Pacific Journal of Mathematics

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Volume 194 No. 2 June 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 194, No. 2, 2000

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Let Γ be a torsion-free lattice in $SO_0(3,1)$, and let $M = \Gamma\backslash\mathbf{H}^3$ be the corresponding hyperbolic 3-manifold. It is well-known that in the presence of a closed, embedded, totally-geodesic surface in M, the canonical flat conformal structure on M can be deformed via the bending construction. Equivalently, the lattice Γ admits non-trivial deformations into $SO_0(4,1)$. We present a new construction of infinitesimal deformations for the hyperbolic Fibonacci manifolds, the smallest of which is non-Haken and contains no immersed totally geodesic surface.

1. Introduction.

This is the first in a series of papers in which we hope to better understand the deformation theory of $SO(n,1)$ lattices, particularly for the case $n = 3$.

We begin with an oriented n-manifold M equipped with a complete Riemannian metric of constant curvature -1, and set $\pi = \pi_1(M)$. The holonomy gives a discrete and faithful representation $\rho_0 : \pi \to SO_0(n,1)$ into the identity component of $O(n,1)$ which is well-defined up to conjugation. When the volume of M is finite and $n \geq 3$, Mostow Rigidity says that any other discrete, faithful, finite covolume representation $\rho_1 : \pi \to SO_0(n,1)$ is conjugate to ρ_0. On the other hand, we can compose ρ_0 with the inclusion $SO_0(n,1) \hookrightarrow SO_0(n+1,1)$, and attempt to deform in the larger group. When $n = 2$, this reduces to the well-understood theory of quasi-Fuchsian deformations of Fuchsian groups.

More generally, consider the inclusion of a lattice $\rho_0 : \Gamma \hookrightarrow G$ in a simple algebraic group G defined over $\mathbb{R}$, and suppose we have an inclusion $G \hookrightarrow H$ into some other algebraic group H over $\mathbb{R}$. The space of representations $\mathrm{Hom}(\Gamma, H)$ is a real algebraic variety in a natural way, and the Zariski tangent space at ρ_0 is identified with the space of cocycles $Z^1(\Gamma, \mathfrak{h})$. Here the coefficients lie in the Lie algebra $\mathfrak{h}$ of H, which is made into a $\mathbb{Z}\Gamma$-module via ρ_0, the inclusion of G in H, and the adjoint action of H on $\mathfrak{h}$. Trivial deformations (conjugation in H) have Zariski tangent vectors which are coboundaries; thus we may think of the group cohomology $H^1(\Gamma, \mathfrak{h})$ as the space of *infinitesimal deformations* of Γ in H. An infinitesimal deformation is *integrable* if it is tangent to a non-trivial curve in $\mathrm{Hom}(\Gamma, H)$.

Raghunathan's vanishing theorem [**25**] shows that if G is not locally isomorphic to $SO(n,1)$ or $SU(n,1)$, then a uniform lattice Γ admits no infinitesimal deformations in H, while a similar local rigidity result is obtained for $G = SU(n,1)$, $H = SU(n+1,1)$, and $n \geq 2$ in [**8**]. We will be concerned with the case of $G = SO_0(n,1)$, $H = SO_0(n+1,1)$, and $n \geq 3$, where the first examples of infinitesimal deformations (for $n = 3$) were given by Apanasov [**1**], [**3**]. Around the same time, Thurston introduced the related notion of "bending" a Fuchsian group along a geodesic lamination to obtain quasi-Fuchsian groups [**32**, §8.7.3]; more general discussions of bending deformations can be found in [**12**] and [**16**]. There are further examples due to Apanasov which typically arise from intersecting totally geodesic surfaces or surfaces with a common boundary geodesic (see [**2**], [**4**] and [**30**]). All of these deformations are integrable.

M. Kapovich conjectured in [**13**] that a closed hyperbolic 3-orbifold admits a non-trivial (integrable) deformation in $O(4,1)$ if and only if it contains an embedded quasi-Fuchsian suborbifold. In [**14**] he proves this conjecture for reflection orbifolds, and shows that infinitely many surgeries on a two-bridge knot are locally rigid. Kapovich's conjecture was one of the starting points for our work. We were also motivated by the close relationship with moduli problems for constant curvature Lorentzian spacetimes, as in [**24**] and [**29**].

Our main result (Theorem 4.3) gives a new construction of infinitesimal deformations in $SO_0(4,1)$ for an infinite family of closed, two-generator, hyperbolic 3-manifolds, the smallest of which is non-Haken and contains no immersed totally geodesic surface.

I would like to thank Alan Reid, Misha Kapovich, and John Millson for useful discussions concerning early versions of this work. Thanks also to John Hempel, Geoff Mess, and Mike Wolf for their help and encouragement.

2. Preliminaries.

Consider a group π with a fixed presentation $\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_p \rangle$ and suppose V is a $\mathbb{Z}\pi$-module. We will compute the group cohomology $H^1(\pi, V)$ in terms of the standard resolution; thus, a 1-cocycle is a function $c : \pi \to V$ satisfying $c(gh) = c(g) + gc(h)$ for all $g, h \in \pi$, and a 1-coboundary is a 1-cocycle of the form $c(g) = (1-g)w$ for some $w \in V$. Writing $\mathbb{F}^n$ for the free group on n generators, we can make V into a $\mathbb{Z}\mathbb{F}^n$-module in a natural way. There is an isomorphism between V^n and $Z^1(\mathbb{F}^n, V)$ given in terms of the Fox derivatives [**7**]

$$(v_1, \ldots, v_n) \mapsto \left(g \mapsto \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} v_i \right).$$

From this one obtains:

Lemma 2.1 ([7]).

$$Z^1(\pi, V) \cong \left\{ (v_1, \dots, v_n) \in V^n \Big| \sum_{i=1}^{n} \frac{\partial r_j}{\partial x_i} v_i = 0 \;\; for \;\; j = 1, \dots, p \right\}.$$

Under this isomorphism, the subspace $B^1(\pi, V)$ of coboundaries consists of all elements of V^n of the form $((1-x_1)w, (1-x_2)w, \dots, (1-x_n)w)$ for some element w of V.

The next lemma is an immediate consequence of the local rigidity theorems for lattices, first proved in this case by Calabi.

Lemma 2.2 ([12]). *Fix a representation $\rho_0 : \pi \to SO_0(3,1) \hookrightarrow SO_0(4,1)$. The Lie algebra $\mathfrak{so}(4,1)$ splits as an $SO_0(3,1)$-module $\mathfrak{so}(4,1) \cong \mathfrak{so}(3,1) \oplus \mathbb{R}^4_1$, inducing a splitting in cohomology*

$$H^*(\pi, \mathfrak{so}(4,1)) \cong H^*(\pi, \mathfrak{so}(3,1)) \oplus H^*(\pi, \mathbb{R}^4_1).$$

When ρ_0 is an isomorphism onto a uniform lattice in $SO_0(3,1)$, we have

$$H^1(\pi, \mathfrak{so}(4,1)) \cong H^1(\pi, \mathbb{R}^4_1).$$

3. Fibonacci Manifolds and Turk's Head Links.

The *Fibonacci groups* are defined by the presentation

$$F(2,n) = \{a_1, \dots, a_n \mid a_i a_{i+1} = a_{i+2} \pmod{n}\}.$$

Determining the structure of these groups has proved to be quite difficult; indeed, only in the last ten years has it been determined exactly which values of n yield a finite group. A nice overview of the results in this area can be found in [31]. Most important to us is the fact that the group $F(2, 2m)$, $m \geq 4$, is the fundamental group of a closed orientable hyperbolic 3-manifold F_m which can be obtained by side-pairings on an appropriate polyhedron in $\mathbb{H}^3$. We call these manifolds F_m the *Fibonacci manifolds*, first described in [9] (a preprint of which appeared in 1989) and discussed from a similar point of view in [17]. From this result we obtain a discrete and faithful representation $\rho_0 : F(2, 2m) \cong \Gamma_m \subset SO_0(3,1)$ which can be written down explicitly as in [31]. We will only need to indicate here one or two special features of ρ_0.

The defining polyhedron for F_m has an order m rotational symmetry about its central axis which induces the group automorphism $a_i \mapsto a_{i+2}$. We may conjugate so that the invariant axis of this rotation fixes 0 and ∞ in $\mathbb{C}P^1 \approx \partial\mathbb{H}^3$. It follows that the automorphism $a_i \mapsto a_{i+1}$ which cyclically permutes the generators is given by an orientation-reversing isometry t of

$\mathbb{H}^3$ which interchanges the points 0 and ∞ in $\mathbb{C}P^1$. This isometry can be written

$$t(z) = \frac{r^2\xi_{2m}}{\bar{z}}, \quad t^{-1}a_i t = a_{i+1} \pmod{2m}$$

where r is an appropriate positive real number and $\xi_{2m} = \cos(\frac{\pi}{m}) + i\sin(\frac{\pi}{m})$. After conjugating the representation of [31] to our desired position, the element $\rho_0(a_1)$ is given by $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in PSL(2,\mathbb{C})$ with

$$\alpha = 1 - \frac{\psi}{1-\xi_{2m}^{-2}}, \quad \delta = 1 - \frac{\bar{\psi}}{1-\xi_{2m}^{-2}},$$

where

$$\psi = \frac{3}{2} - \cos\left(\frac{2\pi}{m}\right) + \frac{i\sqrt{\rho}}{2}$$

and

$$\rho = \left(3 - 2\cos\left(\frac{2\pi}{m}\right)\right)\left(1 + 2\cos\left(\frac{2\pi}{m}\right)\right).$$

The property of $\rho_0(a_1)$ which we need for our main theorem is that $\omega = \alpha\bar{\delta}$ is real; this can be verified by a direct calculation (it turns out that $\omega = -\frac{1}{4}\csc^2(\frac{\pi}{m})$). Note that the condition $\omega \in \mathbb{R}$ is stable under conjugation by t, indeed under conjugation by any isometry leaving $\{0,\infty\} \subset \mathbb{C}P^1$ invariant.

The group $PSL(2,\mathbb{C})$ acts on 2×2 Hermitian matrices by $A\cdot Q = AQ\bar{A}^t$. This action gives the usual identification of $PSL(2,\mathbb{C})$ with $SO_0(3,1)$ by identifying a point $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4_1$ with the Hermitian matrix

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22}\end{pmatrix} = \begin{pmatrix} x_3+x_4 & x_1+ix_2 \\ x_1-ix_2 & x_4-x_3\end{pmatrix}.$$

We choose to work directly with the basis $q_{12}, q_{21}, q_{11}, q_{22}$ with respect to which we have

$$t = \begin{pmatrix} \xi_{2m} & 0 & 0 & 0 \\ 0 & \xi_{2m}^{-1} & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & r^{-1} & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} \omega & \beta\bar{\gamma} & \alpha\bar{\gamma} & \beta\bar{\delta} \\ \gamma\bar{\beta} & \omega & \gamma\bar{\alpha} & \delta\bar{\beta} \\ \alpha\bar{\beta} & \beta\bar{\alpha} & |\alpha|^2 & |\beta|^2 \\ \gamma\bar{\delta} & \delta\bar{\gamma} & |\gamma|^2 & |\delta|^2 \end{pmatrix}.$$

Here and in all that follows we suppress the representation ρ_0.

These manifolds can be realized as branched coverings of links in $\mathbb{S}^3$ in several different ways. For $k \geq 2$, let B_k denote the k^{th} Turk's head link; this is the closed 3-braid in $\mathbb{S}^3$ given by $(\sigma_1\sigma_2^{-1})^k$ in the usual notation for the braid group. The link B_k has three components when $k \equiv 0 \pmod 3$ and is a knot otherwise. In particular, B_2 is the figure-eight knot 4_1, B_3 is the Borromean rings 6^3_2, B_4 is the Turk's head knot 8_{18}, and $B_5 = 10_{123}$ in the standard tables. Let $T_k = \mathbb{S}^3 \setminus B_k$. We have the following:

Proposition 3.1. *F_m is the m-fold cyclic branched cover of the figure-eight knot and also the 2-fold branched cover of B_m. The branched covering over B_m induces the automorphism $a_1 \mapsto a_1^{-1}, a_2 \mapsto a_2^{-1}$ of $F(2,2m)$.*

Proof. We give a sketch of the proof following [**20**]. Let a and b be a pair of standard generators for π_1 of a once-punctured torus, and write R and L for Dehn twists on these curves in the usual fashion, so that $R \cdot b = ab$ and $L \cdot a = ba$. The fundamental group of the once-punctured torus bundle with monodromy $(RL)^m$ is generated by a, b, and an element t such that conjugation by t induces the monodromy. Let Φ_m be the manifold obtained by surgery on this bundle killing the generator t, so that

$$\pi_1(\Phi_m) \cong \langle a, b \mid a = (RL)^m a, \quad b = (RL)^m b\rangle.$$

Clearly Φ_m is an m-fold cyclic branched cover of $M_1 \approx \mathbb{S}^3$ branched over the core of the attached solid torus (the figure-eight knot). It is easy to check that sending $a_{2i+1}^{-1} \mapsto (RL)^i b$ and $a_{2i+2}^{-1} \mapsto (RL)^i a$ defines an isomorphism of $F(2,2m)$ and $\pi_1(\Phi_m)$. Since it is well-known that the m-fold cyclic branched cover of 4_1 is hyperbolic if and only if $m \geq 4$ [**10**], it follows from Mostow Rigidity that Φ_m is isometric to F_m in its polyhedral description [**9**].

The involution of $T^2 \backslash \{pt\}$ given by $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$ gives a branched covering of D^2 branched over three points p_1, p_2, p_3. This extends fiber-wise to an involution of the once-punctured torus bundle and then to the surgered manifold Φ_m. Via the explicit isomorphism above, this involution acts by $a_1 \mapsto a_1^{-1}$ and $a_2 \mapsto a_2^{-1}$ on $F(2,2m)$. The quotient of Φ_m under this map is an orbifold $\hat{T}_m \approx S^3$ with branched set coming from images of the $p_i \times I$. By choosing orientations carefully, the twist R induces the braid group element σ_1 on D^2 and L induces σ_2^{-1}, so the branched set is precisely the link B_m. □

4. Cohomology Computations.

As in the proof of Proposition 3.1, $\hat{T}_m$ is the π-orbifold branched on the Turk's head link B_m, and we will write Δ_m for the orbifold fundamental group of $\hat{T}_m$, a split $\mathbb{Z}_2$-extension of Γ_m.

Proposition 4.1. $H^1(\Gamma_m, \mathbb{R}^4_1) \cong H^1(\Delta_m, \mathbb{R}^4_1)$.

Proof. Using the transfer operator in group cohomology, $H^1(\Delta_m, \mathbb{R}^4_1)$ is isomorphic to the subspace of $H^1(\Gamma_m, \mathbb{R}^4_1)$ of cohomology classes invariant under the covering translation. Now Γ_m is a two-generator group so an element $[c] \in H^1(\Gamma_m, \mathbb{R}^4_1)$ is given by a pair $(v_1, v_2) \in \mathbb{R}^4_1 \times \mathbb{R}^4_1$. We may integrate this class to a deformation ρ_t of $\mathbb{F}^2$. For any value of t, there exists an orientation-preserving involution inverting $\rho_t(a_1)$ and $\rho_t(a_2)$ which can be obtained by a rotation of π about an axis in $\mathbb{H}^4$ perpendicular to the invariant axes of $\rho_t(a_1)$ and $\rho_t(a_2)$. This defines a deformation of Δ_m to first order, hence an element of $H^1(\Delta_m, \mathbb{R}^4_1)$. □

Corollary 4.2. $\dim_{\mathbb{R}} H^1(\Gamma_m, \mathbb{R}^4_1) \leq 2$.

Proof. Since the centralizer of Δ_m in $SO_0(4,1)$ is trivial, the dimension of $B^1(\Delta_m, \mathbb{R}^4_1)$ is 4, so it suffices to show that $\dim Z^1(\Delta_m, \mathbb{R}^4_1) \leq 6$. Because the Turk's head links are closed 3-braids, the group Δ_m is generated by three order-two elliptics $\gamma_1, \gamma_2, \gamma_3 \in SO_0(3,1)$. Any cocycle is trivial when restricted to the $\mathbb{Z}_2$ generated by one of the γ_j, so it is determined by a triple of the form $((1-\gamma_1)v_1, (1-\gamma_2)v_2, (1-\gamma_3)v_3)$ for $v_1, v_2, v_3 \in \mathbb{R}^4_1$. But an order-two element in $SO_0(3,1)$ has 1 as an eigenvalue of multiplicity two, so the space of such triples is 6-dimensional. $\square$

Next we will state and prove our main result. The proof proceeds by a direct computation, exploiting the symmetry of the manifolds involved to reduce the cohomology calculation to a tractable linear algebra problem.

Theorem 4.3. *For all $m \geq 4$, $\dim_{\mathbb{R}} H^1(\Gamma_m, \mathfrak{so}(4,1)) = 2$.*

Proof. By Corollary 4.2 and Lemma 2.2 it suffices to show $\dim H^1(\Gamma_m, \mathbb{R}^4_1) \geq 2$. Throughout the proof, all indices i will be taken modulo $2m$.

Writing R_i for the relator $a_i a_{i+1} a_{i+2}^{-1}$, we have the Fox derivatives

$$\frac{\partial R_i}{\partial a_i} = 1, \quad \frac{\partial R_i}{\partial a_{i+1}} = a_i,$$

$$\frac{\partial R_i}{\partial a_{i+2}} = -a_i a_{i+1} a_{i+2}^{-1} = -1.$$

A cocycle is thus given by a $2m$-tuple of vectors $v_i \in \mathbb{R}^4_1$ satisfying $v_i + a_i v_{i+1} = v_{i+2}$ for each i. With this in mind, we define 8×8 real matrices

$$A_i = \begin{pmatrix} 0 & I \\ I & a_i \end{pmatrix} \text{ and } T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

so that a cocycle is the same thing as a 1-eigenvector $(v_1, v_2) \in \mathbb{R}^4_1 \times \mathbb{R}^4_1$ for the matrix

$$A_{2m} A_{2m-1} \cdots A_1 = T^{-(2m-1)} A_1 T^{2m-1} \cdots T^{-1} A_1 T A_1 = (TA_1)^{2m}.$$

Our plan is to find six distinct eigenvalues of TA_1 which are $2m^{th}$-roots of unity; four of these will correspond to the space of coboundaries. We claim that these are exactly the eigenvalues of t (namely $1, -1, \xi_{2m}$, and ξ_{2m}^{-1}). To see this, let v be a λ-eigenvector for t. Then

$$\begin{pmatrix} 0 & t \\ t & ta_1 \end{pmatrix} \begin{pmatrix} (1-a_1)v \\ (1-a_2)v \end{pmatrix} = \begin{pmatrix} t(1-a_2)v \\ t(1-a_1a_2)v \end{pmatrix} = \begin{pmatrix} (1-a_1)tv \\ (1-a_2)tv \end{pmatrix} = \lambda \begin{pmatrix} (1-a_1)v \\ (1-a_2)v \end{pmatrix}.$$

We now claim that $-\xi_{2m}$ and hence $-\xi_{2m}^{-1}$ are eigenvalues of TA_1, which suffices to prove the theorem. For $\lambda \neq 0$ to be an eigenvalue of TA_1, we need (v_1, v_2) such that

$$\begin{pmatrix} 0 & t \\ t & ta_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}.$$

Taking $v_1 = \lambda^{-1} t v_2$, this reduces to finding v_2 such that $(\lambda^{-1}t^2 + ta_1 - \lambda I)v_2 = 0$. Multiplying through by t^{-1} and setting $E(\lambda) = \lambda^{-1}t - \lambda t^{-1}$, this is equivalent to showing

$$\det(a_1 + E(\lambda)) = 0.$$

With respect to our chosen representation of Γ_m, we have

$$E(\lambda) = \begin{pmatrix} \lambda^{-1}\xi_{2m} - \lambda\xi_{2m}^{-1} & 0 & 0 & 0 \\ 0 & \lambda^{-1}\xi_{2m}^{-1} - \lambda\xi_{2m} & 0 & 0 \\ 0 & 0 & 0 & (\lambda^{-1} - \lambda)r \\ 0 & 0 & (\lambda^{-1} - \lambda)r^{-1} & 0 \end{pmatrix}.$$

The result will follow from the symmetry $\lambda \leftrightarrow -\lambda^{-1}$; in particular, we already know that $\det(a_1 + E(\xi_{2m}^{-1})) = 0$ since $\lambda = \xi_{2m}^{-1}$ corresponds to a coboundary. The matrices $E(\xi_{2m}^{-1})$ and $E(-\xi_{2m})$ only differ by reversing the first two diagonal entries, so these determinants can be compared easily. Setting

$$C = \xi_{2m} - \xi_{2m}^{-1} = 2i \sin\left(\frac{\pi}{m}\right),$$

and collecting terms we have

$$\begin{aligned} \det(a_1 + E(-\xi_{2m})) &= \det(a_1 + E(-\xi_{2m})) - \det(a_1 + E(\xi_{2m}^{-1})) \\ &= \big(\xi_{2m}^2 - \xi_{2m}^{-2})(2\mathrm{Im}\,(\alpha\bar{\gamma}\gamma\bar{\delta})(Cr + |\beta|^2) \\ &\qquad + 2\mathrm{Im}\,(\beta\bar{\delta}\alpha\bar{\beta})(Cr^{-1} + |\gamma|^2) \\ &\qquad - 2|\alpha|^2\mathrm{Im}\,(\beta\bar{\delta}\gamma\bar{\delta}) - 2|\delta|^2\mathrm{Im}\,(\alpha\bar{\gamma}\alpha\bar{\beta})\big). \end{aligned}$$

Now $\alpha\bar{\gamma}\gamma\bar{\delta} = \omega|\gamma|^2 \in \mathbb{R}$ so the first term vanishes, and similarly the second term. The last two terms combine to give:

$$-2\mathrm{Im}\,(|\alpha|^2\beta\bar{\delta}\gamma\bar{\delta} + |\delta|^2\alpha\bar{\gamma}\alpha\bar{\beta}) = -2\omega\mathrm{Im}\,(\bar{\alpha}\bar{\delta}\beta\gamma + \alpha\delta\bar{\beta}\bar{\gamma}) = 0.$$

□

5. Concluding Remarks.

Since F_m is obtained as a branched cover of the closed 3-braid T_m, the results of [18] imply that while F_m contains a closed incompressible surface for $m \geq 5$, the manifold F_4 is non-Haken. In particular, F_4 contains no closed, embedded, totally geodesic surface. Even better, it is known [9], [11] that Γ_m is arithmetic for $m = 4, 5, 6, 8$, and 12 and the results of [19] applied to the case $m = 4$ show that F_4 contains no non-elementary Fuchsian subgroups at all (the invariant trace field and quaternion algebra for Γ_4 are computed in [27]). Our result should also be contrasted with the fact that the complex structure on $\Gamma\backslash SL(2,\mathbb{C})$ is rigid if and only if the first Betti number of Γ is zero [6], [26].

It is shown in [22] that for $m \geq 2$, the hyperbolic volume of F_{2m} is equal to the volume of T_m. In particular, the volume of F_4 is equal to the volume of the figure-eight knot complement, $2.02988\ldots$ (this result for F_4 also appears in [27]). The alert reader may note [21], [27] that F_4 double covers vol_3, the manifold obtained by (3, -2; 6, -1) surgery on the Whitehead link. The notation is meant to indicate that vol_3 has the third smallest volume among known hyperbolic 3-manifolds. Although vol_3 is obtained by surgery on a two-bridge link, it does not follow directly from the results in [14] that it is locally rigid. We have verified however that the cohomology classes given by the main theorem do not transfer to this smaller manifold, indeed by direct computation that $H^1(\pi_1(vol_3), \mathfrak{so}(4,1)) = 0$. We conjecture that F_4 is the smallest volume hyperbolic 3-manifold admitting infinitesimal deformations in $SO_0(4,1)$. We believe the best previously known example is the smallest volume hyperbolic 3-manifold containing a closed embedded totally geodesic surface [5] which has volume $6.4519\ldots$

The deformation theory of the Turk's head links appears quite interesting in its own right, and we hope to return to the question of computing $H^1(\pi, \mathfrak{so}(4,1))$ at various points along the $SL(2,\mathbb{C})$ character variety. A better understanding of these cohomology groups for hyperbolic knots in $\mathbb{S}^3$ can be viewed as an approach to the Menasco-Reid conjecture [23], which states that no hyperbolic knot complement in $\mathbb{S}^3$ contains a closed embedded totally geodesic surface. In particular, computations similar to the ones in this paper, [14], and [23] can be used to verify the conjecture for knots of up to 10 crossings. A related rigidity theorem for closed manifolds obtained by hyperbolic Dehn surgery is proved in [28].

We do not know if the cohomology classes constructed in the main theorem are integrable. An easy calculation shows that the quadratic obstruction to integrability of $[c] \in H^1(\pi, \mathbb{R}^4_1)$ vanishes in general, but recent examples of Kapovich and Millson [15] lead one to believe that there may be higher order obstructions in some cases.

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Received September 21, 1998

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