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ON NON-VANISHING OF TWISTED SYMMETRIC AND EXTERIOR SQUARE L-FUNCTIONS FOR $GL(n)$

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Dedicated to Olga Taussky-Todd

1. Introduction.

Let π be a cuspidal representation of $GL_n(\mathbb{A}_F)$, where $\mathbb{A}_F$ is the ring of adeles of a number field F. Write $\pi = \otimes_v \pi_v$, where each π_v is an irreducible unitary representation of $GL_n(F_v)$. Here F_v is the completion of F with respect to v. Let v be a place of F such that π_v is spherical. There exist unramified quasicharacters $\mu_{1,v}, \ldots, \mu_{n,v}$ of F_v^* such that π_v is a constituent of the representation $I(\mu_v)$, $\mu_v = \mu_{1,v} \otimes \cdots \otimes \mu_{n,v}$, induced from character μ_v of T_v, a Cartan subgroup of $GL_n(F_v)$ which for simplicity is taken to be the subgroup of diagonal matrices, extended trivially along the subgroup of upper triangular unipotent matrices U_v of $GL_n(F_v)$. Let A_v be the diagonal matrix $\mathrm{diag}(\mu_{1,v}(\varpi_v), \ldots, \mu_{n,v}(\varpi_v))$ where ϖ_v is a uniformizing parameter, $|\varpi_v|_v = q_v^{-1}$. Here q_v is the cardinality of the residue field O_v/P_v, where O_v and P_v are the ring of integers of F_v and its unique maximal ideal, leading to the normalization $|\varpi_v|_v = q_v^{-1}$ of $|\ |_v$. The class of π_v is uniquely determined by the conjugacy class of A_v in $GL_n(\mathbb{C})$, the L-group of GL_n.

Let χ be an arbitrary character of $F^* \setminus \mathbb{A}_F^*$. Write $\chi = \otimes_v \chi_v$. Let S be a finite set of places of F such that π_v and χ_v are both unramified for every $v \notin S$. Let Λ^2 and Sym^2 denote the exterior and the symmetric square representations of $GL_n(\mathbb{C})$, respectively. Let $s \in \mathbb{C}$. Set

$$\begin{aligned} L(s, \pi_v, \Lambda^2 \otimes \chi_v) &= \det(I - \Lambda^2(A_v)\chi_v(\varpi_v)q_v^{-s})^{-1} \\ &= \prod_{1 \le i < j \le n} (1 - \mu_{i,v}\mu_{j,v}\chi_v(\varpi_v)q_v^{-s})^{-1} \end{aligned}$$

and

$$\begin{aligned} L(s, \pi_v, \mathrm{Sym}^2 \otimes \chi_v) &= \det(I - \mathrm{Sym}^2(A_v)\chi_v(\varpi_v)q_v^{-s})^{-1} \\ &= \prod_{1 \le i \le j \le n} (1 - \mu_{i,v}\mu_{j,v}\chi_v(\varpi_v)q_v^{-s})^{-1}. \end{aligned}$$

Finally, set

$$L_S(s,\pi,\Lambda^2\otimes\chi)=\prod_{v\notin S}L(s,\pi_v,\Lambda^2\otimes\chi_v)$$

and

$$L_S(s,\pi,\mathrm{Sym}^2\otimes\chi)=\prod_{v\notin S}L(s,\pi_v,\mathrm{Sym}^2\otimes\chi_v).$$

The basic analytic properties of these two L-functions for arbitrary n are studied and proved by several authors (cf. [**2**, **4**, **8**, **9**] for exterior square and [**3**, **8**, **9**] for symmetric square; in the context of [**3**] the twisted case has been taken up by William Banks).

When χ is the square of another character χ_0, then

$$L_S(s,\pi,\Lambda^2\otimes\chi)=L_S(s,\pi\otimes\chi_0,\Lambda^2)$$

and the twisted L-function is nothing new. This is in particular the case if χ is a character of order $2m+1$, since

$$L_S(s,\pi,\Lambda^2\otimes\chi)=L_S(s,\pi\otimes\chi^{-m},\Lambda^2).$$

On the other hand, if χ is a non-square character of even order, then $L_S(s,\pi,\Lambda^2\otimes\chi)$ is completely new. Similarly for Sym^2. The purpose of this paper is to prove:

Theorem 1.1. *Both L-functions $L_S(s,\pi,\Lambda^2\otimes\chi)$ and $L_S(s,\pi,\mathrm{Sym}^2\otimes\chi)$ are non-zero for* $\mathrm{Re}(s)=1$.

While the theorem readily follows from Theorem 5.1 of [**8**] by considering GL_n as Siegel Levi subgroups of special orthogonal groups when $\chi=1$, the case of L-functions twisted by an arbitrary character is more delicate.

Applications of Theorem 1.1 that we know of are all in the case that n is even and, in fact, the even case is deeper (see the next paragraph). But for the sake of completeness, we treat the odd case as well.

The proofs for even and odd n are completely different. The even case follows from applying Theorem 5.1 of [**8**] to simply connected coverings (spin groups) of special orthogonal groups and sits within the theory developed in [**8**] and [**9**]. On the other hand the odd case needs to be proved indirectly and spin groups imply nothing about twisted L-functions with $\chi\neq 1$. This is due to the fact that the Siegel Levi subgroup of the corresponding spin group Spin_{2n} remains GL_n if n is odd (Remark 2.2). Proofs still rely on

Theorem 5.1 of [8], but not entirely. One needs to use the holomorphy of $L_S(s,\pi,\Lambda^2\otimes\chi)$ on Re(s)=1 (Theorem 9.6.2 of [4]) as well, a result which is valid since n is odd. Because $L_S(s,\pi,\Lambda^2\otimes\chi)$ may have a pole at $s=1$ when n is even, the method of odd case does not apply to the even case.

The twisted case seems to have some further interesting applications in number theory and was suggested by Dinakar Ramakrishnan. (See Remark 2.5.) I would like to thank him for the suggestion and useful discussions on the problem during my stay at Caltech in the Spring of 97. Thanks are also due to Don Blasius for useful conversations which led to a proof in the odd case for which we have included two proofs. The other one, which uses base change [1], is due to Ramakrishnan. Finally, I would like to thank David Rohrlich for some useful communications.

2. The Even Case.

In this section we prove Theorem 1 when n is even. Let $\mathbf{G}$ be either Spin_{2n} or Spin_{2n+1}, simply connected coverings of groups SO_{2n} or SO_{2n+1}, split special orthogonal groups in $2n$ and $2n+1$ variables, respectively. The field of definition F could be either local or global. Let $\mathbf{B}=\mathbf{TU}$ be a Borel subgroup of $\mathbf{G}$ with a maximal torus $\mathbf{T}$ and the unipotent radical $\mathbf{U}$. We may and shall assume $\mathbf{T}$ projects onto the subgroups of diagonal elements in SO_{2n} or SO_{2n+1}, respectively. Denote by $\Delta=\{\alpha_1,\cdots,\alpha_n\}$ the set of simple roots of $\mathbf{T}$ in the Lie algebra of $\mathbf{U}$. Let $\mathbf{M}_\theta=\mathbf{M}\supset\mathbf{T}$ be the Levi subgroup of $\mathbf{G}$ generated by $\theta=\{\alpha_1,\cdots,\alpha_{n-1}\}$ and let $\mathbf{P}=\mathbf{MN}$, $\mathbf{N}\subset\mathbf{U}$, be the corresponding standard parabolic subgroup of $\mathbf{G}$. The group $\mathbf{G}$ being simply connected implies $\mathbf{M}_D=SL_n$, where $\mathbf{M}_D$ is the derived group of $\mathbf{M}$. Let R be the set of kth roots of 1, where $n=2k$. We have

Lemma 2.1. $\mathbf{M}\cong(GL_1\times SL_n)/R.$

Proof. Given $\alpha\in\Delta$, let H_α be the corresponding coroot. Let $n=2k$ and let $\mathbf{A}$ be the connected component of the center of $\mathbf{M}$. Then $\mathbf{A}\subset\mathbf{T}$. Assume $\mathbf{G}=\mathrm{Spin}_{2n}$. Being the connected component of the center of $\mathbf{M}$, $\mathbf{A}=(\bigcap_{\alpha\in\theta}\ker\alpha)^0$ and a simple calculation shows that

$$\mathbf{A}=\left\{\prod_{j=1}^{n-2}H_{\alpha_j}(\lambda^j)H_{\alpha_{n-1}}(\lambda^{k-1})H_{\alpha_n}(\lambda^k)|\lambda\in\overline{F}^*\right\},$$

where $n=2k$. Although we do not need to know, the center of $\mathbf{M}$ is $\mathbf{A}\cup c\mathbf{A}$, where $c=H_{\alpha_{n-1}}(-1)H_{\alpha_n}(-1)$. Moreover center $Z(\mathbf{G})$ of $\mathbf{G}$ is

$$Z(\mathbf{G})=\{1,z,c,cz\},$$

where

$$z = \prod_{j=1}^{n-2} H_{\alpha_j}((-1)^j) H_{\alpha_{n-1}}(-1)$$

if k is even, and

$$z = \prod_{j=1}^{n-2} H_{\alpha_j}((-1)^j) H_{\alpha_n}(-1)$$

otherwise.

The center of $\mathbf{M}_D = SL_n$ is

$$\{H_{\alpha_1}(\lambda) H_{\alpha_2}(\lambda^2) \cdots H_{\alpha_{n-2}}(\lambda^{n-2}) H_{\alpha_{n-1}}(\lambda^{n-1}) | \lambda^n = 1\}.$$

In the intersection of $\mathbf{M}_D$ and $\mathbf{A}$, $\lambda^k = 1$ and $\mathbf{A} \cap \mathbf{M}_D = R$, completing the lemma for Spin_{2n}.

On the other hand, if $\mathbf{G} = \mathrm{Spin}_{2n+1}$, $\mathbf{M}_D = SL_n$, and

$$\mathbf{A} = \{H_{\alpha_1}(\lambda) H_{\alpha_2}(\lambda^2) \cdots H_{\alpha_{n-1}}(\lambda^{n-1}) H_{\alpha_n}(\lambda^k) | \lambda \in \overline{F}^*\},$$

and again $\mathbf{A} \cap \mathbf{M}_D = R$. In passing we note that the center $Z(\mathbf{M})$ of $\mathbf{M}$ equals $\mathbf{A} \cup \mathbf{A} H_{\alpha_n}(-1)$. The lemma is now complete.

Remark 2.2. It can be shown that if n is odd, then $\mathbf{M} \cong GL_n$. If $\mathbf{G} = \mathrm{Spin}_{2n+1}$, the other component of $Z(\mathbf{M})$ is $\eta\mathbf{A}$, where η is a generator of $\mathbb{Z}/n\mathbb{Z}$, the center of SL_n. In other words,

$$\eta = H_{\alpha_1}(\xi) H_{\alpha_2}(\xi^2) \cdots H_{\alpha_{n-1}}(\xi^{n-1}),$$

where ξ is a primitive nth root of 1.

Now assume F is a non-archimedean local field. Let σ be a spherical representation of $GL_n(F)$ whose central character is ω. Let χ be an unramified character of F^*. Consider χ as a character of the F-points of the split component of $\mathbf{M}$ whose kernel is $R(F)$. Since χ is trivial on the F-points $R(F)$ of R which is the intersection of the derived group $SL_n(F)$ of M with $GL_1(F)$, it can be extended to a character of M which we still denote by χ.

There is a natural F-surjection

$$\mathbf{M} \to GL_n \to 0$$

defined by the covering map from Spin_{2n} onto SO_{2n}. It defines an isomorphism onto SL_n upon restriction to $\mathbf{M}_D$. The representation σ then lifts to a spherical representation τ of M. More precisely, τ is any irreducible constituent of this lift since this map may no longer be a surjection at the level of F-points. We can now consider $\tau \otimes \chi$ as a spherical representation of M.

Let ${}^L M = \widehat{M}$ be the L group of $\mathbf{M}$, with the trivial action of Galois group disregarded. Then $\widehat{M}_D$, the derived group of $\widehat{M}$ is $\widehat{M}_D \cong SL_n(\mathbb{C})/\{\pm 1\}$. One quick way of seeing this is to consider $\widehat{M}$ as a Levi subgroup of $\widehat{G} = PSO_{2n}(\mathbb{C})$ or $PSp_{2n}(\mathbb{C})$ according as $\mathbf{G} = \mathrm{Spin}_{2n}$ or Spin_{2n+1}, respectively. Then

$$\widehat{M} = GL_1(\mathbb{C})(SL_n(\mathbb{C})/\{\pm 1\}).$$

Let $\widehat{\widehat{M}} = GL_n(\mathbb{C})$ denote the GL_n-Levi subgroup of $SO_{2n}(\mathbb{C})$. The covering map φ from $SO_{2n}(\mathbb{C})$ onto $PSO_{2n}(\mathbb{C}) = \widehat{G}$ leads to a surjection from $\widehat{\widehat{M}}$ onto $\widehat{M}$ which we still denote by φ. Its restriction from $SL_n(\mathbb{C})$ onto $SL_n(\mathbb{C})/\{\pm 1\}$ is their corresponding covering map. We define a similar map for the case $\mathbf{G} = \mathrm{Spin}_{2n+1}$ using the covering map $\varphi : Sp_{2n}(\mathbb{C}) \to PSp_{2n}(\mathbb{C})$.

Let $A_\sigma \subset GL_n(\mathbb{C})$ be the conjugacy class attached to σ. By abuse of notation, we use A_σ to also denote a representative of A_σ as in the introduction. One has a surjection

$$\widehat{\widehat{M}} \xrightarrow{\varphi} \widehat{M} \longrightarrow 0$$

sending

$$A_\sigma \mapsto \varphi(A_\sigma) = A_\tau$$

with obvious meaning for A_τ.

In general, characters of F-points of a connected reductive group are parametrized by 1-cocycles of Weil group of $\overline{F}/F$ into the center of the connected component of its L-group [7]. The group $\mathbf{M}$ being split, the parameter attached to χ becomes χ itself, now again as a character of F^*, if one interprets $GL_1(\mathbb{C})$ as the center of $\widehat{M}$. The class attached to χ is then $\chi(\varpi) \in GL_1(\mathbb{C})$, the center of $\widehat{M}$. If $\chi(\varpi) = -1$, it is given by

$$\mathrm{diag}(i, \dots i, i^{-1}, \dots, i^{-1}) \mod (\pm I),$$

$i = \sqrt{-1}$, in $PSO_{2n}(\mathbb{C})$ (in the case $\mathbf{G} = \mathrm{Spin}_{2n}$), the element of order 2 in the standard Cartan subgroup of $PSO_{2n}(\mathbb{C})$. A similar element in $SO_{2n}(\mathbb{C})$ is $-I$ whose adjoint action is trivial.

Let r and r_0 be the adjoint actions of $\widehat{M}$ and $\widehat{\widehat{M}}$ on the Lie algebra $\widehat{\mathfrak{n}}$ of $\widehat{N}$, repsectively. Then $r \cdot \varphi = r_0$. Moreover $r(\chi(\varpi))$ is multiplication by $\chi(\varpi)$.

On the other hand

$$\begin{aligned} r(A_\tau) &= r \cdot \varphi(A_\sigma) \\ &= r_0(A_\sigma) \\ &= \Lambda^2(A_\sigma) \end{aligned}$$

if $\mathbf{G} = \mathrm{Spin}_{2n}$, while

$$r(A_\tau) = \mathrm{Sym}^2(A_\sigma)$$

for $\mathbf{G} = \mathrm{Spin}_{2n+1}$. Consequently

$$r(A_\tau \chi(\varpi)) = \chi(\varpi)\Lambda^2(A_\sigma)$$

or

$$r(A_\tau \chi(\varpi)) = \chi(\varpi)\mathrm{Sym}^2(A_\sigma)$$

according as $\mathbf{G} = \mathrm{Spin}_{2n}$ or Spin_{2n+1}. We have therefore proved:

Lemma 2.3. *Let σ be a spherical representation of $GL_n(F)$ and χ an unramified character of F^*. Consider χ as a character of M and lift σ to a spherical representation τ of $M = \mathbf{M}(F)$. Then*

$$L(s, \tau \otimes \chi, r) = L(s, \sigma, \Lambda^2 \otimes \chi)$$

if $\mathbf{G} = \mathrm{Spin}_{2n}$, while

$$L(s, \tau \otimes \chi, r) = L(s, \sigma, \mathrm{Sym}^2 \otimes \chi)$$

if $\mathbf{G} = \mathrm{Spin}_{2n+1}$.

Theorem 2.4. *Assume n is even. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_n(\mathbb{A}_E)$. Let $\chi = \otimes_v \chi_v$ be a character of $F^* \setminus \mathbb{A}_F^*$. Let S be a finite set of places of F such that if $v \notin S$, then both π_v and χ_v are unramified. Then both $L_S(s, \pi, \Lambda^2 \otimes \chi)$ and $L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ are non-zero for $\mathrm{Re}(s) = 1$.*

Proof. We lift π to a cuspidal representation τ of $M = \mathbf{M}(\mathbb{A}_F)$, using the natural F-surjection

$$\mathbf{M} \to GL_n \to 0$$

defined by the covering map from Spin_{2n} onto SO_{2n} as in the local case. Similarly for Spin_{2n+1} and SO_{2n+1}. Also consider χ as a character of M. Then

$$L_S(s, \tau \otimes \chi, r) = L_S(s, \pi, \Lambda^2 \otimes \chi)$$

by Lemma 2.3. Similarly for $L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi)$. The theorem now follows by applying Theorem 5.1 of [8] to $(\mathrm{Spin}_{2n}, \mathbf{M})$ or $(\mathrm{Spin}_{2n+1}, \mathbf{M})$, accordingly.

Remark 2.5. Theorem 2.4 can be used to prove cyclic base change for globally generic cuspidal representations of $GSp_4(\mathbb{A}_F)$ for any number field F. The idea is due to Ramakrishnan and is as follows. Let K/F be a cyclic extension of F and fix a globally generic cuspidal representation π of $GSp_4(\mathbb{A}_F)$. By the unpublished results of Jacquet, Piatetski-Shapiro, and Shalika, as well as Theorem 8.1 of [6], there exists a cuspidal representation π' of $GL_4(\mathbb{A}_F)$, where Langlands classes agree at all the unramified places with those of π with respect to the embedding ${}^LGSp_4 = GSp_4(\mathbb{C}) \subset GL_4(\mathbb{C}) = {}^LGL_4$. (To use the results of Kudla-Rallis-Soudry [6], one needs to first appeal to Theorem 5.1 of [8] to conclude that the degree 5 standard L-function $L_S(s, \pi, r_5)$ has no zero at $s = 1$.) Now let π'_K be the base change lift of π' to $GL_4(\mathbb{A}_K)$ as in [1]. Then

$$L_S(s, \pi'_K, \Lambda^2) = L_S(s, \pi', \Lambda^2) L_S(s, \pi', \Lambda^2 \otimes \chi),$$

where χ is the character of $F^* \setminus \mathbb{A}_F^*$ correspsonding to K/F by class field theory. But using the identities

$$\begin{aligned} L_S(s, \pi', \Lambda^2) &= L_S(s, \pi, \Lambda^2) \\ &= L_S(s, \pi, r_5) L_S(s, 1), \end{aligned}$$

$L_S(s, \pi', \Lambda^2)$ has a pole at $s = 1$, as $L_S(s, \pi, r_5)$ has no zero at $s = 1$. By Theorems 2.4, $L_S(s, \pi', \Lambda^2 \otimes \chi)$ is non-zero at 1. Then $L_S(s, \pi'_K, \Lambda^2)$ has a pole at $s = 1$ and by the unpublished results of Jacquet, Piatetski-Shapiro, and Shalika quoted above, π'_K descends to an automorphic representation π_K of $GSp_4(\mathbb{A}_K)$. This π_K is the base change lift of π to K/F.

3. The Odd Case.

In view of Remark 2.5, the machinery of previous section does not work. As we shall see one can use a trick to reduce the problem to the even case. But there is one case that we can immediately dispose of. Let $\pi = \otimes_v \pi_v$ be a cuspidal representation of $GL_n(\mathbb{A}_F)$, where n is now odd. Let χ be a character of $F^* \setminus \mathbb{A}_F^*$. As before, let S be a finite set of places of F for which $v \notin S$ implies both π_v and χ_v are unramified. For simplicity, in the proofs we shall assume χ is quadratic. We have

Proposition 3.1. *Suppose n is odd. Then the L-function $L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi) \neq 0$ for* $\mathrm{Re}(s) = 1$.

Proof. We have

$$L_S(s, \pi \times (\pi \otimes \chi)) = L_S(s, \pi, \Lambda^2 \otimes \chi) L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi), \tag{1}$$

where the L-function on the right is the Rankin-Selberg product L-function for π and $\pi \otimes \chi$ (cf. [5, 8]). By Theorem 5.1 of [8], $L_S(s, \pi \times (\pi \otimes \chi)) \neq 0$ for $\mathrm{Re}(s) = 1$. But by Theorem 9.6.2 of [4] $L_S(s, \pi, \Lambda^2 \otimes \chi)$ is holomorphic for $\mathrm{Re}(s) = 1$. The assertion now follows.

For non-vanishing of $L_S(s, \pi, \Lambda^2 \otimes \chi)$ at $\mathrm{Re}(s) = 1$ we include two proofs.

Proposition 3.2. *Suppose n is odd. Then the L-function $L_S(s, \pi, \Lambda^2 \otimes \chi)$ is non-zero for* $\mathrm{Re}(s) = 1$.

First proof: Let E/F be the quadratic extension attached to χ and let Π be the base change lift of π defined by Arthur and Clozel [1]. Then

$$L_S(s, \Pi, \Lambda^2) = L_S(s, \pi, \Lambda^2) L_S(s, \pi, \Lambda^2 \otimes \chi). \tag{2}$$

In a moment we will show that Π is cuspidal. Then the left-hand side of (2) is non-vanishing for $\mathrm{Re}(s) = 1$ by Theorem 5.1 of [8] applied to GL_n-Levi subgroup of $SO_{2n}(\mathbb{A}_E)$. The proposition follows from the holomorphy of $L_S(s, \pi, \Lambda^2)$ at $\mathrm{Re}(s) = 1$ (Theorem 9.6.2 of [4]) since n is odd.

Suppose Π is not cuspidal. Then by Theorem 3.4.2.b of [1], $\pi \cong \pi \otimes \chi$. Let ω_π be the central character of π. Then $\omega_\pi = \omega_\pi \chi^n$ or $\chi^n = 1$. Since n is odd, $\chi = 1$, a contradiction.

Second proof: This is a proof which does not use base change and is consequently longer. It came out of conversations with Blasius before the first proof was suggested to us by Ramakrishnan.

Let η be an arbitrary (unitary) character of $F^* \backslash \mathbb{A}_F^*$. Consider $\sigma = (\pi \otimes \chi) \otimes (\pi \otimes \eta)$ as a cuspidal representation of $GL_n(\mathbb{A}_F) \times GL_n(\mathbb{A}_F)$. Let $\mathbf{M}$ be the Levi subgroup of Spin_{4n} generated by simple roots $\{\alpha_1, \ldots, \alpha_{2n-1}\}$. Denote by $\mathbf{M}' \subset \mathbf{M}$, the Levi subgroup generated by $\{\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n+1}, \ldots, \alpha_{2n-1}\}$. There are natural F-surjections

$$\mathbf{M} \to GL_{2n} \to 0$$

and

$$\mathbf{M}' \to GL_n \times GL_n \to 0,$$

obtained by restricting the covering map from Spin_{4n} onto SO_{4n}. Their restrictions to derived groups of $\mathbf{M}$ and $\mathbf{M}'$ are isomorphisms onto SL_{2n} and $SL_n \times SL_n$, respectively.

The representation of σ then lifts to a cuspidal representation τ of $M' = \mathbf{M}'(\mathbb{A}_F)$. There is a natural map from

$$I(\sigma) = \mathrm{Ind}(GL_n(\mathbb{A}_F) \times GL_n(\mathbb{A}_F), GL_{2n}(\mathbb{A}_F), \sigma)$$

into $I(\tau) = \mathrm{Ind}(M', M, \tau)$. Consequently

$$L_S(s, I(\tau) \otimes \chi, r) = L_S(s, I(\sigma), \Lambda^2 \otimes \chi),$$

where r is the adjoint action of $\widehat{M}$ on the Lie algebra of $\widehat{N}$, the L-group of $\mathbf{N}$, and $\mathbf{MN}$ is the standard parabolic subgroup of Spin_{4n} whose Levi is $\mathbf{M}$.

The non-vanishing of $L_S(s, I(\tau) \otimes \chi, r)$ for $\mathrm{Re}(s) = 1$ follows by considering the Eisenstein series built from the standard parabolic subgroup of $\mathbf{G} = \mathrm{Spin}_{4n}$ whose Levi subgroup is $\mathbf{M}'$, no longer a maximal parabolic subgroup, and the cuspidal representation τ of M'. The comlex parameter on A'/A will be set equal to zero. Here $\mathbf{A}$ and $\mathbf{A}'$ are split components of $\mathbf{M}$ and $\mathbf{M}'$, respectively. Lemma 2.1 of [10] applies, leading to Theorem 5.1 of [8]. This implies the non-vanishing of $L_S(s, I(\tau) \otimes \chi, r)$ and therefore $L_S(s, I(\sigma), \Lambda^2 \otimes \chi)$ for $\mathrm{Re}(s) = 1$.

Write $\pi = \otimes_v \pi_v$, $\chi = \otimes_v \chi_v$, and $\eta = \otimes_v \eta_v$, with all unramified for $v \notin S$. Let A_v be the class in $GL_n(\mathbb{C})$ attached to π_v and set

$$B_v = \mathrm{diag}(\chi_v(\varpi_v), \eta_v(\varpi_v)) \in GL_2(\mathbb{C}).$$

Then

$$\begin{aligned} L(s, I(\sigma_v), \Lambda^2 \otimes \chi_v) &= L(s, \Lambda^2(A_v \otimes B_v) \otimes \chi_v(\varpi_v)) \\ &\overset{\mathrm{def}^n}{=} \det(I - \Lambda^2(A_v \otimes B_v)\chi_v(\varpi_v) q_v^{-s})^{-1}. \end{aligned}$$

Using

$$\Lambda^2(A_v \otimes B_v) = \Lambda^2 A_v \otimes \mathrm{Sym}^2 B_v \oplus \mathrm{Sym}^2 A_v \otimes \Lambda^2 B_v,$$

we have

$$\begin{aligned} L(s, \Lambda^2(A_v \otimes B_v) \otimes \chi_v(\varpi_v)) &= L(s, \Lambda^2 A_v \otimes \mathrm{Sym}^2 B_v \otimes \chi_v(\varpi_v)) \\ &\quad \cdot L(s, \mathrm{Sym}^2 A_v \otimes \Lambda^2 B_v \otimes \chi_v(\varpi_v)). \end{aligned}$$

But now

$$\mathrm{Sym}^2 B_v = \mathrm{diag}(\chi_v^2(\varpi_v), \chi_v\eta_v(\varpi_v), \eta_v^2(\varpi))$$

and

$$\Lambda^2 B_v = \chi_v(\varpi_v)\eta_v(\varpi_v).$$

Then

$$\begin{aligned}L(s, I(\sigma_v), \Lambda^2 \otimes \chi_v) = L(s, \Lambda^2 A_v \otimes \chi_v(\varpi_v))L(s, \Lambda^2 A_v \otimes \eta_v(\varpi_v))\\ \cdot L(s, \Lambda^2 A_v \otimes \chi_v\eta_v^2(\varpi_v))L(s, \mathrm{Sym}^2 A_v \otimes \eta_v(\varpi_v)).\end{aligned}$$

Taking the product for all $v \notin S$, this implies:

$$L_S(s, \pi, \Lambda^2 \otimes \chi)L_S(s, \pi, \Lambda^2 \otimes \eta)L_S(s, \pi, \Lambda^2 \otimes \chi\eta^2)L_S(s, \pi, \mathrm{Sym}^2 \otimes \eta) \neq 0$$

for $\mathrm{Re}(s) = 1$. All the L-functions attached to twists of Λ^2 are holomorphic for $\mathrm{Re}(s) = 1$, again by Theorem 9.6.2 of [4], n being odd. Suppose $\eta = \theta^2$, $\theta \in \widehat{F^* \setminus \mathbb{A}_F^*}$, so that

$$L_S(s, \pi, \mathrm{Sym}^2 \otimes \eta) = L_S(s, \pi \otimes \theta, \mathrm{Sym}^2).$$

Let ω_π be the central character of π. Then $\omega_\pi\theta^n$ becomes the central character of $\pi \otimes \theta$. Suppose $\omega_\pi^2\theta^{2n} \neq 1$. Then by Theorem 7.5 of [3] or [5] applied to Equation (1) with $\chi = 1$, $L_S(s, \pi \otimes \theta, \mathrm{Sym}^2)$ is holomorphic for $\mathrm{Re}(s) = 1$. Consequently

$$L_S(s, \pi, \Lambda^2 \otimes \chi) \neq 0$$

for $\mathrm{Re}(s) = 1$, completing the proposition.

Corollary 3.3. *Suppose n is odd. Then $L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ has a (simple) pole at $s = 1$ if and only if $\pi \otimes \chi \cong \widetilde{\pi}$.*

Proof. By

$$L_S(s, \pi \times (\pi \otimes \chi)) = L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi)L_S(s, \pi, \Lambda^2 \otimes \chi),$$

Propostion 3.2 of this paper and Theorem 9.6.2 of [4], $L_S(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ has a (simple) pole at $s = 1$ if and only if $L_S(s, \pi \times (\pi \otimes \chi))$ does. But by [5] the last L-function has a pole at $s = 1$ if and only if $\pi \otimes \chi \cong \widetilde{\pi}$.

4. The Completed L-Functions.

Suppose n is even. For each $v \in S$, let $L(s, \pi_v, \Lambda^2 \otimes \chi)$ and $L(s, \pi_v, \mathrm{Sym}^2 \otimes \chi)$ be the local L-functions attached to $(\tau_v \otimes \chi_v, r)$, where r is the adjoint action of $\widehat{M}$ on $\widehat{\mathfrak{n}}$ with $\mathbf{G} = \mathrm{Spin}_{2n}$ and Spin_{2n+1}, respectively, as in [9]. Here $\tau = \otimes_v \tau_v$, where τ is as in Section 2. We may and will use the local base change identity

$$L(s, \Pi_v, \Lambda^2) = L(s, \pi_v, \Lambda^2)L(s, \pi_v, \Lambda^2 \otimes \chi_v)$$

(cf. Ramakrishnan's proof for the global one) to define $L(s, \pi_v, \Lambda^2 \otimes \chi_v)$ if n is odd. This can be proved using the local global arguments in [9]. We can use the same type of identity to define $L(s, \pi_v, \mathrm{Sym}^2 \otimes \chi_v)$. Uniqueness argument of [9] shows that even if n is even, our L-functions, alluded to at the beginning of the paragraph, satisfy such identities. We now set

$$L(s, \pi, \Lambda^2 \otimes \chi) = \prod_v L(s, \pi_v, \Lambda^2 \otimes \chi_v)$$

and

$$L(s, \pi, \mathrm{Sym}^2 \otimes \chi) = \prod_v L(s, \pi_v, \mathrm{Sym}^2 \otimes \chi_v).$$

We then have (Theorem 7.7 of [9]).

Theorem 4.1. *The completed L-functions $L(s, \pi, \Lambda^2 \otimes \chi)$ and $L(s, \pi, \mathrm{Sym}^2 \otimes \chi)$ both sastisfy a standard functional equation, sending s to $1 - s$. Consequently they are both non-zero for $\mathrm{Re}(s) = 0$ and 1.*

References

[1] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies, Vol. 120, Princeton University Press, Princeton, 1989.

[2] D. Bump and S. Friedberg, *The "exterior square" automorphic L-functions on $GL(n)$*, in 'Festschrift' in honor of I. Piatetski-Shapiro (Editors: S. Gelbart, R. Howe, P. Sarnak), Israel Math. Conf. Proc., Weizmann, Jerusalem, 1990.

[3] D. Bump and D. Ginzburg, *Symmetric square L-functions on $GL(r)$*, Ann. of Math., **136** (1992), 137-205.

[4] H. Jacquet and J.A. Shalika, *Exterior square L-functions*, in 'Automorphic Forms, Shimura Varieties, and L-functions', Vol. 2 (Editors: L. Clozel and J. Milne), Academic Press, New York, 1990, 143-226.

[5] ———, *On Euler products and the classification of automorphic representations* II, Amer. J. Math., **103** (1981), 777-815.

[6] S. Kudla, S. Rallis, and D. Soudry, *On the degree 5 L-function for* Sp(2), Invent. Math., **107** (1992), 483-541.

[7] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups*, in 'Representation Theory and Harmonic Analysis on Semisimple Lie Groups' (Editors: P.J. Sally, Jr. and D.A. Vogan), Mathematical Surveys and Monographs, AMS, Vol. 31, 1989, 101-170.

[8] F. Shahidi, *On certain L-functions*, Amer. J. Math., **103** (1981), 297-355.

[9] ———, *A proof of Langlands' conjecture on Plancherel measures; Complementary series for p-adic groups*, Ann. of Math., **132** (1990), 273-330.

[10] ———, *Functional equation satisfied by certain L-functions*, Comp. Math., **37** (1978), 171-207.

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