_{\underline{nn}}.$$

Applying (A-2), we obtain the desired formula as

$$U_{i,j} = D_j U_i + n_j n_p D_i U_p + 2 n_j n_p \Omega_{ip} + n_i n_j D_{\underline{nn}}. \tag{A-7}$$

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[4] We extend the definition of $\boldsymbol{n}$ to a neighborhood of $\partial\mathfrak{D}$ by parallel transport along the normal.

[5] A repeated underlined index implies no summation. Here $D_{\underline{nn}} = D_{pq} n_p n_q$.

The vector $n_p\Omega_{ip}$ is obviously situated on the tangent plane to $\partial\mathfrak{D}$; moreover,

$$n_p\Omega_{ip} = -\varepsilon_{ipk}\omega_k n_p = -\varepsilon_{ipk}\widetilde{\omega}_k n_p,$$

where $\widetilde{\omega}_k$ denotes the tangential component of the vector ω_k. We can also write (A-7) in the form

$$U_{i,j} = D_j U_i + n_j n_p D_i U_p - 2n_j\varepsilon_{ipk}\widetilde{\omega}_k n_p + n_i n_j D_{\underline{nn}}. \tag{A-8}$$

A.2. ***Transformation of Equation (30).*** Let us start with the term arising from the triply contracted product of two tensors of order 3. We have, taking into consideration the symmetry of μ_{ijk},

$$\begin{aligned}\mu_{ijk}K_{ijk} = \mu_{ijk}U_{i,jk} &= (\mu_{ijk}U_{i,j})_{,k} - \mu_{ijk,k}U_{i,j}\\ &= (\mu_{ijk}U_{i,j})_{,k} - (\mu_{ijk,k}U_i)_{,j} + \mu_{ijk,jk}U_i,\end{aligned}$$

and a subsequent integration and application of the divergence theorem yield

$$\int_{\mathfrak{D}} \mu_{ijk}K_{ijk}\,dv = \int_{\mathfrak{D}} \mu_{ijk,jk}U_i\,dv - \int_{\partial\mathfrak{D}} \mu_{ijk,k}n_j U_i\,da + \int_{\partial\mathfrak{D}} \mu_{ijk}U_{i,j}n_k\,da.$$

It remains to transform the last integral. To this end, we write the integrand, using (A-8), as

$$\mu_{ijk}U_{i,j}n_k = D_j U_i(\mu_{ijk}n_k + \mu_{jpk}n_i n_p n_k) - 2\varepsilon_{ikq}\mu_{jkp}n_j n_p n_q\widetilde{\omega}_i + \mu_{ijk}n_i n_j n_k D_{\underline{nn}}.$$

We now integrate by noting that the first term is of the form $D_j q_j$ and applying (A-6). Finally, we obtain

$$\begin{aligned}\int_{\mathfrak{D}} \mu_{ijk}K_{ijk}\,dv = \int_{\mathfrak{D}} &\mu_{ijk,jk}U_i\,dv\\ &- \int_{\partial\mathfrak{D}} \{\mu_{ijk,k}n_j + (D_j - n_j(D_p n_p))(\mu_{ijk}n_k + \mu_{jlk}n_i n_l n_k)\}U_i\,da\\ &+ \int_{\partial\mathfrak{D}} (\mu_{ijk}n_i n_j n_k D_{\underline{nn}} - 2\varepsilon_{ikq}\mu_{kjp}n_j n_p n_q\widetilde{\omega}_i)\,da\\ &+ \int_{\Gamma\frown} \varepsilon_{jmq}\tau_m [\![\mu_{ijk}n_k n_q + \mu_{jpk}n_i n_p n_q n_k]\!] U_i\,ds.\end{aligned} \tag{A-9}$$

We proceed in a similar fashion with the term arising from the doubly contracted product of the tensors μ_{ij} and K_{ij}:

$$\begin{aligned}\mu_{ij}K_{ij} = -\tfrac{1}{2}\varepsilon_{ipq}U_{p,qj}\mu_{ij} &= -\tfrac{1}{2}(\varepsilon_{ipq}\mu_{ij}U_{p,q})_{,j} + \tfrac{1}{2}\varepsilon_{ipq}\mu_{ij,j}U_{p,q}\\ &= -\tfrac{1}{2}(\varepsilon_{ipq}\mu_{ij}U_{p,q})_{,j} + \tfrac{1}{2}(\varepsilon_{ipq}\mu_{ij,j}U_p)_{,q} - \tfrac{1}{2}\varepsilon_{ipq}\mu_{ij,jq}U_p.\end{aligned}$$

Integrating, we obtain

$$\int_{\mathfrak{D}} \mu_{ij}K_{ij}\,dv \\ = \tfrac{1}{2}\varepsilon_{ipq}\left\{\int_{\mathfrak{D}} \mu_{pj,jq}U_i\,dv - \int_{\partial\mathfrak{D}} \mu_{pj,j}n_q U_i\,da - \int_{\partial\mathfrak{D}} \mu_{ij}n_j U_{p,q}\,da\right\}. \tag{A-10}$$

It is convenient to write the last integral in a more suitable form. Let us set

$$A_i = \mu_{ij} n_j, \qquad \tilde{A}_i = A_i - n_i A_k n_k,$$

so that $\tilde{A}_i$ denotes the tangential component of A_i and that we can write the integrand as

$$-\tfrac{1}{2}\varepsilon_{ipq} U_{p,q} A_i = \omega_i A_i = \widetilde{\omega}_i \tilde{A}_i + n_j n_k \omega_j A_k.$$

Moreover,

$$\tilde{A}_i = (\delta_{ik} - n_i n_k) A_k = \mu_{iq} n_q - n_i \mu_{kq} n_k n_q = \mu_{iq} n_q - n_i \mu_{\underline{nn}},$$

where $\mu_{\underline{nn}}$ denotes the doubly normal component of μ_{ij}. Furthermore,

$$\begin{aligned} n_j n_k \omega_j A_k &= -\tfrac{1}{2}\varepsilon_{ipq} U_{p,q} A_k n_k n_i = -\tfrac{1}{2}\varepsilon_{ipq} A_k n_k n_i D_q U_p \\ &= -\tfrac{1}{2} D_q(\varepsilon_{ipq} A_k n_k n_i U_p) + \tfrac{1}{2} D_q(\varepsilon_{ipq} A_k n_k n_i) U_p \\ &= -\tfrac{1}{2} D_j(\varepsilon_{ipj} n_i \mu_{\underline{nn}} U_p) + \tfrac{1}{2} D_j(\varepsilon_{ipj} n_i \mu_{\underline{nn}}) U_p. \end{aligned}$$

Invoking (A-5), we can write

$$\begin{aligned} \int_{\partial\mathfrak{D}} n_j n_k \omega_j A_k \, da &= \frac{1}{2}\int_{\mathfrak{D}} D_j(\varepsilon_{ipj} n_i \mu_{\underline{nn}}) U_p \, da - \frac{1}{2}\int_{\Gamma_{\frown}} [\![\nu_j \varepsilon_{pij} \mu_{\underline{nn}} n_p]\!] U_i \, ds \\ &= \frac{1}{2}\int_{\partial\mathfrak{D}} D_j(\varepsilon_{pij} n_p \mu_{\underline{nn}}) U_i \, da + \frac{1}{2}\int_{\Gamma_{\frown}} [\![\mu_{\underline{nn}}]\!] \tau_i U_i \, ds. \end{aligned}$$

Finally, collecting the various partial results, we obtain

$$\begin{aligned} \int_{\mathfrak{D}} \mu_{ij} K_{ij} \, dv = \frac{1}{2}\int_{\mathfrak{D}} &\varepsilon_{ipq} \mu_{pj,jq} U_i \, dv \\ &+ \frac{1}{2}\int_{\partial\mathfrak{D}} \{\varepsilon_{pij} D_j(\mu_{\underline{nn}} n_p) - \varepsilon_{ipq} \mu_{pj,j} n_q\} U_i \, da \\ &+ \int_{\partial\mathfrak{D}} (\mu_{iq} n_q - n_i \mu_{\underline{nn}}) \widetilde{\omega}_i \, da + \frac{1}{2}\int_{\Gamma_{\frown}} [\![\mu_{\underline{nn}}]\!] \tau_i U_i \, ds. \end{aligned} \tag{A-11}$$

Note that we can write

$$\varepsilon_{pij} D_j(\mu_{\underline{nn}} n_p) = \varepsilon_{pij} n_p \mu_{\underline{nn},j} \tag{A-12}$$

since $D_j n_p = D_p n_j$. This permits us to give a different form to the surface integral that appears on the right-hand side of (A-11).

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