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PERIOD RELATIONS AND CRITICAL VALUES OF L-FUNCTIONS

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To the memory of Olga Taussky-Todd

Introduction.

I.1. In this paper we present a general conjecture concerning the arithmetic of critical values of the L-functions of algebraic automorphic forms. While individual critical values seem almost always transcendental, the evidence of Shimura (c.f., esp. [**Sh1**], [**Sh2**]) shows that interesting relations between values at different critical integers, and between values of L-functions related by "twisting", do exist. Furthermore, a general recipe due to Deligne ([**D**]) enables one to predict many such relations. This recipe further allows one to derive reciprocity laws which certain conjectually algebraic numbers, formed essentially as ratios of critical values, ought to obey.

To give an example, let K be a quadratic imaginary extension of $\mathbf{Q}$, and let $\chi : \mathbb{A}_K^* \to \mathbb{C}^*$ be a Hecke character of K with $\chi(k_\infty) = k_\infty^{-w}$ for $k_\infty \in \mathbb{C}^* \hookrightarrow \mathbb{A}_K^*$, and $w \in \mathbb{Z}$. Let ψ be a Hecke character of finite order of K, and let T be the finite extension of K generated by the values of χ and ψ on the finite idèles of K. If $w > 0$, both $L(\chi, s)$ and $L(\chi\psi, s)$ are critical at $s = 0$. The ratio (defined if $L(\chi, 0) \neq 0$) is algebraic and satisfies the reciprocity law

$$\sigma\left(\frac{L(\chi\psi, 0)}{L(\chi, 0)}\right) = (\psi\mid_T)(\sigma)\left(\frac{L(\chi\psi, 0)}{L(\chi, 0)}\right)$$

for $\sigma \in \operatorname{Gal}(\overline{K}/T)$ and where we regard ψ as a Galois character. In fact, for any σ,

$$\sigma\left(\frac{L(\chi\psi, 0)}{L(\chi, 0)}\right)$$

can be given exactly if we employ as well the conjugate L-series $L(\tau(X), s)$, $L(\tau(X\psi), s)$ for the various $\tau : T \to \mathbb{C}$, and use a non-abelian reciprocity law. Note that, over T, the ratio is a Kummer generator of the class field attached to $\psi\mid_T$. This type of law was first deduced, for K a CM field, from the results of [**B1**], themselves in part a refinement of some aspects of [**Sh3**]. However, the Deligne formalism shows clearly a) that such results in no way require that one restrict to Hecke characters (GL_1), and b) that even so, all such reciprocity laws already occur in the setting of Hecke L-series.

On the other hand, if f is a new form of weight $k \geq 2$, and ψ is an even Dirichlet character, then Shimura showed

$$\frac{D(f\psi, k)}{D(f,k)} = \alpha G(\psi)$$

where $\alpha \in (T_f \cdot T_\psi)^*$, and $G(\psi)$ is the Gaussian sum attached to ψ. Of course, if $\sigma \in \operatorname{Gal}(\overline{\mathbf{Q}}/T_f T_\psi)$, then $\sigma(G(\psi)) = \psi(\sigma)G(\psi)$, and so we have again a ratio which satisfies a reciprocity law. This paper constructs a common framework for unifying the known results of this sort and for predicting new ones.

The paper has 3 sections. In Section M, we introduce the motivic language, review the construction of Deligne's periods $c^{\pm}(M)$, establish some elementary facts about the action of the endomorphism algebra on M_{DR}^{+}, factor $c^{\pm}(M)$ into more basic periods $c_v^{\pm}(M)$ attached to each place v of K, and establish the reciprocity law which relates a suitable monomial $c_{\varepsilon_\pi}^{\alpha}(M)$ in the $c_v^{\pm}(M)$ to $c^{\pm}(M \otimes \pi)$ where π is an Artin motive. At the end of the section, we give some examples. In Section L, we introduce L-functions and, invoking Deligne's general conjecture, obtain our main conjecture. This says that, for a k which is critical for both M and $M \otimes \pi$,

$$(1 \otimes 2\pi i)^{k d_\alpha(M\otimes\pi)} \frac{L(M \otimes \pi, T; k)}{c_{\varepsilon_\pi}^{\alpha}(M)} = c(\delta_*^{\alpha}, \det(\pi))$$

where $c(\delta_*^{\alpha}, \det(\pi))$ is a quantity characterized $\mod (T(M)T(\pi))^*$ by a reciprocity law depending on δ_*^{α} and $\det(\pi)$. Here δ_*^{α} is a character of $T(M)^*$ and $\alpha = (-1)^k$. Finally, we undertake the formal exercise of transcribing the main conjecture into the setting of algebraic cusp forms, replacing M by an algebraic $\prod$ on $GL_N(\mathbb{A}_K)$ and supposing now π is on $GL_m(\mathbb{A}_K)$, of Galois type. Since Deligne's conjecture is known for Hecke L-series of CM fields, this conjecture is a theorem for K CM, and $N = m = 1$.

A last Section H, gives some special results which arise in the case of GL_1.

This paper is a revised version of a 1987 MSRI preprint. Since then, Hida [**Hi**] has obtained basic results for $GL_2(\mathbb{A}_K)$ where K is any number field; we leave to the reader the task of confirming that Hida's work is consistent with what we conjecture. Also, both Harris ([**Ha**]) and Yoshida [**Y**] have, with independent motivations, pursued the calculation of various period relations.

I would like to thank D. Ramakrishnan for encouraging me to look again at this work and publish it in this revised form.

M. Motives and period relations.

M.1. We briefly review our notations and basic objects. See [CV] for more details. Let M be a motive (for absolute Hodge cycles) of pure weight w defined over K and having coefficients in T, both number fields. Then

$$M = \{M_{DR}, M_B, M_f, I_\infty, I_f\}$$

with $T \otimes K$ module M_{DR}, a T vector space M_B, and a $T \otimes \mathbf{Q}_f$ module M_f, all free of the same finite rank. Here, for a number field L, L_f denotes the finite ideles. These modules are related by T-linear isomorphisms

$$I_\infty : M_B \otimes_B \mathbf{C} \to M_{DR} \otimes_K \mathbf{C}$$
$$I_f : M_B \otimes \mathbf{Q}_f \to M_f.$$

For L as above, G_L denotes the Galois group $\mathrm{Gal}(\overline{L}/L)$. Then M_f is a G_K module, and the collection $\{M_\lambda \mid \lambda = \text{finite place of } T\}$, where $M_\lambda \subset M_f$ is the subspace associated to $T_\lambda \subset T \otimes \mathbf{Q}_f$, is a compatible system of rational λ-adic representations of G_K.

Also, M_B has a Hodge decomposition

$$M_B \otimes \mathbf{C} = \bigoplus_{\substack{p,q\in\mathbf{Z}\\ p+q=w}} M^{p,q}$$

with $(M^{p,q})^{1\otimes\rho} = M^{q,p}$, where ρ denotes complex conjugation. For each $p \in \mathbf{Z}$, there is a K subspace $F^pM_{DR} \subseteq M_{DR}$ such that

$$I_\infty\left(\bigoplus_{p'\geq p} M^{p',q}\right) = F^pM_{DR} \otimes_K \mathbf{C}.$$

The Hodge decomposition is stable for the action of T.

M.2. Throughout this paper, we make the strong assumption that for two motives M and N, defined over K, if M_ℓ is isomorphic to N_ℓ, as a G_K module, for a single prime ℓ, then M is isomorphic to N. This hypothesis is a motivic version of Tate's isogeny conjecture, proved for the H^1 of abelian varieties by Faltings. See L.3 below for a variant of the conjecture.

M.3. Special motives. For a number field L, let J_L and $J_{L,\mathbf{R}}$ denote the embeddings of L into $\mathbf{C}$ and $\mathbf{R}$, respectively. Let $J_{L,\mathbf{C}} = \langle 1,\rho\rangle\backslash(J_L - J_{L,\mathbf{R}})$, i.e. $J_{L,\mathbf{C}}$ is the set of complex places of L. For each $\sigma \in J_{K,\mathbf{R}}$, let $F_\sigma : (\sigma M)_B \to (\sigma M)_B$ denote the action of complex conjugation on the conjugate by σ, σM, of the motive M defined over K. Set $(\sigma M)_B = (\sigma M)_B^+ \oplus (\sigma M)_B^-$ where $(\sigma M)_B^\pm = \ker(F_\sigma \mp 1)$. If $w \in 2\mathbf{Z}$, assume

$M^{w/2,w/2} = \{0\}$ unless K is totally real. In this case, assume that each F_σ acts on $M^{w/2,w/2}$ by a scalar $\varepsilon = \pm 1$, independent of σ. We call such motives *special*, and henceforth any motive denoted by M will be special, defined over K, and with coefficients in T.

Define spaces $F^{\pm}M_{DR} \subseteq M_{DR}$ by

$$\begin{aligned} F^{\pm}M_{DR} &= F^{w/2}M_{DR} \qquad (w \in 2\mathbf{Z},\ \varepsilon = \pm 1)\\ F^{\pm}M_{DR} &= F^{w/2+1}M_{DR} \qquad (w \in 2\mathbf{Z},\ \varepsilon = \mp 1)\\ F^{+}M_{DR} &= F^{-}M_{DR} = F^{\frac{w+1}{2}}M_{DR} \qquad (w \text{ odd}).\end{aligned}$$

Put $M^{\pm}_{DR} = M_{DR}/F^{\mp}M_{DR}$. Let $d = \dim_T M_B$ and define $d^{\pm} : J_T \to \mathbf{Z}$ by $d^{\pm}(\sigma)$ = dimension of σ-eigenspace for action of T in $M^{\pm}_{DR} \otimes_K \mathbf{C}$. Note that if $\tau \in J_K$, $(\tau M_{DR})^{\pm} = \tau(M^{\pm}_{DR})$.

M.4. Periods. Let $R_{K/\mathbf{Q}}M$ be the motive over $\mathbf{Q}$ obtained by applying the restriction of scalars functor to M. Following Deligne ([D]), define $c^{\pm}(M,T) \in (T \otimes \mathbf{C})^*$ by

$$c^{\pm}(M,T) = \det\nolimits_T\ (I^{\pm})$$

where $I^{\pm}$ is the composite

$$(R_{K/\mathbf{Q}}M)^{\pm}_B \otimes \mathbf{C} \xrightarrow{I_\infty} (R_{K/\mathbf{Q}}M_{DR}) \otimes \mathbf{C} \to (R_{K/\mathbf{Q}}M_{DR})^{\pm} \otimes \mathbf{C}$$

and the determinant is computed relative to T bases of each side. When convenient, we will write $c^{\pm}(M) = c^{\pm}(M,T)$, leaving T implicit.

M.5. Let $\mathrm{End}(M)$ denote the algebra of all endomorphisms of M which are defined over K.

Proposition 1. *If M is simple, i.e. has no non-trivial submotives, then* $\mathrm{End}(M)$ *is a division algebra with positive involution.*

Proof. The proof follows, exactly as for abelian varieties, from the existence of K-rational polarization of M. □

Remark. Hence, by Albert's classification, $\mathrm{End}(M)$ is 1) a totally real number field F, or 2) a quaternion algebra B over F with $B \otimes_{\mathbf{Q}} \mathbf{R}$ isotypic, or 3) a division algebra over a CM field. (We recall here that a CM field is a totally imaginary quadratic extension of a totally real field.)

Let $\delta^{\pm}$ denote the isomorphism class of the representation of T on $M^{\pm}_{DR}$. For a number field L, let I_L denote the ring of functions from J_L to $\mathbf{Z}$. The elements of I_L with non-negative values are identified with the $\overline{\mathbf{Q}}$-rational representations of L as a $\mathbf{Q}$-algebra. If $\Phi \in I_L$ and $\tau \in G_{\mathbf{Q}}$, define $\tau\Phi$ by $(\tau\Phi)(\eta) = \Phi(\tau^{-1}\eta)$ for $\eta \in J_L$.

Proposition 2.

(a) *If either T or K contains no CM subfield, then $\delta^{\pm}$ is a multiple of the regular representation. In any case, $\delta^{\pm} + \rho\delta^{\pm}$ is such a multiple.*

(b) *If $\delta^{\pm}$ is not a multiple of the regular representation, then $\delta^{+} = \delta^{-}$ and T contains a CM subfield E such that $\delta^{\pm}(\eta_1) = \delta^{\pm}(\eta_2)$ if η_1 and η_2 agree on E. Further, for $K_0 = \mathbf{Q}(Tr\ \delta^{\pm}(t) \mid t \in T)$, $\tau\delta^{\pm} = \delta^{\pm}$ if and only if $\tau \in G_{K_0}$.*

Proof. Assume that M is simple as a motive with coefficients in T. Then $\delta^{\pm}$ is the restriction to $T \subseteq \mathrm{End}(M)$ of a representation of $\mathrm{End}(M)$. Let T' be a maximal subfield of $\mathrm{End}(M)$ which contains T. Let $\delta^{\pm}_{T'}$ be the representation of T' on $M^{\pm}_{DR}$. Then

$$\delta^{\pm}(\sigma) = \sum_{\substack{\eta \in J_{T'} \\ \eta|_T = \sigma}} \delta^{\pm}_{T'}(\eta).$$

Regard these functions as defined on $G_{\mathbf{Q}}$. Then $\delta^{\pm}_{T'}(\rho g) = \delta^{\pm}_{T'}(g\rho)$ for all $g \in G_{\mathbf{Q}}$. It follows that the same property holds for $\delta^{\pm}$, and hence, if $E \subseteq T$ denotes the field attached to $H^{\pm} \subseteq G_{\mathbf{Q}}$, the right stabilizer of $\delta^{\pm}$, then E is one of these types of field. Since the period map

$$(M_B \oplus (\rho M)_B)^{\pm} \otimes \mathbf{C} \xrightarrow{\sim} (M^{\pm}_{DR} \otimes \mathbf{C} \oplus (\rho M)^{\pm}_{DR} \otimes \mathbf{C})$$

is $T \otimes \mathbf{C}$ linear, $\delta^{\pm} + \rho\delta^{\pm}$ is a multiple of the regular representation R. If E is totally real, then $\delta^{\pm} = \rho\delta^{\pm}$, and hence $\delta^{\pm}$ is itself a multiple of R, and $E = K_0 = \mathbf{Q}$. If E is a CM field, then K_0 is a CM field, K can contain no real place, and hence $M^{+}_{DR} = M^{-}_{DR}$, i.e. $\delta^{+} = \delta^{-}$. If $\tau \in G_{\mathbf{Q}}$, then τ fixes K_0 if and only if $(\tau\delta^{\pm})(t) = \delta^{\pm}(t)$ for all $t \in T$. This happens if and only if $\tau\delta^{\pm} = \delta^{\pm}$. □

Remark. The example of an abelian variety of CM type shows that $\delta^{\pm}$ is not always a multiple of R.

M.6. Basic periods. Let M be simple, and let Z be the center of $\mathrm{End}(M)$ with $[\mathrm{End}(M) : Z] = n^2$. Let $T \supseteq Z$ be a maximal subfield of $\mathrm{End}(M)$. Then $[T : Z] = n$. Let L be a Galois extension of Z which contains T. Let $N = R_{K/\mathbf{Q}}M$. Then

$$N \otimes_Z L = \oplus_{\sigma \in \mathrm{Hom}_Z(T,L)} N \otimes_{T,\sigma} L$$

where $N \otimes_{T,\sigma} L$ denotes the extension of coefficients of N via $\sigma : T \to L$. The $N \otimes_{T,\sigma} L$ are all L linearly isomorphic since each is isomorphic to the image of $M \otimes_Z L$ by a minimal idempotent of $\mathrm{End}(M) \otimes_Z L = M_n(L)$. Hence, the quantities $c^{\pm}(M,T) \otimes_{T,\sigma} 1 \in L \otimes \mathbf{C}$ all lie in the same coset modulo L^*, i.e. the class of $c^{\pm}(M,T) \otimes 1$ in $(L \otimes \mathbf{C})^*/L^*$ is fixed under the action of $\mathrm{Gal}(L/Z)$ via the first factor. Put $z^{\pm}(\sigma) = (\sigma \otimes 1)(c^{\pm}(M,T) \otimes 1)/c^{\pm}(M,T) \otimes 1 \in L^*$. Then $z^{\pm}(\sigma)$ is a 1-cocycle for the action of $\mathrm{Gal}(L/Z)$ on L^*. Hence there exists $b^{\pm} \in L^*$ such that $b^{\pm} = (\sigma b^{\pm}) z^{\pm}(\sigma)$. Put $c_0^{\pm}(M) = b^{\pm} c^{\pm}(M,T)$. Then $(\sigma \otimes 1) c_0^{\pm}(M) = c_0^{\pm}(M)$, i.e., $c_0^{\pm}(M)$ belongs to $Z \otimes \mathbf{C}$. Since the map $(Z \otimes \mathbf{C})^*/Z^* \to (L \otimes \mathbf{C})^*/L^*$ is injective for every L, we see easily that $c_0^{\pm}(M)$ depends only on M and not the auxiliary choices of T and L.

M.7. Relations. If F is a field contained in $\mathrm{End}(M)$, $c^{\pm}(M,F) \sim N_{T/F}(c_0^{\pm}(M)) \bmod (F^*)$ where $T \supseteq F$ is a maximal subfield, $N_{T/F} : (T \otimes \mathbf{C})^* \to (F \otimes \mathbf{C})^*$ is the norm map, and, for any field L, and $\alpha, \beta \in L \otimes \mathbf{C}$, with β a unit, $\alpha \sim \beta \bmod (L^*)$ means $\alpha\beta^{-1} \in L = L \otimes 1 \hookrightarrow L \otimes \mathbf{C}$.

Proposition. *Let M be a simple motive defined over K with coefficients in T. Let Z be the center of* $\mathrm{End}(M)$*, with* $[\mathrm{End}(M) : Z] = n$. *Then*

$$c^{\pm}(M,T) \sim N_{TZ/T}(c_0^{\pm}(M))^{n[TZ:Z]^{-1}} \bmod (T^*).$$

Proof. $c^{\pm}(M,T) \sim N_{L/T}(c_0^{\pm}(M))$ where $L \supseteq T$ is a maximal subfield of $\mathrm{End}(M)$. Since $N_{L/T} = N_{TZ/T} \circ N_{L/TZ}$, $c^{\pm}(M,T) \sim N_{TZ/Z}(c_0^{\pm}(M))^{[L:TZ]}$, because $c_0^{\pm}(M) \in Z \otimes \mathbf{C}$. Since $[L : TZ][TZ : Z] = n$, the result follows. $\square$

Although the periods $c_0^{\pm}(M)$ are more fundamental, we shall work throughout the paper with the quantities $c^{\pm}(M,T)$.

M.8. Factorization of $c^{\pm}(M,T)$. Let $P_K = \langle 1, e \rangle \backslash J_k$ denote the set of infinite places of K. Given M and $T \subseteq \mathrm{End}(M)$, we will define, for $v \in P_K$ periods $c_v^{\pm}(M,T) \in (T \otimes \mathbb{C})^*$ such that

$$c^{\pm}(M,T) = \alpha^{\pm}(M) \cdot \prod_{v \in P_K} c_v^{\pm}(M,T)$$

for an elementary computable factor $\alpha(M) \in (T \otimes \mathbf{Q})^*$ depending only certain choices of differentials and characterized $\bmod T^*$ by a Galois recprocity law.

In the following $\Phi \subseteq J_K$ denotes a G_T orbit of embeddings of K and Ψ denotes a G_K orbit of embeddings of T. Identifying, according to context, a set of embeddings $\Psi \subseteq J_T$ with the sum $\Sigma_{\sigma\in\Psi}\sigma \in I_T$, also denoted Ψ, we have

$$\delta^{\pm} = \sum_{\Psi} n^{\pm}(\Psi)\Psi \qquad (n(\Psi) \geq 0).$$

The algebra $T \otimes K$ is isomorphic to a direct sum of fields indexed by the orbits of $G_{\mathbf{Q}}$ in $J_T \times J_K$: If $\Psi \subseteq J_T$, we define $\Psi^* \subseteq J_K$ by

$$\Psi^* = \pi_2((G_{\mathbf{Q}}(\Psi \times 1_K)) \cap (1_T \times J_T))$$

where $\pi_2 : J_T \times J_K \to J_K$ denotes projection on the second factor. Similarly, we can define $\Psi^* \subseteq J_T$ given $\Psi \subseteq J_K$. For a G_T orbit Ψ (resp. G_K orbit Ψ) let $\widetilde{\sigma} \in \Psi$ (resp. $\widetilde{\tau} \in \Psi$) be a representative. Then the decomposition becomes

$$\begin{aligned} T \otimes K \ &\xrightarrow{\sim} \bigoplus_{\Phi}(T \otimes K)^{\Phi} \xrightarrow{\sim} \bigoplus_{\widetilde{\sigma}} T\widetilde{\sigma}(K) \\ &\xrightarrow{\sim} \bigoplus_{\Psi}(T \otimes K)^{\Psi} \xrightarrow{\sim} \bigoplus_{\widetilde{\tau}} \widetilde{\tau}(T)K, \end{aligned}$$

and the $T \otimes K$-module $M^{\pm}_{DR}$ decomposes as $T \otimes K$-module:

$$M^{\pm}_{DR} \xrightarrow{\sim} \bigoplus_{\Psi}(M^{\pm}_{DR})^{\Psi}$$

where

$$(M^{\pm}_{DR})^{\Psi} \xrightarrow{\sim} ((T \otimes K)^{\Psi})^{n^{\pm}(\Psi)}.$$

Let $(\delta^{\pm})^* = \sum_{\Psi} n^{\pm}(\Psi)\Psi^*$, and let $d(\Psi^*)$ denote the number of elements in Ψ^*. Then $(T \otimes K)^{\Psi} = (T \otimes K)^{\Psi *} \xrightarrow{\sim} T\widetilde{\tau}(K)$ is an extension of T of degree $d(\Psi^*)$, and hence $(M^{\pm}_{DR})^{\Psi}$ is a $(T \otimes K)^{\Psi}$ vector space of dimension $n^{\pm}(\Psi)$.

Let $\Omega^{\pm}(\Psi) = \{\omega^{\pm}_1(\Psi), \dots, \omega^{\pm}_{n^{\pm}(\Psi)}(\Psi)\}$ be a $(T \otimes K)^{\Psi}$-basis of $(M^{\pm}_{DR})^{\Psi}$, and let $\Omega^{\pm} = \bigcup_{\Psi}\Omega^{\pm}(\Psi)$. Let $\eta \in J_K$.

Case 1. Suppose $\eta \in J_{K_1,\mathbb{R}}$. The $\delta^{\pm}$ is a multiple n of the regular representation, and $\eta(\Omega^{\pm}) = \bigcup_{\Psi}\eta\Omega^{\pm}(\Psi)$ is a T-basis of $\eta M^{\pm}_{DR}$. Let $\Gamma^{\pm}(\eta) = \{\gamma^{\pm}_1(\eta), \dots, \gamma^{\pm}_n(\eta)\}$ be a T-basis of $(\eta M)_B$. Let

$$I^{\pm}_{\infty}(\eta) : (\eta M)^{\pm}_B \otimes \mathbb{C} \xrightarrow{\sim} (\eta M)^{\pm}_{DR} \otimes_{\eta K} \mathbb{C}$$

be the $T \otimes \mathbb{C}$ linear period map, and let

$$p(\eta(\Omega^{\pm}(\Psi)), \Gamma^{\pm}(\eta)) = \det_{T\otimes\mathbb{C}}(P(\eta\omega^{\pm}(\Psi)_i,\ \gamma^{\pm}(\eta)_j)) \quad (1 \leq i, j \leq n),$$

where $p(\omega^{\pm}(\Psi)_i, \gamma^{\pm}_j(\eta))$ is defined by

$$I^{\pm}_{\infty}(\eta)(\eta\omega^{\pm}(\Psi)_i) = \sum_{j=1}^{n} p(\eta\omega^{\pm}(\Psi)_i, \gamma^{\pm}(\eta)_j) \cdot \gamma^{\pm}(\eta)_j.$$

Setting $v = \eta$, define

$$c_v^{\pm}(M) = \prod_{\Psi} p(\eta(\Omega^{\pm}(\Psi)), \Gamma^{\pm}(\eta)).$$

If we change $\Omega^{\pm}$ and $\Gamma^{\pm}(\eta)$, then $c_v^{\pm}(M)$ undergoes a change $c_v^{\pm}(M) \to (t \otimes \eta(k))c_v^{+}(M)$ with at $t \in T^*$, and a k in K which is independent of η.

Case 2. (η complex). The $M_{DR}^{+} = M_{DR}^{-}$. Let

$$\Lambda(\Psi_\eta) = \{\lambda_1, \dots, \lambda_{n(\Psi)}, \quad \lambda_1', \dots, \lambda_{n(\Psi)}'\}$$

where

$$\begin{aligned} \lambda_j &= \eta\omega_j(\Psi) + \rho\eta\omega_j(\Psi) \\ \lambda_j' &= (1 \otimes i)(\eta\omega_j(\Psi) - (\rho\eta)\omega_j(\Psi)) \end{aligned}$$

for $i = \sqrt{-1}$, $\omega_j \in \Omega(\Psi)$. Let $\Gamma^{\pm}(\eta) = \{\gamma_1(\eta) + \rho\gamma_1(\eta), \dots, \gamma_d(\eta) + \rho\gamma_d(\eta),\ (1 \otimes i)(\gamma_1(\eta) - \rho\gamma_1(\eta)), \dots, (1 \otimes i)(\gamma_d(\eta) - \rho\gamma_d(\eta))\}$ where $\gamma_1(\eta), \dots, \gamma_d(\eta)$ is a T-basis of $(\eta M)_B$. Let

$$\begin{aligned} &(I_\infty^{\pm}(\eta) \oplus I_\infty^{\pm}(\rho\eta)) : (((\eta M)_B \otimes \mathbb{C})^{\eta\Psi} \oplus \\ &\qquad ((\rho\eta)(M)_B \otimes \mathbb{C})^{\rho\eta\Psi})^{\pm} \to (\eta M_{DR}^{\pm} \otimes_{\eta(K)} \mathbb{C} \oplus (\rho\eta) M_{DR}^{\pm} \otimes_{\rho\eta(K)} \mathbb{C}) \end{aligned}$$

be the period isomorphism, and let, as before

$$p(\Lambda(\Psi, \eta),\ \Gamma^{\pm}(\eta))$$

be the $T \otimes \mathbb{C}$-linear determinant of the matrix representing $I_\infty^{\pm}(\eta) \oplus I_\infty(\rho\eta)$ relative to be the bases $\Lambda(\Psi, \eta)$ and $\Gamma^{\pm}(\eta)$. Let $c_v^{\pm}(M) = \prod_\Psi p(\Lambda(\Psi, \eta), \Gamma^{\pm}(\eta))$.

Then $c_v^{+}(M)$ undergoes a change of the form $c_v^{+}(M) \to (t \otimes 1)c_v^{+}(M)$ if we change the basis of $(\eta M)_B^{\pm}$, and a change of the form $c_v^{+}(M) \to (1 \otimes \eta(k)(\rho\eta)(k))c_v^{+}(M)$ if we change the $\Omega(\Psi)$. Finally, if we replace η by $\rho\eta, c_v^{\pm}(M)$ is unchanged.

For any $\varepsilon : P_K \to \mathbb{Z}$ and any $d \in \mathbb{Z}$ such that $\varepsilon(v) = d$ if v is complex, let

$$\begin{aligned} c_\varepsilon^{\pm}(M) = &\left(\prod_{v \in P_{K,\mathbb{R}}} c_v^{\pm}(M)^{\varepsilon(v)} c_v^{\mp}(M)^{d-\varepsilon(v)} \right) \\ &\left(\prod_{v \in P_K \setminus P_{K,\mathbb{R}}} c_v^{\pm}(M) \right)^d . \end{aligned}$$

Then, because M is special, $c_\varepsilon^{\pm}(M, T)$ is independent of choices up multiplication by an element $t \otimes 1$ for $t \in T$.

M.9. We retain the notation of the previous paragraph. Let

$$\{\alpha_1(\Psi), \dots, \alpha_{d(\Psi^*)}(\Psi)\} = A(\Psi)$$

be a basis of $(T\otimes K)^{\Psi^*}$ over T. Then $A\Omega(\Psi) \overset{\text{def}}{=} \{\alpha_1(\Psi)\Omega(\Psi), \alpha_2(\Psi)\Omega(\Psi), \dots, \alpha_{d(\Psi^*)}\Omega\Psi\}$ is a basis of $(M^{\pm}_{DR})^{\Psi}$ as a T vector space. Then $A\Omega = \cup_\Psi A\Omega(\Psi)$ is an unordered basis of $M^{\pm}_{DR}$ as a T-vector space.

Since the T-vector space $(RM)^{\pm}_{DR}$ is canonically identified, ignoring the K action on the latter, with $M^{\pm}_{DR}$ as T-vector space; $A\Omega$ is a T-basis of $RM^{\pm}_{DR}$. Since $(RM)^{\pm}_{B} = \left(\underset{\eta\in J_K}{\oplus}(\eta M)_B\right)^{\pm}$, the family of $\Gamma^{\pm}(\eta)$'s from (M.8) provides a basis of $(((RM)_B)\otimes\mathbb{C})^{\pm}$. Computing $c^{\pm}(M,T)$ using these bases, and letting $\varepsilon = 1$, the ratio

$$c_1^{\pm}(M,T)/c^{\pm}(M,T) = b^{\pm}(M)$$

is well-defined modulo T^*, and is computed as the $T\otimes\overline{\mathbf{Q}}$-linear determinant of the matrix which expresses the basis

$$\left(\underset{\Psi}{\cup}\ \underset{\eta\in J_{K,\mathbb{R}}}{\cup} \eta\Omega(\Psi)\right) \cup \left(\underset{\Psi}{\cup}\ \underset{\eta\in R_{K,\mathbb{C}}}{\cup} \Lambda(\Psi,\eta)\right)$$

(where $R_{K,\mathbb{C}}$ is a set of representatives in J_K for $\langle 1,\rho\rangle\backslash(J_K\backslash J_{K,\mathbb{R}})$), in terms of $A\Omega = \underset{\Psi}{\cup} A\Omega(\Psi)$.

To do this, note that if we use the $\Omega(\Psi)$ bases to identify $M^{\pm}_{DR}$ with $\underset{\Psi}{\oplus}((T\otimes K)^{\Psi})^{n(\Psi)}$, then our task relates to that of calculating, for each Ψ, the inverse $b(\Psi)$ of the determinant of the matrix in $M_{d(\Psi^*)}(T\otimes\overline{\mathbf{Q}})$ giving the canonical isomorphism

$$J(\Psi): (T\otimes K)^{\Psi}\otimes\overline{\mathbf{Q}} \xrightarrow{\sim} (T\otimes\overline{\mathbf{Q}})^{\Psi^*},$$

where we regard the right member as the $T\otimes\overline{\mathbf{Q}}$-algebra of maps $\Psi^*\to T\otimes\overline{\mathbf{Q}}$, relative to the bases given by $A(\Psi)$ and the set of idempotents 1_η, satisfying $1_\eta(\eta) = 1_{T\otimes\overline{\mathbf{Q}}}, 1_\eta(\eta') = 0$ if $\eta'\neq\eta$. Note that

$$J(\Psi)(\alpha_j(\Psi)) = \sum_{\eta\in\Psi^*} \widetilde{\eta}(\alpha_j(\Psi))\cdot 1_\eta,$$

where $\widetilde{\eta}$ is the T-linear extension of η to $(T\otimes K)^{\Psi}$.

Hence,

$$b(\Psi) = \det(\widetilde{\eta}(\alpha_j(\Psi)))_{1\le j\le d(\Psi^*),\ \eta\in\Psi^*}$$

and

$$b^{\pm} = \prod_{\Psi} b(\Psi)^{n^{\pm}(\Psi)}.$$

Then b depends only on T, K and $\delta^{\pm}$ modulo T^*.

This quantity is characterized by a reciprocity law which is easy to compute: Since each Ψ^* is stable under the action of G_T, the signature $\pi_\Psi(\tau)$ defines a character $\pi_\Psi(\tau) : G_T \to \{\pm 1\}$. Define

$$\pi_{\delta^{\pm}} = \prod_{\Psi} \pi_\Psi(\tau)^{n^{\pm}(\tau)}$$

and let $b(\delta^{\pm}) = (b(\delta^{\pm})_\sigma)_{\sigma \in J_T}$. Then

$$\tau b(\delta^{\pm}) = \pi_{\delta^{\pm}}(\tau) b(\delta^{\pm})_1$$

for $\tau \in G_T$, and, letting $R_{\delta^{\pm}} = \mathrm{Ind}_T^{\mathbf{Q}}(\pi_{\delta^{\pm}})$, we have

$$(1 \otimes \tau) b(\delta^{\pm}) = R(\delta^{\pm})(\tau) b(\delta^{\pm}).$$

Note that $b^+ \sim b^-$ unless K is totally real and $M_B^{p,p} \neq \{0\}$ for some p. Several special cases are worth noting.

1. If $T = \mathbf{Q}$, $\delta^{\pm}$ is a multiple $n^{\pm}$ of the regular representation of K and $\pi(\delta^{\pm}) = \pi(J_K)^{n^{\pm}}$. Hence $b^{\pm} \sim (\sqrt{D_K})^n$ where D_K is the discriminant.
2. For more general T, but if $\delta^{\pm}$ is a multiple of the regular representation, then $\pi(\delta^{\pm}) = (\mathrm{sgn}(J_K) \mid_T)^n$, and again

$$b^{\pm} \sim 1 \otimes \sqrt{D_K}^n.$$

3. If K is a CM field, and $\delta^{\pm}$ is a multiple of a CM type, then G_T acts on $(\delta^{\pm})^*$ via the same permutation as it acts on the embeddings J_{K_0}, since the restriction $\delta^{\pm} \to J_{K_0}$ is a bijection, where K_0 is the maximal real subfield of K. Hence $\pi(\delta^{\pm}) = (\mathrm{sgn}_{J_F}) \mid_T$ and so $b \sim 1 \otimes \sqrt{D_F}$.

M.10. Artin motives. Let $\pi : G_K \to \mathrm{Aut}(V)$ be a representation of G_K on the rational vector space V. Set $\pi_B = V$, $\pi_f = V \otimes \mathbf{Q}_f$, and $\pi_{DR} = (V \otimes \overline{\mathbf{Q}})^{G_K}$. Let $I_\infty : \pi_{DR} \otimes_K \mathbf{C} = \pi_B \otimes \mathbf{C}$ be the identity map, and define I_f similarly. The structure $\pi = (\pi_{DR}, \pi_B, \pi_f, I_\infty, I_f)$ is called an Artin motive ([**D**]). We have $\pi_B \otimes \mathbf{C} = \pi^{0,0}$, and π admits a field T as coefficients exactly when we can embed T into $\mathrm{End}(\pi_B)$ with image in the commutant of the image of G_K. Assuming π has coefficients in T, for each $\sigma \in J_{K,\mathbf{R}}$, let $\varepsilon_\pi(\sigma) = \frac{1}{2}(d_\pi + Tr(F_\sigma \mid (\sigma\pi)_B))$, where $d_\pi = \dim_T \pi_B$. For v complex, put $\varepsilon_\pi(v) = d_\pi$. (Here, for $\sigma \in G_{\mathbf{Q}}$, $\sigma\pi$ is the Artin motive attached to the representation $\tau \to \pi(\sigma^{-1}\tau\sigma)$, $\tau \in G_{\sigma K}$.)

M.11. Reciprocity laws. Let H, U, and V be subgroups of a topological group G, with H finite, and U and V of finite index, and $U \subseteq V$. Put

$J_U = G/U$, $J_V = G/V$. Let $\psi : J_V \to \mathbf{Z}$ satisfy, for an H invariant subset $S \subseteq J_V$,

$$\sum_{\tau \in H\sigma} \psi(\tau) = \begin{cases} c & \sigma \in S \\ 0 & \sigma \notin S \end{cases}$$

with a constant c depending only upon ψ. Via the natural map $J_U \to J_V$, regard ψ as a function on J_U. Let G_H be the subgroup of $\tau \in G$ for which $\tau H\sigma = H\tau\sigma$ for all $\sigma \in J_V$. For each $\sigma \in J_U$, choose a representative $w_\sigma \in G$, and arrange that $hw_\sigma = w_{h\sigma}$ for all $h \in H$. Let $H_S = \{g \in G \mid gS = S\}$. Define for $\tau \in G_H \cap H_S$, $t_\psi(\tau) \in U^{ab}$ by

$$t_\psi(\tau) = \prod_{\sigma \in J_K} (w_{\tau\sigma}^{-1} \tau w_\sigma)^{\psi(\sigma)} \bmod U^c$$

where U^c denotes the closure of the commutator subgroup.

Proposition 1. *For $\tau \in G_H \cap H_S$, $t_\psi(\tau)$ is well-defined.*

Proof. If $w'_\sigma = w_\sigma u_\sigma$ for a $u_\sigma \in U$, then $u_{h\sigma} = u_\sigma$ for all $h \in H$. Hence, it is enough to show

$$\prod_{\sigma \in J_U} u_{\tau\sigma}^{\psi(\sigma)} \equiv \prod_{\sigma \in J_U} u_\sigma^{\psi(\sigma)} \bmod U^c.$$

Since $\tau H\sigma = H_{\tau\sigma}$ for each $\sigma \in J_V$, the result will follow if

$$\sum_{\substack{\eta \in J_U \\ \eta \to \sigma}} \sum_{h \in H\tau\eta} \psi(h) = \sum_{\substack{\eta \in J_U \\ \eta \to \sigma}} \sum_{h \in H\eta} \psi(h).$$

Since $\tau S = S$ and $\tau(J_V - S) = J_V - S$, each side above is either $[V : U]c$ or 0, simultaneously. □

Next, let $H_\psi = \{g \in G \mid g\psi = \psi\}$.

Proposition 2. *If $\tau_1, \tau_2 \in H_\psi$, $t_\psi(\tau_1\tau_2) = t_\psi(\tau_1)t_\psi(\tau_2)$.*

Proof. Let $X \subseteq J_U$ be a set of representatives for $H_\psi \backslash J_U$. Then

$$(*) \qquad t_\psi(\tau) = \prod_{x \in X} \left[\prod_{y \in H_\psi x} (w_{\tau y}^{-1} \tau w_y) \right]^{\psi_0(x)}$$

where $\psi_0 : H_\psi \backslash J_U \to \mathbf{Z}$ is the function defined by ψ. Fixing x, each w_y belongs to $H_\psi w_x U = w_x(w_x^{-1} H_\psi w_x \cdot U)$, and hence $w_y = w_x z_y$ with $z_y \in w_x^{-1} H_\psi w_x \cdot U$. Hence, if $\tau \in H_\psi$,

$$\prod_{y \in H_\psi x} (w_{\tau y}^{-1} \tau w_y) = \prod_{y \in H_\psi x} (z_{\tau y}^{-1} (w_x^{-1} \tau w_x) z_y).$$

Now let A and B be subgroups of finite index inside a group C, and let $\{z_\alpha \in A{\cdot}B \mid \alpha \in A{\cdot}B/B\}$ be a set of representatives in $A{\cdot}B$ for the quotient $A{\cdot}B/B$. If $a \in A$, define $t(a) \in B^{ab}$ by

$$t(a) = \prod_{\alpha\in A\cdot B/B} (z_{a\alpha}^{-1} a z_\alpha) \bmod B^c.$$

Then $t(a)$ is well-defined, and

$$\begin{aligned} t(a_1a_2) &= \prod_{\alpha\in A\cdot B/B} (z_{a_1a_2\alpha}^{-1} a_1a_2 z_\alpha) \bmod B^c \\ &= \prod_{\alpha\in A\cdot B/B} (z_{a_1a_2\alpha}^{-1} a_1 z_{a_2\alpha})(z_{a_2\alpha}^{-1} a_2 z_\alpha) \bmod B^c \\ &= t(a_1)t(a_2) \bmod B^c. \end{aligned}$$

Applying this remark to $a_1 = w_x^{-1}\tau_1 w_x$, $a_2 = w_x^{-1}\tau_2 w_x$, $A = w_x^{-1}H_\psi w_x$ and $B = U$, we see that inner term of (*) is a homomorphism and we are done. □

Note that if ψ is a constant, taking the value d, then $H_\psi = G$ and $t_\psi : G^{ab} \to U^{ab}$ is the d-th power of the usual transfer map. However, in general, for $\tau \notin H_\psi$, t_ψ is not a homomorphism. Rather, we find $t_\psi(\tau_1\tau_2) = t_{\tau_2\psi}(\tau_1)t_\psi(\tau_2)$ for general $\tau_1, \tau_2 \in G$.

Below, we apply the results with $H = \langle 1, \rho\rangle$, $U = G_K$ and $V = G_{K_{cm}}$, where $K_{cm} \subseteq K$ is the maximal CM subfield, or $\mathbf{Q}$, if K contains no CM subfield. Hence, if $S = J_V$, $G_H = H_S = G_{\mathbf{Q}}$.

M.12. If L and T are number fields with $L \supseteq T$, we regard any $\delta \in I_T$ as a function in I_L via the map $J_L \to J_T$. Let L be a Galois extension of $\mathbf{Q}$ which contains T. Then, for $\delta^\pm$ as in M.5, there exists a unique $\delta_*^\pm \in I_{K_{cm}}$ (c.f. Prop. M.5.2.) such that $\delta_*^\pm(\sigma) = \delta^\pm(\sigma^{-1})$ for all $\sigma \in J_L$. If $\tau \in G_{\mathbf{Q}}$, then $\tau\delta_*^\pm$ depends only upon the image of τ in J_T.

The $\psi \in I_K$ which are of interest to us are of the form $\delta_*^\pm$, and hence satisfy

(M.12.1)
$$\psi(\rho\sigma) + \psi(\sigma) = w \qquad (\sigma \in J_K).$$

(M.12.2)
$$\psi(\sigma_1) = \psi(\sigma_2) \qquad \text{if } \sigma_1 = \sigma_2 \text{ on } K_{cm}.$$

(M.12.3)
$$\tau\psi = \psi \qquad (\tau \in J_T).$$

Clearly, for such ψ, the above procedure defines $\psi_* \in I_T$, and we have $(\psi_*)_* = \psi$.

M.13. Let ψ be as in M.12, and define $r_\psi : G_{\mathbf{Q}} \to (G_K^{ab})^{J_T}$ by $r_{\psi,\eta} = t_{\eta\psi}$, for each $\eta \in J_T$. Then, if $\varphi : G_K^{ab} \to T^*$ is a character, define, for $\tau \in G_{\mathbf{Q}}$, $\varphi_* r_\psi(\tau) \in (T \otimes_{\mathbf{Q}} \overline{\mathbf{Q}})^*$ by

$$\varphi_* r_\psi(\tau)_\eta = (\eta\psi) \circ r_{\psi,\eta}(\tau) \qquad (\eta \in J_T).$$

Proposition. *There exists an element $c(\psi,\varphi) \in (T \otimes \overline{\mathbf{Q}})^*$, unique up to a change of the form $c(\psi,\varphi) \mapsto (t \otimes 1)c(\psi,\varphi)$ with $t \in T^*$, such that for all $\tau \in G_{\mathbf{Q}}$,*

$$(1 \otimes \tau)c(\psi,\varphi) = \varphi_* r_\psi(\tau) c(\psi,\varphi).$$

Proof. Let π_φ be an Artin motive, defined over K, with coefficients in $\mathbf{Q}(\varphi)$, the field generated by the values of φ. Set $\pi = \pi_\varphi \otimes_{\mathbf{Q}(\varphi)} T$. Then $\dim_T \pi_B = 1$, and hence there exists $p(\varphi) \in (T \otimes K^{ab})^*$ such that $\gamma = p(\varphi){\cdot}w$ where $0 \neq \gamma \in \pi_B$, and w is a basis of the free $T \otimes K$ module π_{DR}. If $\tau \in G_K$, it follows, from the definition of π_{DR}, that $(1 \otimes \tau)p(\varphi) = (\varphi(\tau) \otimes 1)p(\varphi)$, and $p(\varphi)$ is independent of choices, up to change of the form $p(\varphi) \to \alpha p(\varphi)$ with an $\alpha \in (T \otimes K)^*$.

Now let $\psi : J_T \to \mathbf{Z}$ be as in M.12. with the roles of T and K reversed, and for an $a \in (T \otimes \overline{\mathbf{Q}})^*$, define $\psi(a) \in T \otimes \overline{\mathbf{Q}}$ by $\psi(a)_\sigma = a_\sigma^{\psi(\sigma)}$. For $\sigma \in J_K$, put $p(\varphi)_\sigma = (1 \otimes w_\sigma)p(\varphi)$, and set

$$P(\psi,\varphi) = \prod_{\sigma \in J_K} (\sigma\psi)(p(\varphi)_\sigma).$$

Note that, for $\tau \in G_{\mathbf{Q}}$, we have $(1 \otimes \tau)(\psi(a)) = (\tau\psi)(\tau a)$. Then

$$\begin{aligned}(1 \otimes \tau)P(\psi,\varphi) &= \prod_{\sigma \in J_K} (\tau\sigma\psi)((1 \otimes w_{\tau\sigma}k(\tau,\sigma))p(\varphi)) \\ &= \prod_{\sigma \in J_K} (\tau\sigma\psi)((\varphi(k(\tau,\sigma)) \otimes w_{\tau\sigma})p(\varphi)) \\ &= \left[\prod_{\sigma \in J_K} (\tau\sigma\psi)(\varphi(k(\tau,\sigma)) \otimes 1)\right] P(\psi,\varphi),\end{aligned}$$

where we have set $\tau w_\sigma = w_{\tau\sigma}k(\tau,\sigma)$ with $k(\tau,\sigma) \in G_K$.

Now, for $\eta \in J_T$,

$$\begin{aligned}\left[\prod_{\sigma \in J_K} (\tau\sigma\psi)(\varphi(k(\tau,\sigma)) \otimes 1)\right]_\eta &= (\eta\varphi)\left(\prod_{\sigma \in J_K} k(\tau,\sigma)^{(\tau\sigma\psi)(\eta)}\right) \\ &= (\eta\varphi)\left(\prod_{\sigma \in J_K} k(\tau,\sigma)^{(\tau^{-1}\eta\psi_*)(\sigma)}\right) \\ &= \eta\varphi \circ t_{\tau^{-1}\eta\psi_*}(\tau).\end{aligned}$$

Thus $(1 \otimes \tau)P(\psi,\varphi) = \varphi * r_{\psi_*}(\tau)P(\psi,\varphi)$.

If we now start with a $\psi : J_K \to \mathbf{Z}$ as in the hypothesis of the Proposition, we prove the above result with the roles of ψ and ψ_* reversed, using $(\psi_*)_* = \psi$. Put $c(\psi,\varphi) = P(\psi_*,\varphi)$. Then we are done since the reciprocity law clearly characterizes $c(\psi,\varphi)$ up to multiplication by $t \otimes 1$. □

M.14.

Corollary 1. *Define $c(\psi,\varphi) \in (T \otimes \overline{\mathbf{Q}})^*$ as in* M.13. *then*

(i) *if $\varphi^n = 1$, then $c(\psi,\varphi)^n \in T^*$,*

(ii) *let $E \subseteq T$ be the field corresponding to H_ψ. Then $c(\psi,\varphi)_1 \in E^{ab}$,*

(iii) *over $E(\varphi)$, $c(\psi,\varphi)_1$ generates the Kummer extension attached to the character $\varphi \circ t_\psi$ of G_E^{ab}.*

Proof. Part (i) is obvious, part (ii) follows from Prop. M.13 upon direct calculation of the actions of $\tau_1\tau_2$ and $\tau_2\tau_1$ for $\tau_1,\tau_2 \in G_E$, and part iii) is elementary. □

Corollary 2. *The numbers $c(\psi,\varphi)_\eta \quad (\eta \in J_T)$ generate abelian extensions of CM fields.*

Proof. By Prop. M.12., $E = \mathbf{Q}$ or E is a CM field, and the result follows from (ii) above. □

M.15. Let E_ψ be the field generated over E by the elements $c(\psi,\varphi)_1$ as φ varies among the characters of finite order of G_K^{ab}.

Proposition.

(i) *E_ψ is the subfield of E^{ab} corresponding to the subgroup of E_f^* consisting of the elements e for which $\psi_*(e)$ belongs to the connected component of K_f^*/K_+^*, where $K_+^* \subseteq K^*$ is the subgroup of totally positive elements.*

(ii) *Suppose that ψ is a CM type, i.e. $\psi(\sigma) + \psi(\rho\sigma) = 1$, and $\psi(\sigma) \geq 0$ for all $\sigma \in J_K$, and let F be the maximal totally real subfield of the field E.*

Let $z \in F \otimes \mathbf{C} - F \otimes \mathbf{R}$ be a CM point of type (E,ψ_*) (c.f. [**CV**]). *Then E_ψ is the field generated over E by the values at z of all arithmetic Hilbert modular functions (e.g. elements of $A_0(\mathbf{Q}_{ab})$ in the notation of* [**Sh3**]) *which are defined at z.*

Proof. The argument of Tate in [**L**] extends immediately to our case to show that $r_K(\psi_*(e)) = t_\psi(\sigma)$ if $r_E(e) = \sigma \in G_E^{ab}$, and where, for any number field L, $r_L : L_f^* \to G_L^{ab}$ denotes the Artin reciprocity law. Thus, (i) is immediate. To see (ii), recall that the field generated by the given values is the compositum of E with the fields of moduli $k(P)$ of the collection of PEL structures $P = (A, C, \vartheta, v_1, \ldots, v_N)$, where (A, C) is a fixed polarized abelian variety with an action ϑ of $\mathcal{O}_F$, and $v_1, \ldots, v_N$ denote a variable set of torsion points. By [**SH2**], Prop. 5.17, $Ek(P)$ is the field of moduli of (A, C, ϑ_*), where $\vartheta_* : E_* \to \mathrm{End}(A) \otimes \mathbf{Q}$ is an extension of ϑ. By [**Sh2**], Prop. 5.16, $\cup_P\ Ek(P)$ is the class field attached to

$$\cap_{n\geq 1}\ (\psi_*(K^*U_K(n)))^{-1} = (\psi_*(\cap_{n\geq 1}(K^*U_K(n))))^{-1}$$

where $U_K(n)$ denotes the subgroup of local units in K_f^* whose elements are congruent to 1 modulo n. Since $\cap_{n\geq 1}\ K^*U_K(n)/K^*$ is the connected component of K_f^*/K_+^*, we are done. □

M.16. For M special, irreducible and π an Artin motive, $M \otimes \pi$ is special if and only if i) $M^{w/2,w/2} = \{0\} \quad (w \in 2\mathbf{Z})$, or ii), if $w \in 2\mathbf{Z}$ and $M^{w/2,w/2} \neq \{0\}$, then K is totally real, F_τ acts as a scalar independent of τ on each $\tau\pi$, for all $\tau \in J_K$, and $tr(F_\tau, (\tau M)_B)$ is independent of τ. Assume that these conditions are satisfied.

Theorem. *For a special motive M and π, as above, both with coefficients in T and defined over K,*

$$c_1^{\pm}(M \otimes \pi) \sim c_{\varepsilon_\pi}^{\pm}(M)c(\delta_*^{\pm}, \det\ \pi) \mod (T^*)$$

where ε_π is defined in M.10, $\det\ \pi$ is the maximal T linear exterior power, and we have omitted notation referring to T.

Proof. For $\sigma \in J_K$, let $\xi(\sigma) = (\xi_1(\sigma), \ldots, \xi_{d_\pi}(\sigma))$ be a basis of $(\sigma\pi)_B$. Define $\Delta_\sigma : (\sigma M)^{d_\pi} \to \sigma(M \otimes \pi)$ by

$$\Delta_{\sigma,B}(v_1, \ldots, v_{d_\pi}) = \sum_{i=1}^{d_\pi} v_i \otimes \xi_i(\sigma)$$

where $v_1, \ldots, v_{d_\pi}$ belong to $(\sigma M)_B$. Passing to quotients, define $\Delta_\sigma^{\pm} : ((\sigma M)_{DR}^{\pm} \otimes_{\sigma K} \overline{\mathbf{Q}})^{d_\pi} \to (\sigma(M \otimes \pi))_{DR}^{\pm\alpha} \otimes_{\sigma K} \overline{\mathbf{Q}}$, via $\Delta_{\sigma,DR}$, where $\alpha = 1$ unless $M^{w/2,w/2} \neq \{0\}$ and $\pi(\rho) = -1$, in which case $\alpha = -1$.

We consider two commutative diagrams. First, if $\sigma \in J_{K,\mathbf{R}}$, assume that $\xi(\sigma)$ has been chosen so that $\xi(\sigma)_i \in (\sigma\pi)_B^+$ if $1 \leq i \leq \varepsilon(\sigma)$ and $\xi(\sigma)_i \in (\sigma\pi)_B^-$ if $\varepsilon(\sigma) + 1 \leq i \leq d_\pi$. Then,

$$\begin{array}{ccc}
(((\sigma M)_B^{\pm})^{\varepsilon(\sigma)} \oplus ((\sigma M)_B^{\mp})^{d_\pi - \varepsilon(\sigma)}) \otimes \mathbf{C} & \xrightarrow{J_\sigma^{\pm}} & ((\sigma M)_{DR}^{\pm} \otimes_{\sigma K} \mathbf{C})^{d_\pi} \\
\downarrow \Delta_{\sigma,B} & & \downarrow \Delta_\sigma^{\pm} \\
(\sigma(M \otimes \pi))_B^{\pm} \otimes \mathbf{C} & \xrightarrow{I_\sigma^{\pm}} & (\sigma(M \otimes \pi))_{DR}^{\pm} \otimes_{\sigma K} \mathbf{C}
\end{array}$$

commutes, where $J_\sigma^\pm = (I_\sigma^\pm)^{\varepsilon(\sigma)} \times (I_\sigma^\mp)^{d_\pi - \varepsilon(\sigma)}$. Hence,

$$c_\sigma^\pm(M \otimes \pi) \sim c_\sigma^\pm(M)^{\varepsilon(\sigma)} c^\mp(M)^{d_\pi - \varepsilon(\sigma)} \times \det(\Delta_\sigma^\pm),$$

since $\det(\Delta_{\sigma,B}) \in T^*$. If $\sigma \in J_{K,\mathbf{C}}$, assume $\xi(\rho\sigma) = \rho\xi(\sigma)$. Then

$$\begin{array}{ccc}
((\sigma M)_B^{d_\pi} \oplus (\rho\sigma M)_B^{d_\pi})^\pm \otimes \mathbf{C} & \xrightarrow{I_\sigma^{d_\pi} \oplus I_{\rho\sigma}^{d_\pi}} & \begin{array}{c}((\sigma M)_{DR}^\pm \otimes_{\sigma K} \mathbf{C})^{d_\pi} \oplus \\ ((\rho\sigma M)_{DR}^\pm \otimes_{\rho\sigma K} \mathbf{C})^{d_\pi}\end{array} \\
\Big\downarrow{\scriptstyle \Delta_{\sigma,B} \oplus \Delta_{\rho\sigma,B}} & & \Big\downarrow{\scriptstyle \Delta_\sigma^\pm \oplus \Delta_{\rho\sigma}^\pm} \\
((\sigma(M \otimes \pi))_B \oplus (\rho\sigma(M \otimes \pi))_B)^\pm \otimes \mathbf{C} & \xrightarrow{I_\sigma \oplus I_{\rho\sigma}} & \begin{array}{c}((\sigma(M \otimes \pi))_{DR}^\pm \otimes_{\sigma K} \mathbf{C}) \oplus \\ ((\rho\sigma(M \otimes \pi))_{DR}^\pm \otimes_{\rho\sigma K} \mathbf{C})\end{array}
\end{array}$$

commutes. Hence, if v denotes the place of K corresponding to the pair $(\sigma, \rho\sigma)$, then

$$c_v^\pm(M \otimes \pi) \sim c_v^\pm(M)^{d_\pi} \det(\Delta_\sigma^\pm) \det(\Delta_{\rho\sigma}^\pm),$$

where we agree that $\det(\Delta_\sigma^\pm)$ has component 1 at any $\eta \in J_T$ which fails to occur in $\sigma M_{DR}^\pm$. Thus, we must show

$$\prod_{\sigma \in J_K} \det(\Delta_\sigma^\pm) \sim c(\delta_*^\pm, \det \pi).$$

From the definition of Δ_σ, we see that if $w_1, \ldots, w_{d_\pi}$ belong to $M_{DR}^\pm$,

$$\Delta_\sigma^\pm(\sigma w_1, \ldots, w_{d_\pi}) = \sum_{i=1}^{d_\pi} \sigma w_i \otimes \xi_i(\sigma).$$

Let $\mu_1, \ldots, \mu_{d_\pi}$ be a basis of the free $T \otimes K$ module π_{DR}, then

$$\xi_i(\sigma) = \sum_{j=1}^{d_\pi} a_{ij}(\sigma)\sigma(\mu_j),$$

with $a_{ij}(\sigma) \in T \otimes \overline{\mathbf{Q}}$. Thus,

$$\Delta_\sigma^\pm(\sigma w_1, \ldots, \sigma w_{d_\pm}) = \sum_{i=1}^{d_\pi} \sum_{j=1}^{d_\pi} \sigma(w_i \otimes \mu_j) a_{ij}(\sigma).$$

Decomposing this map into its η eigencomponents for $\eta \in J_T$, we see that

$$(\det(\Delta_\sigma^\pm))_\eta = \det(A(\sigma))_\eta^{\sigma\delta^\pm(\eta)}$$

where $A(\sigma) = (a_{ij}(\sigma))_{1 \le i,j \le d_\pi}$.

Without loss of generality, assume that $K \subset \mathbf{R}$, if K has a real embedding. Choose $\xi(1)$ as above and let $\xi(\sigma) = w_\sigma \xi(1)$. Then for each real σ, $\xi(\sigma)$ satisfies our hypothesis, and $\rho\xi(\sigma) = \xi(\rho\sigma)$ if σ is complete. Now, for any Artin motive π, defined over K, and any $\tau \in G_{\mathbf{Q}}$, the map $\tau : \pi_B \to (\tau\pi)_B$ is the restriction of $\tau : \pi_{DR} \otimes_K \overline{K} \to (\tau\pi)_{DR} \otimes_{\tau K} \overline{\tau K}$, defined by $\tau(\mu \otimes k) = \tau\mu \otimes \tau k$ for $\mu \in \pi_{DR}$ and $k \in \overline{K}$. Thus,

$$\xi_i(\sigma) = \sum_{i=1}^{d_\pi} (1 \otimes w_\sigma) a_{ij}(1) \cdot w_\sigma \mu_j.$$

Hence $\det A(\sigma) = (1 \otimes w_\sigma) \det A(1)$, and so

$$\prod_{\sigma \in J_K} \det(\Delta_\sigma^\pm) \sim \prod_{\sigma \in J_K} (\sigma\delta^\pm)((1 \otimes w_\sigma) \det(A(1))).$$

As to $\det(A(1))$, note that $(1 \otimes \tau)(\xi_1(1) \wedge \cdots \wedge \xi_{d_\pi}(1)) = \det(\pi)\, \xi_1(1) \wedge \cdots \wedge \xi_{d_\pi}(1)$ for $\tau \in G_K$, and so, from the definition of π_{DR},

$$(1 \otimes \tau) \det(A(1)) = ((\det(\pi))(\tau) \otimes 1) \det(A(1)).$$

Setting $\det(A(1)) = p(\det(\pi))$, as in the proof of Prop. M.13, and recalling that argument, we are done. □

M.17. Applications.

Corollary 1. *Suppose that K is totally complex. Then*

$$c^\pm(M \otimes \pi) \sim c^\pm(M)^{d_\pi} c(\delta_*^\pm, \det \pi) \qquad \bmod (T^*).$$

Proof. Apply Theorem M.16. and Prop. M.9. □

For the next applications, let L be a finite extension of K. For each $\sigma \in J_K$, let $r(\sigma)$ the number of real embeddings of L which extend σ, and define $\varepsilon_{L/K} : J_K \to \mathbf{Z}$ by

$$\varepsilon_{L/K}(\sigma) = \begin{cases} \frac{[L:K]+r(\sigma)}{2} & \sigma \text{ real} \\ [L:K] & \sigma \text{ not real.} \end{cases}$$

Let $M \times_K L$ denote the motive over L obtained from M by extending scalars.

Corollary 2. *Suppose that $M \times_K L$ is special. Let $\pi_{L/K}$ be the character of G_K which gives the sign of the permutation given by the action of G_K on G_K/G_L. Then, with $\varepsilon = \varepsilon_{L/K}$,*

$$c^\pm(M \times_K L) \sim c_\varepsilon^\pm(M) c(\delta_*^\pm, \pi_{L/K}) \qquad \bmod (T^*).$$

Corollary 3. *Suppose that K is totally complex. Then*

$$c^{\pm}(M \ \times_K \ L) \ \sim \ c^{\pm}(M)^{[L:K]}c(\delta_*^{\pm}, \pi_{L/K}) \qquad mod \ (T^*).$$

Proof. Apply Corollary 2 and Prop. M.9. □

It is easy to give the quantities $c(\delta_*^{\pm}, \pi_{L/K})$ an explicit form. Let $\ell_1, \ldots, \ell_t$ be a basis of L as a K vector space. Put

$$p_{L/K} \ = \ \det(\sigma(\ell_i))_{1 \leq i \leq t, \sigma \ \in \ G_K/G_L}.$$

This determinant is well defined up to an element of K^*, and $\tau p_{L/K} \ = \ \pi_{L/K}(\tau)p_{L/K}$. As in the proof of Prop. M.13, we can put

$$c(\delta_*^{\pm}, \pi_{L/K}) \ = \ \prod_{\sigma \in J_K} (\sigma\delta^{\pm})((1 \ \otimes \ w_\sigma)(1 \ \otimes \ p_{L/K})).$$

Suppose next that $\delta^{\pm}$ is a multiple $m^{\pm}$ of the regular representation of T. Then, for $\varphi : \ G_K \ \to \ T^*$, the reciprocity law is

$$(1 \ \otimes \ \tau)c^{\pm}(\delta_*^{\pm}, \varphi) \ = \ (\varphi \ \circ \ tr_{K/\mathbf{Q}}(\tau)^{m^{\pm}} \ \otimes \ 1)c^{\pm}(\delta_*^{\pm}, \varphi)$$

where $tr_{K/\mathbf{Q}} : \ G_{\mathbf{Q}} \ \to \ G_K^{ab}$ is the transfer homomorphism. Let $\widetilde{\varphi} : \ K_f^* \ \to \ T^*$ be the character of K_f^* associated to φ via r_K. Let $\psi : \ K_f \ \to \ \mathbf{C}^*$ be a non-trivial additive character. Define

$$G(\varphi) \ = \ G(\varphi, \psi) \ = \ \int_{U_K} \widetilde{\varphi}(u) \ \otimes \ \psi(u)du \ \in \ (T \ \otimes \ \mathbf{Q}_{ab})^*$$

with the Haar measure du on K_f which assigns measure 1 to the integral adeles. Then the identity

$$(1 \ \otimes \ r_{\mathbf{Q}}(z))G(\varphi) \ \sim \ (\widetilde{\varphi}(z) \ \otimes \ 1)G(\varphi)$$

is immediate, provided we recall that $r_{\mathbf{Q}}(z)(e^{2\pi i/m}) \ = \ e^{2\pi i a/m}$ where $a \ \in \ (\mathbf{Z}/(m))^*$ is the inverse of the image of z. Since the diagram

$$\begin{array}{ccc} \mathbf{Q}_{\mathbf{A}}^* & \longrightarrow & G_{\mathbf{Q}}^{ab} \\ \downarrow & & \downarrow \mathrm{tr}_{K/\mathbf{Q}} \\ K_{\mathbf{A}}^* & \longrightarrow & G_K^{ab} \end{array}$$

commutes, we see that

$$(1 \ \otimes \ \tau)G(\varphi) \ = \ (\varphi \ \circ \ \mathrm{tr}_{K/\mathbf{Q}}(\tau) \ \otimes \ 1)G(\varphi)$$

for all $\tau \in G_{\mathbf{Q}}$. Thus we have

$$c(\delta_*^{\pm}, \varphi) \sim G(\varphi)^{m^{\pm}} \qquad \text{mod } (T^*)$$

for $\delta^{\pm} = m^{\pm}$ times the regular representation.

Corollary 4. *Let K have a real place. Then*

$$c^{\pm}(M \otimes \pi) \sim c^{\pm}_{\varepsilon_\pi}(M)(1 \otimes D_K^{1/2})^{d^{\pm\alpha} \cdot d_\pi} G(\det(\pi))^{m^{\pm\alpha}} \qquad \text{mod } (T^*)$$

where $\alpha = +1$ unless $M^{w/2,w/2} \neq \{0\}$, in which case α is the scalar by which ρ acts on π.

Proof. $\delta^{\pm}$ is $m^{\pm}$ times the regular representation by Prop. M.5.2, since K contains no CM subfield. □

Corollary 5. *Let K be totally real. Let L be a totally complex finite extension of K of degree $2a$. Suppose that $M^{w/2,w/2} = \{0\}$ if $w \in \lambda\mathbf{Z}$. Then $M \times_K L$ is special, and*

$$c^{\pm}(M \times_K L) \sim (c^+(M)c^-(M))^a G(\pi_{L/K})^{m^{\pm}} \qquad \text{mod } (T^*).$$

Corollary 6. *Let K be totally real. Let L be a totally real extension of K of degree a. Then $M \times_K L$ is special, and*

$$c^{\pm}(M \times_K L) \sim c^{\pm}(M)^a G(\pi_{L/K})^{m^{\pm}} \qquad \text{mod } (T^*).$$

Example. Let L be a cubic extension of $\mathbf{Q}$. Then either L is cyclic over $\mathbf{Q}$ or N, the Galois closure of L, has for its Galois group the symmetric group on 3 letters. In the first case, $\pi_{L/\mathbf{Q}} \equiv 1$. In the second, there is a unique quadratic extension of $\mathbf{Q}$ contained in M such that $\pi_{L/\mathbf{Q}}$ is the non-trivial quadratic character of $\mathrm{Gal}(N/\mathbf{Q})$ associated to this extension. L is either totally real, or has one real and one complex place. Hence, Theorem M.16 provides the following possibilities:

$$c^{\pm}(M \times_{\mathbf{Q}} L) \sim \begin{cases} c^{\pm}(M)^3(1 \otimes D_{L/\mathbf{Q}}^{1/2})^{d^{\pm}} & (L \text{ totally real, non-cyclic}), \\ c^{\pm}(M)^3 & (L \text{ totally real, cyclic}), \\ c^{\pm}(M)^2 c^{\mp}(1 \otimes D_{L/\mathbf{Q}}^{1/2})^{d^{\pm}} & (L \text{ not totally real}). \end{cases}$$

L. L-functions.

L.1. For a finite place v of K, let D_v, I_v, and Φ_v denote a decomposition group, an inertia subgroup of D_v, and a (geometric) Frobenius coset in D_v/I_v, respectively. In order to define L-functions we henceforth assume that each structure on M as motive with coefficients in T is *strictly compatible*: For each finite plane λ of T, the polynomial

$$L_v(M,T;X)^{-1} = \det_T(1 - \Phi_v X, M_\lambda^{I_v}) \in T_\lambda[X]$$

belongs to $T[X]$ and is independent of the choice of λ. Let $L_v(M,T;s)$ be the image of L_v under the map $T[X] \to T \otimes \mathbf{C}[N_v^{-s}]$ where N_v denotes the norm of v and $s \in \mathbf{C}$. Then define

$$L(M,T;s) = \prod_v L_v(M,T;s).$$

If we assume the Riemann hypothesis for M, i.e. that the roots of $N_{T/\mathbf{Q}}(L_v(M,T;\ X))^{-1} \in \mathbf{Q}[X]$ all have absolute value $(N_v)^{w/2}$, then $L(M,T;s)$ converges for $\mathrm{Re}(s) > \frac{w+1}{2}$ and takes values, for such s, in $T \otimes \mathbf{C}$. It is standard to conjecture that $L(M,T;s)$ continues to a meromorphic function on $\mathbf{C}$. Further, if M is simple, the continuation should be entire, unless $M = \mathbf{Q}(-w/2)$, in which case $L(M,\mathbf{Q};s) = \zeta_K(s - w/2)$ where ζ_K denotes the Dedekind zeta function of K.

L.2. The various $L(M,T;s)$ attached to M for different coefficient structures are related by the following elementary result.

Proposition. *Let M be a simple motive defined over K with coefficients in T. Let F be the center of the division algebra* $\mathrm{End}(M)$. *Then there exists a unique Dirichlet series $L_0(M,s) = \prod_v L_{0,v}(M,s)$, $L_{0,v}(M,s) = L_{0,v}(M,X)|_{X=N_v^{-s}}$ with $L_{0,v}(M,X)^{-1} \in F[X]$, such that*

$$L(M,T;s) = N_{TF/T}(L_0(M,s))^{n/d}$$

where $n^2 = [\mathrm{End}(M) : F]$ and $d = [TF : F]$. In particular, for any $T \supseteq F$ with $[T:F] = n$, $L(M,T;s) = L_0(M,s)$.

Proof. Suppose first that $T \supseteq F$, $[T:F] = n$, and let $T' \subseteq T$ be a Galois extension of F. Let v and λ be finite planes of K and F, respectively, whose restrictions to $\mathbf{Q}$ are distinct. Then

$$Tr_{T\otimes F_\lambda}(\Phi_v, M_\lambda^{I_v}) = Tr_{T\otimes_F T'\otimes F_\lambda}(\Phi_v, M_\lambda \otimes_F T') =$$
$$Tr_{(T')^n\otimes_F F_\lambda}(\Phi_v, (N_\lambda^{I_v})^n) = Tr_{T'\otimes F_\lambda}(\Phi_v, N_\lambda^{I_v}) \in T' \subseteq (T')^n,$$

where $N_\lambda = e(M_\lambda \otimes_F T')$ for a minimal idempotent of $T \otimes_F T'$. The first term belongs to $T \otimes 1$ while the last term belongs to $1 \otimes T'$. Since $T \otimes 1 \cap 1 \otimes T' = F$ (inside $T \otimes_F T'$), $L(M,T;s)$ is an Euler product formed of polynomials with coefficients in F. Now $\mathrm{End}(M \otimes_F T') = \mathrm{End}(M) \otimes_F T' = M_n(T')$ and $T \otimes_F T' \simeq (T')^n \hookrightarrow M_n(T')$ is a maximal semisimple commutative subalgebra. Hence, N_λ is independent of the choice of T, and depends only upon the choice of T' which splits $\mathrm{End}(M)$. Thus $L(M,T;s) = L(M;T_1;s)$ if $T_1 \supseteq F$ is any coefficient structure with $[T_1 : F] = n$.

Put $L_0(M,s) = L(M,T;s)$, with T as above. Next let T_2 be any coefficient structure for M, and assume, changing T if necessary, that $T \supseteq T_2$. Then $L(M,T_2;s) = N_{T/T_2}(L_0(M,s)) = N_{T_2F/T_2}(N_{T/T_2F}(L_0(M,s))) = N_{T_2F/T_2}(L_0(M,s))^{n/d}$ with $d = [T_2F : F]$, since $[T : T_2F] = [T : F][T_2F : F]^{-1}$. Q.E.D. □

L.3. We say that M satisfies the *Tate conjecture* if, for each prime ℓ, the $\mathbf{Q}_\ell$ subalgebra of $\mathrm{End}(M_\ell)$ generated by the image of G_K is the commutant of $\mathrm{End}(M) \otimes \mathbf{Q}_\ell$.

Proposition. *Let M be a simple motive over K with coefficients in T. Let F be the center of* End(M)*. Let F^T be the field attached to the subgroup of $G_{\mathbf{Q}}$ which stabilizes G_TG_F/G_F. Then* i) *the polynomials $L_v(M,T;X)^{-1}$ have coefficients in F^T and,* ii) *if M satisfies the Tate conjecture, the coefficients of the $L_v(M,T;X)^{-1}$ generate F^T.*

Proof. From the previous proposition, we see that the coefficients lie in the field generated by the elements $Tr_{TF/T}(\alpha)$, where $\alpha \in F$ varies among the coefficients of the $L_{0,v}(M,X)^{-1}$. Since the map $G_T/G_{TF} \to G_{\mathbf{Q}}/G_F$ is injective, with image G_TG_F/G_F, this field is F^T. □

To see ii), let A be the commutant of $\mathrm{End}(M)$ inside M_B. Then the center of A is F. By assumption, and the Cebotarev density theorem, the $\mathbf{Q}_\ell$ span of the Frobenius elements Φ_v is $A \otimes \mathbf{Q}_\ell$, for any prime ℓ. Hence the $\mathbf{Q}_\ell$ span of their reduced traces $Tr_{A/F} \otimes 1 : Q \otimes \mathbf{Q}_\ell \to F \otimes \mathbf{Q}_\ell$ is $F \otimes \mathbf{Q}_\ell$. Let $F_0 \subseteq F$ be the field generated over $\mathbf{Q}$ by the coefficients of $L_0(M,s)$. Then F_0 coincides with the field generated by the above traces. Thus $F_0 \otimes \mathbf{Q}_\ell = F \otimes \mathbf{Q}_\ell$, for all ℓ. Hence $F = F_0$. Now it follows from the analysis above for part i) that the coefficients of the $L_v(M,T;X)$ generate F^T. The last claim is obvious.

L.4.

Remarks. There exists an abelian variety A defined over $\mathbf{Q}$ for which $\mathrm{End}(A) \otimes \mathbf{Q}$ is a non-commutative division algebra. Set $M = H^1(A)$.

Then $L(M,s) = N_{F/\mathbf{Q}}(L_0(M,s))^n$ where $n^2 = [\mathrm{End}(A) \otimes \mathbf{Q} : F]$ and F is the center of $\mathrm{End}(M)$. $L(M,s)$ is the usual Hasse-Weil L-function in degree one of A. $N_{F/\mathbf{Q}}(L_0(M,s))$ cannot itself occur as $L(N,s)$ for $N = H^1(B)$ with an abelian variety B, because then $L(N,s)^n = L(M,s)$, and so by Tate's isogeny conjecture, proved for the motives attached to H^1 of abelian varieties, A is isogenous to B^n, contrary to hypothesis.

L.5. Critical strip. Let M be a critical motive defined over K, and put $N = R_{K/\mathbf{Q}}M$. If $N_B \otimes \mathbf{C} = N^{w/2,w/2}$ with $w \in 2\mathbf{Z}$, put $I_1 = \mathbf{Z}$. Otherwise, put $I_1 = \{P+1, \ldots, Q\}$ with

$$P = \max_{p<q}\{p | N^{p,q} \neq 0\}.$$

Recall that if $N^{w/2,w/2} \neq 0$, then F_∞ acts on $N^{w/2,w/2}$ as a scalar $(-1)^\varepsilon$ for $\varepsilon = 0$ or 1. The *critical strip* for M is $C(M) = C_\ell(M) \cup C_r(M)$ where

$$C_r(M) = \left\{\lambda \in I_1 \mid \lambda \leq \frac{w}{2} \text{ and } \lambda \not\equiv \varepsilon(2)\right\}.$$
$$C_\ell(M) = \left\{\lambda \in I_1 \mid \lambda > \frac{w}{2} \text{ and } \lambda \equiv \varepsilon(2)\right\}.$$

Recall (M.6) that we have attached to M a pair of basic periods $c_0^\pm(M)$.

L.6. The following conjectures are due to Deligne [**D**].

Conjecture 1. *Let M be a simple special motive defined over K. Then, for each $k \in C(M)$,*

$$L_0(M,k) \sim (1 \otimes 2\pi i)^{ke_\alpha/n} c_0^\alpha(M) \qquad \mathrm{mod}\ (F^*)$$

where $\alpha = (-1)^k$, F is the center of $\mathrm{End}(M)$, $[\mathrm{End}(M) : F] = n^2$, *and* $e_\alpha = \dim_F((R_{K/\mathbf{Q}}M)_B^\alpha)$.

Conjecture 1 implies the following assertions.

Conjecture 2. *Let M be a special motive over K with coefficients in T. Then, for each $k \in C(M)$,*

$$L(M,T;k) \sim (1 \otimes 2\pi i)^{kd_\alpha} c^\alpha(M,T) \qquad \mathrm{mod}\ (T^*)$$

where $d_\alpha = \dim_T((R_{K/\mathbf{Q}}M)_B^\alpha)$.

Conjecture 3. *If M is K-simple, and $k \in C(M)$,*

$$L(M,T;k) \sim (1 \otimes 2\pi i)^{kd_\alpha} c^\alpha(M,T) \qquad \mathrm{mod}\ ((F^T)^*).$$

L.7. Results. Conjecture 2 is known when $M = M(\chi)$ is the motive of CM type attached to an algebraic Hecke character χ of a CM field K, by the principal result of [**CV**]. A method of Harder ([**H**]) will establish the theorem for general K, by reduction to the former case. See H.12 below. Also, the truth of Conjecture 2, for all T coefficient structures on M implies Conjecture 1 for M, by an easy argument. Thus, all three conjectures are known in this case. Conjecture 2 is also known for the motives attached to classical holomorphic modular forms ([**D**]) and for the tensor product of two such motives ([**Sh1**], [**BO**]). For a partial result in the case of a triple tensor product, see [**BO**].

L.8. Main conjecture. Let π be an Artin motive with coefficients in T. From Conjecture L.6.2 and Theorem M.16, we obtain:

Conjecture 4. *Let $k \in C(M) \cap C(M \otimes \pi)$, then*

$$L(M \otimes \pi, T; k) \sim b^m(1 \otimes 2\pi i)^{kd_\alpha(M\otimes\pi)} c^\alpha_{\varepsilon_\pi}(M) c(\delta^\alpha_*, \det(\pi)) \mod (T^*)$$

where $\alpha = (-1)^k$, where $b = b(\delta^\pm)$ unless $M_B^{w/2,w/2} \neq \{0\}$, in which case $b = b(\delta^{\alpha sgn(\pi)})$.

This conjecture is known for Hecke L-series provided $\dim_T \pi = 1$, and results compatible with this conjecture have been obtained by Shimura in [**Sh1**]. If M is K simple with $\dim_T M > 1$, then for all cases in which it is known, $\delta^\pm$ is a multiple of the regular representation.

L.9.1. Automorphic L-functions. Fix K and let $\prod$ be a cuspidal automorphic representation of $GL_N(\mathbb{A}_K)$ whose infinite part $\pi_\infty = \underset{v\in P_K}{\otimes} \pi_v$ is algebraic in the sense of Clozel ([**C**]). It is now standard to conjecture (c.f. [**C**]) that there exists a motive M defined over K, with coefficients in some field T, such that

$$e_1(L_v(M, T; s)) = L_v(\pi, s)$$

for all places v of K; here $e_1 : T \otimes \mathbb{C} \to \mathbb{C}$ is the projection determined by $1_T \in J_T$. It is thus reasonable to ask for a reformulation of L.8 in purely automorphic terms, invoking additional hypotheses as needed.

L.9.2. Let $v \in P_K$ and let W_v denote the Weil group of K_v. To each $\prod_v$ is attached a representation

$$R(\pi_v) : W_v \to GL_N(\mathbb{C})$$

whose isomorphism class we denote by $[R(\prod_v)]$; the restriction of $R(\prod_v)$ to $R^* > 0 \subseteq W_v$ is a scalar $c \mapsto r^{-w}$ for $w \in \mathbb{Z}$ which is independent of v. For

v a complex place of K, let $\sigma \in J_K$ be an embedding determining v. Then σ determines an isomorphism

$$\sigma^{-1} : \mathbb{C}^* \xrightarrow{\sim} K_v^*.$$

We can write

$$R\left(\prod{}_v\right) \cdot \sigma^{-1} \xrightarrow{\sim} \operatorname{Diag}(z^{-a_1(\sigma)}\overline{z}^{-(w-a_1(\sigma))}, \ldots, z^{-a_N(\sigma)}\overline{z}^{-(w-a_N(\sigma))})$$

with integers $a_i(\sigma)$.

On the other hand, if $v = \sigma$ is real, the class of $R(\prod_v) \circ \widetilde{\sigma}^{-1} : \mathbb{C}^* \to GL_N(\mathbb{C})$ is independent of the choice of isomorphism $\widetilde{\sigma}^{-1} : \mathbb{C} \xrightarrow{\sim} \overline{K}_v$, and $R(\prod_v) \circ \widetilde{\sigma}^{-1}$ can be diagonalized as above.

L.9.3. Our first task is to define the critical strip $C(\prod)$ of $\prod$. To do this, let

$$k_{\min} = 1 + \max_{\sigma,j} \{a_j(\sigma) \mid a_j(\sigma) < w/2\}$$
$$d = w - 2k_{\min} + 1$$

and let

$$C_1\left(\prod\right) = \{k_{\min},\ w - k_{\min} + 1\}.$$

If there is a σ and a j for which $a_j(\sigma) = \frac{w}{2}$, then $C(\prod) = \emptyset$ unless K is totally real and a signature condition is satisfied: For each 1-dimensional factor χ of $R(\prod_v)$, the sign $\chi(j)(=\pm 1)$ is independent of v and χ in $R(\prod_v)$. (Here $j \in W_v$ satisfies $j^2 = -1$.) We denote this sign by $\operatorname{sgn}(\prod)$ when it is defined.

Now let

$$C_\ell\left(\prod\right) = \left\{k \in C_1\left(\prod\right) \Big|\, k < \left[\frac{w}{2}\right],\ \operatorname{sgn}\left(\prod\right) = (-1)^{k+1}\right\}$$
$$C_r\left(\prod\right) = \left\{k \in C_1\left(\prod\right) \Big|\, k \geq \left[\frac{w}{2}\right],\ \operatorname{sgn}\left(\prod\right) = (-1)^{k}\right\}.$$

Then $C(\prod) = C_1(\prod)$ if no $a_j(\sigma) = \frac{w}{2}$, $= C_1(\prod) \cap C_\ell(\prod) \cap C_r(\prod)$ if some $a_j(\sigma) = \frac{w}{2}$, K totally real, and $\operatorname{sgn}(\prod)$ is defined, $= \emptyset$ otherwise.

L.9.4. Definition of $\delta_*^{\pm}(\prod)$, $d^{\pm}(\prod)$. Let

$$d^{\pm}\left(\prod\right) = \frac{1}{2} \sum_{v \in P_{K,\mathbb{R}}} \left(N \pm \operatorname{tr}\left(R\left(\prod{}_v\right)(j)\right)\right) + \sum_{v \notin P_{K,\mathbb{R}}} N.$$

Let $\delta_*^{\pm} = d^{\pm}\left(\prod\right)\left(\sum_{\sigma\in J_T}\sigma\right)$ unless K is totally complex.

If K is totally complex, $v \in P_K$, and $\sigma \in J_K$ determines v, let $f(\sigma)$ be the number of indices i $(1 \le i \le N)$ such that $a_i(\sigma) < w - a_i(\sigma)$, and define a function $F : J_T \times J_K \to \mathbb{Z}$ by $F(1,\sigma) = f(\sigma)$ and $F(\tau|_T, \tau\sigma) = F(1,\sigma)$ for all $\tau \in$ Gal$(\overline{\mathbf{Q}}/\mathbf{Q})$. Finally put

$$\delta^{\pm}(\pi) = \sum_{\sigma\in J_T} F(\sigma,1)\cdot\sigma.$$

L.9.5. Galois forms. Let π be a cuspidal algebraic representation of $GL_m(\mathbb{A}_K)$ such that $\sigma(\pi_v)$ has finite image for each v in P_K. Such a π is called *of Galois type* since $L(\pi, s)$ is conjecturally an Artin L-function. Let ω_π denote the central character of π.

Define $\varepsilon_\pi : P_K \to \mathbb{N}$ by

$$\begin{aligned} \varepsilon_\pi(v) &= m && \text{if } v \text{ complex} \\ \varepsilon_\pi(v) &= \frac{1}{2}(m + tr(R(\pi_v)(j))) && \text{if } v \text{ is real.} \end{aligned}$$

L.9.6. Tensor product $\prod\otimes\pi$. Let $L_S(\prod\otimes\pi, s)$ denote the usual Rankin product L-function of $\prod$ and π without Euler factors for $v \in S$, a finite set of places including the infinite ones. Define

$$d^{\pm}\left(\prod\otimes\pi\right) = \frac{1}{2}\sum_{v\in P_K,\mathbb{R}}\left(N_m \pm trR(\pi_v)(j)\,\mathrm{tr}\left(R\left(\prod_v\right)(j)\right)\right) + \sum_{v\notin P_K,\mathbb{R}} N_m$$

and define $C\left(\prod\otimes\pi\right)$ using the tensor product representations $R\left(\left(\prod\otimes\pi\right)_v\right) \overset{\text{def}}{=} R\left(\prod_v\right)\otimes R(\pi_v)$ instead of $R\left(\prod_v\right)$. As before, if some $R\left(\prod_v\right)$ contains an abelian representation, we must have $\varepsilon(\pi_v) = \pm m$, with a sign independent of v, if $C\left(\prod\otimes\pi\right) \neq \emptyset$. Let $\mathrm{sgn}(\pi) \in \{\pm 1\}$ be this sign.

L.9.7. Conjugates. Let $\tau \in$ Aut$(\mathbb{C})$. Then ${}^{\tau}\prod_f$ is defined, where $\prod_f$ is the "finite part" of $\prod = \prod_\infty \otimes \prod_f$. If $\prod$ is algebraic, $\prod_f$ should be definable over a finite extension of $\mathbf{Q}$ and we let $T\left(\prod\right)$ be fixed field of $H = \left\{\tau \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \mid \tau\prod_j \xrightarrow{\sim} \prod_f\right\}$. Let $L_s^*\left(\prod, s\right)$ be the $T\otimes\mathbb{C}$ valued L-function such that $e_\tau(L_s^*(\pi, s)) = L_s\left({}^{\tau}\prod_f, s\right)$ where $e_\tau : T\otimes\mathbb{C} \to \mathbb{C}$ is the projector attached to $\tau \in J_T$. Similar remarks and definitions apply to the formal product $\prod_f \otimes \pi_f$, and we let now T be any number field containing $T\left(\prod\right)T(\pi)$.

L.9.8. Conjecture. There exist quantities $c_v^{\pm}\left(\prod\right) \in \left(T\left(\prod\right)\otimes\mathbb{C}\right)^*$ $(v \in P_K)$ such that for all cuspidal π of $GL_m(\mathbb{A}_K)$ of Galois type, all $k \in C\left(\prod\right) \cap$

$C\left(\prod\otimes\pi\right)$, and any finite S containing P_K,

$$L_S^*\left(\prod\otimes\pi,k\right)\sim b(\delta_*^{\alpha sgn(\pi)})(1\otimes 2\pi i)^{kd^\alpha(\prod\otimes\pi)}C^\alpha_{\varepsilon_\pi}\left(\prod\right)C(\delta_*^\alpha,w_\pi)$$
$$\bmod\ \left(T\left(\prod\right)T(\pi)\right)^*$$

where $\alpha=(-1)^k$ and

$$C^\alpha_{\varepsilon_\pi}\left(\prod\right)=\left(\prod_{v\in P_{K,\mathbb{R}}}C_v^\alpha\left(\prod\right)^{\varepsilon_\pi(v)}C_v^{-\alpha}\left(\prod\right)^{m-\varepsilon(v)}\right)\times\left(\prod_{v\notin P_{K,\mathbb{R}}}C_v^\alpha\left(\prod\right)\right)^m.$$

H. CM Motives.

H.1. In this chapter, we employ a method of [**CV**] to derive special period relations which obtain between periods of motives of CM type with respect to T, i.e. $\dim_T M_B = 1$. Some proofs are only sketched or omitted since they are easy consequences of the methods of [**CV**] and the earlier results of this paper. Let $\chi: K_f^*\to T^*$ be an (algebraic) Hecke character (cf. [**CV**], 3.2). Recall that from χ, we construct a motive $M=M(\chi)$, over K with coefficients in T, whose T linear isomorphism class depends only upon χ and T. $M(\chi)$ is special if and only if, for all $\alpha\in K^*\subset K_f^*$, putting

$$(*)\qquad\qquad \chi(\alpha)=\prod_{\sigma\in J_K}\sigma(\alpha)^{n(\sigma)},$$

we have $n(\sigma)\neq n(\rho\sigma)$ for any $\sigma\in J_K$. As in [**CV**], (5.1), we say that χ is critical if $k=0$ belongs to $C(M)$. Assuming χ is critical, put $c^\pm(\chi)=c^\pm(M)$, and $L(\chi,k)=L(M,T;k)$. Here $L(M,T;s)$ is simply to $T\otimes\mathbf{C}$ valued series whose components are the Hecke L-series $L(\eta\chi,s)$ for $\eta\in J_T$. Write $L(\chi)=L(\chi,0)$.

H.2. Assume that K is not totally real until H.5. Let $K_{CM}\subseteq K$ be the maximal CM subfield. Define $\psi\in I_K$ by $\psi(\sigma)=n(\sigma)$, with the $n(\sigma)$ as in H.1. Then $\psi\in I_{K_{CM}}\subseteq I_K$. If χ is critical, then either ψ or $1-\psi$ (with $1(\sigma)=1$ for all $\sigma\in J_K$) equals $w\Phi+\mu\rho-\mu$ where $\mu,\Phi\in I_{K_{CM}}$, $\Phi(\sigma)+\Phi(\rho\sigma)=1$, $\Phi(\sigma)\geq 0$ and $\mu(\sigma)\geq 0$ for all σ *in* J_K, and $\mu(\sigma)=0$ unless $\Phi(\sigma)=1$. Let E be the field attached to the group $H_\Phi\subseteq G_{\mathbf{Q}}$, with H_Φ as in M.11. Then $\Phi_*\in I_E$ is defined. Let $r_{\Phi_*}: E_f^*\to K_f^*$ be defined by $r_{\Phi_*}=\det(\Phi_*)$. For $\eta\in J_E$ and $\tau\in G_{\mathbf{Q}}$, let $\varepsilon_{\eta\Phi}(\tau)\in\{1,-1\}$ be the sign of the permutation of $<1,\rho>\backslash J_K$ obtained via the composition

$$\langle 1,\rho\rangle\backslash J_K\simeq|\eta\Phi|\simeq|\tau\eta\Phi|\simeq\langle 1,\rho\rangle\backslash J_K$$

where $|\ \ |$ denotes support. Note that $\varepsilon_\Phi : G_E \to \{1, -1\}$ is a character. For $\eta \in J_T$, let $e_\Phi(\eta) = (-1)^{t(\eta)}$ where $t(\eta)$ is the number of elements in $|\rho\Phi| \cap |\eta\Phi| \subseteq J_K$. Let $\chi_a : K_f^* \to T_f^*$ be the map defined by $(*)$.

H.3. The following result summarizes a basic construction of [**CV**].

Theorem. *Let χ be a critical Hecke character of K whose values on K_f^* lie in T. Let N be a motive defined over E with coefficients in T attached to the algebraic Hecke character $\chi \circ r_{\Phi_*} \cdot \varepsilon_\Phi$. Then*

a) *there exists a collection $\{\gamma_\eta^\pm \mid \eta \in J_E, 0 \neq \gamma_\eta^\pm \in (\eta N)_B\}$ such that for all $\eta \in J_E$, $\tau \in G_{\mathbf{Q}}$, and $k \in K_f^*$ satisfying $r_K(k) = \tau$ on K_{ab},*

$$\tau(I_f(\gamma_\eta^\pm)) = \chi_a(k)^{-1}\chi(k)\varepsilon_{\eta\Phi}(\tau)e_\Phi(\eta)e_\Phi(\tau\eta)I_f(\gamma_{\tau\eta}^\pm)$$

this collection is uniquely characterized by this formula up to a change $\gamma_\eta^\pm \to t\gamma_\eta^\pm$ for a $t \in T^*$ which is independent of $\eta \in J_E$ and the choice of sign.

b) *Put*

$$\gamma^\pm = \sum_{\eta \in J_E} \gamma_\eta^\pm$$

and define $F^* = F^s(R_{E/\mathbf{Q}}N)_{DR}$ where $F^s \neq (R_{E/\mathbf{Q}}N)_{DR}$ but $F^{s-1}(R_{E/\mathbf{Q}}N)_{DR} = (R_{E/\mathbf{Q}}N)_{DR}$. Then $\dim_T(R_{E/\mathbf{Q}}N)_{DR}/F^* = 1$, and if $I^* : (R_{E/\mathbf{Q}}N)_B \otimes \mathbf{C} \to ((R_{E/\mathbf{Q}}N)_{DR}/F^*) \otimes \mathbf{C}$ denotes the map constructed from I_∞, we have, for $0 \neq \omega \in (R_{E/\mathbf{Q}}N)_{DR}/F^*$

$$I^*(\gamma^\pm) \sim c^\pm(\chi) \cdot \omega \qquad \mathrm{mod}\ (T^*).$$

Proof. The proof is given, in [**CV**], 4 and 5, for an analogous result for the dual motive, $\{\gamma_\eta^+ \mid \eta \in J_E\}$, and where K is a CM field. For the case here, the construction of $\{\gamma_\eta^- \mid \eta \in J_E\}$ and $c^-(\chi)$ follows by the same method, if we use elements $\gamma_\sigma - \gamma_{\rho\sigma}$ $(\sigma \in J_K)$ starting from (5.2.4) of [**CV**]. For general K, the proof is the same as in [**CV**], but employs the $\varepsilon_{\eta\Phi}$ where [**CV**] employed a simpler character $\sim \varepsilon$ of $G_{\mathbf{Q}}$. □

H.4.

Proposition. *Let M be a motive associated to the χ of Theorem H.3. Then*

$$c^+(\chi) \sim e_\Phi \cdot c^-(\chi)$$

where $e_\Phi = \{e_\Phi(\eta) \mid \eta \in J_E\} \in E \otimes \overline{\mathbf{Q}} \subseteq T \otimes \overline{\mathbf{Q}}$.

Proof. From the construction of the γ_η out of vectors γ_σ, $\sigma \in J_K$, we see at once that $\gamma_\eta^- = e_{\Phi(\eta)}\gamma_\eta^+$, and the result follows from H.3. □

H.5. Let $C(\chi) = C(M(\chi))$, the critical strip.

Corollary. *For $m, n \in C(\chi)$,*

$$(1 \otimes 2\pi i)^{-mg} L(\chi, m) \sim (1 \otimes 2\pi i)^{-ng} L(\chi, n) e_{\Phi}^{(m-n)},$$

where $g = \frac{1}{2}[K : \mathbf{Q}]$, unless K is totally real, when $g = [K : \mathbf{Q}]$.

H.6. If K is totally real, let $\Phi \in I_K$ be the regular representation. Let $\varepsilon_\Phi : G_{\mathbf{Q}} \to \{1, -1\}$ be defined by $\tau D_F^{1/2} = \varepsilon_\Phi(\tau) D_F^{1/2}$. Let $c(K, \Phi) \in (T \otimes \overline{\mathbf{Q}})^*$ satisfy $\tau c(K, \Phi)_\eta = \varepsilon_{\eta\Phi}(\tau) c(K, \Phi)_{\tau\eta}$ for all $\tau \in G_{\mathbf{Q}}$ and $\eta \in J_T$.

Proposition. *Let $0 < \eta \in \mathbf{Z}$, and let χ be a critical Hecke character. Then*

$$c^{\pm}(\chi^n) \sim c^{\pm}(\chi)^n c(K, \Phi)^{n-1} \qquad \mathrm{mod}\ (T^*).$$

Proof. Suppose that K is not totally real. If $c^{\pm}(\chi)$ is defined via $\{\gamma_\eta^{\pm} \mid \eta \in J_E\}$ and $\omega \in (R_{E/\mathbf{Q}} N)_{DR}/F^*$, as in Theorem H.3, then $c^{\pm}(\chi)^n$ is defined using $\{\gamma_\eta^{\pm \otimes n} \mid \eta \in J_E\}$ and $\omega^{\otimes n}$ on $R_{E/\mathbf{Q}} N^{\otimes n}$. On the other hand, let N_n be the motive attached to χ^n as in Theorem H.3. Then N_n is attached to the character $\chi^n \circ r_{\Phi_*} \cdot \varepsilon_\Phi$, and $c^{\pm}(\chi^n)$ is defined via a system $\{\gamma_\eta^{\pm}(n) \mid \eta \in J_E\}$ and w_n. The map sending $\gamma_\eta^{\pm \otimes n}$ to $\gamma_{\eta}^{\pm}(n)$ is a T-linear isomorphism $\lambda^{\pm} : R_{E/\mathbf{Q}}(N^n) \times \overline{\mathbf{Q}} \to R_{E/\mathbf{Q}}(N_n) \times \overline{\mathbf{Q}}$. Since $N^{\otimes n}$ is attached to the character $\chi^n \circ r_{\Phi_*} \cdot \varepsilon_\Phi^n$, $\lambda(\omega^{\otimes n}) \sim c(K, \Phi)^n \cdot \omega_n$, and the claim is proved. If K is totally real, the result is an elementary calculation. □

H.7.

Corollary. *Let χ be a critical Hecke chracter and let $0 < n \in \mathbf{Z}$. Then*

$$L(\chi^n) \sim L(\chi)^n c(K, \Phi)^{n-1} \qquad \mathrm{mod}\ (T^*).$$

H.8.

Proposition. *Let χ be a critical Hecke character of K. Suppose that L is a finite extension of K for which $\chi \circ N_{L/K}$ is critical. Then*

$$c^{\pm}(\chi \circ N_{L/K}) \sim c^{\pm}(\chi)^{[L:K]} c(\Phi, \pi_{L/K})$$

with $\pi_{L/K} : G_K \to \{1, -1\}$ as in M.17.

Proof. The proposition just restates M.17, Corollary 3 and Corollary 6, since these are the only possible cases. We use here that $\Phi = \delta_*^{\pm}$. □

See M.17., below Corollary 3, for a discussion of $c(\Phi, \pi_{L/K})$.

H.9.

Corollary. *Let χ be a critical Hecke character of K, and assume that Conjecture L.6.2 holds for $M(\chi)$ and $M(\chi \circ N_{L/K})$, with a finite extension L of K for which $\chi \circ N_{L/K}$ is critical. Then*

$$L(\chi \circ N_{L/K}) \sim L(\chi)^{[L:K]} c(\Phi, \pi_{L/K}).$$

H.10. Now let L be a subfield of K such that $\chi_a \in I_K$ lies in the image of I_L. Then:

i) If χ is critical, $\chi|_{L_f^*}$ is also critical.

ii) The following formula holds:

$$c^{\pm}(\chi) \sim c^{\pm}(\chi|_{L_f^*}) c(K, \Phi) c(L, \phi) \qquad \mod (T^*).$$

Proof. If K is complex, then, we compute $c^{\pm}(\chi)$ and $c^{\pm}(\chi|_{L_f^*})$ by means of Theorem H.3. as periods of motives $R_{E/\mathbf{Q}}(M(\chi \circ r_{\Phi_*} \varepsilon_\Phi))$ and $R_{E/\mathbf{Q}}(M(\chi \circ r_{\Phi_*} \cdot \varepsilon_{\Phi|_L}))$, respectively, where $\varepsilon_{\Phi|_L}$ is the character obtained by regarding Φ as an element of I_L. The proof now concludes as in the proof of Proposition H.6. If K is totally real, then the result follows easily from the identity $c^{\pm}(\chi) \sim c^{\pm}(\chi \circ tr_{K/\mathbf{Q}} \cdot \pi_{K/\mathbf{Q}})$ with $\pi_{K/\mathbf{Q}}$ as in M.17. □

H.11.

Remark. Let $K_{CM} = L$ be the maximal CM subfield of K, in the case where K is totally complex. Then $c(K_{CM}, \Phi) \sim 1 \otimes D_F^{1/2} \mod (T^*)$, where F is the maximal real subfield of L. It is not hard to check that

$$c(K, \Phi) \sim (1 \otimes D_F^{1/2})^{[K:L]} \beta \qquad \mod (T^*)$$

with $\beta = \{\beta_\eta \mid \eta \in J_T\}$ and

$$\beta_\eta = \prod_{\sigma \in J_F} \beta_\eta(\sigma)$$
$$\beta_\eta(\sigma) = \det(\beta_i^\tau)_{1 \le i \le [K:L], \tau \in S_{\eta,\sigma}}$$

for a basis $\beta_1, \ldots, \beta_{[K:L]}$ of K over L, and where

$$S_{\eta,\sigma} = \{\tau \in |\eta\Phi| \text{ such that } \tau|_F = \sigma\}$$

and is ordered by first imposing an order upon $\langle 1, \rho\rangle \backslash J_K$, and ordering $|\eta\Phi|$ via its image in this set.

H.12.

Theorem. *Let χ be a critical Hecke character of a totally complex field K with maximal CM subfield L. Let T be the field generated by the values of χ on K_f^*. Suppose that $[K:L] > 1$. Then Conjecture L.6.2 is true for $M(\chi)$ if and only if*

$$L(\chi) \sim L(\chi|_{L_f^*})(1 \otimes D_F^{1/2})^{([K:L]-1)}\beta \qquad \mathrm{mod}\ (T^*)$$

with the notations introduced above.

Proof. The hypothesis ensures that $L(\chi|_{L_f^*}) \in (T \otimes \mathbf{C})^*$. Since Theorem 9.3.1 of [**CV**] establishes L.6.2 for $\chi|_{L_f^*}$, the theorem follows from H.10 and H.11. □

H.13. It appears likely that the method of Harder ([**H**]) will establish H.12. The paper [**H**] treats the case where $[K:L] = 2$.

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Note: Corollary 3 in Section M.17 was missing from the paper version. Also, the references there to L.5.2 should be to L.6.2.