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L-FUNCTIONS FOR GSp_4

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Dedicated to Olga Taussky-Todd

1. Introduction.

To a classical modular cusp form $f(z)$ with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

one can associate the Dirichlet series

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Hecke showed that $L(f,s)$ converges absolutely in some right half plane, has an analytic continuation to the whole s-plane, and satisfies a nice functional equation. In addition, Hecke discovered that if $f(z)$ is an eigenfunction for the Hecke operators, then the associated Dirichlet series has an Euler product expansion

$$L(f,s) = \prod_{p \text{ prime}} \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1}.$$

Siegel introduced a theory of modular forms on the domains

$$H_n = \{Z = X + iY \in M_n(\mathbb{C}) |\, {}^tZ = Z, Y > 0\},$$

called the Siegel upper half plane of genus n. If $f(Z)$ is a Siegel modular cusp form, then $f(Z)$ has a Fourier expansion

$$f(Z) = \sum_{\substack{T>0 \\ \text{integral, symmetric}}} a_T e^{2\pi i tr(TZ)},$$

and one can associate to $f(Z)$ a Dirichlet series

$$L(f,s) = \sum_{\substack{T \text{ modulo} \\ \text{integral} \\ \text{equivalence}}} \frac{a_T}{|T|^s}.$$

Hecke's original proof of the analytic continuation and functional equation for the one dimensional case goes over with little change to the higher dimensional case, giving the analytic continuation and functional equation for Dirichlet series attached to Siegel modular cusp forms of any genus. However, Hecke's proof of the Euler product expansion for the Dirichlet series attached to eigenfunctions of the Hecke operators on the upper half plane completely fails in the higher dimensional case. It is clear that these Dirichlet series do not have Euler product expansions.

For a Siegel modular cusp form $f(Z)$ of genus $n = 2$, Andrianov suggested the following construction [1].

Define

$$L(f,T_0,s) = \sum_{m=1}^{\infty} \frac{a_{mT_0}}{m^s},$$

where T_0 is a positive definite integral symmetric matrix. He was then able to prove that $L(f,T_0,s)$ converges in some half plane, has a meromorphic continuation to the whole s-plane, and satisfies a nice functional equation. Moreover, if $f(Z)$ is an eigenfunction of the Hecke operators, then the Dirichlet series $L(f,T_0,s)$ have Euler product expansions.

At the same time, Shimura was studying modular forms of half-integral weight [15]. A modular cusp form of half-integral weight has a Fourier expansion of the usual type

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

Consider the analogue of the classical Dirichlet series attached to $f(z)$

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

As in Hecke, $L(f,s)$ has an analytic continuation to the whole s-plane and satisfies a nice functional equation. But $L(f,s)$ has an Euler product expansion only in very special cases. Then Shimura considered a family of Dirichlet series attached to $f(z)$

$$L(f,t_0,s) = \sum_{n=1}^{\infty} \frac{a_{t_0 n^2}}{n^s}$$

with t_0 a positive integer. As in the Siegel modular case, Shimura was able to prove that the $L(f, t_0, s)$ converge in some half plane, have meromorphic continuation to the whole s-plane, satisfy a nice functional equation, and all have Euler product expansions.

Both Andrianov and Shimura obtained the meromorphic continuation of their Dirichlet series by using the Rankin-Selberg method. In this method, the analytic continuation of a Dirichlet series follows from the analytic continuation of a certain Eisenstein series.

Andrianov's construction, like Hecke's, does not provide a definition of the local L- and ϵ-factors. In [**12**] we suggested a reformulation of Andrianov's construction which, like J. Tate's thesis for GL_1 or Jacquet-Langlands for GL_2, allows for the definition of L and ϵ factors for GSp_4 over local and global fields. Our construction was based on a generalization of a Whittaker model. The uniqueness of this generalized model was proved by Novodvorsky and the author in [**10**]. Later on it was considered by Rodier [**14**]. The complete proof of uniqueness in all cases was given by Novodvorsky.

The computation of the L and ϵ factors, however, was not done explicitly. This seems to be an interesting problem. One of the aims of this paper is to present the results of these computations in some cases. For instance, let π be a representation of $\mathrm{GSp}_4(k)$ induced from a Borel subgroup (k a local non-archimedean field). Denote by ${}^L\pi$ the representation of $\mathrm{GL}_4(k)$ which corresponds to π under the Langlands correspondence associated with the natural embedding of ${}^L\mathrm{GSp}_4 = \mathrm{GSp}_4(\mathbb{C})$ in ${}^L\mathrm{GL}_4 = \mathrm{GL}_4(\mathbb{C})$ [**3**]. Then we have that our L and ϵ facotrs for π are equal to the standard L and ϵ for ${}^L\pi$. Let τ be a mysterious cuspidal nongeneric representation of PGSp_4. (For a finite field this representation was found by Mrs. Srinivasan and labelled θ_{10} [**16**].) We prove that if ${}^L\tau$ exists then it should be an induced representation π_K which will be defined at the end of Section 4. During the discussion of this result with R. Langlands, he pointed out that he had suggested a different approach to the computation of ${}^L\tau$ [**9**].

There is a striking difference between L-functions for GL_n and L-functions for GSp_4. The L-function of a representation of GL_n depends only upon embeddings in representations induced from the appropriate parabolic subgroups. The same is true for generic representations of GSp_4. However, for nongeneric representations of GSp_4, embeddings in representations induced from some reductive subgroups give a contribution to the L-function. This implies that cuspidal representations of GSp_4 can have nontrivial L-functions.

In the proceedings of the Corvallis conference, there is an article by Novodvorsky where similar questions were studied by him and me [**11**]. It also contains a study of the L-function corresponding to $\pi \times \sigma$, where π is a representation of GSp_4 as before and σ a representation of GL_2.

We would like to express our gratitude to R. Howe, R. Langlands, A. Weil, and G. Zuckerman for interesting and stimulating discussions. This paper is based on two lectures given in G. D. Mostow's "Lie Group Seminar" at Yale in the late 1970's. Note taking and final editing were done by J. Cogdell, to whom we are very grateful. This article has circulated as a preprint for a number of years. We would like to thank Dinakar Ramakrishnan for giving us the opportunity to publish it in this volume dedicated to Olga Taussky-Todd.

2. $\mathbf{GSp_4}$ and its subgroups.

Let k be any field of characteristic $\neq 2$. Let $J = \left(\begin{smallmatrix} 0 & I_2 \\ -I_2 & 0 \end{smallmatrix}\right)$ be the standard skew symmetric matrix of dimension 4. Then define

$$\mathrm{GSp}_4 = \{g \in \mathrm{GL}_4(k) | {}^t g J g = \lambda J \text{ for some } \lambda \in k^\times\},$$

which is an algebraic group defined over k. We let C denote the center and B denote the standard Borel subgroup of GSp_4.

Let S denote the subgroup of GSp_4 comprised of all matrices of the form

$$\begin{pmatrix} I_2 & A \\ 0 & I_2 \end{pmatrix} \text{ with } {}^t A = A.$$

Then S is an abelian group and is equal to the unipotent radical of the parabolic subgroup

$$P = \left\{ g \in \mathrm{GSp}_4 \middle| g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.$$

Let $P = MS$ be the Levi decomposition of P, M being the reductive part.

Any homomorphism of S into k can be written in the form $s \mapsto \mathrm{tr}(\beta s)$ with $\beta \in M_2(k)$ with ${}^t\beta = \beta$. A linear form $s \mapsto \mathrm{tr}(\beta s)$ on S is called nondegenerate if $\det(\beta) \neq 0$.

Now fix a non degenerate linear form $\ell(s) = \mathrm{tr}(\beta s)$ on S. Let D denote the connected component of the stabilizer of ℓ in M. Then there is a unique semisimple algebra K over k, with $(K : k) = 2$, such that $D \simeq K^\times$ as an algebraic group over k. It is known that either $K = k \oplus k$ or K is a quadratic extension of k. The subgroup of GSp_4 which will be most important in our investigation of L-functions will be $R = DS$. We introduce

$$N = \{s \in S \mid \mathrm{tr}(\beta s) = 0\}.$$

If K is the semisimple algebra over k determined by D, as above, then let $V = K^2$ and consider the group

$$G = \{g \in \mathrm{GL}_2(K) | \det(g) \in k^\times\}.$$

We will write vectors of V in row form and let G act on the right. On V we consider the skew symmetric form

$$\rho(x, y) = \mathrm{Tr}_{K/k}(x_1 y_2 - x_2 y_1),$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are elements of V. Then it is easy to see that G preserves ρ up to a factor in $k^\times$. (More precisely, the action of $g \in G$ changes $\mathrm{Tr}_{K/k}(x_1 y_2 - x_2 y_1)$ by $\det(g) \in k^\times$.) If we then consider V as a 4 dimensional vector space over k, we get a natural embedding

$$G \hookrightarrow \mathrm{GSp}_\rho = \{g \in \mathrm{GL}_4(k) | \rho(xg, yg) = \lambda\rho(x, y)\}.$$

If we define the k-linear transformation ι on V by

$$\iota : (x_1, x_2) \mapsto (\bar{x}_1, \bar{x}_2)$$

then ι preserves ρ and gives us a well defined element of GSp_ρ. (Here $x \mapsto \bar{x}$ denotes the non-trivial automorphism of K/k.)

Proposition 2.1. *There exists an isomorphism* $\mathrm{GSp}_\rho \simeq \mathrm{GSp}_4$ *such that* $G \cap R = DN$.

We fix such an isomorphism and consider G as a subgroup of GSp_4.

Let H be the subgroup of G defined by $H = \{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) | \alpha \in k^\times\}$.

Lemma. *Let K be a field and let $L^1 = \{\ell \in \mathrm{Hom}_k(S, k) \,|\, \ell$ is a nontrivial linear functional but $\ell|_N$ is trivial$\}$. Then H acts simply transitively on L^1.*

Remark. All the groups defined above are algebraic groups defined over k and all statements made are true in this context. In particular, these constructions and results are valid for local and global fields and also for the ring of adeles.

3. Local L and ϵ factors.

In this section we let k denote a local field.

The Jacquet-Langlands construction of local L and ϵ factors for GL_2 is based on the definition of a Whittaker model. For a representation π of $\mathrm{GSp}_4(k)$ there does not always exist a Whittaker model defined with respect to the maximal unipotent subgroup. For $\mathrm{GSp}_4(k)$ we use a generalized Whittaker model defined with respect to the subgroup R introduced in Section 2.

Let ν be a character on D_k and ψ a nondegenerate character on S_k, i.e., $\psi(s) = \psi_0(\mathrm{tr}(\beta s))$ with $\det(\beta) \neq 0$, ${}^t\beta = \beta$ and ψ_0 a nontrivial character of k. Then we get a character $\alpha_{\nu,\psi}$ on R_k by defining

$$\alpha_{\nu,\psi}(r) = \alpha_{\nu,\psi}(ds) = \nu(d)\psi(s)$$

where $r = ds \in R_k$, $d \in D_k$, $s \in S_k$. We have $D_k \simeq K^\times$.

Theorem 3.1. *Let (π, V_π) be an irreducible smooth admissible preunitary representation of $\mathrm{GSp}_4(k)$ and let $\alpha_{\nu,\psi}$ be a character on R_k as above. Then there exists at most one (up to scalar multiple) linear functional $\ell : V_\pi \longrightarrow \mathbb{C}$ such that $\ell(\pi(r)\xi) = \alpha_{\nu,\psi}(r)\ell(\xi)$ for all $r \in R_k$, $\xi \in V_\pi$.*

We will call such a linear functional an $\alpha_{\nu,\psi}$-eigenfunctional. (If k is archimedean, we also require that ℓ be continuous in the C^∞-topology.)

Using the well-known Gelfand-Kazhdan method, this theorem was proved by Novodvorsky and the author for the case K a field [**10**] and for cuspidal representations by Rodier for arbitrary k [**14**]. Novodvorsky later proved the theorem for arbitrary K and nonarchimedean local fields using the same method (unpublished). The archimedean case is even simpler to handle by using the fact that in this case any representation is a quotient of a principal series representation.

R. Howe has proven results from which it follows that, if $k \neq \mathbb{C}$, then for any infinite dimensional representation (π, V_π) of $\mathrm{GSp}_4(k)$ there exists a nontrivial $\alpha_{\nu,\psi}$-eigenfunctional on V_π for an appropriate choice of ν, ψ, and R [**5**]. If $k = \mathbb{C}$, the only representations for which this fails will be the Weil representations. One can prove that these representations never occur as components of cuspidal automorphic representations.

Let (π, V_π) be an irreducible smooth representation of $\mathrm{GSp}_4(k)$ and let ℓ be an $\alpha_{\nu,\psi}$-eigenfunctional. For a vector $\xi \in V_\pi$ we define the generalized Whittaker function $W_\xi(g)$ on $\mathrm{GSp}_4(k)$ by

$$W_\xi(g) = \ell(\pi(g)\xi).$$

If we let $\mathcal{W}^{\nu,\psi}$ denote this space of functions and let $\mathrm{GSp}_4(k)$ act on $\mathcal{W}^{\nu,\psi}$ by right translations, then the representation of $\mathrm{GSp}_4(k)$ on $\mathcal{W}^{\nu,\psi}$ is equivalent to the representation π. From the defining property of ℓ we see that the functions $W_\xi(g)$ satisfy

$$W_\xi(rg) = \alpha_{\nu,\psi}(r)W_\xi(g)$$

for $r \in R_k$. We call the space $\mathcal{W}^{\nu,\psi}$ a generalized Whittaker model for π. If we let $\overline{W}_\xi(g) = W_\xi(\iota g)$, then the space $\overline{\mathcal{W}}$ comprised of the functions $\overline{W}_\xi(g)$ for $\xi \in V_\pi$ is the generalized Whittaker model associated to the character $\alpha_{\bar{\nu},\psi}$, where $\bar{\nu}$ is the character on D_k defined by $\bar{\nu}(d) = \nu(\bar{d})$.

Let $V = K^2$ be the vector space upon which the group G, defined in Section 2, acts. Let $\Phi \in S(V)$, the space of Schwartz-Bruhat functions on V, let ν be the character of $K^\times \simeq D_k$ chosen above, and let μ be a character of $k^\times$. Then for $s \in \mathbb{C}$ we define a function on G by

$$f^\Phi(g;\mu,\nu,s) = \mu(\det(g))|\det(g)|_k^{s+\frac{1}{2}} \int_{K^\times} \Phi((0,t)g)|t\bar{t}|_k^{s+\frac{1}{2}}\mu(t\bar{t})\nu(t)d^\times t.$$

Then $f^\Phi(g;\mu,\nu,s) \in \mathrm{ind}_{B'}^G(\chi)$ where B' is the Borel subgroup of G and χ is the character on B' defined by

$$\chi\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar{t} & 0 \\ 0 & t\end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}\right) = \mu(x)|x|^{\frac{1}{2}+s}\nu^{-1}(t).$$

Now let $W \in \mathcal{W}^{\nu,\psi}$. Then we define

$$\begin{aligned} L(W,\Phi,\mu,s) &= \int_{DN\backslash G} W(g) f^\Phi(g;\mu,\nu,s)\,dg \\ &= \int_{N\backslash G} W(g)\Phi((0,1)g)\mu(\det g)|\det g|_k^{s+\frac{1}{2}}\,dg. \end{aligned}$$

In the usual manner, the integral defining $L(W,\Phi,\mu,s)$ converges in some half plane $\mathrm{Re}(s) > s_0$ and has a meromorphic continuation to the whole s-plane.

As in Tate's thesis [**17**], [**8**], or [**7**], the $L(W,\Phi,\mu,s)$ admit a greatest common denominator for all $W \in \mathcal{W}^{\nu,\psi}$ and $\Phi \in S(V)$. This allows us to define the function $L(\pi,\mu,s)$ by the property that $\frac{L(W,\Phi,\mu s)}{L(\pi,\mu,s)}$ is an entire function for all $W \in \mathcal{W}^{\nu,\psi}$ and $\Phi \in S(V)$. In our definition, $L(\pi,\mu,s)$ depends upon the choice of ν, ψ, and R. It is easy to see that for a fixed R, $L(\pi,\mu,s)$ does not depend on ψ. In the most important cases we can prove that it does not depend on the choice of ν or R either. See, for example, Theorem 4.4 below and [**13**]. For simplicity we do not include ν in our notation for $L(\pi,\mu,s)$, which is called the local L-factor, or the local Euler factor, associated to the pair (π,μ).

Proposition 3.2. *There is a (local) functional equation*

$$\epsilon(\pi,\mu,\psi,s)\frac{L(W,\Phi,\mu,s)}{L(\pi,\mu,s)} = \frac{L(\overline{W},\hat{\Phi},\mu^{-1},1-s)}{L(\hat{\pi},\mu^{-1},1-s)}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ with respect to the trace form on V, $\hat{\pi}$ is the contragredient representation to π, and $\epsilon(\pi,\mu,\psi,s)$ is an entire function without zeros.

Proof. Denote by (τ, W_τ) the representation $\mathrm{ind}_{B'}^{G}\chi$ where χ is the character on B' defined by

$$\chi\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar{t} & 0 \\ 0 & t\end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}\right) = \mu(x)|x|^{\frac{1}{2}+s}\,\nu^{-1}(t).$$

Then the map $\beta : \Phi \mapsto f^{\Phi}(g;\mu,\nu,s)$ defines a homomorphism $\beta : S(V) \longrightarrow W_\tau$. Let $\mathcal{T} = \ker(\beta)$. Then $L(W,\Phi,\mu,s)$ is a bilinear G quasi-invariant functional on $V_\pi \times S(V)$ which is equal to zero for any $\Phi \in \mathcal{T}$, and hence is actually a bilinear functional on $V_\pi \times W_\tau$. Here, being G quasi-invariant means that for $(\xi, w) \in V_\pi \times W_\tau$ we have

$$(\pi(g)\xi,\ \tau(g)w) = \mu^{-1}(g)|\det g|^{-\frac{1}{2}-s}(\xi, w).$$

It is clear that $L(\overline{W},\hat{\Phi},\mu^{-1},1-s)$ can also be considered as a bilinear functional on $V_\pi \times S(V)$. Using properties of intertwining operators, it is possible to show that $L(\overline{W},\hat{\Phi},\mu^{-1},1-s)$ is zero for the same $\Phi \in \mathcal{T}$. In fact, the composition map $[\Phi \mapsto \hat{\Phi} \mapsto f^{\hat{\Phi}}(g,\mu^{-1},\bar{\nu},1-s)]$ also has kernel $\mathcal{T}$. Hence $L(\overline{W},\hat{\Phi},\mu^{-1},1-s)$ is also a bilinear G-quasi-invariant form on $V_\pi \times W_\tau$. Then, using the uniqueness of our generalized Whittaker models, it is easy to show that such bilinear G quasi-invariant linear forms on $V_\pi \times W_\tau$ are unique up to scalar multiples. Letting s vary, we obtain a meromorphic function $\gamma(\pi,\mu,\psi,s)$ such that

$$L(\overline{W},\hat{\Phi},\mu^{-1},1-s) = \gamma(\pi,\mu,\psi,s)L(W,\Phi,\mu,s).$$

Then $\epsilon(\pi,\mu,\psi,s)$ is given by

$$\epsilon(\pi,\mu,\psi,s) = \gamma(\pi,\mu,\psi,s)\frac{L(\pi,\mu,s)}{L(\hat{\pi},\mu^{-1},1-s)}.$$

□

4. Computation of the local L and ϵ factors.

In this section we restrict our attention to the case where k is a non-archimedean local field. Let $S_0(V) = \{\Phi \in S(V) | \Phi((0,0)) = 0\}$. Then we divide the poles of $L(\pi,\mu,s)$ into two types. We call a pole of $L(\pi,\mu,s)$ regular if it is a pole of some $L(W,\Phi,\mu,s)$ with $\Phi \in S_0(V)$. A pole of

$L(\pi, \mu, s)$ is called exceptional if it is not a pole of any $L(W, \Phi, \mu, s)$ when Φ is restricted to lie in $S_0(V)$.

We assume that π is in general position in the natural meaning. Consider the collection of all characters η of H such that there exists a linear functional ℓ on V_π such that

$$(*) \qquad \ell(\pi(hts)\xi) = \eta(h)|h|^{3/2}\nu(t)\ell(\xi)$$

where $h \in H, t \in D$, and $s \in S$. It is not difficult to prove that the collection Δ of characters for which this linear functional is nontrivial is independent of ν. Of course, we consider here only the ν such that for some nondegenerate character ψ of S there exists a nontrivial $\alpha_{\nu,\psi}$-eigenfunctional.

Theorem 4.1. *The regular part of $L(\pi, \mu, s)$ is equal to $\prod_{\eta\in\Delta} L(s, \eta\mu)$, where the $L(s, \eta\mu)$ are the standard Tate L-functions.*

Proof. It is easy to see that all regular poles come from the asymptotic behavior of the $W_\xi(h)$ when $h \in H$ is small.

Recall that $h \in H$ has the form $h = \begin{pmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ for $x \in k^\times$. By h small we mean x approaching 0 in k. For small h, we can write $W_\xi(h)$ in the form

$$W_\xi(h) = \Sigma_i C_i(\xi)\eta_i(h)|h|^{3/2}.$$

Then it is easy to see that the $C_i(\xi)$ are linear functionals satisfying $(*)$. □

For π not in general positiion a similar result holds [**13**].

Theorem 4.2. *If s_0 is an exceptional pole of $L(\pi, \mu, s)$, then there exists a linear functional ℓ on V_π such that*

$$\ell(\pi(g)\xi) = \mu^{-1}(\det g)|\det g|^{-s_0-\frac{1}{2}}\ell(\xi)$$

for all $g \in G, \xi, \in V_\pi$.

Proof. Assume that $L(\pi, \mu, s)$ has an exceptional pole at s_0, and assume that for some integral $L(W, \Phi, \mu, s)$ we have

$$L(W, \Phi, \mu, s) = \frac{A(W, \Phi)}{s - s_0} + \cdots .$$

Then it is obvious that A is linear in Φ and satisfies $A(W, \Phi) = 0$ if $\Phi((0, 0)) = 0$. Hence $A(W, \Phi)$ is of the form $A(W, \Phi) = \Phi((0, 0))A(W)$. It is also clear that $A(W)$ is a linear function of W. Since $L(W, \Phi, \mu, s)$ is G quasi-invariant, we see that $A(W)$ will be a linear functional of the desired type. □

It is likely that this condition is also sufficient for s_0 to be an exceptional pole of $L(\pi, \mu, s)$, but this has not been proven.

If our representation has additional structure we are sometimes able to say more about the form of $L(\pi, \mu, s)$. We recall that π is called generic if π has a standard Whittaker model, that is, a Whittaker model defined with respect to a maximal unipotent subgroup U of GSp_4.

Theorem 4.3. *If π is generic, then $L(\pi, \mu, s)$ has only regular poles.*

Proof. It suffices to show that for any generic representation π there cannot exist any linear functional on V_π which is G quasi-invariant. The proof is based on a description of the double cosets $P\backslash\mathrm{GSp}_4/G$. This set consists of exactly two double cosets. If g_0 is any representative of the double coset not containing 1, then $P \cap g_0 G g_0^{-1} \cong \mathrm{GL}_2(k)$. This fact is true over an arbitrary field k.

In particular, we can take k to be a finite field. Then if π is a generic representation, denote by ξ a standard Whittaker vector and consider $\varphi(g) = \ell(\pi(g)\xi)$ where $g \in \mathrm{GSp}_4$ and ℓ is a linear functional on V_π which is G quasi-invariant. Then it is easy to see that $\varphi(g)$ satisfies the functional equations

$$\varphi(g_1 g) = \beta(g_1)\varphi(g)$$
$$\varphi(gu) = \psi(u)\varphi(g)$$

where $g \in \mathrm{GSp}_4$, $g_1 \in G$, β a character on G, $u \in U$, a maximal unipotent subgroup of GSp_4, and ψ a generic character on U. Then using the above double coset decomposition one can prove that any function satisfying such functional equations must be identically 0.

Now take k to be a non-archimedean local field. We can similarly define a distribution $B(\varphi)$ on the space of smooth functions with compact support on $\mathrm{GSp}_4(k)$ satisfying

$$B(L_g\varphi) = \beta(g)B(\varphi)$$
$$B(R_u\varphi) = \psi(u)B(\varphi)$$

with $g \in G, u \in U$, and where L_g and R_u are operators of left and right translations respectively. Then, using the standard Gelfand-Kazhdan technique and the double coset decomposition above, we can prove that such a distribution must be identically 0. □

Now let χ be any character of a Cartan subgroup of GSp_4 and extend it trivially on U to all of B. For $x \in k^\times$, define characters

$$\chi_1(x) = \chi\left(\begin{pmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right) = \chi_3(x)^{-1},$$
$$\chi_2(x) = \chi\left(\begin{pmatrix} x & & & \\ & 1 & & \\ & & 1 & \\ & & & x \end{pmatrix}\right) = \chi_4(x)^{-1}.$$

Theorem 4.4. *If $\pi = \mathrm{ind}_B^{\mathrm{GSp}_4} \chi$ and π is irreducible, then*

$$L(\pi, \mu, s) = \prod_{i=1}^{4} L(s, \mu\chi_i), \qquad \epsilon(\pi, \mu, \psi, s) = \prod_{i=1}^{4} \epsilon(s, \mu\chi_i, \psi),$$

where the $L(s, \mu\chi_i)$ and $\epsilon(s, \mu\chi_i, \psi)$ are the Tate L and ϵ factors, [**17**]. *In particular, $L(\pi, \mu, s)$ and $\epsilon(\pi, \mu, \psi, s)$ are independent of ν and R.*

The computation of $L(\pi, \mu, s)$ and $\epsilon(\pi, \mu, \psi, s)$ when π is a Weil lift from a split GO_4 can be found in [**13**].

Consider now a representation τ coming from the dual reductive pair $\mathrm{Sp}_4 \times \mathrm{O}_2$ [**4**], which corresponds to the one dimensional nontrivial representation of O_2. It is known that if O_2 is anisotropic then the corresponding representation of Sp_4 is cuspidal [**2**], [**6**]. The anisotropic O_2 correspond in a 1-1 manner to the quadratic extensions K of k. Since the Weil representation depends on the choice of ψ, and for $\psi(x)$ and $\psi(\lambda x)$, $\lambda \in N_{K/k}(K^\times)$, we get isomorphic representations, we see that to any quadratic extension K of k we can attach two cuspidal representations of Sp_4. If we extend these representations to representations of $\mathrm{PGSp}_4(k)$, we will get two irreducible cuspidal representations. Take τ to be either of these two representations. It is possible to show that τ has the same L-factor as the following representation π_K of $\mathrm{GL}_4(k)$. Denote by σ_0 the special representation of GL_2 which is the uniquely defined subrepresentation of $\mathrm{ind}_{B_0}^{\mathrm{GL}_2}\alpha$, where B_0 is the standard Borel subgroup of GL_2 and $\alpha\left(\begin{smallmatrix} b_1 & x \\ 0 & b_2 \end{smallmatrix}\right) = \left|\frac{b_1}{b_2}\right|^{\frac{1}{2}}$. Denote by $P_{2,2}$ the parabolic subgroup

$$P_{22} = \left\{ \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \,\middle|\, g_{ij} \in M_2(k) \right\} \cap \ \mathrm{GL}_4(k).$$

Then $\pi_K = \mathrm{ind}_{P_{2,2}}^{\mathrm{GL}_4}(\sigma_0 \otimes (\sigma_0 \otimes \beta_K(\det g)))$ where β_K is the character of $k^\times$ given by

$$\beta_K(x) = \begin{cases} 1 & \text{if } x \in N_{K/k}(K^\times) \\ -1 & \text{if } x \notin N_{K/k}(K^\times) \end{cases}.$$

One can prove that π_K is the only generic unitary representation of $\mathrm{GL}_4(k)$ with the same L-function as τ. A very simple proof of this statement was communicated to the author by H. Jacquet. One can also prove, by using other properties of the conjectured Langlands correspondence, that for any other cuspidal representation σ of $\mathrm{PGSp}_4(k)$ the representation ${}^L\sigma$ of $\mathrm{GL}_4(k)$ must be generic and cuspidal. Hence if ${}^L\tau$ exists, it equals π_K. There is a very interesting discussion of how to find ${}^L\tau$ in [9]. The approach in [9] is completely different from ours.

5. Eisenstein Series and Global L-functions.

In this section we let k be a global field.

Let ψ denote a nondegenerate character on $S_{\mathbb{A}}$ and ν a character on $D_{\mathbb{A}} \cong I_K$. Let $V_{\mathbb{A}}$ be the adelic points of the vector space on which $G_{\mathbb{A}}$ acts. Take $\Phi \in S(V_{\mathbb{A}})$, the Schwartz-Bruhat functions on $V_{\mathbb{A}}$. Let μ be a Gröβencharacter on k (i.e., a character on I_k, trivial on the principal ideles). Then to Φ we can associate a function on $G_{\mathbb{A}}$ defined by

$$f^\Phi(g;\mu,\nu,s) = \mu(\det g)|\det g|^{s+\frac{1}{2}} \int_{I_K} \Phi((0,t)g)|t\bar{t}|^{s+\frac{1}{2}}\mu(t\bar{t})\nu(t)\, d^\times t$$

where $|\ |$ denotes the idele norm on I_K. Note that $f^\Phi(g;\mu,\nu,s) \in \mathrm{ind}_{B'_{\mathbb{A}}}^{G_{\mathbb{A}}}\chi$, where χ is a character on $B'_{\mathbb{A}}$ defined by

$$\chi\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \bar{t} & 0 \\ 0 & t\end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix}\right) = \mu(x)|x|^{s+\frac{1}{2}}\nu^{-1}(t).$$

We can then form the Eisenstein series

$$E^\Phi(g;\mu,\nu,s) = \sum_{\gamma \in B'_k \backslash G_k} f^\Phi(\gamma g;\mu,\nu,s).$$

Theorem 5.1 ([7]). *$E^\Phi(g;\mu,\nu,s)$ is a holomorphic function of s, $\forall s \in \mathbb{C}$, except for a finite number of poles, and satisfies the functional equation*

$$E^\Phi(g;\mu,\nu,s) = E^{\hat{\Phi}}(g;\mu^{-1}\nu^{-1},\bar{\nu},1-s)$$

where ν^{-1} is the Gröβencharacter on k obtained by restriction from I_K to I_k, $\bar{\nu}$ is the character of I_K defined by $\bar{\nu}(a) = \nu(\bar{a})$, and $\hat{\Phi}$ is the Fourier *transform of* Φ*. Suppose that* μ *and* ν *are a normalized pair,* i.e., $\mu(t\bar{t})\nu(t) = |t\bar{t}|^\alpha$ *implies* $\alpha = 0$*. Then if* $\mu(t\bar{t})\nu(t) \neq 1$*,* $E^\Phi(g;\mu,\nu,s)$ *has no poles. If* $\mu(t\bar{t})\nu(t) \equiv 1$*, the poles are only at* $s = -\frac{1}{2}$*, with residue* $\Phi(0)\mu(\det g)$*, and at* $s = \frac{3}{2}$*, with residue* $\hat{\Phi}(0)\mu^{-1}(\det g)\nu^{-1}(\det g)$*.*

We now turn to the study of global L-functions. So let (π, V_π) be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$. Then there exists a pair (ν,ψ), where ψ is a nondegenerate character on $S_\mathbb{A}$ and ν a character on the associated group $D_\mathbb{A} \cong I_K$, such that for the character $\alpha_{\nu,\psi}$ on $R_\mathbb{A} = D_\mathbb{A}S_\mathbb{A}$, there exists a cusp form $\varphi \in V_\pi$ such that

$$\int_{C_\mathbb{A}R_k\backslash R_\mathbb{A}} \varphi(r)\alpha_{\nu,\psi}^{-1}(r)dr \neq 0.$$

This again is a consequence of the results of Howe referred to above and the fact that $B_k \setminus B_\mathbb{A}$ is dense in $\mathrm{GSp}_4(k)\backslash \mathrm{GSp}_4(\mathbb{A})$. We fix now such a pair (ν,ψ) and the associated adelic groups as in Section 2.

The nonvanishing of the above integral allows us to define a global generalized Whittaker model for π. If $\varphi \in V_\pi$ we define the function W_φ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$W_\varphi(g) = \int_{C_\mathbb{A}R_k\backslash R_\mathbb{A}} \varphi(rg)\alpha_{\nu,\psi}^{-1}(r)dr.$$

These functions satisfy

$$W_\varphi(rg) = \alpha_{\nu,\psi}(r)W_\varphi(g)$$

for $r \in R_\mathbb{A}$. Denote this space of functions by $\mathcal{W}^{\nu,\psi}$. The representation of $\mathrm{GSp}_4(\mathbb{A})$ on $\mathcal{W}^{\nu,\psi}$ via right translation is then equivalent to π. For any local component $\pi_\mathcal{P}$ of π there exists a nontrivial $\alpha_{\nu_\mathcal{P},\psi_\mathcal{P}}$-eigenfunctional as in Section 3, where $\nu_\mathcal{P}$ and $\psi_\mathcal{P}$ are the local components of the characters ν and ψ respectively. The global (generalized) Whittaker model will then factor as a restricted tensor product of the local (generalized) Whittaker models associated to the $\alpha_{\nu_\mathcal{P},\psi_\mathcal{P}}$ as in Section 3. The uniqueness of the global model will then follow from the uniqueness of the associated local Whittaker models.

Once we have a global (generalized) Whittaker model, we can make constructions analogous to those in Section 3 for local fields. In particular for $W_\varphi(g) \in \mathcal{W}^{\nu,\psi}$ and $\Phi \in S(V_\mathbb{A})$ we define

$$\begin{aligned} L(W_\varphi,\Phi,\mu,s) &= \int_{D_\mathbb{A}N_\mathbb{A}\backslash G_\mathbb{A}} W_\varphi(g)f^\Phi(g;\mu,\nu,s)\,dg \\ &= \int_{N_\mathbb{A}\backslash G_\mathbb{A}} W_\varphi(g)\Phi((0,1)g)\mu(\det g)|\det g|^{\frac{1}{2}+s}\,dg. \end{aligned}$$

Theorem 5.2. *Assume that π is a cuspidal automorphic representation and $\varphi \in V_\pi$. Then $L(W_\varphi, \Phi, \mu, s)$ converges in some half plane* $\mathrm{Re}(s) > s_0$ *and has a meromorphic continuation to the whole s-plane. Further, if $\mu(t\bar{t})\nu(t)$ is nontrivial (μ and ν a normalized pair), then $L(W_\varphi, \Phi, \mu, s)$ is holomorphic everywhere. If $\mu(t\bar{t})\nu(t) \equiv 1$, then $L(W_\varphi, \Phi, \nu, s)$ has poles at $s = -\frac{1}{2}$, with residue*

$$\Phi(0) \int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mu(\det g)\varphi(g)\, dg,$$

and at $s = 3/2$ with residue

$$\hat{\Phi}(0) \int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \mu^{-1}(\det g)\nu^{-1}(\det g)\varphi(g)\, dg.$$

Proof. The convergence for s in some half plane is standard. The meromorphic continuation follows immediately from the identity

$$\int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \varphi(g) E^{\Phi}(g; \mu, \nu, s)\, dg = \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_\varphi(g) f^{\Phi}(g; \mu, \nu, s)\, dg,$$

and the meromorphic continuation of $E^{\Phi}(g; \mu, \nu, s)$. To prove this identity, we let I denote the left hand side above and expand the Eisenstein series. Then

$$I = \int_{C_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \varphi(g) \sum_{\gamma \in B'_k \backslash G_k} f^{\Phi}(\gamma g; \mu, \nu, s)\, dg = \int_{C_{\mathbb{A}} B'_k \backslash G_{\mathbb{A}}} \varphi(g) f^{\Phi}(g; \mu, \nu, s)\, dg,$$

since $\varphi(g)$ is an automorphic form on $\mathrm{GSp}_4(\mathbb{A})$ and thus invariant under $\mathrm{GSp}_4(k) \supseteq G_k$. Next we expand φ in its Fourier expansion

$$\varphi(g) = \sum_{\substack{\psi \in \mathrm{Char}(S_k \backslash S_{\mathbb{A}}) \\ \psi \text{ nontrivial}}} \varphi_\psi(g),$$

where the Fourier coefficients are given by

$$\varphi_\psi(g) = \int_{S_k \backslash S_{\mathbb{A}}} \varphi(sg)\psi^{-1}(s) ds.$$

If we substitute this Fourier expansion in the above expression for I, we note from the left invariance of $f^{\Phi}(g; \mu, \nu, s)$ under $N_{\mathbb{A}}$, that the Fourier coefficients involving characters ψ which are not trivial on $N_{\mathbb{A}}$ will be killed upon integration. If we let Ω denote the set of nontrivial characters on $S_k \backslash S_{\mathbb{A}}$ which are trivial on $N_{\mathbb{A}}$, then we get

$$I = \int_{C_{\mathbb{A}} H_k D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \sum_{\psi \in \Omega} \varphi_\psi(g) f^{\Phi}(g; \mu, \nu, s)\, dg.$$

Here we have used the decomposition $B' = \mathrm{HDN}$. Now, from our lemma at the end of Section 2, it follows that H_k acts simply transitively on Ω. Then, since $f^{\Phi}(g;\mu,\nu,s)$ is invariant under H_k by our choice of μ, we can rewrite the above as

$$I = \int_{C_{\mathbb{A}} H_k D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \sum_{h \in H_k} \varphi_{\psi}(hg) f^{\Phi}(g;\mu,\nu,s)\, dg$$

where ψ is our character on $S_k \setminus S_{\mathbb{A}}$ fixed above. Then

$$\begin{aligned}
I &= \int_{C_{\mathbb{A}} D_k N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \varphi_{\psi}(g) f^{\Phi}(g;\mu,\nu,s)\, dg \\
&= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \left(\int_{C_{\mathbb{A}} D_k \backslash D_{\mathbb{A}}} \varphi_{\psi}(tg) f^{\Phi}(tg;\mu,\nu,s)\, dt \right) dg \\
&= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \left(\int_{C_{\mathbb{A}} D_k \backslash D_{\mathbb{A}}} \varphi_{\psi}(tg) \nu^{-1}(t)\, dt \right) f^{\Phi}(g;\mu,\nu,s)\, dg \\
&= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \left(\int_{C_{\mathbb{A}} D_k \backslash D_{\mathbb{A}}} \int_{S_k \backslash S_{\mathbb{A}}} \varphi(tsg) \psi^{-1}(s) \nu^{-1}(t)\, ds\, dt \right) f^{\Phi}(g;\mu,\nu,s)\, dg \\
&= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} \left(\int_{C_{\mathbb{A}} R_k \backslash R_{\mathbb{A}}} \varphi(rg) \alpha_{\nu,\psi}^{-1}(r)\, dr \right) f^{\Phi}(g;\mu,\nu,s)\, dg \\
&= \int_{D_{\mathbb{A}} N_{\mathbb{A}} \backslash G_{\mathbb{A}}} W_{\varphi}(g) f^{\Phi}(g;\mu,\nu,s)\, dg,
\end{aligned}$$

as desired. □

This method of obtaining the meromorphic continuation of $L(W,\Phi,\mu,s)$ through the continuation of an Eisenstein series is what is commonly called the Rankin-Selberg method [7], [12].

Once we have the meromorphic continuation of $L(W,\Phi,\nu,s)$ we can define $L(\pi,\mu,s)$ as in Section 3, i.e., $L(\pi,\mu,s)$ is a function defined so that $\frac{L(W,\Phi,\mu,s)}{L(\pi,\mu,s)}$ is entire for all choices of $W \in \mathcal{W}^{\nu,\psi}$ and $\Phi \in S(V_{\mathbb{A}})$.

As noted above, to the local factors $\pi_{\mathcal{P}}, \nu_{\mathcal{P}}$, and $\psi_{\mathcal{P}}$ of π, ν and ψ respectively, we can associate the local (generalized) Whittaker model $\mathcal{W}^{\nu_{\mathcal{P}},\psi_{\mathcal{P}}}$. Then we can form the local integrals $L(W_{\mathcal{P}},\Phi_{\mathcal{P}},\mu_{\mathcal{P}},s)$ for $W_{\mathcal{P}} \in \mathcal{W}^{\nu_{\mathcal{P}},\psi_{\mathcal{P}}}$ and $\Phi_{\mathcal{P}} \in S(V_{\mathcal{P}})$ and thereby define the local L and ϵ-factors $L(\pi_{\mathcal{P}},\mu_{\mathcal{P}},s)$ and $\epsilon(\pi_{\mathcal{P}},\mu_{\mathcal{P}},\psi_{\mathcal{P}},s)$.

Theorem 5.3. $L(\pi,\mu,s) = \prod_{\mathcal{P}} L(\pi_{\mathcal{P}},\mu_{\mathcal{P}},s)$. *Then:*

1) *The product converges in some half plane* $\mathrm{Re}(s) > s_0$.

2) *The product has a meromorphic continuation to the whole s-plane.*
3) *We have the functional equation*

$$L(\pi,\mu,s)=\epsilon(\pi,\mu,s)\;L(\hat{\pi},\mu^{-1},1-s)$$

where $\epsilon(\pi,\mu,s)=\prod_{\mathcal{P}}\epsilon(\pi_{\mathcal{P}},\mu_{\mathcal{P}},\psi_{\mathcal{P}},s)$ *and, as usual, this product does not depend on* ψ.
4) *If for some* $\mathcal{P}_0,\pi_{\mathcal{P}_0}$ *is generic, then* $L(\pi,\mu,s)$ *is holomorphic.*

Proof. Choose $\varphi\in V_\pi$ and $\Phi\in S(V_{\mathbb{A}})$, which are factorizable. Then we can decompose the integral

$$\int_{D_{\mathbb{A}}N_{\mathbb{A}}\backslash G_{\mathbb{A}}}W_\varphi(g)f^{\Phi}(g;\mu,\nu,s)\,dg=\prod_{\mathcal{P}}\int_{D_{\mathcal{P}}N_{\mathcal{P}}\backslash G_{\mathcal{P}}}W_{\varphi_{\mathcal{P}}}(g)f^{\Phi_{\mathcal{P}}}(g;\mu_{\mathcal{P}},\nu_{\mathcal{P}},s)\,dg$$

which gives $L(W,\Phi,\mu,s)=\prod_{\mathcal{P}}L(W_{\mathcal{P}},\Phi_{\mathcal{P}},\mu_{\mathcal{P}},s)$. Then the convergence and meromorphic continuation of this product follows from the previous theorem. Statement 3) follows quickly from this. In order to prove 4) we have to use the following local result: If π is generic, then an embedding of π into $\mathrm{ind}_G^{\mathrm{GSp}_4}\omega$ cannot exist for any one dimensional representation ω of G. The main idea of the proof of this statement was given in the proof of Theorem 4.3. □

References

[1] A.N. Andrianov, *Zeta functions and the Siegel modular forms*, Proc. Summer School Bolya-Janos Math. Soc. (Budapest, 1970), Halsted, New York, 1975, 9-20 .

[2] C. Asmuth, *Weil representations of symplectic p-adic groups*, Amer. J. Math., **101**(**4**) (1979), 885-908.

[3] A. Borel, *Automorphic L-functions*, Proc. Sympos. Pure Math., **33**(**2**) (1979), 27-61.

[4] R. Howe, *θ-series and invariant theory*, Proc. Sympos, Pure Math., **33**(**1**) (1979), 275-285.

[5] ———, *On a notion of rank for unitary representations of the classical groups*, C.I.M.E. Summer School on Harmonic Analysis and Group Representations, Cortona, 1980, 223-331.

[6] R. Howe and I.I. Piatetski-Shapiro, *A counterexample to the "generalized Ramanujan conjecture" for (quasi-) split groups*, Proc. Sympos. Pure Math., **33**(**1**) (1979), 315-322.

[7] H. Jacquet, *Automorphic Forms on* GL(2) II, Lecture Notes in Math., **278**, Springer, New York, 1972.

[8] H. Jacquet and R. Langlands, *Automorphic Forms on* GL(2), Lecture Notes in Math., **114**, Springer, New York, 1970.

[9] R. Langlands, Letter to Roger Howe, March 24, 1974.

[10] M.E. Novodvorsky and I.I. Piatetski-Shapiro, *Generalized Bessel models for the symplectic group of rank* 2, Math. Sb., **90(2)** (1973), 246-256.

[11] M.E. Novodvorsky, *Automorphic L-functions for the symplectic group* GSp(4), Proc. Symp. Pure Math., **33(2)** (1979), 87-95.

[12] I.I. Piatetski-Shapiro, *Euler subgroups,* Proc. Summer School Bolya-Janos Math. Soc. (Budapest, 1970), Halsted, New York, 1975, 597-620.

[13] I.I. Piatetski-Shapiro and D. Soudry, *L and ε factors for* GSp_4, J. Fac. Sci. Univ. Tokyo, Sec. IA, **28** (1981), 505-530.

[14] F. Rodier, *Le représentations de* $\mathrm{GSp}(4,k)$ *où k est un corps local*, C. R. Acad. Sci. Paris Sér. A, **283** (1976), 429-431.

[15] G. Shimura, *On modular forms of half integral weight*, Ann. of Math., **97** (1973), 440-481.

[16] B. Srinivasan, *The characters of the finite symplectic group* $\mathrm{Sp}(4,q)$, Trans. AMS, **131** (1968), 488-525.

[17] J. Tate, *Fourier Analysis on Number Fields and Hecke's Zeta Functions*, Thesis, Princeton, 1950, in 'Algebraic Number Theory', Proceedings of the Brighton Conference, Academic Press, New York, 1968, 305-347.

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