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Given a compact manifold M, we prove that every critical Riemannian metric g for the functional "first eigenvalue of the Laplacian" is $\boldsymbol{\lambda_1}$-minimal (i.e., $(\boldsymbol{M}, \boldsymbol{g})$ can be immersed isometrically in a sphere by its first eigenfunctions) and give a sufficient condition for a $\boldsymbol{\lambda_1}$-minimal metric to be critical. In the second part, we consider the case where $\boldsymbol{M}$ is the 2-dimensional torus and prove that the flat metrics corresponding to square and equilateral lattices of $\mathbb{R}^2$ are the only $\boldsymbol{\lambda_1}$-minimal and the only critical ones.

Introduction.

Many recent works concerning the spectrum of compact Riemannian manifolds have pointed out the importance of a particular class of Riemannian metrics which we called in [**5**] λ_1-minimal. Recall that a metric g on a compact m-dimensional manifold M is λ_1-minimal if the eigenspace $E_1(g)$ associated to the first nonzero eigenvalue $\lambda_1(g)$ of the Laplacian of g contains a family $f_1, \dots f_k$ of functions satisfying: $\sum_{1\le i\le k} df_i \otimes df_i = g$. It follows from a well known result of Takahashi [**8**] that this last condition is equivalent to the fact that the map $f = (f_1, \dots f_k)$ is a minimal isometric immersion from (M, g) into the Euclidean sphere $\mathbb{S}_r^{k-1}$ of radius $r = \sqrt{\frac{m}{\lambda_1(g)}}$.

The best known examples of λ_1-minimal metrics are the standard metrics of rank one compact symmetric spaces (i.e., spheres and projective spaces). More generally, any Riemannian irreducible homogeneous space is λ_1-minimal. Also, Yau [**9**] conjectured that a minimal embedded hypersurface of a Euclidean sphere, carrying the induced metric, must be λ_1-minimal.

In [**2**], Berger showed that the λ_1-minimality of a metric g is strongly related to the extremality of g for a spectral functional involving the k-smallest eigenvalues of the Laplacian (where k is the multiplicity of $\lambda_1(g)$). Recently, Nadirashvili [**7**] considered the functional $\lambda_1 : g \mapsto \lambda_1(g)$ defined on the set of Riemannian metrics of given area on a compact surface M and showed that the extremal metrics of this functional are λ_1-minimal (here extremality is defined in a generalized sense because of the non-differentiability of λ_1).

In the first part of this paper we generalize Nadirashvili's theorem to higher dimensions (Theorem 1.1). We also give a sufficient condition for a λ_1-minimal metric to be extremal for λ_1 (Proposition 1.1).

Using results established by us in [**4**] about λ_1-minimal metrics we deduce that (Corollary 1.1), if g is an extremal metric of the λ_1 functional then:

(i) The multiplicity of $\lambda_1(g)$ is at least equal to $m+1$ and equality holds only for the standard metric of Euclidean spheres.
(ii) The restriction of the λ_1 functional to the conformal class of g achieves its maximum at g. In particular, the λ_1 functional has no local minima.
(iii) The metric g is, up to dilatation, the unique extremal metric in its conformal class.
(iv) If g is not isometric to the standard metric of a Euclidean sphere then any conformal diffeomorphism of (M, g) is an isometry.

The second part of this paper deals with the classification of λ_1-minimal metrics and of the extremal metrics of the λ_1 functional. The only manifold for which this classification was available is the 2-dimensional sphere. Indeed, on $\mathbb{S}^2$ the standard metric is (up to dilatation) the only one to be λ_1-minimal and the only extremal metric for λ_1 (this follows from the uniqueness of the conformal class on $\mathbb{S}^2$ and property (iii) above).

The main theorem of Section 2 (Theorem 2.1) states that in genus one (i.e., on the torus $\mathbf{T}^2$) there exists, up to dilatation, exactly two λ_1-minimal metrics: The Clifford metric g_{cl} and the equilateral metric g_{eq} induced from the Euclidean metric respectively on $\mathbb{R}^2/\mathbf{Z}^2$ and $\mathbb{R}^2/\Gamma_{eq}$ with $\Gamma_{eq} = \mathbf{Z}(1,0) \oplus \mathbf{Z}(1/2, \sqrt{3}/2)$. These two metrics are also the only extremal metrics for λ_1 (Corollary 2.2). Moreover, we prove that for each of them, the standard embedding (in $\mathbb{S}^3$ for g_{cl} and $\mathbb{S}^5$ for g_{eq}) is, up to equivalence, the only full (minimal) isometric immersion by the first eigenfunctions.

Note that a first step towards this classification was achieved by Montiel and Ros [**6**] who proved that the only minimal torus immersed in $\mathbb{S}^3$ by its first eigenfunctions is the Clifford torus. They deduced that if the aforementioned conjecture of Yau is true, then the Clifford torus is the only minimally embedded torus in $\mathbb{S}^3$ (Lawson's conjecture).

1. Extremal metrics for the λ_1 functional.

Let M be a compact smooth manifold of dimension $m \geq 2$. Denote by $\mathcal{R}_0(M)$ the set of Riemannian metrics of volume 1 on M. For any $g \in \mathcal{R}_0(M)$, we denote by $0 < \lambda_1(g) \leq \lambda_2(g) \leq \cdots \leq \lambda_k(g) \leq \cdots$ the increasing sequence of eigenvalues of the Laplacian Δ_g of g. The functional:

$$\begin{array}{rccc} \lambda_1 & : \mathcal{R}_0(M) & \to & \mathbb{R} \\ & g & \mapsto & \lambda_1(g) \end{array}$$

is continuous but not differentiable in general. However, for any family $(g_t)_t$ of metrics, analytic in t, $\lambda_1(g_t)$ has right and left derivatives w.r.t. t. Indeed, if $(g_t)_{t\in]-\delta,\delta[}$ is such a family and if k is the multiplicity of $\lambda_1(g_0)$, then there exists k analytic families $\Lambda_{1,t},\dots,\Lambda_{k,t}$ of real numbers and k analytic families of smooth functions $u_{1,t},\dots,u_{k,t}$ such that: $\forall i\le k$ and $\forall t$, $\Delta_{g_t}u_{i,t}=\Lambda_{i,t}u_{i,t}$, $\Lambda_{i,0}=\lambda_1(g_0)$ and $\{u_{1,t},\dots,u_{k,t}\}$ is $L_2(g_t)$-orthonormal (see [**1**] and [**2**] for details). Moreover, Berger [**2**] gave the following formula for the derivative of $\Lambda_{i,t}$:

$$\frac{d}{dt}\Lambda_{i,t}\Big|_{t=0}=-\int_M\langle q(u_i),h\rangle\,\nu_{g_0},$$

where ν_{g_0} is the Riemannian volume element of g_0, $u_i=u_{i,0}$, $h=\frac{d}{dt}g_t\big|_{t=0}$, $\langle\ ,\ \rangle$ is the inner product induced by g_0 on the space $S^2(M)$ of symmetric covariant 2-tensors of M and where for any $u\in C^\infty(M)$,

$$q(u)=du\otimes du+\frac{1}{4}\Delta_{g_0}(u^2)g_0.$$

From the continuity of $\lambda_i(g_t)$ and $\Lambda_{i,t}$ w.r.t. t , we have for t small enough $\{\Lambda_{i,t}\}_{1\le i\le k}=\{\lambda_i(g_t)\}_{1\le i\le k}$ and thus $\lambda_1(g_t)=\min_{1\le i\le k}\{\Lambda_{i,t}\}$. This proves the left and right differentiability of $\lambda_1(g_t)$ and gives:

$$\frac{d}{dt}\lambda_1(g_t)\Big|_{t=0^+}=\min_{1\le i\le k}\frac{d}{dt}\Lambda_{i,t}\Big|_{t=0}=-\max_{1\le i\le k}\int_M\langle q(u_i),h\rangle\,\nu_{g_0},$$

and

$$\frac{d}{dt}\lambda_1(g_t)\Big|_{t=0^-}=\max_{1\le i\le k}\frac{d}{dt}\Lambda_{i,t}\Big|_{t=0}=-\min_{1\le i\le k}\int_M\langle q(u_i),h\rangle\,\nu_{g_0}.$$

This suggests the following definition:

Definition 1.1. A metric $g\in\mathcal{R}_0(M)$ is said to be extremal for the λ_1 functional if for any analytic deformation $(g_t)_t\subset\mathcal{R}_0(M)$, with $g_0=g$, the left and right derivatives of $\lambda_1(g_t)$ at $t=0$ have opposite signs, i.e.,

$$\frac{d}{dt}\lambda_1(g_t)\Big|_{t=0^+}\le 0\le\frac{d}{dt}\lambda_1(g_t)\Big|_{t=0^-}.$$

This last condition is equivalent to :

$$\lambda_1(g_t)\le\lambda_1(g)+o(t)\ \text{ as }\ t\to 0.$$

Hence our definition of extremality is a equivalent formulation of Nadirashvili's one [**7**].

The main result of this section is:

Theorem 1.1. *If a Riemannian metric $g\in\mathcal{R}_0(M)$ is extremal for λ_1 then it is λ_1-minimal.*

In the 2-dimensional case this result was proved by Nadirashvili [**7**]. Some of the arguments in our proof are inspired by his. However, the use of the aforementioned result of Berger makes the proof of this theorem simpler and more transparent.

Lemma 1.1. *If a metric $g \in \mathcal{R}_0(M)$ is extremal for λ_1 then for any $h \in S_0^2(M) = \{h \in S^2(M);\ \int_M tr_g h \nu_g = 0\}$ there exists $u \in E_1(g)\backslash\{0\}$ such that:*

$$\int_M \langle q(u), h\rangle \, \nu_g = 0.$$

Proof. Suppose that g is extremal for λ_1 and let $h \in S_0^2(M)$. We let, for small t, $g_t = \frac{g+th}{V(g+th)^{2/m}} \in \mathcal{R}_0(M)$, where $V(g+th)$ is the Riemannian volume of $g+th$. Since $\frac{d}{dt}V(g+th)\big|_{t=0} = \frac{1}{2}\int_M tr_g h\nu_g = 0$, we find $\frac{d}{dt}g_t\big|_{t=0} = h$. The extremality condition implies that the quadratic form $u \in E_1(g) \mapsto \int_M \langle q(u), h\rangle \, \nu_g$ takes on both nonpositive and nonnegative values, and therefore it admits at least one isotropic direction. $\square$

Proof of Theorem 1.1. Let K be the convex hull in $S^2(M)$ of $\{q(u),\ u \in E_1(g)\}$. The set $K \cup \{g\}$ is contained in a finite dimensional subspace of $S^2(M)$. We claim that $g \in K$. Indeed, if $g \notin K$ then, since K is a convex cone, the Hahn-Banach theorem implies the existence of $s \in S^2(M)$ such that:

$$\int_M \langle s, g\rangle \, \nu_g > 0 \ \text{ and for every } \ l \in K\backslash\{0\},\ \int_M \langle l, s\rangle \, \nu_g < 0.$$

The 2-tensor $\tilde{s} = s - \frac{\left(\int_M \langle s,g\rangle \nu_g\right)}{mV(g)} g$ belongs to $S_0^2(M)$ and, for any $u \in E_1(g)\backslash\{0\}$,

$$\begin{aligned}&\int_M \langle q(u), \tilde{s}\rangle \, \nu_g \\ &= \int_M \langle q(u), s\rangle \, \nu_g - \frac{1}{mV(g)}\left(\int_M \langle s, g\rangle \, \nu_g\right)\left(\int_M |du|^2\nu_g\right) < 0.\end{aligned}$$

By Lemma 1.1, this contradicts the extremality of g.

Thus $g \in K$ and there exists $w_1, ...w_d \in E_1(g)$ such that:

$$\begin{aligned} g &= \sum_{1\le i\le d} q(w_i) = \sum_{1\le i\le d} dw_i \otimes dw_i + \frac{1}{4}\left(\sum_{1\le i\le d} \Delta w_i^2\right) g \\ &= \sum_{1\le i\le d}\left(dw_i \otimes dw_i + \frac{1}{2}\left(\lambda_1(g) w_i^2 - |dw_i|^2\right) g\right). \end{aligned}$$

The traceless part of the last member of this equation must be zero. Therefore,

$$\sum_{1\le i\le d}\left(dw_i\otimes dw_i-\frac{|dw_i|^2}{m}g\right)=0,$$

and then:

$$\frac{\lambda_1}{2}\sum_{1\le i\le d}w_i^2=1+\left(\frac{m-2}{2m}\right)\sum_{1\le i\le d}|dw_i|^2. \tag{1}$$

The λ_1-minimality of g will follow from the fact that $\sum_{1\le i\le d}|dw_i|^2$ is constant and equal to m. Indeed, set $f=\left(\sum_{1\le i\le d}w_i^2\right)-\frac{m}{\lambda_1(g)}$. From (1) we get:

$$(m-2)\Delta_g f=2(m-2)\left(\lambda_1(g)\left(\sum_{1\le i\le d}w_i^2\right)-\sum_{1\le i\le d}|dw_i|^2\right)=-4\lambda_1(g)f.$$

This implies that $f=0$ (the Laplacian being a positive operator). Therefore $\left(\sum_{1\le i\le d}w_i^2\right)=\frac{m}{\lambda_1(g)}$. Replacing in (1) we obtain $\sum_{1\le i\le d}|dw_i|^2=m$. $\square$

In [**4**] we showed that λ_1-minimal metrics satisfy certain remarkable conformal properties. Theorem 1.1 tells us that all these properties are still true for extremal metrics:

Corollary 1.1. *Let $g\in\mathcal{R}_0(M)$ be an extremal metric for λ_1.*

(i) *The multiplicity of $\lambda_1(g)$ satisfies: $mult(\lambda_1(g))\ge m+1$, where equality holds if and only if g is isometric to a standard metric of a Euclidean sphere.*
(ii) *For any $g'\in C_0(g)=\{g'\in\mathcal{R}_0(M)\ ;\ g'\ \ conformal\ \ to\ \ g\}$ we have $\lambda_1(g')\le\lambda_1(g)$, and equality holds if and only if g' is isometric to g. In particular, the functional λ_1 does not admit a local minimum in $\mathcal{R}_0(M)$.*
(iii) *The metric g is, up to isometry, the only extremal metric of λ_1 in $C_0(g)$.*
(iv) *If (M,g) is not isometric to a Euclidean sphere then any conformal diffeomorphism of (M,g) is an isometry.*

The following is a converse to Theorem 1.1.

Proposition 1.1. *Let $g\in\mathcal{R}_0(M)$ and assume there exists an $L_2(g)$-orthonormal basis $\{\phi_1,\dots,\phi_k\}$ of $E_1(g)$ such that the 2-tensor $\sum_{1\le i\le k}d\phi_i\otimes d\phi_i$ is proportional to g. Then g is extremal for λ_1.*

Proof. Let $(g_t)_t \subset \mathcal{R}_0(M)$ be a family of metrics analytic in t with $g_0 = g$ and set $h = \frac{d}{dt} g_t\big|_{t=0}$. With the same notation as above we have for small t:

$$\sum_{1 \le i \le k} \lambda_i(g_t) = \sum_{1 \le i \le k} \Lambda_{i,t}.$$

Therefore, $\sum_{1 \le i \le k} \lambda_i(g_t)$ is differentiable at $t = 0$ and

$$\frac{d}{dt} \sum_{1 \le i \le k} \lambda_i(g_t) \Bigg|_{t=0} = \frac{d}{dt} \sum_{1 \le i \le k} \Lambda_{i,t} \Bigg|_{t=0} = \operatorname{trace} Q_h,$$

where Q_h is the quadratic form defined on $E_1(g)$ by:

$$Q_h(u) = \int_M \langle q(u), h \rangle \, \nu_g,$$

and where the trace of Q_h is taken w.r.t the L_2 inner product induced by g. Now

$$\begin{aligned} \operatorname{trace} Q_h &= \sum_{1 \le i \le k} Q_h(\phi_i) \\ &= \int_M \left\langle \sum_{1 \le i \le k} d\phi_i \otimes d\phi_i, h \right\rangle \nu_g + \frac{1}{4} \sum_{1 \le i \le k} \int_M \left\langle \Delta \phi_i^2, h \right\rangle \nu_g. \end{aligned}$$

Since $\sum_{1 \le i \le k} d\phi_i \otimes d\phi_i$ is proportional to g and $\int_M \langle g, h \rangle \, \nu_g = 2 \, \frac{d}{dt} V(g_t)\big|_{t=0} = 0$ we have $\int_M \left\langle \sum_{1 \le i \le k} d\phi_i \otimes d\phi_i, h \right\rangle \nu_g = 0$. Moreover, by Takahashi's theorem $\sum_{1 \le i \le k} \phi_i^2$ is constant. Therefore, $\operatorname{trace} Q_h = 0$ and $\frac{d}{dt} \sum_{1 \le i \le k} \lambda_i(g_t)\Big|_{t=0} = 0$. The extremality of g then follows from the inequality $\lambda_1(g_t) \le \frac{1}{k} \sum_{1 \le i \le k} \lambda_i(g_t)$ which is an equality at $t = 0$. □

Remarks.

1) It is known that compact irreducible homogeneous Riemannian spaces satisfy the hypothesis of Proposition 1.1 (see [**8**]). Thus, their standard metrics are extremal for λ_1.
2) We restricted ourselves to λ_1. Nevertheless, the results of this paragraph can be carried over to the case of higher eigenvalues.

2. λ_1-minimal and extremal metrics on the torus.

Let (M, g) be an orientable compact surface of genus one endowed with a Riemannian metric g. It is well known that there exists a lattice Γ of $\mathbb{R}^2$ such that (M, g) is conformally equivalent to the torus $\left(\mathbb{R}^2/\Gamma, g_\Gamma\right)$, where g_Γ is the flat metric induced from the Euclidean metric on $\mathbb{R}^2$. The Clifford torus $\left(\mathbf{T}^2_{cl} = \mathbb{R}^2/\Gamma_{cl}, g_{cl} = g_{\Gamma_{cl}}\right)$ with $\Gamma_{cl} = \mathbf{Z}(1,0) \oplus \mathbf{Z}(0,1)$, and the equilateral torus $\left(\mathbf{T}^2_{eq} = \mathbb{R}^2/\Gamma_{eq}, g_{eq} = g_{\Gamma_{eq}}\right)$ with $\Gamma_{eq} = \mathbf{Z}(1,0) \oplus \mathbf{Z}(1/2, \sqrt{3}/2)$, each admit a natural homothetic minimal embedding into a sphere. These embeddings, denoted by ϕ_{cl} and ϕ_{eq}, are those induced on $\mathbf{T}^2_{cl}$ and $\mathbf{T}^2_{eq}$ from $\tilde{\phi}_{cl} : \mathbb{R}^2 \to \mathbb{S}^3$, where $\tilde{\phi}_{cl}(x, y) = \frac{1}{\sqrt{2}}\,(\exp 2i\pi x, \exp 2i\pi y)$, and $\tilde{\phi}_{eq} : \mathbb{R}^2 \to \mathbb{S}^5$, where $\tilde{\phi}_{eq}(x, y) = \frac{1}{\sqrt{3}}\left(\exp 4i\pi y/\sqrt{3}, \exp 2i\pi(x - y/\sqrt{3}), \exp 2i\pi(x + y/\sqrt{3})\right)$.

Theorem 2.1. *Let (M, g) be a compact orientable surface of genus one and suppose that there exists a full isometric immersion $\phi = (\phi_1, \ldots , \phi_{n+1})$ from (M, g) in the n-dimensional unit sphere $\mathbb{S}^n$ such that $\forall i \le n+1$, $\phi_i \in E_1(g)$. Then either:*

(i) *(M, g) is isometric to the normalized Clifford torus $(\mathbf{T}^2_{cl}, 2\pi^2 g_{cl})$, $n = 3$ and ϕ is equivalent to ϕ_{cl}, or*

(ii) *(M, g) is isometric to the normalized equilateral torus $(\mathbf{T}^2_{eq}, \frac{8\pi^2}{3} g_{eq})$, $n = 5$ and ϕ is equivalent to ϕ_{eq}.*

Recall that an immersion ϕ into $\mathbb{S}^n$ is *full* if its image is not contained in a great sphere of $\mathbb{S}^n$. Two immersions ϕ and ψ into $\mathbb{S}^n$ are called *equivalent* if there exists an isometry R of $\mathbb{S}^n$ such that $\phi = R \circ \psi$. A direct consequence of Theorem 2.1 is:

Corollary 2.1. *A compact genus one orientable surface (M, g) is λ_1-minimal if and only if it is homothetic to $(\mathbf{T}^2_{cl}, g_{cl})$ or $(\mathbf{T}^2_{eq}, g_{eq})$.*

As the metrics g_{cl} and g_{eq} trivially satisfy the hypothesis of Proposition 1.1 we have the following:

Corollary 2.2. *Let M be a compact orientable surface of genus one. A metric g on M is extremal for λ_1 if and only if (M, g) is homothetic to $(\mathbf{T}^2_{cl}, g_{cl})$ or $(\mathbf{T}^2_{eq}, g_{eq})$.*

The proof of Theorem 2.1 is based on the following Propositions 2.1 and 2.2 which are valid in a more general setting.

Proposition 2.1. *Let (M, g) be a n-dimensional compact Riemannian homogeneous manifold non homothetic to $\mathbb{S}^n$. If a metric $g = fg_0$, conformal to g_0, is λ_1-minimal, then f is constant on M.*

Proof. As (M, g) is λ_1-minimal non homothetic to $\mathbb{S}^n$ then any conformal diffeomorphism of (M, g) is an isometry (cf. [**4**]). It follows that any isometry of (M, g_0) is also an isometry of (M, g). Thus the function f is invariant under the isometry group of (M, g_0). The result follows from the homogeneity of (M, g_0). $\square$

Proposition 2.2. *Let $\eta_1, \eta_2, \ldots , \eta_N$ be N continuous functions on a domain Ω of $\mathbb{R}^m$ and assume that the N^2 functions: $2\eta_j$ $(1 \le j \le N)$, $\eta_k + \eta_l$ and $\eta_k - \eta_l$ $(1 \le k < l \le N)$ are non-constant and mutually distinct modulo 2π. If $\phi = (\phi_1, \ldots , \phi_{n+1})$ is a map from Ω to $\mathbb{S}^n$ such that all its components ϕ_i are in the vector space generated by $\{\cos \eta_j, \sin \eta_j,\ 1 \le j \le N\}$,* *then there exists an isometry R of $\mathbb{S}^n$ such that*

$$R \circ \phi = (\alpha_1 \exp i\eta_{j_1}, \alpha_2 \exp i\eta_{j_2}, \dots, \alpha_r \exp i\eta_{j_r}, 0, \dots, 0),$$

where $r \le (n+1)/2$, $j_1, \dots, j_r \in \{1, \dots, N\}$ and $\alpha_1, \dots, \alpha_r$ are positive constants satisfying $\sum_{1 \le j \le r} \alpha_j^2 = 1$. In particular, $R(\phi(\Omega)) \subset \mathbb{S}^1(\alpha_1) \times \mathbb{S}^1(\alpha_2) \times \cdots \times \mathbb{S}^1(\alpha_r) \times \{0\}$.

The proof of this proposition is quite elementary and can be omitted.

Proof of Theorem 2.1. In view of Proposition 2.1, it suffices to consider the case where the metric g is flat. It is well known that there exists $(a,b) \in \mathbb{R}^2$; $0 \le a \le \frac{1}{2}$, $b > 0$ and $a^2 + b^2 \ge 1$, such that (M, g) is homothetic to $\left(\mathbf{T}^2_{a,b} = \mathbb{R}^2/\Gamma(a,b), g_{ab} = g_{\Gamma(a,b)}\right)$ with $\Gamma(a,b) = \mathbf{Z}(1,0) \oplus \mathbf{Z}(a,b)$ (cf. [**3**]). Now the existence of an *isometric* immersion from (M,g) into the unit sphere by the first eigenfunctions implies that $\lambda_1(g) = 2$. Since $\lambda_1(g_{ab}) = \frac{4\pi^2}{b^2}$, (M,g) is in fact isometric to $\left(\mathbf{T}^2_{a,b}, \frac{2\pi^2}{b^2} g_{ab}\right)$. Let $E_{a,b}$ be the first eigenspace of g_{ab} and $\phi : \left(\mathbf{T}^2_{a,b}, \frac{2\pi^2}{b^2} g_{ab}\right) \to \mathbb{S}^n$ a full isometric immersion whose components $\phi_i \in E_{a,b}$.

- If $a^2 + b^2 > 1$ then the dimension of $E_{a,b}$ is 2 and there is no such ϕ.
- If $a^2 + b^2 = 1$ and $(a,b) \ne (1/2, \sqrt{3}/2)$ then $E_{a,b}$ is generated by $\cos \eta_j$, $\sin \eta_j$, $j \le 2$, with $\eta_1(x,y) = \frac{2\pi y}{b}$ and $\eta_2(x,y) = 2\pi\left(x - \frac{ay}{b}\right)$. From Proposition 2.2, it follows that $n = 3$ and, up to an isometry of $\mathbb{S}^3$, ϕ has the form $\phi = (\alpha_1 \exp(i\eta_1), \alpha_2 \exp(i\eta_2))$ with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1^2 + \alpha_2^2 = 1$. As ϕ is isometric we deduce that $a = 0$, $b = 1$ and $\alpha_1 = \alpha_2 = \sqrt{2}/2$. Thus (M,g) is isometric to $(\mathbf{T}^2_{cl}, 2\pi^2 g_{cl})$ and ϕ is equivalent to ϕ_{cl}.
- If $(a,b) = (1/2, \sqrt{3}/2)$ then $E_{a,b}$ is generated by $\cos \eta_j$, $\sin \eta_j$, $j \le 3$, where $\eta_1(x,y) = 4\pi y/\sqrt{3}$, $\eta_2(x,y) = 2\pi\left(x - \frac{y}{\sqrt{3}}\right)$ and $\eta_3(x,y) = 2\pi\left(x + \frac{y}{\sqrt{3}}\right)$. As before $n \le 5$ and, up to isometry, $\phi = (\alpha_1 \exp(i\eta_1), \alpha_2 \exp(i\eta_2), \alpha_3 \exp(i\eta_3))$ where α_1, α_2 and α_3 are nonnegative constants such that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. As ϕ is isometric we obtain $\alpha_1 = \alpha_2 = \alpha_3 = \sqrt{3}/3$. Thus ϕ is equivalent to ϕ_{eq}.

$\square$

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