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Symplectic homology is studied on closed symplectic manifolds where the class of the symplectic form and the first Chern class vanish on the second homotopy group. Critical values of the action functional are associated to cohomology classes of the manifold. Those lead to continuous sections in the action spectrum bundle. The action of the cohomology ring via the cap-action and the pants-product on the set of critical values is studied and a bi-invariant metric on the group of Hamiltonian symplectomorphisms is defined and analyzed. Finally, a relative symplectic capacity is defined which is bounded below by the π_1-sensitive Hofer-Zehnder capacity. As an application it is proven that a Hamiltonian automorphism whose support has finite such capacity has infinitely many nontrivial geometrically distinct periodic points.

1. Overview of the Results.

It is a well-known problem in symplectic geometry and Hamiltonian dynamics to study the existence of fixed points of Hamiltonian diffeomorphisms and to relate them to invariants from symplectic topology. The aim of this paper is to study the existence of "homologically visible" critical values of the action functional and their dependence on the Hamiltonian automorphism in the case of symplectically aspherical closed manifolds. The methods are provided by the theory of Floer homology. The initial aim of this paper is to consider a version of Floer homology refined by a filtration via the action functional. This version has been introduced by Floer and Hofer as so-called symplectic homology for open subsets of $\mathbb{R}^{2n}$, [**4**]. Here we study its generalization for closed symplectic manifolds which satisfy the property that

$$\omega_{|\pi_2(M)} = 0 \quad \text{and} \quad c_{1|\pi_2} = 0. \tag{A}$$

This condition forms the simplest case for which Floer homology was studied initially. Note that, very recently, examples have been constructed of closed manifolds satisfying (A) but having nontrivial second homotopy group, [**6**], [**9**]. Observe that a closed symplectic manifold satisfying (A) is necessarily non-simply connected and $\pi_1(M)$ contains elements of infinite order. From now on we call a symplectic manifold (M,ω) satisfying (A) **symplectically aspherical**.

In terms of Floer theory, condition (A) implies that for any given Hamiltonian $H\colon [0,1]\times M \to \mathbb{R}$, the associated action functional $\mathcal{A}_H$ on the space of free contractible loops is real-valued. Having in mind that the full Floer homology can already be uniquely associated to the time-1-map ϕ_H^1 generated by the Hamiltonian H one naturally asks how the action spectrum depends on the choice of H. By the **action spectrum** one denotes the set of critical values of the action functional. In fact, it is easy to prove that any two Hamiltonian functions H and K generating the same automorphism ϕ and homotopic to each other with respect to this property have the same action associated to the fixed points provided that they are normalized as follows

$$\text{(norm)} \qquad \int_M H(t,\cdot)\omega^n = 0 \quad \text{for all } t\,.$$

However, in general, two different homotopy classes of Hamiltonians generating the same time-1-map might have action spectrum differing by a quantity $I([H][K]^{-1})$ associated to the difference of the homotopy classes. In fact, one can define a group homomorphism

$$I\colon \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}$$

describing this obstruction. Here, $\mathrm{Ham}(M,\omega)$ is the group of Hamiltonian automorphisms. For a definition of I one considers $g\in\pi_1(\mathrm{Ham})$ and glues the trivial symplectic fibre bundles $D^2\times M$ with reversed orientations of the disk along their boundaries using a loop in $\mathrm{Ham}(M,\omega)$ representing g. The resulting symplectic fibre bundle E over S^2 carries the so-called coupling form ω_E defined by ω on $D^2\times M$ and one defines

$$I(g) = \int_{S^2} s^*\omega_E$$

for any section s of E. It follows from Floer theory that such a section exists and (A) implies that $I(g)\in\mathbb{R}$ does not depend on the choice of s. As an immediate consequence of a theorem by P. Seidel, [**22**], we observe:

Theorem 1.1. *The obstruction homomorphism $I\colon \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R}$ vanishes if (M,ω) is a closed symplectically aspherical manifold.*

We obtain for each $\phi\in\mathrm{Ham}(M,\omega)$ a well-defined action spectrum Σ_ϕ which as a whole provides the **action spectrum bundle** $\Sigma\to\mathrm{Ham}(M,\omega)$. The main result of this paper is a construction of continuous sections of this bundle associated to cohomology classes of M. Here the topology on $\mathrm{Ham}(M,\omega)$ is given by the bi-invariant Hofer-metric

$$\text{(1)} \qquad d_H\colon \mathrm{Ham}\times\mathrm{Ham}\to[0,\infty), \quad d_H(\phi,\mathrm{id}) = \inf\{\,\|H\| \mid \phi=\phi_H^1\,\}$$

where $\|H\| = \int_0^1 \operatorname{osc}_M H(t,\cdot)dt$.

The result of this paper was predominantly motivated by a remark by M. Bialy and L. Polterovich in [1], cf. 1.5.B. They already suggested a generalization of Hofer's minimax selector γ from the theory on $\mathbb{R}^{2n}$ to more general manifolds using Floer theory and an idea by C. Viterbo. An intrinsic motivation was the problem of infinitely many periodic points of a Hamiltonian automorphism, which is partly analyzed by Theorem 1.4 below.

Theorem 1.2. *Let (M,ω) be symplectically aspherical. Then for each non-zero cohomology class $\alpha \in H^*(M)$ there exists a section $c(\alpha)$ of the action spectrum bundle $\Sigma \to \operatorname{Ham}(M,\omega)$ which is continuous with respect to the metric from (1). These sections satisfy:*

1) *$c(\lambda\alpha) = c(\alpha)$ for all $\lambda \in \mathbb{R}$ and $\alpha \in H^*(M)$ with $\lambda\alpha \neq 0$.*
2) *$c([M]) \leq c(\alpha) \leq c(1)$ for all $0 \neq \alpha \in H^*(M)$ where $[M] = [\omega^n] \in H^{2n}(M)$ and $1 \in H^0(M)$ are the canonical generators. If $\alpha \in H^k(M)$ for $0 < k < 2n$ then we have strict inequality over the regular automorphisms.*
3) *$c(1;\phi) - c([M];\phi) \leq d_H(\phi, \mathrm{id}) = \inf\{\, \|H\| \mid \phi = \phi_H^1 \,\}$, $\phi \in \operatorname{Ham}(M,\omega)$.*
4) *If $\alpha \cup \beta \neq 0$ then $c(\alpha\cup\beta; \psi\circ\phi) \leq c(\alpha;\phi) + c(\beta;\psi)$ for all $\phi, \psi \in \operatorname{Ham}(M,\omega)$.*
5) *$c([M];\phi) = -c(1;\phi^{-1})$ for all $\phi \in \operatorname{Ham}(M,\omega)$.*

Note that the obstruction homomorphism I can be viewed as the *monodromy map* of the action spectrum bundle with respect to any of the sections $c(\alpha)$. This observation should be relevant for a generalization to the non-aspherical situation which will be studied in a separate paper.

The construction of the sections $c(\alpha)$ is based on Viterbo's idea for generating functions defined in the context of a cotangent bundle $M = T^*P$, cf. [23]. There, $c(\alpha)$ is a critical value of a generating function. In our case of closed symplectically aspherical manifolds we use the construction of an explicit isomorphism between Floer homology and standard cohomology of M which was introduced in [16]. A detailed description of this isomorphism will appear in [18]. The critical values of the Hamiltonian action functional then are defined as the infimum of all action levels below which the specified Floer homology class is still nontrivial.

The problem of finding a nontrivial continuous section in the action spectrum bundle was first introduced and treated by Hofer and Zehnder [7] in the context of open subsets of $\mathbb{R}^{2n}$ and a capacity for such symplectic manifolds. This so-called Hofer-Zehnder capacity, denoted by c_{HZ} below, is defined in terms of periodic solutions for compactly supported Hamiltonians on $\mathbb{R}^{2n}$. Using a variational minimax method for the associated action functional they constructed a so-called selector $\gamma\colon \operatorname{Ham}_{\mathrm{cpt}} \to \mathbb{R}$ which is continuous in the Hofer metric. For a detailed treatment see [8]. It is this continuous selector γ in the context of compactly supported Hamiltonians on $\mathbb{R}^{2n}$ which Theorem 1.2 replaces by the family of continuous sections $c(\alpha)$ in the case of closed symplectically aspherical manifolds. In the case of the open symplectic manifold given by a cotangent bundle, the result corresponding to Theorem 1.2 has been obtained by Y.-G. Oh in [**14**], [**15**]. There the finite-dimensional concept of C. Viterbo from [**23**] finding critical levels of generating functions has been applied in Floer homological context replacing the action functional for the generating function.

As a consequence of Theorem 1.2 we can consider the difference between $c(1;\phi)$ and $c([M];\phi)$ which is a continuous function of $\phi \in \mathrm{Ham}(M,\omega)$.

Theorem 1.3. *Given a closed symplectically aspherical manifold* (M,ω) *there exists a unique nonnegative function*

$$\gamma\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}_+, \quad \textit{such that } \gamma(\phi) = c(1;\phi) - c([M];\phi)$$

for all $\phi \in \mathrm{Ham}(M,\omega)$ *which is continuous in Hofer's metric. Moreover,* $d_\gamma(\phi,\psi) = \gamma(\phi\psi^{-1})$ *is a bi-invariant metric on* $\mathrm{Ham}(M,\omega)$ *bounded above by Hofer's metric.*

Since γ measures the maximal action difference of homologically visible 1-periodic solutions of the Hamilton equation, this difference can be related to the oscillation of H if there exists no non-constant 1-periodic solutions. We show in Theorem 5.11 below that γ coincides with the Hofer distance $\|\cdot\|$ for quasi-autonomous Hamiltonians which are admissible in the sense that they admit no non-constant contractible periodic solutions which are "fast", i.e., of period $T \leq 1$. This shows the close relationship between γ and the Hofer-Zehnder capacity defined via the maximal oscillation of an autonomous Hamiltonian admitting no non-constant periodic solutions, cf. [**8**]. Here, this capacity has to be refined as a π_1-sensitive capacity c^o_{HZ} with respect to the nontrivial fundamental group. We consider the larger set of admissible Hamiltonians which admit no non-constant *contractible* 1-periodic solutions. Based on γ we can define a relative capacity for subsets of (M,ω) which is monotone and invariant under global automorphisms of (M,ω). Defining a γ-capacity for subsets of M by

$$\begin{aligned} c_\gamma(A) = \sup\{\, \gamma(\phi) \mid \exists H \text{ s.t. } \phi = \phi^1_H, & \\ \operatorname{supp} X_H(t,\cdot) \subset A \text{ for all } t \in [0,1] \,\}&, \end{aligned}$$

we have the estimate

$$c_{\mathrm{HZ}}(U) \leq c^o_{\mathrm{HZ}}(U) \leq c_\gamma(U) \leq 2e(U)$$

for all open subsets $U \subset M$. Here, $e(U)$ is the displacement energy as considered in [**11**]. The idea of using symplectic homology in order to construct symplectic capacities was first carried out in the series of papers by Floer and Hofer et al., cf. [**4**], [**5**].

In the context of such a capacity based on the Floer homological approach we prove the following conditional existence result for infinitely many periodic points:

Theorem 1.4. *Assume that* $\phi \in \mathrm{Ham}(M,\omega)$ *admits a Hamiltonian function* H *such that* $\phi = \phi_H^1 \neq \mathrm{id}$ *and there exists a uniform bound on the* γ*-capacity of the support for all* ϕ_H^t, $t \in [0,1]$,

$$c_\gamma(\operatorname{supp}\phi_H^t) \leq m < \infty, \quad \text{for all } t \in [0,1].$$

Then ϕ *has infinitely many geometrically distinct non-constant periodic points corresponding to contractible solutions.*

There are clearly examples for such a bounded capacity of the support of ϕ_H^t, for example if the support can be separated from itself by a Hamiltonian isotopy. However, in general, in view of Theorem 1.3 it is obvious that such a uniform bound cannot exist if $\gamma(\phi^n) \to \infty$ for $n \to \infty$. Hence our method of finding infinitely many periodic points via the action spectrum is closely related to the question of the diameter of $\mathrm{Ham}(M,\omega)$ in the Hofer-metric. It is in fact conjectured that this diameter should always be infinite. Using the methods developed in this paper we are also able to reproduce several known examples of such infinite diameter.

It is clearly an interesting question how far one obtains similar results if $\omega_{|\pi_2} \neq 0$. This will be studied in a sequel.

Organization of the paper. In Section 2 we present the construction of symplectic homology on a closed symplectically aspherical manifold for the purpose of this paper. For a more thorough exposition in the case of $\mathbb{R}^{2n}$ see [**4**]. We associate critical levels of the Hamiltonian action functional to given cohomology classes of M using the explicit isomorphism between standard cohomology of M and Floer homology for a given Hamiltonian function H. It is shown that these homologically visible critical levels depend continuously on H in the C^0-norm.

In Section 3 we analyze more closely the action spectrum, i.e., the critical levels of the action functional for a given Hamiltonian and we show that these levels are already uniquely associated to the Hamiltonian equivalence class $\phi = [H] \in \widetilde{\mathrm{Ham}}(M,\omega)$ and do not depend on the specific Hamiltonian provided that we use a suitable normalization. We consider a more intrinsic version of Floer homology. Namely, Floer homology is already canonically associated to the automorphism $\phi \in \widetilde{\mathrm{Ham}}_{\mathrm{reg}}(M,\omega)$ such that we still have the filtration by the action functional. This requires methods of J-holomorphic sections in symplectic fibre bundles over Riemann surface which are used by P. Seidel in [**22**], where a monodromy action of $\pi_1(\mathrm{Ham}(M,\omega))$ on this intrinsically defined Floer homology is analyzed.

Using these methods, we obtain in Section 4 a concise proof for the "sharp energy estimate" for the pair-of-pants multiplication on Floer homology.

This estimate expresses the compatibility of symplectic homology with this multiplication provided that it is viewed as a product

$$HF_*(\phi) \times HF_*(\psi) \to HF_*(\phi\psi)\,.$$

The result of this estimate is the nontrivial fact that the nonnegative function $\gamma\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ satisfies the triangle inequality. As a consequence of the pair-of-pants multiplication we also obtain the continuity of the sections $c(\alpha)$ with respect to Hofer's metric.

In the last section we draw conclusions from the constructed metric γ. We define the γ-capacity, compare it to the π_1-sensitive Hofer-Zehnder capacity and deduce the results on the infinite number of periodic points.

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2. Symplectic homology and critical levels for the Hamiltonian action.

2.1. Symplectic homology. Let (M,ω) be a closed symplectically aspherical manifold, i.e., satisfying condition (A). Throughout this section we assume $H\colon S^1 \times M \to \mathbb{R}$ to be a regular Hamiltonian, i.e., every fixed point $x \in \mathrm{Fix}\,\phi_H^1$ is non-degenerate, where $\phi_H^1\colon \mathbb{R} \times M \to M$ is the time-1-map for the flow of the non-autonomous Hamiltonian vector field X_H defined by

$$\omega(X_H,\cdot) = -dH\,.$$

Denoting the space of contractible free loops by $\Omega^o(M) \subset C^\infty(\mathbb{R}/\mathbb{Z}, M)$ we define the set of 1-periodic contractible solutions of the Hamilton equation by

$$\mathcal{P}_1(H) = \{\, x \in \Omega^o(M) \,|\, \dot{x}(t) = X_H(t,x)\,\}.$$

We have $\mathcal{P}_1(H) = \mathrm{Crit}\,\mathcal{A}_H$ for the action functional

$$\mathcal{A}_H(x) = \int_{D^2} \bar{x}^*\omega - \int_{S^1} H(t,x)dt$$

where $\bar{x}\colon D^2 \to M$ is any extension of x to the unit disk. Note that $\mathcal{A}_H$ is real-valued due to assumption (A). In view of the same condition we have the integral grading by the Conley-Zehnder index $\mu\colon \mathcal{P}_1(H) \to \mathbb{Z}$, where we choose the normalization such that for an autonomous C^2-small Morse function H we have $\mu(x) = n - \mu_{\mathrm{Morse}}(x)$ for stationary $x \in \mathcal{P}_1(H) = \mathrm{Crit}\,H$, cf. [**17**].

Given a generic ω-compatible S^1-dependent almost complex structure $J(t,p)$, $(t,p) \in S^1 \times M$, i.e., $\omega \circ (\mathrm{id} \times J) = g_J$ is a Riemannian metric on $TM \to S^1 \times M$, we obtain the moduli spaces of Floer trajectories with the component-wise structure of a smooth manifold

$$\mathcal{M}_{y,x}(J,H) = \big\{ u\colon \mathbb{R} \times S^1 \to M \,|\, \partial_s u + J(t,u)\big(\partial_t u - X_H(t,u)\big) = 0, \\ u(-\infty,\cdot) = y,\ u(+\infty,\cdot) = x \big\}$$

where $u(\pm\infty,\cdot)$ denotes the uniform limit in $C^0(S^1,M)$ as $s \to \pm\infty$. A standard computation in Floer theory is:

Lemma 2.1. *The dimension satisfies* $\dim \mathcal{M}_{y,x}(J,H) = \mu(x) - \mu(y)$ *and for* $u \in \mathcal{M}_{y,x}(J,H)$ *we have*

$$E(u) \overset{\mathsf{def}}{=} \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2 ds dt \;=\; \mathcal{A}_H(x) - \mathcal{A}_H(y) \geq 0.$$

In particular, if the flow energy $E(u)$ *vanishes then* $x \equiv y$ *and* $u(s,t) = x(t)$ *for all* $(s,t) \in \mathbb{R} \times S^1$.

Heuristically, in view of the chosen sign conventions, the Floer trajectories correspond to the positive gradient flow of $\mathcal{A}_H$.

The full Floer homology is based on the chain complex obtained from the $\mathbb{Z}$-module $C_*(H) = \mathrm{Crit}_* \mathcal{A}_H \otimes \mathbb{Z}$ which is graded by μ. The boundary operator is defined as

$$\partial\colon C_k(H) \to C_{k-1}(H),$$
$$\partial x = \sum_{\mu(y)=k-1} \#_{\mathsf{alg}} \widehat{\mathcal{M}}_{y,x}(J,H)\, y,$$

where $\#_{\mathsf{alg}}$ denotes counting the finitely many unparameterized Floer trajectories with sign determined by a coherent orientation, cf. [3]. Here, $\widehat{\mathcal{M}}_{y,x} = \mathcal{M}_{y,x}/\mathbb{R}$ is the space of unparameterized trajectories after dividing out the free $\mathbb{R}$-action from shifting the trajectories in the s-variable. Floer's central theorem is that $\partial \circ \partial = 0$ and that the thus defined homology

$$HF_*(J,H) = H_*(C_*(H), \partial(J,H))$$

is canonically isomorphic to the singular cohomology of M.

Symplectic homology was developed by Floer and Hofer, cf. [4] using the additional filtration of the complex $C_*(H)$ by the action functional.

Definition 2.2. For any $a \in \mathbb{R}$ we define the the $\mathbb{Z}$-module

$$C_k^a(H) = \left\{ \sum_{\mu(x)=k} a_x\, x \in C_k(H) \,|\, a_x = 0 \quad \text{for}\ \ \mathcal{A}_H(x) > a \right\}.$$

In view of Lemma 2.1, $\big(C^a_*(H),\partial\big)$ is obviously a sub-complex of $(C_*(H),\partial)$. Denoting $C^{(a,b]}_*(H)=C^b_*(H)/C^a_*(H)$ for $a<b$ we have the short exact sequence of chain complexes

$$0\to C^a_*\xrightarrow{\ i\ }C^b_*\xrightarrow{\ j\ }C^{(a,b]}_*\to 0,\quad a<b\le\infty.$$

We use the convention that $C^\infty_*(H)=C_*(H)$ and $C^{(-\infty,a]}_*=C^a_*$. Consequently we obtain for the homology groups

$$HF^{(a,b]}_*=\big(C^{(a,b]}_*(H),\partial(J,H)\big)$$

the long exact sequence associated to $-\infty\le a<b<c\le\infty$,

(2)
$$\ldots\to HF^{(b,c]}_{k+1}\xrightarrow{\partial_*}HF^{(a,b]}_k\xrightarrow{i_*}HF^{(a,c]}_k\xrightarrow{j_*}HF^{(b,c]}_k\xrightarrow{\partial_*}HF^{(a,b]}_{k-1}\to\ldots.$$

Lemma 2.3. *It holds that $HF^{(-\infty,a]}_*(H)=\{0\}$ for $a\in\mathbb{R}$ small enough and $HF^{(b,\infty]}_*(H)=\{0\}$ for $b\in\mathbb{R}$ large enough.*

Proof. This follows immediately from the regularity of H which implies that $\mathcal{P}_1(H)$ is finite. So, choose $a<\min\{\,\mathcal{A}_H(x)\,|\,x\in\mathcal{P}_1(H)\,\}$ and $b>\max\{\,\mathcal{A}_H(x)\,|\,x\in\mathcal{P}_1(H)\,\}$. $\square$

Let us now analyze within the context of symplectic homology the concrete realization of the canonical isomorphism between Floer homology and singular cohomology which was introduced in [**16**]. We first define the isomorphism

$$\Phi\colon H^k(M)\xrightarrow{\cong}HF^{(-\infty,\infty]}_{n-k}(H).$$

We represent the standard cohomology of M in terms of Morse cohomology, see [**19**]. That is, let f be an auxiliary Morse function on M and g a generic Riemannian metric. We choose a smooth cut-off function

$$\beta^-(s)=\begin{cases}1, & s\le -1,\\ 0, & s\ge -\frac{1}{2},\end{cases}$$

and define the solution spaces of mixed type associated to $x\in\operatorname{Crit}_k f=\{x\in\operatorname{Crit} f|\mu_{\mathrm{Morse}}(x)=k\}$ and $y\in\mathcal{P}_1(H)$,

$$\begin{aligned}\mathcal{M}^-_{y;x}(H,J;f)=\Big\{(u,\gamma)\,\Big|\,&u\colon\mathbb{R}\times S^1\to M,\ \gamma\colon[0,\infty)\to M,\\ &\partial_s u+J\big(\partial_t u-\beta^-(s)X_H(u)\big)=0,\\ &\int_{-\infty}^{\infty}|\partial_s u|^2dsdt<\infty,\ \dot\gamma+\nabla_g f\circ\gamma=0,\\ &u(-\infty)=y,\quad u(+\infty)=\gamma(0),\quad \gamma(+\infty)=x\Big\}.\end{aligned}$$

Note that the condition of finite flow energy of u together with the cut-off function β^- in the perturbed Cauchy-Riemann equation implies by removable singularities that u has a continuous extension to $u(+\infty)$. For regularity, we have to allow even explicit dependence of J on the variable $s \in \mathbb{R}$ in order that for a generic J, f and g, $\mathcal{M}^-_{y;x}$ is again component-wise a manifold of dimension

$$\dim \mathcal{M}^-_{y;x}(H, J; f, g) = n - \mu_{\text{Morse}}(x) - \mu(y)\,.$$

(See also [**20**].) Moreover, in view of (A) and standard bubbling-off analysis these solution spaces are compact in dimension 0, i.e., finite and the following $\mathbb{Z}$-module homomorphism is well-defined,

$$\begin{aligned} &\Phi_\bullet \colon C^k(f) \to C_{n-k}(H), \\ &\Phi_\bullet(x) = \sum_{\mu(y)+\mu_{\text{Morse}}(x)=n} \#_{\mathsf{alg}} \mathcal{M}^-_{y;x}(H, J; f, g)\, y. \end{aligned} \tag{3}$$

A theorem analogous to Floer's central theorem states that $\Phi_\bullet$ commutes with the boundary operator $\partial(H, J)$ on $C_*(H)$ and the coboundary operator of the Morse cochain complex associated to f and g. Hence we obtain the induced homomorphism

$$\Phi \colon H^k(M) \to HF_{n-k}(H, J).$$

Details are carried out in [**18**]. There it is shown that Φ is indeed an isomorphism. We have an analogous representation of Φ^{-1}. Define in the reversed order for $x \in \operatorname{Crit} f$ and $y \in \mathcal{P}_1(H)$

$$\begin{aligned} \mathcal{M}^+_{x;y}(H, J; f) = \Big\{ (\gamma, u) \,\Big|\, & u \colon \mathbb{R} \times S^1 \to M,\ \gamma \colon (-\infty, 0] \to M, \\ & \partial_s u + J\big(\partial_t u - \beta^-(-s) X_H(u)\big) = 0, \\ & \int_{-\infty}^{\infty} |\partial_s u|^2 ds dt < \infty,\ \dot\gamma + \nabla_g f \circ \gamma = 0, \\ & \gamma(-\infty) = x, \quad \gamma(0) = u(-\infty), \quad u(+\infty) = y. \end{aligned}$$

This implies for generic J, f and g manifolds of dimension $\mu(y) + \mu_{\text{Morse}}(x) - n$, so that we obtain

$$\begin{aligned} &\Psi_\bullet \colon C_k(H) \to C^{n-k}(f), \\ &\Psi_\bullet(y) = \sum_{\mu(y)+\mu_{\text{Morse}}(x)=n} \#_{\mathsf{alg}} \mathcal{M}^+_{x;y}(H, J; f, g)\, x. \end{aligned} \tag{4}$$

In [**16**] it is shown that the induced homomorphism

$$\Psi \colon HF_k(H) \to H^{n-k}(M)$$

equals Φ^{-1}.

2.2. Critical levels of the action functional. Following an idea of Viterbo, which in [**23**] was used to define symplectic capacities via generating functions, we will associate critical levels $c(\alpha; H)$ for the action functional $\mathcal{A}_H$ associated to given cohomology classes α in M. The same idea was also employed in [**14**].

Definition 2.4. Given the Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ we define

$$\begin{aligned} E_+(H) &= -\int_0^1 \inf_M H(t,\cdot)dt, \\ E_-(H) &= -\int_0^1 \sup_M H(t,\cdot)dt, \\ \text{and} \quad \|H\| &= E_+(H) - E_-(H). \end{aligned}$$

Note that $\|H\|$ is a semi-norm with $\|H-K\| \le \|H-K\|_{C^0(M\times S^1)}$. Consider now $x \in \mathrm{Crit}_k(f)$ and $\Phi_\bullet(x) = \sum_{\mu(y)=n-k} a_y\, y$. It is a straightforward computation (compare the energy estimate, Lemma 4.1 in [**21**] and the proof of Lemma 2.12) to show that

$$\mathcal{A}_H(y) \le E_+(H) \quad \text{for } a_y \ne 0. \tag{5}$$

Analogously, we have

$$\mathcal{A}_H(y) \ge E_-(H) \quad \text{whenever } \Psi_\bullet(y) \ne 0. \tag{6}$$

This shows:

Lemma 2.5. *The isomorphisms Φ and Ψ factorize as*

$$\Phi\colon H^k(M) \xrightarrow{\tilde\Phi} HF_{n-k}^{(-\infty,E_+(H)]}(H) \xrightarrow{i_*} HF_{n-k}(H),$$

$$\Psi\colon HF_k(H) \xrightarrow{j_*} HF_k^{(E_-(H)-\epsilon,\infty]} \xrightarrow{\tilde\Psi} H^{n-k}(M), \textit{ for all } \epsilon > 0.$$

Consider the diagram for given $a \in \mathbb{R}$

$$\begin{tikzcd} & H^k(M) \arrow[d, "\Phi"'] & \\ HF_{n-k}^{(-\infty,a]}(H) \arrow[r, "i_*^a"] & HF_{n-k}^{(-\infty,\infty]}(H) \arrow[r, "j_*^a"] \arrow[d, "\Psi"'] & HF_{n-k}^{(a,\infty]}(H) \\ & H^k(M). & \end{tikzcd}$$

In other terms, Lemma 2.5 says that

$$\begin{cases} j_*^a \circ \Phi = 0, \text{ i.e., } \operatorname{im}\Phi \subset \operatorname{im} i_*^a, & \text{if } a \ge E_+(H), \\ \Psi \circ i_*^a = 0, \text{ i.e., } \operatorname{im}\Phi \cap \operatorname{im} i_*^a = \{0\} & \text{if } a < E_-(H). \end{cases} \tag{7}$$

This lends itself to the following definition:

Definition 2.6. Given any nonzero cohomology class $\alpha \in H^*(M)$ we define

$$c(\alpha; H) = \inf\{\, a \in \mathbb{R} \mid j_*^a(\Phi(\alpha)) = 0\}.$$

In view of Lemma 2.5 and (7) the $c(\alpha; H)$ are finite real numbers which are obviously critical values of $\mathcal{A}_H$. They satisfy

$$E_-(H) \le c(\alpha; H) \le E_+(H) \quad \text{for all } 0 \ne \alpha \in H^*(M). \tag{8}$$

In the following we discuss the behaviour of $c(\alpha; H)$ with respect to variations of H leading to first continuity properties, and with respect to variations of α, in terms of the cohomology ring structure on $H^*(M)$.

2.3. The cap-action of $H^*(M)$. Let us recall from [**2**] the definition of the cap-action

$$\cap\colon H^l(M) \times HF_k(H) \to HF_{k-l}(H). \tag{9}$$

Given a generic Morse function f and a Riemannian metric g as above we denote for a critical point $x \in \operatorname{Crit} f$

$$W_x^s(f) = \left\{\, \gamma\colon [0,\infty) \to M \mid \dot\gamma + \nabla_g f \circ \gamma = 0,\ \gamma(+\infty) = x \,\right\}.$$

Recall the definition of the Floer trajectory space $\mathcal{M}_{z,y}(J, H)$ from above. Given $y, z \in \mathcal{P}_1(H)$ we set

$$\mathcal{M}_{z;x;y}(H, J; f, g) = \left\{\, (u, \gamma) \in \mathcal{M}_{z,y} \times W_x^s \mid u(0,0) = \gamma(0) \,\right\}.$$

As before, for generic J, f and g, this is a manifold of dimension

$$\dim \mathcal{M}_{z;x;y} = \mu(y) - \mu(z) - \mu_{\mathrm{Morse}}(x),$$

compact in dimension 0. Thus, we can define

$$\cap\colon C^l(f) \times C_k(H) \to C_{k-l}(H),$$

$$x \cap y = \sum_{\mu(z) = \mu(y) - \mu_{\mathrm{Morse}}} \#_{\mathsf{alg}} \mathcal{M}_{z;x;y}(H; f)\, z.$$

Again, $\cap$ commutes with the boundary operators in the standard way so that we obtain the cap-action (9) of the standard cohomology ring $H^*(M)$ on Floer homology. The first of the following crucial relations is essentially due to Floer and has been studied in full details in [**13**].

Proposition 2.7.

1) *The operation* $\cap$ *is a ring operation, i.e.,*

$$(u \cap \alpha) \cap \beta = u \cap (\alpha \cup \beta),$$

for all $\alpha, \beta \in H^*(M)$, $u \in HF_*(H)$.

2) *It is compatible with the isomorphism* $\Phi\colon H^*(M) \to HF_*(H)$,

$$\Phi(\alpha) \cap \beta = \Phi(\alpha \cup \beta).$$

The proof of the second property is carried out in [**18**].

It should be no surprise that this cap-action is also compatible with the refined structure as symplectic homology. We have:

Lemma 2.8. *For generic J, f and g and $\mu_{Morse}(x) > 0$, a non-empty solution space $\mathcal{M}_{z;x;y}(H, J; f, g)$ implies*

$$\mathcal{A}_H(z) \le \mathcal{A}_H(y) - \epsilon(H)$$

with $0 < \epsilon(H) \stackrel{\mathsf{def}}{=} \min\{0 < \mathcal{A}_H(y) - \mathcal{A}_H(z) | y, z \in \mathcal{P}_1(H)\}$.

Proof. Suppose the assertion is not true. By definition of $\epsilon(H)$ we have $\mathcal{A}_H(y) = \mathcal{A}_H(z)$. Hence, due to Lemma 2.1, $u \in \mathcal{M}_{z,y}$ must be constant, i.e., $u(0,0) = y(0) = z(0) \in \operatorname{Fix}\phi_H^1$. However, $u(0,0) \in W_x^s(f, g)$, but for generic f and g, W_x^s is a manifold of codimension at least 1 and does not intersect the finite set $\operatorname{Fix}\phi_H^1$. □

We obtain:

Proposition 2.9. *Given $\alpha, \beta \in H^*(M)$ with $\alpha \cup \beta \neq 0$ and $\beta \neq 1$ we have the estimate*

$$c(\alpha \cup \beta; H) \le c(\alpha; H) - \epsilon(H),$$

in particular, $c(\alpha \cup \beta; H) < c(\alpha; H)$.

Proof. In view of the definition of $c(\cdot; H)$ suppose that $j_*^a(\Phi(\alpha)) = 0$. Due to Lemma 2.8 it follows that $\Phi(\alpha) \cap \beta \in \operatorname{im} i_*^{a-\epsilon(H)}$. Hence by Proposition 2.7 we have $j_*^{a-\epsilon(H)}\big(\Phi(\alpha \cup \beta)\big)$. □

We have the following a priori estimate for the critical levels $c(\alpha; H)$, compare (8).

Proposition 2.10. *Let $1 \in H^0(M)$ and $[M] \in H^{2n}(M)$ be the two canonical classes in the ring $H^*(M)$. We have*

$$E_-(H) \le c([M]; H) \;<\; c(1; H) \le E_+(H),$$

in particular,

$$c([M]; H) \;<\; c(\alpha; H) \;<\; c(1; H) \quad \textit{for any } \alpha \in H^k(M),\ 0 < k < 2n.$$

The upper estimate follows directly from Proposition 2.9, the lower estimate is due to the Poincaré duality which will be discussed below.

Following Viterbo in [**23**] we can make the following:

Definition 2.11. Given any regular Hamiltonian H we define the positive number $\gamma(H)$,

$$0 < \epsilon(H) \le \gamma(H) \stackrel{\mathsf{def}}{=} c(1; H) - c([M]; H) \le \|H\|.$$

2.4. First continuity properties. Let H and K be any two Hamiltonians and consider the associated canonical isomorphism between the associated Floer homologies (cf. [**17**], [**20**])

$$\Phi_{KH}\colon HF_*(H) \xrightarrow{\cong} HF_*(K).$$

Namely, we choose a monotone increasing smooth cut-off function $\beta(s)$, say $\beta(s)=0$ for $s\le -1$ and $\beta(s)=1$ for $s\ge 1$, and we define the homotopy of Hamiltonians

$$G_s(t,x) = K(t,x) + \beta(s)\big(H(t,x)-K(t,x)\big).$$

Given $y\in\mathcal{P}_1(K)$ and $x\in\mathcal{P}_1(H)$ we recall the associated moduli space of homotopy trajectories

$$\mathcal{M}_{y,x}(G_s,J) = \{\, u \,|\, \partial_s u + J(\partial_t u - X_{G_s}(u)) = 0,\ u(-\infty)=y,\ u(+\infty)=x \,\}.$$

We have the following energy estimate.

Lemma 2.12. *If $u\in\mathcal{M}_{y,x}(G_s)$ then*

$$\mathcal{A}_K(y) \le \mathcal{A}_H(x) + E^+(H-K).$$

Recall the definition of $E^+(f) = -E^-(-f)$ in 2.4.

Proof. A simple computation shows

$$\begin{aligned}
0 &\le \int_{-\infty}^{\infty}\int_0^1 |\partial_s u|^2 dsdt = \iint_{\mathbb{R}\times S^1} u^*\omega - \int_{-\infty}^{\infty}\int_0^1 \omega(u_s, X_{G_s})dsdt \\
&= \mathcal{A}_H(x) - \mathcal{A}_K(y) + \int_{-\infty}^{\infty}\beta'(s)\int_0^1 \big(H(t,u(s,t)) - K(t,u(s,t))\big)dtds \\
&\le \mathcal{A}_H(x) - \mathcal{A}_K(y) + \int_0^1 \sup_p \big(H(t,p)-K(t,p)\big)dt.
\end{aligned}$$

$\square$

Thus it follows that the canonical homomorphism is already defined as

$$\Phi_{KH}\colon HF_*^{(-\infty,a]}(H) \to HF_*^{(-\infty,a+E^+(H-K)]}(K).$$

What is more, it is compatible with the long exact sequence of symplectic homology (2), i.e., we have the commutative diagram with $e^+ = E^+(H-K)$,

(10)

$$\begin{array}{ccccccc}
HF^{(-\infty,a]}(H) & \xrightarrow{i_*^a} & HF^{(-\infty,\infty)}(H) & \xrightarrow{j_*^a} & HF^{(a,\infty]}(H) & \xrightarrow{\partial_*} & \\
\Big\downarrow \Phi_{KH} & & \cong\Big\downarrow \Phi_{KH} & & \Big\downarrow \Phi_{KH} & & \\
HF^{(-\infty,a+e^+]}(K) & \xrightarrow{i_*^{a+e^+}} & HF^{(-\infty,\infty)}(K) & \xrightarrow{j_*^{a+e^+}} & HF^{(a+e^+,\infty]}(K) & \xrightarrow{\partial_*} & .
\end{array}$$

Moreover, recall from the explicit isomorphism between Morse homology and Floer homology that for $\Phi_H \colon H^*(M) \xrightarrow{\cong} HF_{n-*}(H)$ we have

$$\Phi_K = \Phi_{KH} \circ \Phi_H. \tag{11}$$

Thus, if $j_*^a\big(\Phi_H(\alpha)\big) = 0$ for some $\alpha \in H^*(M)$ and $a \in \mathbb{R}$ it follows by (10) and (11) that also $j_*^{a+e^+}\big(\Phi_K(\alpha)\big) = 0$. We obtain:

Lemma 2.13. *For any two Hamiltonians H and K and any nonzero cohomology class $\alpha \in H^*(M)$ we have the estimate*

$$c(\alpha; K) \le c(\alpha; H) + E^+(H - K).$$

This immediately implies the property that the map $H \mapsto c(\alpha; H) \in \sigma(\phi_H^1) \subset \mathbb{R}$ is continuous with respect to the semi-norm

$$\|H\| = \int_0^1 \operatorname{osc}_x H(t,x)dt \;=\; E_+(H) - E_-(H),$$

namely

$$\big|c(\alpha; H) - c(\alpha; K)\big| \le \|H - K\| \tag{12}$$

for each $0 \neq \alpha \in H^*(M)$. By this first continuity result we see that $c(\alpha, H)$ is in fact well-defined for all Hamiltonians. Denoting by $\mathcal{H}$ the set of all Hamiltonians and by $\mathcal{H}_{reg}$ the dense subset of regular ones we have:

Proposition 2.14. *We have a well-defined function*

$$c\colon H^*(M) \setminus \{0\} \times \mathcal{H} \to \mathbb{R}, \quad (\alpha, H) \mapsto c(\alpha, H),$$

which is continuous in H with respect to the semi-norm $\|H\|$.

Proof. This follows immediately from (12) recalling from [**17**] that $\mathcal{H}_{reg} \subset \mathcal{H}$ is C^0-dense □

From now on we do not assume regularity of H without further notice.

3. The action spectrum.

The fact known for $(\mathbb{R}^{2n}, \omega_o)$ that the action spectrum is well-defined already for a Hamiltonian automorphism with compact support regardless of the chosen Hamiltonian will now be transfered to the case of a symplectically aspherical closed manifold. However, a priori we have to lift the analysis to the universal covering of $\mathrm{Ham}(M, \omega)$. Choosing a suitable normalization for the generating Hamiltonian functions H we observe that the above functions $c(\alpha, \cdot)$ are uniquely defined on this covering group and we will finally show that they are continuous sections in the so-called action spectrum bundle over $\widetilde{\mathrm{Ham}}(M,\omega)$ with respect to Hofer's bi-invariant metric. Recall the definition of this metric on $\mathrm{Ham}(M,\omega)$,

$$d_H(\phi,\psi) = d_H(\phi\psi^{-1},\mathrm{id}), \quad d_H(\phi,\mathrm{id}) = \inf\{\,\|H\| \mid \phi = \phi_H^1\,\},$$
$$\|H\| = \int_0^1 \operatorname*{osc}_{x\in M} H(t,x)\,dt.$$

From now on we assume that the symplectic manifold M is connected and that $\big(\mathrm{Ham}(M,\omega), d_H\big)$ is the topological group with Hofer's bi-invariant metric.

3.1. The action spectrum bundle. Let H and K be two Hamiltonians which generate the same symplectomorphism,

$$\phi_H^1 = \phi_K^1 \stackrel{\mathsf{def}}{=} \phi.$$

Without loss of generality we can assume that $H(1,\cdot) = K(1,\cdot) = 0$. Namely, replace H by $H_\alpha(t,\cdot) = \alpha'(t)H(\alpha(t),\cdot)$ where α is any monotone map $\alpha\colon [0,1] \to [0,1]$ satisfying $\alpha(0) = 0$, $\alpha(1) = 1$ and α' has compact support in $(0,1)$. Consider the loop in $\mathrm{Ham}(M,\omega)$ based at the identity

$$g_t = \begin{cases} \phi_H^t, & 0 \le t \le 1, \\ \phi_K^{2-t}, & 1 \le t \le 2\,. \end{cases} \tag{13}$$

Clearly, g_t is the flow associated to the Hamiltonian

$$\varphi^t = \phi_G^t, \quad G(t,\cdot) = \begin{cases} H(t,\cdot), & 0 \le t \le 1, \\ -K(2-t,\cdot), & 1 \le t \le 2\,. \end{cases}$$

Let $\chi\colon M \to \Omega(M)$ be the induced continuous map into the free loop space $\chi(p) = (g_t(p))_{t\in[0,2]}$. Thus, $\chi(p_o)$ is contractible exactly if all $\chi(p)$ are contractible. From Floer theory we know that for any k-periodic Hamiltonian function $G(t+k,p) = G(t,p)$ there exists at least one contractible k-periodic solution of the associated Hamiltonian equation. Therefore the map χ has its image in the component of contractible loops and we obtain:

Proposition 3.1. *Given any two Hamiltonians H and K generating the same symplectomorphism $\phi_H^1 = \phi_K^1$, there is a canonical identification of the contractible* 1*-periodic solutions*

$$\mathcal{P}_1(H) \cong \mathcal{P}_1(K) \neq \emptyset.$$

Hence, we can define for $\phi \in \mathrm{Ham}(M,\omega)$

$$\mathrm{Fix}^o(\phi) = \big\{\, x(0) \mid x \in \mathcal{P}_1(H),\ \phi = \phi_H^1\,\big\}. \tag{14}$$

In the following we use the same notation for $x \in \mathrm{Fix}^o(\phi_H^1)$ and $x \in \mathcal{P}_1(H)$.

In order to associate a well-defined action already to the fixed point $x \in \mathrm{Fix}^o(\phi)$ we consider the following function $I_g\colon M \to \mathbb{R}$. Let $g\colon [0,1] \to \mathrm{Ham}(M,\omega)$ be a closed loop based at the identity

$$g_0 = g_1, \quad g_t = \phi_G^t, \quad G(t+1,p) = G(t,p).$$

Given any $p \in M$ denote by $\widetilde{g_t(p)}$ an extension of the contractible loop to the unit disk. Since $\omega_{|\pi_2} = 0$, the function

$$I(g,p) = \int_{D^2} \widetilde{g_t(p)}^* \omega - \int_0^1 G(t, g_t(p))dt$$

is well-defined.

Definition 3.2. We call a Hamiltonian function $H\colon \mathbb{R} \times M \to \mathbb{R}$ **normalized** if

$$\int_M H(t,\cdot)\omega^n = 0 \quad \text{for all } t \in \mathbb{R}.$$

Obviously, for any Hamiltonian isotopy ϕ_H^t, H can be normalized,

$$H_{\mathrm{norm}}(t,x) = H(t,x) - \frac{1}{\int_M \omega^n} \int_M H(t,\cdot)\omega^n, \tag{15}$$

and the action changes by a constant,

$$\mathcal{A}_H = \mathcal{A}_{H_{\mathrm{norm}}} - \int_0^1 \int_M H(t,x)\omega^n dt.$$

Lemma 3.3. *The function $I(g,p)$ does not depend on $p \in M$ and $I(g)$ is invariant under homotopies of the loop g if the Hamiltonian G is homotoped through normalized functions.*

Proof. Let $s \mapsto p(s)$, $s \in (-\epsilon, \epsilon)$ be any differentiable arc with $p'(0) = \xi$, then

$$\begin{aligned} \frac{d}{ds}\big(I(g,p(s))\big)\big|_{s=0} &= \int_0^1 \omega\big(Dg_t(p)\xi, X_G(t, g_t(p))\big)\, dt \\ &\quad - \int_0^1 dG(t, g_t(p)) \circ Dg_t(p)\xi\, dt = 0, \end{aligned}$$

hence $I(g,p) = I(g)$ for all $p \in M$. Let $g_{s,t} = \phi_{G_s}^t$, $G(s,t+1,p) = G(s,t,p)$, $s \in (-\epsilon,\epsilon)$, be a 1-parameter family of Hamiltonian loops based at id_M. Then a straightforward computation shows

$$\frac{d}{ds}\big(I(g(s),p)\big) = -\int_0^1 \partial_s G(s,t,g_{s,t}(p))\, dt \quad \text{f.a.}\, p \in M.$$

Integration over M with respect to ω^n shows that $\frac{d}{ds}I(g(s)) = 0$. □

Clearly, we obtain group homomorphisms

$$I\colon \pi_1(\mathrm{Ham}(M,\omega)) \to \mathbb{R} \tag{16}$$

describing the obstruction to a well-defined action spectrum for each $\phi \in \mathrm{Ham}(M,\omega)$.

Remark 3.4. If every element of $\pi_1(\mathrm{Ham}(M,\omega))$ is of finite order, then $I \equiv 0$. In general, if $\omega_{|\pi_2} \neq 0$, we only have $I\colon \pi_1(\mathrm{Ham}) \to \mathbb{R}/\omega(\pi_2(M))$ and this argument fails.

Given any point $p_o \in M$ consider the evaluation map $\mathrm{ev}_{p_o}\colon \mathrm{Ham}(M,\omega) \to M$, $\phi \mapsto \phi(p_o)$. Like in the proof of Proposition 3.1 we know that the induced homomorphism $\pi_1(\mathrm{Ham},\mathrm{id}) \to \pi_1(M,p_o)$ vanishes, $\mathrm{ev}_* = 0$. Moreover, ev_{p_o} is a locally trivial fibration with typical fibre $\mathrm{Ham}_{p_o} = \left\{ \phi \in \mathrm{Ham} \mid \phi(p_o) = p_o \right\}$. We obtain the exact homotopy sequence

$$\ldots \to \pi_2(M,p_o) \to \pi_1(\mathrm{Ham}_{p_o},\mathrm{id}) \to \pi_1(\mathrm{Ham},\mathrm{id}) \to 0.$$

We conclude:

Remark 3.5. For each $\{g\} \in \pi_1(\mathrm{Ham},\mathrm{id})$ and $p_o \in M$ there exists a normalized generating Hamiltonian $G(t,p)$ such that $\nabla G(t,p_o) = 0$ for all $t \in S^1$ and thus

$$I(\{g\}) = -\int_0^1 G(t,p_o)\,dt.$$

Example 3.6. Consider (S^2,ω_o) with $\omega_o(S^2) = 1$ and $\{g\} \in \mathrm{Ham}(S^2,\omega_o) = \mathbb{Z}_2$ generated by the normalized autonomous Hamiltonian G generating the rotation around the axis through the poles. Its values at the poles are $\pm 1/2$. Hence for the homomorphism $I\colon \mathbb{Z}_2 \to \mathbb{R}/\mathbb{Z}$ we have $I(\{g\}) = I(1) = 1/2 \pmod 1$.

Let the Hamiltonians H and K generate the same automorphism $\phi_H^1 = \phi_K^1$. We call them *equivalent*, $H \sim K$ if there exists a homotopy $(G_s)_{s\in[0,1]}$ of Hamiltonians such that each G_s generates the same automorphism, $\phi_{G_s}^1 = \phi_H^1 = \phi_K^1$ for all $s \in [0,1]$. It is easy to see that the group of equivalence classes is the universal covering of $\mathrm{Ham}(M,\omega)$,

$$\widetilde{\mathrm{Ham}}(M,\omega) = \left\{ [H] \mid H \sim K,\ H,K \text{ normalized} \right\}.$$

From the previous analysis it follows that we obtain a well-defined action spectrum bundle over the group $\widetilde{\mathrm{Ham}}$,

$$\Sigma = \bigcup_{\bar\phi \in \widetilde{\mathrm{Ham}}} \{\bar\phi\} \times \Sigma_{\bar\phi}, \quad \Sigma_{\bar\phi} = \{\mathcal{A}_H(x) | x \in \mathrm{Fix}^o(\phi_H^1),\ \bar\phi = [H]\}.$$

The action $\pi_1(\mathrm{Ham}) \times \widetilde{\mathrm{Ham}} \to \widetilde{\mathrm{Ham}}$, $(\gamma,\bar\phi) \mapsto \gamma\bar\phi$ gives

$$\mathcal{A}_{\gamma\bar\phi}(x) = \mathcal{A}_{\bar\phi}(x) + I(\gamma), \quad \text{for all } x \in \mathrm{Fix}^o(\phi),\ \phi = \pi(\bar\phi).$$

Analogously to the situation of compactly supported automorphisms of $(\mathbb{R}^{2n}, \omega_o)$ as studied in [8] we have:

Proposition 3.7. *The action spectrum $\mathcal{A}_{\bar\phi} \subset \mathbb{R}$ is compact and nowhere dense.*

Proof:

Lemma 3.8. *The regular values of the Hamiltonian action functional $\mathcal{A}_H$ form a residual subset of $\mathbb{R}$.*

Proof. We will construct a smooth function on a finite-dimensional manifold whose set of critical values contains all critical values of $\mathcal{A}_H$. Thus by Sard's theorem the claim follows.

Let $S_2 = \mathbb{R}/2\mathbb{Z}$ and define the function $F\colon H^{1,2}(S_2, M) \to \mathbb{R}$,

$$F(x) = \int_{D^2} \tilde{x}^*\omega - \int_0^1 H(t, x(t)),$$

where $D^2 = \{z \,|\, |z| \le 1\}$, $\tilde{x}\colon D^2 \to M$, $\tilde{x}(e^{\pi i t}) = x(t)$, $t \in [0,2]$. Its differential is given by

$$dF(x)(\xi) = \int_0^2 \big\langle -J\big(\dot{x} - \chi_{[0,1]} X_H(t, x(t))\big), \xi(t)\big\rangle dt$$

for $\xi \in H^{1,2}(x^*TM)$. We obtain the embedding of the critical points of $\mathcal{A}_H$ into the critical set of F,

$$i\colon \operatorname{Crit} \mathcal{A}_H \hookrightarrow \operatorname{Crit} F, \quad i(x)(t) = \begin{cases} x(t), & 0 \le t \le 1, \\ x(1), & 1 \le t \le 2\,. \end{cases}$$

with $\mathcal{A}_H(x) = F(i(x))$ for all $x \in \operatorname{Crit} \mathcal{A}_H$. Clearly, $i(\operatorname{Crit} \mathcal{A}_H) \subset H^{1,2}(S_2, M)$ and F is a smooth function.

Denote now by $U \subset TM$ the injectivity neighbourhood of the exponential map such that

$$\exp\colon U \xrightarrow{\approx} V(\triangle) \subset M \times M, \quad \exp(p, v) = (p, \exp_p(v)).$$

We define the mapping $c\colon X \to H^{1,2}(S_2, M)$ on the open subset $X = \{\, x \in M \,|\, (x, \phi_H^1(x)) \in U \,\}$ by

$$c(p)(t) = \begin{cases} \phi_H^t(p), & 0 \le t \le 1, \\ \exp_p\big((2-t)\exp^{-1}(p, \phi_H^1(p))\big), & 1 \le t \le 2. \end{cases}$$

The mapping c is smooth and we have $c(x(0)) = i(x)$ for all $x \in \operatorname{Crit} \mathcal{A}_H$. Defining

$$f\colon X \to \mathbb{R}, \quad f(p) = F(c(p))$$

we conclude that the critical values of $\mathcal{A}_H$ form a subset of the critical values of the smooth function f. $\square$

The proof of Proposition 3.7 follows immediately from this Lemma and the compactness of the set of critical points of $\mathcal{A}_H$ since it can be identified with the fixed point set $\mathrm{Fix}^o(\phi) \subset M$. $\square$

We now conclude that $c(\alpha; H)$ only depends on the equivalence class $[H] \in \widetilde{\mathrm{Ham}}(M, \omega)$. Consider H and K such that there exists a homotopy $H \overset{G_s}{\simeq} K$, that is $(G_s)_{s\in[0,1]}$, with $\phi^1_H = \phi^1_{G_s} = \phi^1_K$ for all $s \in [0, 1]$. After subdividing G_s in homotopies (G_s), $s \in [s_i, s_{i+1}]$, $0 = s_0 < s_1 < \ldots < s_N = 1$, we have

$$\left|c(\alpha; G_{s_{i+1}}) - c(\alpha; G_{s_i})\right| \leq \|G_{s_{i+1}} - G_{s_i}\|.$$

Choosing the subdivision small enough we obtain in view of (12) and Proposition 3.7 that $c(\alpha; H) = c(\alpha; K)$. This idea of an *adiabatic* homotopy implies that $c(\alpha, \cdot)$ can be viewed as sections of the action spectrum bundle over $\widetilde{\mathrm{Ham}}$.

Proposition 3.9. *For every nonzero cohomology class $\alpha \in H^*(M)$ we obtain a section $c(\alpha)$ of the action spectrum bundle $\Sigma \to \widetilde{\mathrm{Ham}}(M, \omega)$ which is continuous with respect to the Hofer-metric $d_{\tilde{H}}(\bar{\phi}, \mathrm{id}) = \inf \{ \|H\| \mid \bar{\phi} = [H] \}$ on the covering $\widetilde{\mathrm{Ham}}(M, \omega)$.*

Note that the continuity does not follow directly from the C^0-continuity $H \mapsto c(\alpha; H)$ as shown in Lemma 2.13. We will prove continuity in Corollary 4.10 below using an additional structure with respect to the pair-of-pants product.

Since the sections $c(\alpha)$ are only defined over the universal covering $\widetilde{\mathrm{Ham}}$ it is important to determine the action of $\pi_1(\mathrm{Ham})$ on them. Let us introduce a more suitable description of Floer homology which has also been used by P. Seidel in [**22**] in order to study $\pi_1(\mathrm{Ham})$.

3.2. The intrinsic Floer homology. Let $\phi \in \mathrm{Ham}(M, \omega)$ be a given automorphism and consider the associated mapping torus as a symplectic fibre bundle over S^1,

$$M_\phi = \{(t, x) \in \mathbb{R} \times M\}/((t, \phi(x)) \sim (t + 1, x)) \to S^1. \tag{17}$$

Obviously $\omega \in \Omega^2(M_\phi)$ is a closed 2-form and (M, ω) is the typical fibre. Consider now the following path space

$$\mathcal{L}_\phi = \{ x\colon \mathbb{R} \to M \mid \phi(x(t + 1)) = x(t) \},$$

i.e., the space of sections in $M_\phi \to S^1$. We can view the nonempty point set of fixed points $\mathrm{Fix}^o(\phi)$ defined in (14), corresponding to contractible periodic solutions, as a subset of $\mathcal{L}_\phi$. In fact, it is contained entirely in one component of $\mathcal{L}_\phi$, and we define

$$\mathcal{L}^o_\phi = \{ x \in \mathcal{L} \mid x \simeq x_o \}, \quad \text{for any } x_o \in \mathrm{Fix}^o(\phi)\,. \tag{18}$$

Recall that the symplectic action was uniquely associated to an $\bar\phi \in \widetilde{\mathrm{Ham}}$ with $\pi(\bar\phi) = \phi$ and defined on the set $\mathrm{Fix}^o(\phi)$. For each $\bar\phi$ we have a well-defined extension to the component $\mathcal{L}^o_\phi$ by:

Proposition 3.10. *For each $\bar\phi \in \widetilde{\mathrm{Ham}}$ there exists a continuous extension of the Hamiltonian action $\mathcal{A}_{\bar\phi}$ to a continuous function $l_{\bar\phi} \colon \mathcal{L}^o_\phi \to \mathbb{R}$ such that for each differentiable*[1] *path $\gamma \colon [0,1] \to \mathcal{L}^o_\phi$ it holds*

$$l_{\bar\phi}\big(\gamma(1)\big) - l_{\bar\phi}\big(\gamma(0)\big) = \int_0^1 \int_0^1 \gamma^*\omega.$$

Proof. Essentially, it satisfies to realize that given any two paths $\gamma, \bar\gamma \colon [0,1] \to \mathcal{L}^o_\phi$ with coinciding ends, $\gamma(0) = \bar\gamma(0)$, $\gamma(1) = \bar\gamma(1)$, we have

$$\iint \gamma^*\omega = \iint \bar\gamma^*\omega.$$

This follows from $d\omega = 0$ on M_ϕ, $\omega_{|\pi_2(M)} = 0$ and the fact that we restricted $\mathcal{L}_\phi$ to the component of sections which contains the fixed points from contractible periodic solutions. It remains to verify

$$\iint \gamma^*\omega = \mathcal{A}_H(\gamma(1)) - \mathcal{A}_H(\gamma(0)) \tag{19}$$

for a differentiable path between $x, y \in \mathrm{Fix}_o(\phi)$. Set $u(s,t) = \phi^t_H(\gamma(s,t))$, then clearly $u(s,t+1) = u(s,t)$ and one computes

$$\begin{aligned}\int_0^1 \int_0^1 u^*\omega &= \iint \omega(D\phi^t_H \gamma_s, D\phi^t_H\gamma_t + X_H(u))\, dsdt \\ &= \iint \gamma^*\omega + \int_0^1 H(t,u(1,t))dt - \int_0^1 H(t,u(0,t))dt.\end{aligned}$$

This proves (19). □

Recall the definition of the moduli space of Floer trajectories $\mathcal{M}_{x,y}$ associated to a Hamiltonian $H \colon M \times S^1 \to \mathbb{R}$ and an ω-calibrated almost complex structure J on $TM \to M \times S^1$,

$$\begin{gathered} u \colon \mathbb{R} \times S^1 \to M, \\ u_s + J(t,u)\big[u_t - X_H(t,u)\big] = 0, \\ \lim_{s\to-\infty} u(s,\cdot) = x,\ \lim_{s\to\infty} u(s,\cdot) = y, \end{gathered}$$

where we consider $x, y \in \mathcal{P}_1(H)$.

In fact, Floer homology HF_* is already uniquely associated to the symplectomorphism ϕ, regardless of the generating Hamiltonian (for the grading,

[1] i.e., differentiable as map $[0,1] \times \mathbb{R} \to M$.

see the remark below). Namely, we consider maps $v\colon \mathbb{R}\times\mathbb{R}\to M$, satisfying

$$\begin{gathered} u(s,t) = \phi_H^t(v(s,t)), \text{ i.e., } v(s,t) = \phi_H^1(v(s,t+1)), \\ v_s + \bar{J}_H(t,v)v_t = 0, \\ \text{for } \bar{J}_H(t,v) = (D\phi_H^t)^{-1}J(t,\phi_H^t(v))D\phi_H^t, \\ \lim_{s\to-\infty} v(s,t) = x(0),\ \lim_{s\to\infty} v(s,t) = y(0), \quad \text{for all } t\in\mathbb{R}. \end{gathered}$$

We identify v with the section $\bar{v}(s,t) = [s,t,v(s,t)]$ of $E = \mathbb{R}\times M_\phi \to \mathbb{R}\times S^1$. Let us now generalize this Cauchy-Riemann problem for such mappings v by considering merely the time-1-map $\phi = \phi_H^1$ and

$$\begin{aligned} &v\colon \mathbb{R}\times\mathbb{R}\to M, \quad v(s,t) = \phi(v(s,t+1)), \\ &v_s + \bar{J}(t,v)v_t = 0, \quad v(-\infty,\cdot) = x(0),\ v(\infty,\cdot) = y(0), \\ &\text{for } \bar{J} \text{ satisfying} \quad D\phi^{-1}(v)J(t+1,\phi(v))D\phi(v) = \bar{J}(t,v). \end{aligned} \tag{20}$$

As in (19), we have for solutions with $\bar{J} = \bar{J}_H$

$$\iint_{\mathbb{R}\times S^1} \bar{v}^*\omega = \int_{-\infty}^{\infty}\int_0^1 |v_s|^2_{J_H} dsdt = \mathcal{A}_H(y(0)) - \mathcal{A}_H(x(0)). \tag{21}$$

Moreover, as a corollary we obtain the improvement of (10):

Proposition 3.11. *Given any two equivalent Hamiltonians $H\sim K$ generating $\bar{\phi}\in\widetilde{\mathrm{Ham}}$, the associated canonical homomorphism between the Floer homologies is compatible with the long exact sequence and respects the filtration by the action,*

$$\Phi_{KH}\colon HF_*^a(H)\to HF_*^a(K).$$

Proof. We define now $\Phi_{KH}\colon C_*(H)\to C_*(K)$ by means of (20) where we allow the generic almost complex structure $\bar{J}$ on the fibre bundle $\mathbb{R}\times M_\phi \to \mathbb{R}\times S^1$ to be explicitly s-dependent such that

$$\bar{J}(s,\cdot) = \begin{cases} \bar{J}_H, & s\le -T, \\ \bar{J}_K, & s\ge T, \end{cases}$$

for some $T>0$. Observe that such a connecting $\bar{J}$ exists since the space of ω-compatible almost complex structures on the fibre bundle is fibrewise contractible. By virtue of Proposition 3.10 it follows analogously to (7) that connecting trajectories between $y\in\mathcal{P}_1(H)$ and $x\in\mathcal{P}_1(K)$ solving

$$v_s + \bar{J}(s,t,v)v_t = 0 \tag{22}$$

satisfy

$$0\le \iint |v_s|^2_{\bar{J}} dsdt = \iint v^*\omega = \mathcal{A}_K(y(0)) - \mathcal{A}_H(x(0)).$$

Note that such trajectories solving (22) cannot be obtained directly from the "adiabatic" homotopy (G_s) described above since the s-derivative would imply an additional 0-order term. The most general setup formulated in terms of connections will be described in the following section. It is straightforward to see that all operators Φ_{KH} defined on the chain level by such sections in $\mathbb{R} \times M_\phi$ induce identical operators between the homology groups $HF_*(H) \xrightarrow{\cong} HF_*(K)$. $\square$

Remark on $HF_*(\phi)$. Clearly, for any two $H \sim K$ generating the same $\bar{\phi} \in \widetilde{\mathrm{Ham}}$, the canonical isomorphism Φ_{HK} viewed as an automorphism of the Floer homology $HF_*(\phi)$ associated to the time-1-map $\phi = \pi(\bar{\phi})$ has to be the identity. This is not true in general for $\bar{\phi}, \tilde{\phi} \in \widetilde{\mathrm{Ham}}$ with $\pi(\bar{\phi}) = \pi(\tilde{\phi})$. P. Seidel shows in [**22**] that the group $\pi_1(\mathrm{Ham})$ operates on $HF_*(\phi)$ in terms of the quantum cup-product, i.e., using the canonical isomorphism $HF_*(\phi) \cong QH^{n-*}(M, \omega)$ with the quantum cohomology ring (cf.[**16**]),

$$q \colon \pi_1(\mathrm{Ham}) \to \mathrm{Aut}(HF_*(\phi))$$

is given by a group homomorphism $\pi_1(\mathrm{Ham}) \to QH^\times(M, \omega)$ into the group of invertibles of the quantum cohomology ring of homogeneous degree. Hence, under our assumption (A) of symplectic asphericity, the quantum cohomology and thus q is trivial. It can also directly be seen that the grading on $\mathrm{Fix}^o(\phi)$ by the Conley-Zehnder index is already well-defined if $c_{1|\pi_2} = 0$.

4. The pair-of-pants product and Poincaré duality.

We will now show that also the canonical ring structure on Floer homology, the pair-of-pants product which was constructed in [**20**], is compatible with symplectic homology. This requires a "sharp" energy estimate which will be proven along the same lines as Proposition 3.11. As a consequence we obtain a crucial sub-additivity property for the section $c(\cdot, \cdot)$.

In [**20**] it was shown that every topological surface with oriented cylindrical ends gives rise to a multi-linear operation on Floer homology. Namely, choosing a conformal structure on the surface Σ and associating a Hamiltonian function to each end, one can generalize the Cauchy-Riemann type equation from $\mathcal{M}_{x,y}(J, H)$ to mappings from Σ into M with respective 1-periodic solutions as boundary conditions. The full theory with the verification of the axioms of a topological field theory is carried out in [**20**]. From the gluing axiom describing the concatenation of such multi-linear operators on Floer homology it follows that the entire theory is already uniquely determined by the multiplication associated to the surface with two exits and one entry, by the standard cylinder, by the cylinder with two exits and by the disk with only one end. From [**18**] it follows that this graded algebra is naturally isomorphic to the cohomology ring of M, if $\omega_{|\pi_2(M)} = 0$.

In [**20**], the construction of the pair-of-pants product on Floer homology was as follows. Let Σ be a Riemann surface with three cylindrical ends, one entry and two exits. Associate the Hamiltonians H and K to the exits and choose any Hamiltonian L for the entry. Then for $x \in \mathcal{P}_1(H)$, $y \in \mathcal{P}_1(K)$ and $z \in \mathcal{P}_1(L)$ we can define a moduli space $\mathcal{M}_{z;x,y}(L; H, K)$ of pair-of-pants solutions $u\colon \Sigma \to M$ converging towards x, y and z on the respective end. As before, under generic choices of the almost complex structure J on M, the solution space is a manifold, which is compact in dimension 0 and its dimension formula is

$$\dim \mathcal{M}_{z;x,y} = \mu(x) + \mu(y) - \mu(z) - n.$$

The multiplication on Floer homology is induced by

$$*\colon C_k(H) \times C_l(K) \to C_{k+l-n}(L),$$
$$x * y = \sum_{\mu(z)=k+l-n} \#_{\mathsf{alg}} \mathcal{M}_{z;x,y}(L; H, K)\, z,$$

and it is isomorphic to the cup-product under the isomorphisms Φ_H, Φ_K and Φ_L (cf. [**16**] and [**18**]),

$$\Phi_H(\alpha) * \Phi_K(\beta) = \Phi_L(\alpha \cup \beta) \tag{23}$$

for all $\alpha, \beta \in H^*(M)$.

In order to combine this multiplicative structure with the refinement of Floer homology by the action filtration we need a suitable energy estimate. For this purpose we again generalize the definition of the moduli space $\mathcal{M}_{z;x,y}$ in terms of J-holomorphic sections in a suitably defined symplectic fibre bundle over the pair-of-pants surface Σ. We obtain:

Proposition 4.1. *Assume that $[L] = [H] \circ [K]$ in $\widetilde{\mathrm{Ham}}(M, \omega)$. Then, the pair-of-pants product $*$ is compatible with the filtration by the Hamiltonian action and the following diagram commutes,*

$$\begin{array}{ccc}
HF_*^{(-\infty,a]}(H) \otimes HF_*^{(-\infty,b]}(K) & \xrightarrow{i_*^a \otimes i_*^b} & HF_*(H) \otimes HF_*(K) \\
\Big\downarrow * & & \Big\downarrow * \\
HF_*^{(-\infty,a+b]}(L) & \xrightarrow{i_*^{a+b}} & HF_*(L)\,.
\end{array}$$

For example, as a concrete Hamiltonian generating the composition $\phi_K^1 \circ \phi_H^1$ one can choose

$$(H\#K)(t, x) = H(t, x) + K(t, (\phi_H^t)^{-1}(x)) \tag{24}$$

for $\phi_K^t \circ \phi_H^t$. Note that this operation preserves also the normalization.

Suppose $a > c(\alpha; H)$ and $b > c(\beta; K)$ so that $\Phi_H(\alpha) \in \operatorname{im} i_*^a$ and $\Phi_K(\beta) \in \operatorname{im} i_*^b$. Then, Proposition 4.1 implies that $c(\alpha \cup \beta; H\#K) \le a + b$. This proves:

Theorem 4.2. *Given any $\alpha, \beta \in H^*(M)$ with $\alpha \cup \beta \neq 0$ and $\bar\phi, \bar\psi \in \widetilde{\mathrm{Ham}}(M,\omega)$ it holds*

$$c(\alpha \cup \beta; \bar\psi \circ \bar\phi) \leq c(\alpha; \bar\phi) + c(\beta; \bar\psi)\,.$$

Note that by definition it is obvious that

$$c(\lambda\alpha; \bar\phi) = c(\alpha; \bar\phi) \quad \text{for all } \lambda \text{ with } \lambda\alpha \neq 0.$$

In particular, $c(\alpha \cup \beta; \bar\phi) = c(\beta \cup \alpha; \bar\phi)$.

Remark on regularity. In order to prove that, for a generic choice of J, the solution space associated to the pair-of-pants model surface is a manifold, one has to exclude solutions which stay constantly on a fixed periodic solution. Such solutions trivially exist for $H = K$ and $L(t,x) = 2H(2t,x)$. This singular situation can be excluded by choosing a generic pair of regular Hamiltonians H, K which have no fixed points for their time-1-maps in common. Finally, in view of Proposition 2.14 we can approximate the case $H = K$ by such generic pairs such that all conclusions about critical levels remain valid.

4.1. The energy estimate for the pair-of-pants. In order to prove Proposition 4.1 we use a more general formulation of (20) given in terms of a connection on a symplectic fibre bundle.

Let Σ_o be a compact Riemann surface of genus 0 with three boundary components two of which are oriented as exits and one as entry, denoted by $\partial_1^+\Sigma_o$, $\partial_2^+\Sigma_o$ and $\partial_-\Sigma_o$. Assume that $E \to \Sigma_o$ is a smooth locally trivial fibre bundle with a closed 2-form $\bar\omega \in \Omega^2(E)$, $d\bar\omega = 0$, such that $(E_z, \bar\omega_{|T(E_z)})$ is a symplectic manifold and the typical fibre is (M,ω). Recall the symplectic fibre bundle $M_\phi \to S^1$ from (17) associated to $\phi \in \mathrm{Ham}(M,\omega)$. Given three symplectomorphisms ϕ, ψ, η we assume that there are symplectic diffeomorphisms

$$M_\phi \xrightarrow{\sim} E_{|\partial_1^+\Sigma_o},\ M_\psi \xrightarrow{\sim} E_{|\partial_2^+\Sigma_o},\ M_\eta \xrightarrow{\sim} E_{|\partial^-\Sigma_o}. \tag{25}$$

Note that such diffeomorphisms lead to trivializations over S^1 if the symplectomorphisms are isotopic to the identity. The simple but crucial observation is:

Lemma 4.3. *Given ϕ and ψ, such a symplectic fibre bundle $(E, \bar\omega)$ satisfying (25) and such that the fibrewise symplectic form $\bar\omega$ is closed on E exists if $\eta = \psi \circ \phi$.*

Proof. Consider the domain $D \subset \mathbb{C}$ as sketched in Figure 1 and impose an equivalence relation $\sim$ on $D \times M$ by identifying boundary points as indicated using the symplectomorphisms ϕ, ψ and $\psi\phi$. Details are left to the reader.

Note that the induced M-fibre bundle $E = (D \times M)/\sim$ over the pair-of-pants surface Σ_o carries the closed fibrewise symplectic form $\bar\omega \in \Omega^2(E)$ canonically induced from (M,ω). $\square$

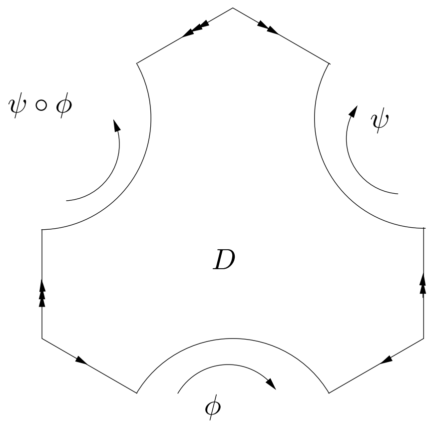

Figure 1. Boundary identification for the pants bundle.

Consider now exactly this fibrewise symplectic form $\bar{\omega}$ and choose an almost complex structure $\bar{J}$ such that the projection map $\pi\colon E \to \Sigma_o$ is $\bar{J}$-holomorphic, and restricts to an ω-compatible almost complex structure on the fibres E_z, $z \in \Sigma_o$. Associated to $\bar{\omega}$ we have the following connection on E,

$$\begin{gathered} T_\xi E = \mathrm{Ver}_\xi \oplus \mathrm{Hor}_\xi, \quad \xi \in E, \\ \mathrm{Ver}_\xi = (\ker D\pi(\xi)\colon T_\xi E \to T_{\pi(\xi)}\Sigma_0), \\ \mathrm{Hor}_\xi = \{\, v \in T_\xi E \,|\bar{\omega}(v,w) = 0 \text{ for all } w \in \mathrm{Ver}_\xi \,\}. \end{gathered}$$

The connection map $K\colon TE \to \mathrm{Ver}$ is the projection w.r.t. this connection. Given a section $v\colon \Sigma_o \to E$ we denote the covariant derivative by

$$\nabla v = K \circ Dv, \quad \nabla v(z) \in \mathrm{Hom}(T_z\Sigma_o, T_{v(z)}(E_z)). \tag{26}$$

Recall that, since we have a fixed conformal structure on Σ_o and a vertically Riemannian structure by $\bar{\omega}(\cdot, \bar{J}\cdot)_{|\,\mathrm{Ver}}$, the L^2-norm $\int_{\Sigma_o} |\nabla v|^2_{\bar{J}}$ of ∇v for any section v is intrinsically defined.

What is crucial for the following is that for the above defined form $\bar{\omega}$ on the bundle $E \to \Sigma_o$ we have

$$\int_{\Sigma_o} |\nabla v|^2_{\bar{J}} = 2\int_{\Sigma_o} v^*\bar{\omega} + \int_{\Sigma_o} |\overline{\partial} v|^2_{\bar{J}} \tag{27}$$

where $\overline{\partial} v = \bar{J}\nabla v\, i + \nabla v$, and consequently

$$\int_{\Sigma_o} v^*\bar{\omega} \geq 0 \quad \text{for } \overline{\partial} v = 0. \tag{28}$$

We call a section v $\bar{J}$-holomorphic if $\overline{\partial} v = 0$. This is not true for an arbitrary closed form $\tilde{\omega} \in \Omega^2(E)$ which restricts fibrewise to ω. The above form $\bar{\omega}$ leads to a so-called *flat* connection.

Let us now combine this energy estimate with the Floer trajectories from (20). Given $\phi, \psi \in \mathrm{Ham}(M, \omega)$ we consider the trivial symplectic fibre bundles

$$M_\phi \times [0, \infty), \quad M_\psi \times [0, \infty), \quad \text{and } M_{\psi\phi} \times (-\infty, 0],$$

and glue them to $E \to \Sigma_o$ along the boundary circles by means of the gluing maps from (25). This gives a symplectic fibre bundle over a Riemann surface Σ, a pair-of-pants, which we denote again by E. We have extensions of the structures on Σ_o, $\bar{\omega}$, $\bar{J}$ and K. Given a continuous section $v \colon \Sigma \to E$ we denote by $v(\partial_i^\pm \Sigma)$ the uniform limits of v as paths in $\mathcal{L}_\phi^o$, $\mathcal{L}_\psi^0$ or $\mathcal{L}_\eta^0$ as we approach the respective ends of Σ, i.e., in cylindrical coordinates $s \to \pm\infty$. Observe that in terms of the cylindrical coordinates we have for a $\bar{J}$-holomorphic section $\overline{\partial} v = 0$

$$\tfrac{1}{2} |\nabla v|^2 = |v_s|^2 ds \wedge dt.$$

Note that the structure $\bar{\omega}$ on E still satisfies the flatness condition (27). The crucial estimate analogous to Proposition 3.10 for this fibre bundle over the surface Σ is:

Proposition 4.4. *Let v be a section of the pair-of-pants bundle $(E \to \Sigma_o, \bar{\omega})$ with boundary values*

$$v_{|\partial\Sigma_o} = (v_\phi, v_\psi, v_{\psi\phi}) \in \mathcal{L}_\phi^o \times \mathcal{L}_\psi^o \times \mathcal{L}_{\psi\phi}^o.$$

Then, we have

$$\int_{\Sigma_o} v^* \bar{\omega} = l_{\bar{\phi}}(v_\phi) + l_{\bar{\psi}}(v_\psi) - l_{\bar{\psi}\bar{\phi}}(v_{\psi\phi}), \quad \textit{for all } \bar{\phi}, \bar{\psi} \in \widetilde{\mathrm{Ham}}(M, \omega).$$

Proof of Proposition 4.4. Observe first that, due to $d\bar{\omega} = 0$ on E and $\omega_{|\pi_2(M)} = 0$, the value of $\int_{\Sigma_o} v^* \bar{\omega}$ only depends on the boundary values $v_{|\partial\Sigma_o}$. The proof is now exactly analogous to the computation of the Fredholm index in [**20**]. Namely, we construct symplectic fibre bundles $E^\pm \to D^\pm$ over the disks with both orientations which restrict respectively to M_ϕ, M_ψ and $M_{\psi\phi}$ over the boundary. Moreover, we have to construct fibrewise symplectic forms on these bundles which are closed. Using the obvious additivity for the integration it thus remains to compute the relevant formula for the symplectic fibre bundle over the disk, if the integration over the closed sphere S^2 as a base gives $\int_{S^2} v^* \bar{\omega} = 0$.

Let us construct the symplectic fibre bundle $E^+ \to D^+$ over the unit disk $D^+ = \{|z| \leq 1\}$ in terms of the coordinates $z = \mathrm{e}^{2\pi(s+it)}$, $(s, t) \in$

$(-\infty, 0] \times S^1$. Let $\beta\colon \mathbb{R} \to [0,1]$ be a smooth cut-off function

$$\beta(s) = \begin{cases} 0, & s \leq -2, \\ 1, & s \geq -1, \end{cases}$$

and choose a generating Hamiltonian $H(t,x)$ for $\phi = \phi_H^1$. Consider the smooth 2-parameter family of symplectomorphisms

$$\phi_s^t = \phi_{\beta(s)H}^t, \quad \text{i.e., } \partial_t \phi_s^t = \beta(s) X_H(t, \phi_s^t).$$

Define the symplectic fibre bundle $E^+ \to D^+$ with fibre (M, ω) by

$$E^+ = (-\infty, 0] \times \mathbb{R} \times M / \{\, (s,t,x) \sim (s, t-1, \phi_s^1(x)) \,\} \to (-\infty, 0] \times S^1$$

which is trivialized by

$$\begin{aligned} \Phi\colon E^+ &\xrightarrow{\sim} (-\infty, 0] \times S^1 \times M \\ [s,t,x] &\mapsto (s, t \pmod 1, \phi_s^t(x)). \end{aligned}$$

Via Φ, sections $\bar{v}(s,t) = [s,t,v(s,t)]$ of E^+ are identified with maps $u\colon (-\infty, 0] \times S^1 \to M$, $u(s,t) = \phi_s^t(v(s,t))$. Consider the following 2-form $\omega_E \in \Omega^2(E^+)$, given by $\omega_E = \Phi^*\alpha$ with $\alpha \in \Omega^2(\mathbb{R} \times S^1 \times M)$,

$$\alpha_{(s,t,x)} = -\beta'(s) H(t,x) ds \wedge dt + \omega_x + i_{\beta(s) X_H(t,x)} \omega_x \wedge dt.$$

One computes

$$\omega_E = -\beta' H ds \wedge dt + \omega - i_{(D\phi)^{-1}\partial_s \phi} \omega \wedge ds. \tag{29}$$

Straightforward computation shows that

$$\begin{aligned} \omega_{E|\mathrm{fibre}} &= \omega, \\ d\omega_E &= 0, \\ \pi_* \omega_E^{n+1} &= 0, \end{aligned} \tag{30}$$

where $\pi_*\colon \Omega^{2n+2}(E) \to \Omega^2(\mathbb{R} \times S^1)$ denotes the fibre integration map. The last property holds if H is normalized.

Lemma 4.5. *Let $E^+ \to (-\infty, 0] \times S^1$ and ω_E be given as above. Then*

$$\int_{D^+} \bar{v}^* \omega_E = l_{[H]}\big(\bar{v}(0)\big)$$

holds for any section $\bar{v}$ in E^+.

Proof. We can assume E^+ as canonically extended over $\mathbb{R} \times S^1$. Since $\omega_{|\pi_2} = 0$ the left hand side does not depend on the section if $\bar{v}(\infty)$ is fixed. In view of Proposition 3.10 we can also assume that $\bar{v}(\infty,\cdot) = x \in \operatorname{Fix}^o \phi$. Denoting $(s,t,u(s,t)) = \Phi(\bar{v}(s,t))$ and $x(t) = u(\infty,t)$ we compute

$$\begin{aligned}\iint \bar{v}^*\omega_E =& \iint w^*\alpha \\ =& \iint \alpha(w_s, w_t)\, ds\, dt \\ =& \iint \alpha(\partial_s + u_s, \partial_t + u_t)\, ds\, dt \\ =& -\iint \beta' H(t,u) dsdt + \iint u^*\omega + \iint i_{\beta X_H}\omega(u_s) dsdt \\ =& \iint u^*\omega - \iint [\beta'(s)H(t,u(s,t)) + \beta(s) dH(u(s,t))u_s] dsdt \\ =& \iint u^*\omega - \iint \frac{d}{ds}\big(\beta(s)H(t,u)\big) dsdt \\ =& \mathcal{A}_H(u(+\infty)).\end{aligned}$$

□

Lemma 4.6. *Let $E \to S^2$ be a symplectic fibre bundle with a form ω_E satisfying (30). Then for any section s of E we have*

$$\int_{S^2} s^*\omega_E = I(\gamma)$$

if E is obtained from gluing two trivial bundles $E^\pm = D^\pm \times M$ along their boundary via a loop representing $\gamma \in \pi_1(\mathrm{Ham})$. In particular, if E is trivial then $\int_{S^2} s^*\omega_E = 0$.

Proof. Since we know that $\mathcal{A}_{\gamma[H]}(x) = \mathcal{A}_{[H]}(x) + I(\gamma)$ it suffices to show that

$$\int_{S^2} s^*\omega_E = 0$$

in case E is trivial. Let $p\colon E \to M$ be the projection map obtained from a trivialization $E \cong S^2 \times M$ and $\pi\colon E \to S^2$ the canonical projection. Denoting by $\sigma \in H^2(S^2)$ the generator normalized by $\sigma(S^2) = 1$ we have

$$\{\omega_E\} = p^*\{\omega\} + a\pi^*\sigma, \quad a = \int_{S^2} s^*\omega_E\,.$$

Then, the last condition of (30) implies that $a = \int_{S^2} \pi_*\omega_E^{n+1} = 0$. □

The proof of Proposition 4.4 now follows from Lemmata 4.5 and 4.6 if we glue the bundles $E^-_{[H]}$, $E^-_{[K]}$ over D^- and $E^+_{[H\#K]}$ over D^+ to $E \to \Sigma_o$ over the pair-of-pants. We obtain a trivial (M,ω)-fibre bundle $\pi\colon \bar{E} \to S^2$ together with a coupling form $\bar{\omega}$ satisfying (30). □

Recalling the energy identity (28) for $\bar{J}$-holomorphic sections in the pair-of-pants bundle $E \to \Sigma_o$ we obtain the energy estimate:

Corollary 4.7. *Every $\bar{J}$-holomorphic section $v\colon \Sigma \to E$ with the boundary condition $v(\partial_1^+\Sigma) = x(0) \in \mathrm{Fix}_o(\phi)$, $v(\partial_2^+\Sigma) = y(0) \in \mathrm{Fix}_o(\psi)$ and $v(\partial^-\Sigma) = z(0) \in \mathrm{Fix}_o(\psi\phi)$ satisfies the energy estimate for automorphisms $\phi = \phi_H^1$, $\psi = \phi_K^1$,*

$$0 \le \int_\Sigma |\nabla v|_{\bar{J}}^2 = \int_\Sigma v^*\bar{\omega} = \mathcal{A}_{[H]}(x) + \mathcal{A}_{[K]}(y) - \mathcal{A}_{[K][H]}(z).$$

Observe that such a positivity estimate for $\bar{J}$-holomorphic sections in the bundle $E^\pm \to D^\pm$ does not hold because the closed form $\bar{\omega}$ has non-vanishing $ds \wedge dt$-components, see (29). There is no equivalent of (27).

Let the moduli space of such $\bar{J}$-holomorphic sections of $E \to \Sigma$ with boundary condition as in Corollary 4.7 replace the originally considered space $\mathcal{M}_{z;x,y}(L; H, K)$. Then, analogously to Proposition 3.11 we obtain a multiplication

$$*\colon C_k(\phi) \times C_l(\psi) \to C_{k+l-n}(\psi\phi)$$

which coincides with the original pair-of-pants product on the level of Floer homology. The sharp energy estimate from Corollary 4.7 concludes the proof of Proposition 4.1.

4.2. Poincaré duality. Let us consider the dual Floer complex associated to symplectic fixed points by applying the Hom-functor. For sake of simplicity we restrict ourselves to coefficients in a field $\mathbb{F}$, e.g., $\mathbb{Z}_2$, $\mathbb{Q}$ or $\mathbb{R}$. Given $a \in \mathbb{R}$ we have the cochain complex

$$C^k_{(a,\infty)}(H) = \{\, x \in \mathcal{P}_1(H) \,|\, \mu(x) = k,\ \mathcal{A}_H(x) > a \,\} \otimes \mathbb{F},$$
$$\delta\colon C^k_{(a,\infty)} \to C^{k+1}_{(a,\infty)}, \quad \delta x = \sum_{\mu(y)=\mu(x)+1} \#_{\mathsf{alg}} \widehat{\mathcal{M}}_{x,y}(J, H)\, y,$$

and we identify

$$C^*(H) = \mathrm{Hom}(C_*(H)), \quad C^*_{(a,\infty)} = \mathrm{Hom}(C_*^{(a,\infty)}),$$
$$\text{and} \quad C^*_{(-\infty,a]} = C^*/C^*_{(a,\infty)} = \mathrm{Hom}\left(C_*^{(-\infty,a]}\right).$$

We observe that the long exact cohomology sequence induced from the short exact sequence of cochain complexes

$$0 \to C^*_{(a,\infty)} \xrightarrow{j_a^\bullet} C^* \xrightarrow{i_a^\bullet} C^*_{(-\infty,a]} \to 0 \tag{31}$$

equals the dual sequence obtained by the Hom-functor from the long exact homology sequence (2). In view of the universal coefficient theorem we have

$$C^*(H) = \mathrm{Hom}(C_*(H), \mathbb{F}), \quad HF^*(H) \cong \mathrm{Hom}(HF_*(H); \mathbb{F}),$$

respectively for $C^*_{(a,\infty)}$, etc.

Let us consider now the dual isomorphism $\Psi^*\colon H_*(M) \to HF^{n-*}(H)$ and the dual generators $[\mathrm{pt}] \in H_0(M)$ and $1 \in H^0(M)$. In view of the long exact cohomology sequence obtained from (31) we have:

Lemma 4.8. *The critical value $c(1;H)$ can be equivalently represented in Floer cohomology by*

$$\inf\bigl\{\, a \in \mathbb{R} \mid j^a_*\bigl(\Phi_H(1)\bigr) = 0 \,\bigr\} \;=\; \sup\bigl\{\, a \in \mathbb{R} \mid i^*_a\bigl(\Psi^*_H([\mathrm{pt}])\bigr) = 0 \,\bigr\}.$$

Proof. We have $\Phi_H(1) \in \operatorname{im} i^a_*$ if and only if there exists a $u_a \in HF_n^{(-\infty,a]}$ such that $\bigl\langle u_a, i^*_a(\Psi^*_H([\mathrm{pt}]))\bigr\rangle \neq 0$. That is,

$$j^a_*\bigl(\Phi_H(1)\bigr) = 0 \quad \text{if and only if} \quad i^*_a\bigl(\Psi^*_H([\mathrm{pt}])\bigr) \neq 0\,.$$

□

We now use the fact that Poincaré duality is represented in terms of Floer homology by the canonical isomorphism between the homology $HF_*(H)$ and the cohomology $HF^*(H^{(-1)})$ of the Hamiltonian generating the inverse symplectomorphism,

$$H^{(-1)}(t,x) = -H(-t,x), \quad \phi^t_{H^{(-1)}} = \phi^{-t}_H.$$

It is straightforward to verify that the identification

$$\mathcal{P}_1(H) \cong \mathcal{P}_1(H^{(-1)}), \quad x^{-1}(t) = x(-t),$$
$$\mathcal{M}_{x,y}(J,H) \cong \mathcal{M}_{y^{-1},x^{-1}}(J,H^{(-1)}), \quad u^{-1}(s,t) = u(-s,-t),$$

provides an identification of the chain complex of H with the cochain complex of $H^{(-1)}$,

$$\bigl(C_*(H),\partial\bigr) \cong \bigl(C^{-*}(H^{(-1)}),\delta\bigr).$$

Note that it holds

$$\mu(x^{-1}) = -\mu(x) \quad \text{and} \quad \mathcal{A}_{H^{(-1)}}(x^{-1}) = -\mathcal{A}_H(x).$$

Hence we have the identification of $C_*^{(-\infty,a]}(H)$ with $C^{-*}_{[-a,\infty)}(H^{(-1)})$, etc., and the short exact sequence of chain complexes

$$0 \to C_*^{(-\infty,a]}(H) \xrightarrow{i^a_*} C_*(H) \xrightarrow{j^a_*} C_*^{(a,\infty)}(H) \to 0$$

becomes isomorphic to the short exact sequence of cochain complexes

$$0 \to C^*_{[-a,\infty)}(H^{(-1)}) \xrightarrow{j^*_{-a}} C^*(H^{(-1)}) \xrightarrow{i^*_{-a}} C^*_{(-\infty,-a)}(H^{(-1)}) \to 0.$$

In [**20**], Poincaré duality was analyzed in terms of a canonical non-degenerate bilinear form β on Floer homology associated to a Riemann surface with two cylindrical ends both oriented as entries. This is equivalent to the identification of the homology for H with the cohomology for $H^{(-1)}$ because the change $H \to H^{(-1)}$ corresponds to the change of orientation of a cylindrical end. Hence, we have the commutative diagram

$$
\begin{tikzcd}
H^k(M) \arrow[r, "\Phi_H"] \arrow[d, "\mathrm{PD}"'] & HF_{n-k}(H) \arrow[d, "="'] \\
H_{2n-k}(M) \arrow[r, "\Psi^*_{H^{(-1)}}"] & HF^{k-n}(H^{(-1)})\,.
\end{tikzcd}
\tag{32}
$$

Proposition 4.9. *The representation of Poincaré duality in Floer homology yields the identity*

$$c([M];\phi) = -c(1;\phi^{-1}), \quad \textit{for all } \phi \in \widetilde{\mathrm{Ham}}(M,\omega).$$

Proof. From (32) we obtain that

$$c([M];H) = \inf\{\, a \mid j^a_*(\Phi_H([M])) = 0\,\} = \inf\{\, a \mid i^*_{-a}(\Psi^*_{H^{(-1)}}([\mathrm{pt}])) = 0\,\}.$$

Hence, the assertion follows by Lemma 4.8. □

An immediate consequence is:

Corollary 4.10. *Given any nonzero cohomology class $\alpha \in H^*(M)$ and $\phi, \psi \in \widetilde{\mathrm{Ham}}(M,\omega)$ we have the estimate*

$$c([M],\psi) \leq c(\alpha;\psi\phi) - c(\alpha;\phi) \leq c(1,\psi)$$

and $\phi \mapsto c(\alpha,\phi)$ is continuous with respect to $d_{\tilde{H}}(\phi,\mathrm{id}) = \inf\{\|H\| | \phi = [H]\}$.

Proof. First we combine

$$c(\alpha;\phi) \leq c(\alpha;\psi\circ\phi) + c(1;\psi^{-1})$$

from Theorem 4.2 with Corollary 4.10. The continuity follows from

$$c(1,[H]) - c([M],[H]) \leq \|H\|.$$

Recall that $c(1,[H]) \leq E_+(H)$ and $0 \leq E_+(H)$ because H is normalized. □

We can now compute the action of $\pi_1(\mathrm{Ham})$ on these continuous sections $c(\alpha)$ in the action spectrum bundle.

Proposition 4.11. *For any $\alpha \in H^*(M)\setminus\{0\}$, $\gamma \in \pi_1(\mathrm{Ham})$, $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$ we have*

$$c(\alpha;\gamma\phi) = c(\alpha;\phi) + I(\gamma).$$

Proof. Considering the covering $\pi\colon \widetilde{\mathrm{Ham}} \to \mathrm{Ham}$ we identify $\pi_1(\mathrm{Ham}) = \pi^{-1}(\mathrm{id})$. Given $\gamma \in \pi^{-1}(\mathrm{id})$ the action spectrum of γ consist only of one value, $\Sigma_\gamma = \{I(\gamma)\}$, hence

$$c(\alpha,\gamma) = I(\gamma) \quad \text{for all } \alpha \in H^*(M)\setminus\{0\}.$$

Then assertion then follows from Corollary 4.10. □

This result allows us to define the following function on $\mathrm{Ham}(M,\omega)$.

Definition 4.12. Given $\phi \in \widetilde{\mathrm{Ham}}(M,\omega)$ we define

$$c_-(\phi) = c([M];\phi), \quad c_+(\phi) = c(1;\phi) \quad \text{and} \quad \gamma(\phi) = c_+(\phi) - c_-(\phi).$$

We obtain $\gamma\colon \mathrm{Ham}(M,\omega) \to \mathbb{R}$ as a continuous function.

Summing up the above results we have

$$0 \le \gamma(\phi) = \gamma(\phi^{-1}) \quad \text{and} \quad \gamma(\phi \circ \psi) \le \gamma(\phi) + \gamma(\psi) \tag{33}$$

for all $\phi, \psi \in \mathrm{Ham}_{p_o}(M,\omega)$. This function γ plays the role of the nontrivial *selector* of Hofer and Zehnder in the case of compactly supported Hamiltonian automorphisms of $(\mathbb{R}^{2n}, \omega_o)$.

4.3. Vanishing of the monodromy I. We now show that the homomorphism $I\colon \pi_1(\mathrm{Ham}) \to \mathbb{R}$ is in fact trivial in the symplectically aspherical case (A).

Let $M^{2n} \overset{i}{\hookrightarrow} E \xrightarrow{\pi} S^2$ be a Hamiltonian fibre bundle and $\omega_E \in \Omega^2(E)$ be a coupling form, i.e., satisfying (30). Let $\omega_{|\pi_2(M)} = c_{1|\pi_2(M)} = 0$ and choose a generic almost complex structure J on E such that J is fibrewise ω-compatible, and the projection π is J-i-holomorphic. Then, the space $\mathcal{M}(J)$ of holomorphic sections $s\colon S^2 \to E$ is a closed manifold of dimension $2n$.

Theorem 4.13 ([**22**])**.** *The evaluation map ev_{z_o} at any point $z_o \in S^2$ induces a homomorphism $ev_*\colon H_{2n}(\mathcal{M}(J),\mathbb{Z}) \to H_{2n}(M,\mathbb{Z})$ of degree* ± 1.

Proof. Given ω_E and a generic J on E, Seidel defines in [**22**] the invariant

$$Q(E,\omega_E,S) = \sum_{\gamma\in\Gamma} \big[\, \mathrm{ev}_{z_o}\big(S(j,J,\gamma+S)\big)\big] \otimes \langle\gamma\rangle \in QH_d(M,\mathbb{Z}_2),$$

where $z_o \in S^2$, $\Gamma = \pi_2(M)/\ker\omega_{|\pi_2} \cap \ker c_{1|\pi_2}$. This is an invariant of a given equivalence class S of a section of E. The degree of the quantum homology class Q is given by $d = 2n + 2c_1(TE^v,\omega_E)(S)$ where TE^v is the vertical subbundle. Since by assumption $\omega_{|\pi_2} = c_{1|\pi_2} = 0$ we can assume coefficients in $\mathbb{Z}$, $\Gamma = 0$, and the quantum homology equals ordinary homology, the invariant

$$Q(E,\omega_E,S) \in H_{2n}(M,\mathbb{Z}) \cong \mathbb{Z}$$

is independent of S. Seidel's main result is that $Q(E,\omega_E)$ is an invertible element in QH_* of homogeneous degree. Hence $Q(E,\omega_E) = \pm 1 \in \mathbb{Z}$. □

Corollary 4.14. *For any section $s\colon S^2 \to E$ it holds $s^*[\omega_E] = 0$ in $H^2(M,\mathbb{R})$.*

Proof. Clearly, the result does not depend on the choice of the section s. Let $u \in \mathcal{M}(J)$ be a holomorphic section for a generic J on E and consider the commutative diagram

$$\begin{tikzcd} \mathcal{M}(J) \times S^2 \arrow[r, "\mathrm{ev}"] & E \arrow[d, "\pi"] \\ S^2 \arrow[u, "\tilde{u}"] \arrow[r, "="] & S^2 \end{tikzcd}$$

where $\mathrm{ev}(w,z) = w(z)$ is a bundle map and $\tilde{u}(z) = (u,z)$ is a constant section in the trivial bundle. We have $u^*[\omega_E] = \tilde{u}^* \mathrm{ev}^*[\omega_E]$ and $\mathrm{ev}^*[\omega_E] = \alpha \times 1 + a 1 \times \sigma$, $a \in \mathbb{R}$, for $\alpha \in H^2(\mathcal{M}(J), \mathbb{R})$ and $\sigma \in H^2(S^2, \mathbb{Z})$ a generator. Hence,

$$u^*[\omega_E] = a\sigma \in H^2(S^2, \mathbb{R})$$

and we have to prove that $a = 0$.

Since the fibre integration homomorphism $\pi_*\colon H^{2n+2}(E) \to H^2(S^2)$ is an isomorphism, the assumption (30) on the form ω_E implies that the class $[\omega_E]^{n+1}$ vanishes, hence $a\alpha^n \times \sigma = 0$. Since $\alpha = \mathrm{ev}_{z_o}^*[\omega]$ for $\mathrm{ev}_{z_o}\colon \mathcal{M}(J) \to \pi^{-1}(z_o) \approx M$ for any fixed $z_o \in S^2$, we have

$$\langle \alpha^n, [\mathcal{M}(J)] \rangle = \deg(\mathrm{ev}_*) = \pm 1$$

by Seidel's theorem. Thus, it follows that $a = 0$. □

This proves Theorem 1.1 on the vanishing of the obstruction homomorphism I in the symplectically aspherical case.

Corollary 4.15. *The sections $c(\alpha)$ in the action spectrum bundle are well-defined over* $\mathrm{Ham}(M,\omega)$ *and continuous in the Hofer-metric* $d_H(\mathrm{id},\phi) = \inf\{\, \|H\| \,|\phi = \phi_H^1 \,\}$.

Proof. Combine Corollary 4.10, Proposition 4.11 and Corollary 4.14. □

This concludes the proof of Theorem 1.2.

5. The bi-invariant metric γ and a relative capacity.

Let us first analyze the relation between the function γ and the well-known displacement energy introduced by Hofer. Let $H\colon [0,1] \times M \to \mathbb{R}$ be a normalized Hamiltonian and $\psi \in \mathrm{Ham}(M,\omega)$ an automorphism separating the support set of ϕ_H^1

$$(34) \qquad \psi(\mathcal{S}(H)) \cap \mathcal{S}(H) = \emptyset \quad \mathcal{S}(H) = \bigcup_{t\in[0,1]} \operatorname{supp} X_H(t,\cdot).$$

As in [8] and [23] we observe that $\mathrm{Fix}(\phi_H^t \circ \psi) = \mathrm{Fix}(\psi \circ \phi_H^t) = \mathrm{Fix}(\psi)$ for all $t \in [0,1]$ and for $x \in \mathrm{Fix}^o\,\psi$, $\psi = \phi_K^1$ we have

$$\mathcal{A}_{[K][H]}(x) = \mathcal{A}_{[K]}(x) + \int_0^1 H(t,x)dt.$$

Setting $a(H) = \int_0^1 H(t,p)dt$ for any $p \notin \mathcal{S}(H)$ we obtain for the action spectra

$$\Sigma_{[K][H]} = \Sigma_{[K]} + a(H).$$

Considering the continuous path $\epsilon \mapsto \epsilon H$, $\epsilon \in [0,1]$ we obtain for every $\alpha \neq 0$ the continuous function

$$\epsilon \mapsto c(\alpha; [\epsilon H][K]) - a(\epsilon H) = c(\alpha; [K][\epsilon H]) - a(\epsilon H)$$

into the nowhere dense action spectrum $\Sigma_{[K]} \subset \mathbb{R}$ which therefore has to be constant. It follows that

$$\gamma(\phi_H^1 \psi) = \gamma(\psi \phi_H^1) = \gamma(\psi). \tag{35}$$

In particular, this implies:

Proposition 5.1. *Given H and ψ as in (34) we have*

$$\gamma(\phi_H^k) \leq 2\gamma(\psi)$$

for all $k \in \mathbb{Z}$, where $\phi_H^k = (\phi_H^1)^k$.

Proof. The triangle inequality for γ (33) yields

$$\gamma(\phi_H^1) \leq \gamma(\psi \circ \phi_H^1) + \gamma(\psi).$$

For $k \in \mathbb{Z}$ use $\phi_H^k = \phi_{H^k}^1$ with $H^k(t,x) = kH(kt,x)$ and that condition (34) holds for all $k \in \mathbb{Z}$ since we can assume $H(t+1,x) = H(t,x)$. □

Moreover, we have:

Proposition 5.2. *Given any open subset $U \subset M$ there exists $\phi_H^1 \in$ Ham(M,ω) such that $\mathcal{S}(H) \subset U$ and $\gamma(\phi_H^1) > 0$.*

Proof. We pick a smooth positive function $H \colon M \to \mathbb{R}$ independent of t with $\operatorname{supp} H \subset U$ such that the only critical point $p \in \operatorname{supp} H$ is a maximum and the C^2-norm of H is small enough so that the only 1-periodic solution are the constant solutions $p \in \mathcal{P}_1(H)$ and $q \in M \setminus \operatorname{supp} H$. Approximating H suitably by C^2-small Morse functions we obtain $c_-(H) = -H(p)$ and $c_+(H) = 0$, i.e., $\gamma(H) = \|H\| > 0$. Recall that here we do not need H to be normalized. □

Combining this observation with Proposition 5.1 we deduce the metric property of γ.

Theorem 5.3. *For every $\phi \in$ Ham(M,ω) we have*

$$\gamma(\phi) > 0 \quad \text{if} \quad \phi \neq \mathrm{id}\,.$$

That is, $\gamma \colon$ Ham$(M,\omega) \to \mathbb{R}_+$ defines a metric d_γ by

$$d_\gamma(\phi,\psi) = \gamma(\phi\psi^{-1}).$$

Moreover, this metric is bi-invariant.

Proof. It remains to show that d_γ is bi-invariant. This follows analogously to d_H in [8] from the fact that for any $\theta \in \mathrm{Aut}(M,\omega)$ we have

$$\theta \circ \phi_H^t \circ \theta^{-1} = \phi_{H_\theta}^t \quad \text{for all } t,$$

where $H_\theta(t,x) = H(t,\theta^{-1}(x))$. □

Remark 5.4. This result implies another proof for the nontrivial fact that Hofer's metric d_H is in fact a metric. This has been shown for all closed symplectic manifolds in [10]. Here we obtain a different Floer-theoretical proof for at least symplectically aspherical manifolds.

An interesting application of this bi-invariant metric is obtained analogously to [8].

Theorem 5.5. *Let $\phi \in \mathrm{Ham}(M,\omega)$ such that there exists a uniform bound*

$$\gamma(\phi^n) \le C \quad \textit{for all } n \in \mathbb{N},$$

then ϕ has infinitely many nontrivial geometrically distinct periodic points corresponding to contractible periodic solutions.

Proof. Let H be a normalized Hamiltonian generating $\phi = \phi_H^1$.Then $\phi^n = \phi^1_{H^{(n)}}$. Assume that ϕ has only finitely many nontrivial periodic points, then without loss of generality the spectra are related by the scaling with n, $\Sigma_{\phi^n} = n \cdot \Sigma_\phi$. But if $\phi \neq \mathrm{id}$ then $c_+(\phi^n) - c_-(\phi^n) > 0$ for all $n \in \mathbb{N}$ and necessarily $\gamma(\phi^n) \to \infty$ contradicting the assumption. □

Clearly examples of such automorphisms with uniformly bounded γ-distance exist. For example, if the support $\bigcup_{t\in[0,1]} \operatorname{supp}\phi^t$ can be disjoined from itself by a Hamiltonian isotopy.

Let us study the following examples which show that, in general however, there is no upper bound on γ.

5.1. Examples for infinite diameter of Ham(M,ω).

Example 5.6. Consider the autonomous Hamiltonian function

$$H\colon S^1 \times S^1 \to \mathbb{R}, \quad H(x,y) = \sin 2\pi x.$$

Since we consider only contractible 1-periodic solutions, $\mathcal{P}_1(H)$ equals the set of critical points of H,

$$\mathcal{P}_1(H) = \{\pi\} \times S^1 \cup \{-\pi\} \times S^1.$$

Moreover, we see that the action spectrum contains only two values, $\sigma(\phi_H^1) = \{-1,1\}$ and since $\gamma(\phi_H^1) > 0$ we have $\gamma(\phi_H^1) = 2$. We conclude that

$$\gamma(\phi_H^k) = 2k \to \infty \quad \text{for } k \to \infty\,.$$

Clearly, it is straightforward to generalize this observation for $\mathbf{T}^2$ to any symplectically aspherical manifold, i.e., $\omega_{|\pi_2} = 0$, admitting an incompressible Lagrangian torus.

A subset $A \subset M$ is called **incompressible** if the inclusion map gives an injection for the fundamental group.

Example 5.7. Assume that $T^n \hookrightarrow M$ is an incompressible Lagrangian torus. By Weinstein's theorem a tubular neighbourhood is symplectomorphic to a neighbourhood of the zero section in T^*T^n which is a product of T^*S^1. We can now find an autonomous Hamiltonian with compact support in this neighbourhood which factorizes according to the product structure and is independent of the base variables. Thus the only nonconstant periodic solutions are non-contractible and we find a Hamiltonian automorphism ϕ_H such that $\gamma(\phi_H^k) \to \infty$ as $k \to \infty$.

Example 5.8. Clearly, this argument also works for a general incompressible Lagrangian submanifold which admits a Riemannian metric all of whose contractible closed geodesics are constant. For example, if it admits a metric of nonpositive sectional curvature. This example recovers the result by Lalonde and Polterovich in [**12**] which is proven without restriction to $\omega_{|\pi_2(M)}$.

Example 5.9. It is also easy to recover here, in the more restrictive case of a symplectically aspherical manifold, the example of products with surfaces of genus ≥ 1 as given in [**11**].

More generally one can consider symplectic fibrations $(M,\omega) \hookrightarrow (E,\omega_E) \xrightarrow{\pi} (B,\sigma)$ where ω_E is a symplectic form such that its restriction to the induced horizontal subbundle of TE with respect to the associated connection equals the pull-back $\pi^*\sigma$. Assume that (B,σ) is a symplectic manifold of the previously mentioned kind with a Hamiltonian of arbitrarily high oscillation norm $\|H\|$ but without nontrivial contractible 1-periodic solutions. Then the Hamiltonian vector field on E of the pull-back π^*H lies in the horizontal sub-bundle of TE and $D\pi$ identifies it with X_H on B. Hence its non-constant periodic solutions have to be non-contractible too.

5.2. Comparison with Hofer-Zehnder capacity. Let us now compare the metric γ with the Hofer-Zehnder capacity for symplectic manifolds which is defined via so-called admissible Hamiltonians.

Definition 5.10. We call a Hamiltonian function $H\colon S^1 \times M \to \mathbb{R}$ **admissible** if its set of 1-periodic contractible solutions $x \in \mathcal{P}_1(H)$ contains only constant solutions $\dot{x} = 0$, i.e., $x(0) = x(t) \in \operatorname{Crit} H(t,\cdot)$ for all $t \in S^1$.

A smooth Hamiltonian $H\colon [0,1] \times M \to \mathbb{R}$ is called **quasi-autonomous** if there exist two points $x_+, x_- \in M$ such that maximum and minimum of H are attained at x_+ and x_- uniformly for all t. Here we will consider a slightly stronger condition:

$$\begin{aligned} \max_x H(t,x) &= H(t,p) \quad \text{for all } t \in [0,1],\ p \in U_+, \\ \min_x H(t,x) &= H(t,p) \quad \text{for all } t \in [0,1],\ p \in U_- \end{aligned} \tag{36}$$

for some disjoint open neighbourhoods $U_-, U_+ \subset M$ of x_-, x_+.

Theorem 5.11. *Every quasi-autonomous Hamiltonian K satisfying (36) which is admissible and homotopic to 0 through admissible Hamiltonians satisfies*

$$\gamma(\phi_K^1) = \|K\|.$$

Throughout this section we do not assume the Hamiltonian functions to be normalized without further notice. From Proposition 2.14 we know that the functions $c(\alpha, H)$ are well-defined and continuous with respect to $\|H\|$ without any normalization condition. We also still have the symmetry from Poincaré duality, $c_+(H^{-1}) = -c_-(H)$. Moreover, in view of (15), also $\gamma(\phi_H^1) = c_+(H) - c_-(H)$ does not depend on the normalization.

Let K be an admissible Hamiltonian such that there exists an open subset $U \subset M$ with

$$K(t,p) = \max_{x \in M} K(t,x) \quad \text{for all } t \in S^1,\ p \in U. \tag{37}$$

Recall from Definition 2.4, $E_-(K) = -\int_0^1 \max_M K(t,\cdot)\,dt$.

Proposition 5.12. *Assume that an admissible Hamiltonian K satisfying (37) is homotopic to 0 through admissible Hamiltonians. Then any regular Hamiltonian $H \in \mathcal{H}_{reg}$ satisfies*

$$c_-(H) \le E_-(K) + \|H - K\|.$$

Clearly, an autonomous Hamiltonian K is homotopic to 0 through admissible Hamiltonians if every T-periodic contractible solution of $\dot{x} = X_K(x)$ for $0 < T \le 1$ is constant, namely take the homotopy $(\tau H)_{\tau \in [0,1]}$.

Proof of Proposition 5.12. The proof is based on a suitable variation of the definition of the map $\Phi_H \colon H^{2n}(M) \to HF_{-n}(H)$ as given in (3). Given any Morse function f, we known that any local maximum $p \in \mathrm{Crit}_{2n} f$ represents the top cohomology class in terms of Morse cohomology, $[M] = \{p\} \in H^{2n}(f)$. Moreover, any negative gradient flow trajectory for f converging towards p has to lie constant in p. Therefore, we can identify the moduli space used for the definition of $\Phi_H([M])$ as

$$\mathcal{M}^-_{y;p}(H, J; f) = \{\, (u,p) \mid u \in \mathcal{M}^-_{y;p}(H,J)\,\},$$

where

$$\mathcal{M}^-_{y;p}(H,J) = \Big\{ u\colon \mathbb{R}\times S^1 \to M \,\Big|\, \partial_s u + J\big(\partial_t u - \beta(s) X_H(u)\big) = 0, \\ \int_{-\infty}^{\infty} |\partial_s u|^2 ds dt < \infty, \\ u(-\infty) = y, \quad u(+\infty) = p \Big\}$$

and $\beta(s) = 1$ for $s \le -1$, $\beta(s) = 0$ for $s \ge 0$. We therefore have

$$\Phi_H([M]) = \sum_{\mu(y)=-n} \#_{\mathsf{alg}} \mathcal{M}^-_{y;p}(H,J)\, y.$$

Let us now consider the homotopy between H and K

$$G_s(t,x) = \beta(s) H(t,x) + \big(1-\beta(s)\big) K(t,x),$$

so that the associated Hamiltonian vector field satisfies $X_{G_s}(t,x) = 0$ for $s \ge 0$ and $x \in U$. The associated Cauchy-Riemann type flow equation reads

$$u\colon \mathbb{R}\times S^1 \to M, \quad u_s + J\big(u_t - X_{G_s}(u)\big) = 0. \tag{38}$$

Analogously to the solutions in $\mathcal{M}^-_y(H,J)$, every finite energy solution u of (38), i.e., $\iint |u_s|^2 ds dt < \infty$, which satisfies $\lim_{s\to\infty} u(s,t) \in U$ for all $t \in S^1$ has a removable singularity at $+\infty$ and can be smoothly extended over $\mathbb{R}\times S^1 \cup \{+\infty\} \cong \mathbb{C}$. We have thus the well-defined solution space

$$\widetilde{\mathcal{M}}^-_{y;p}(H,K) = \{\, u\colon \mathbb{R}\times S^1 \cup \{\infty\} \to M \,|\, u \text{ solves } (38), \\ u(-\infty) = y,\, u(+\infty) = p \,\}.$$

Again, for a generic almost complex structure J, $\widetilde{\mathcal{M}}^-_{y;p}$ is a $(\mu(y)+n)$-dimensional manifold. To be precise, we allow almost complex structures J to be explicitly (s,t)-dependent for $|s| < 2$ and t-dependent for $t \le -2$. (For details cf. [**20**].) Our aim is to define the element $\Phi_{HK} \in HF_{-n}(H)$ by

$$\Phi_{HK} = \sum_{\mu(y)=-n} \#_{\mathsf{alg}} \widetilde{\mathcal{M}}^-_{y;p}(H,K)\, y. \tag{39}$$

The crucial point is to show that the 0-dimensional solution space $\widetilde{\mathcal{M}}^-_{y;p}(H,K)$ is compact. Recall that in view of the asphericity condition $\omega_{|\pi_2} = 0$ the only compactness obstruction can occur by splitting off of cylindrical Floer trajectories at either end. On the negative end, i.e., for $s \to -\infty$ this is prohibited by the regularity assumption on H, the transversality condition and the index restriction that $\dim \widetilde{\mathcal{M}}^-_{y;p} = 0$. It remains to rule out splitting off at the positive end of u. Suppose we have such a weak

C^∞_{loc}-convergence to a split solution for a sequence of suitably reparameterized trajectories in $\widetilde{\mathcal{M}}^-_{y;p}(H,K)$. Then there exists a non-constant solution of

$$v\colon \mathbb{R}\times S^1 \to M, \quad v_s + J(v)\big(v_t - X_K(t,v)\big) = 0$$

with $v(+\infty) = p$ and $\lim_{s\to-\infty} v(s,\cdot) = z \in \mathcal{P}_1(K)$. By assumption of admissibility of K it follows that $z(t) = z(0) = z_o$ for all $t \in S^1$ and we can compute the flow energy of v as

$$\begin{aligned} E(v) = \iint |v_s|^2\, dsdt = \mathcal{A}_K(p) - \mathcal{A}_K(z) \\ = -\int_{S^1} K(t,p)dt + \int_{S^1} K(t,z_o)dt. \end{aligned}$$

Since by assumption $K(t,x) \le K(t,p)$ for all $t \in S^1$, $x \in M$, it follows that $E(v) = 0$, that is, v has to be constant. Altogether it follows that the 0-dimensional manifold $\widetilde{\mathcal{M}}^-_{y;p}(H,K)$ is compact and $\Phi_{HK} \in CF_{-n}(H)$ is a well-defined Floer chain.

By the standard arguments in Floer theory we can show that

$$\partial\Phi_{HK} = 0,$$

that is $\{\Phi_{HK}\} \in HF_{-n}(H)$ is well-defined. In order to show that $\{\Phi_{HK}\} = \Phi_H([M])$ we use the typical homotopy-cobordism argument in Floer theory. For this we need the assumption that K is homotopic to 0 through admissible Hamiltonians so that we have compactness up to splitting off of Floer trajectories at the negative end. By a standard argument from Floer theory we can then define a chain $S \in CF_{-n+1}$ such that

$$\Phi_{HK} - \Phi_H([M]) = \partial S.$$

Hence we have

$$\{\Phi_{HK}\} = \Phi_H([M]) \in HF_{-n}(H). \tag{40}$$

An energy estimate analogous to (5) shows that

$$\mathcal{A}_H(y) \le \mathcal{A}_K(p) + \|H-K\| \quad \text{if } \widetilde{\mathcal{M}}^-_{y;p}(H,K) \neq \emptyset.$$

Therefore, it follows that

$$\{\Phi_{HK}\} \in \operatorname{im} i^a_* \quad \text{for } a = E_-(K) + \|H-K\|.$$

□

The assumption that K is homotopic to 0 through admissible Hamiltonians can be replaced by a simpler condition in view of the following. If we assume that the possibly non-autonomous Hamiltonian K has no non-constant contractible T-periodic solution for $0 < T \le 1$, the homotopy $(\tau K)_{\tau\in[0,1]}$ is a homotopy through Hamiltonians which are admissible apart from the fact that they are not anymore 1-periodic in t. But in view of the nonlinear Fredholm analysis for the quasi-linear Cauchy-Riemann type operator of the type $\bar\partial\colon W^{1,p}\to L^p$-maps used in the cobordism argument for (40) this non-continuity of H at $t=1$ is not essential. However, this point is not carried out in further detail since we will apply Proposition 5.12 to autonomous Hamiltonians in view of the Hofer-Zehnder capacity.

From Proposition 5.12 we can now conclude the:

Proof of Theorem 5.11. Note that $K^{-1}(t,x)=-K(-t,x)$ satisfies the same condition as K. For any regular Hamiltonian H we obtain from Proposition 5.12 applied to $H^{\pm 1}$ and $K^{\pm 1}$,

$$\gamma(H)\ge \|K\|-2\|H-K\|.$$

From the denseness of $\mathcal{H}_{\mathrm{reg}}$ and the continuity of γ we thus obtain

$$\gamma(K)\ge\|K\|$$

which implies the assertion in view of the obvious estimate $\gamma(K)\le\|K\|$. □

In principle, such admissible Hamiltonian functions are the key ingredient of the definition of the Hofer-Zehnder capacity for symplectic manifolds. However, here we consider an alteration by allowing admissible Hamiltonians to exhibit non-contractible non-constant 1-periodic solutions.

Definition 5.13. Let (U,ω) be a symplectic manifold. Consider the function space

$$\mathcal{H}_c(U)=\{\,H\in C_o^\infty(\operatorname{int}U)\,|\,H\ge 0,\ H_{|V}=\sup H \text{ for some open subset } V\,\}.$$

We call H **HZ-admissible** if the corresponding Hamiltonian flow has no contractible non-constant T-periodic solution with period $T\le 1$. Let

$$\mathcal{H}^o_{\mathsf{HZ}}(U,\omega)=\{\,H\in\mathcal{H}_c(U)\,|\,H \text{ is HZ-admissible}\,\}.$$

Then, the π_1**-sensitive Hofer-Zehnder capacity** of (U,ω) is defined as

$$c^o_{\mathrm{HZ}}(U,\omega)=\sup_{H\in\mathcal{H}^o_{\mathsf{HZ}}(U,\omega)}\|H\|.$$

The definition of the original Hofer-Zehnder capacity $c_{\mathrm{HZ}}(U,\omega)$ also excludes the existence of non-contractible slow non-constant periodic orbits. That is, we have

$$c^o_{\mathrm{HZ}}(U,\omega)\ge c_{\mathrm{HZ}}(U,\omega).$$

We now are able to relate the metric γ with this π_1-sensitive Hofer-Zehnder capacity.

Corollary 5.14. *Let U be an open subset of (M,ω). Then*

$$c^o_{HZ}(U)=\sup\{\gamma(\phi)\,|\,\phi=\phi^1_H,\ H\ \textit{HZ-admissible}\}.$$

Proof. This is a direct consequence from the estimate of Theorem 5.11 because every HZ-admissible Hamiltonian is quasi-autonomous, satisfies (36) and is homotopic to 0 through admissible Hamiltonians. □

This comparison result suggests the following definition of a relative capacity based on the Floer-homological approach via γ.

Definition 5.15. Given any subset $A \subset M$ we can define the following relative capacity $c_\gamma(A) \in [0, \infty) \cup \{\infty\}$,

$$c_\gamma(A) = \sup\{\, \gamma(\phi) \mid \phi = \phi_H^1 \in \mathrm{Ham}(M, \omega),\, \operatorname{supp} X_H(t, \cdot) \subset A \text{ for all } t\}.$$

According to its definition, c_γ is a priori only a relative capacity, i.e., invariant under global symplectic automorphisms of (M, ω). Following [**8**] and [**11**] we have the displacement energy

$$e(A) = \inf\{d_H(\phi, \mathrm{id}) \mid \phi \in \mathrm{Ham}(M, \omega),\ \phi(A) \cap A = \emptyset\}.$$

From Corollary 5.14 and Proposition 5.1 we have the inequality:

Corollary 5.16. *For any open subset $U \subset M$ we have the inequality of the capacities*

$$c_{\mathrm{HZ}}(U) \le c^o_{\mathrm{HZ}}(U) \le c_\gamma(U) \le 2e(U).$$

It can, in general, not be expected that $c_\gamma(U)$ and $c^o_{\mathrm{HZ}}(U)$ coincide because the Hofer-Zehnder capacity is based on the exclusion of *any* fast non-constant contractible periodic solution, whereas the γ-capacity only considers periodic solutions which represent the top and bottom cohomology class. However, there might be nontrivial fast solutions representing nontrivial intermediate classes, for example associated to the levels $c([\omega]^k)$ with $0 < k < n$.

As example, consider a symplectic embedding of the standard ball $(B^{2n}(r), \omega_o)$ of radius r into M such that the image lies within a set A. This embedding provides a push-forward of HZ-admissible Hamiltonians on $B^{2n}(r)$ to M and we obtain

$$\pi r^2 = c_{\mathrm{HZ}}(B^{2n}(r)) \le c_\gamma(A) \le 2e(A).$$

This reproduces the result from Theorem 1.1 in [**10**] in our Floer homological setup for symplectically aspherical manifolds.

Example 5.17. In the case of the torus, we have seen that

$$c_\gamma(S^1 \times S^1) = \infty.$$

But by passing to a suitably large finite covering one can show that

$$c_\gamma(S^1 \times S^1 \setminus (\{\mathrm{pt}\} \times S^1 \cup S^1 \times \{\mathrm{pt}\})) < \infty.$$

In fact, since the universal covering is $\mathbb{R}^2$, we have

$$c^o_{\mathrm{HZ}}(S^1 \times S^1 \setminus (\{\mathrm{pt}\} \times S^1 \cup S^1 \times \{\mathrm{pt}\})) = 1.$$

To make the covering argument more precise we conclude with the following observations.

Let $\pi\colon \hat{M} \to M$ be a finite, symplectic covering of degree m. Given a Hamiltonian $H\colon S^1 \times M \to \mathbb{R}$ we obtain the pull-back $\pi^*H\colon S^1 \times \hat{M} \to \mathbb{R}$

and define the notation $\pi^*\phi_H^1 = \phi_{\pi^*H}^1$. We observe that for such a finite covering we obtain

$$\gamma(\pi^*\phi) = \gamma(\phi) \quad \text{for all } \phi \in \mathrm{Ham}(M,\omega). \tag{41}$$

Namely, since π is in particular a local symplectomorphism, considering the pull-back operation on Floer homology for this finite covering we deduce that

$$c(\pi^*\alpha, \pi^*H) = c(\alpha, H) \quad \text{for all } \alpha \in H^*(M), \text{ s.t. } \pi^*\alpha \neq 0.$$

The identity (41) then follows from property 1). in Theorem 1.2.

Now suppose we have an open subset $U \subset M$ such that its preimage $\pi^{-1}(U) = U_1 \cup \ldots \cup U_m$ is a disjoint union of m copies of a lift of U, and suppose that H satisfies the condition $\operatorname{supp} X_H(t,\cdot) \subset U$ for all t. We may assume that $\operatorname{supp} H(t,\cdot) \subset U$ for all t. Then define $\phi_i \in \mathrm{Ham}(\hat{M},\omega)$ by $\phi_i = \phi^1_{\chi_i\pi^*H}$ where χ_i is the characteristic function of U_i. It follows for $\phi = \phi_H^1$ that $\pi^*\phi = \phi_1 \circ \ldots \circ \phi_m$, hence by (41)

$$\gamma(\phi) \leq \sum_{i=1}^{m} \gamma(\phi_i) = m\gamma(\phi_i). \tag{42}$$

In the case that for (M,U) there exists a finite symplectic covering $\pi\colon \hat{M} \to M$ and $\psi \in \mathrm{Ham}(\hat{M}, \pi^*\omega)$ such that $\psi(U_i) \cap U_i = \emptyset$ we obtain

$$c_\gamma(U) \leq 2\gamma(\psi) < \infty$$

thus proving the finiteness assertion in Example 5.17.

Remark 5.18. The estimate (42) should not be sharp. One might expect $\gamma(\pi^*\phi) = \gamma(\phi_i)$. This will be studied more closely elsewhere.

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