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COMPRESSIBLE FLUIDS IN A CAPILLARY TUBE

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We study a mathematical model for a compressible liquid in a capillary tube. We establish necessary and sufficient conditions for existence and for uniqueness or near uniqueness of solutions, and we provide general height estimates for solutions, depending on the geometrical structure of the definition domain. We show that solutions exhibit discontinuous dependence properties in domains with corners, analogous to those that are known for the classical capillarity equation.

1. Introduction

The mathematical theory of capillary surfaces was founded by Young [1805], by Laplace [1805–1806] and by Gauss [1830]. The profound investigations of these authors led to the equation

$$\operatorname{div} Tu = \kappa u + \lambda, \quad Tu = \frac{Du}{\sqrt{1+|Du|^2}} \tag{1}$$

for the rise height $u(x, y)$ in a vertical cylindrical capillary tube of general section $\Omega \subset \mathbb{R}^2$. Here $\kappa = \rho g/\sigma$, with ρ the density change across the surface, g the gravitational acceleration, σ the interfacial tension, and λ a constant to be determined by an eventual volume constraint. On the boundary $\Sigma = \partial\Omega$, and with ν the outer unit normal to Σ, the condition

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

is imposed, which asserts that the free surface S meets the bounding cylinder surface in the (prescribed) angle γ. These relations were established by Young and by Laplace using force balance reasoning that was not clearly defined and in some respects incorrect (see [Finn 2006]), then later obtained independently by Gauss using Johann Bernoulli's "principle of virtual work", under the hypothesis that position variations internal to the bulk fluid do not affect the mechanical energy of the system. That was certainly reasonable to suppose at the time, but nevertheless may now be appropriate to question.

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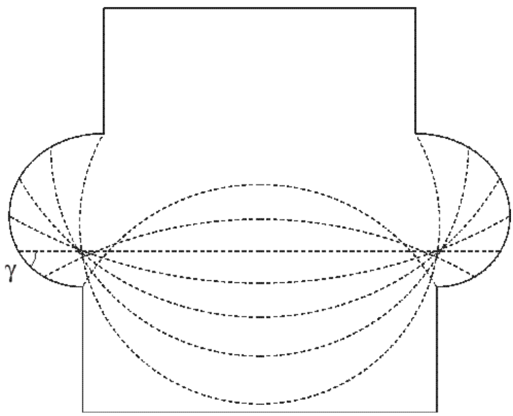

Figure 1. Profiles of static equilibrium surfaces in an "exotic container".

The equations (1)+(2) have served for two centuries, though perhaps not as well as might initially have been hoped, in view of their seemingly intractable nonlinearities. During the initial century, some isolated particular solutions were found by essentially numerical procedures [Bashforth and Adams 1883], and many attempts were made to obtain general information via linearization procedures; these latter attempts led to little information of substantive interest, and in fact to some misconceptions as to the behavior of the solutions. (See also [Finn 1986; 1999] for an overview.)

During the past half century, the problems were attacked anew on the basis of the full nonlinearity of the equations, yielding unexpected predictions of discontinuous behavior; some of these predictions were since verified by experiment; see, for example, [Concus et al. 2000; 1999; Finn 1999, p. 773]. The first existence proofs for (1)+(2) appeared in [Emmer 1973; Ural'tseva 1973; 1975], followed by a number of others under varying conditions.

In this sense, the qualitative validity of (1)+(2) as descriptions of reality was clearly established. Nevertheless, there remain significant questions as to their correctness in quantitative detail. Figure 1 displays profile curves of seven of the continuum of rotationally symmetric equilibrium surfaces in an "exotic container" [Concus et al. 1999]; all of these surfaces bound the same volume of fluid below them, all provide identical mechanical energies in the sense of Gauss, and all of them meet the boundary walls in the same contact angle γ.

Neither the system (1)+(2) nor the variational procedure of Gauss can distinguish among these formal solutions. Nevertheless, there are significant distinctions among the surfaces relative to the physical criteria that underlie those procedures. According to a discovery of Young, there is a pressure jump across each surface

S of magnitude $\delta p = 2\sigma H$, where H is the scalar mean curvature of S. We may assume vacuum ($p = 0$) above each surface in the family. Since H varies widely among the surfaces, so will the fluid pressures, and one must expect corresponding changes in the internal energy of the fluid.

Finn [2001] took an initial step to account for such energy changes, by assuming a slightly compressible fluid, with a phenomenological pressure/density relation $\rho = \rho_0 + \chi(p - p_0)$. By taking account of the thus induced effects of gravity on density, he was led to the equation

$$\operatorname{div} Tu = \frac{\rho_0 g}{\sigma} u - \chi g \cos\omega + \lambda, \tag{3}$$

where ω is the angle between the upward directed surface normal and the vertical, and λ is a Lagrange parameter, depending on an eventual mass constraint. For the problem of a prescribed mass $\mathcal{M}$ in a tube closed at the bottom, Finn found a necessary condition

$$\mathcal{M} < \rho_0 |\Omega| / \chi g \tag{4}$$

on $\mathcal{M}$ for existence of a solution, and he showed that for a circular tube (4) also suffices for existence of a uniquely determined solution.

In the present work we study (3) for domains Ω of general shape in the absence of a mass constraint, and we also consider the equation that arises on taking account of the expansion energy in fluid elements, resulting from density changes. In both cases, although mass is not prescribed, (4) will appear as a general bound for the mass lifted above the rest level $u \equiv 0$; see the discussion in [Finn 2001], which applies to all cases considered here.

The energy released in the expansion of a unit mass of compressible liquid on being raised from the base level 0 to level h is

$$\delta^1 \boldsymbol{E}_e = -\int_{p_0}^{p(h)} p\, d(1/\rho) = \frac{p_0}{\rho_0} - \frac{p(h)}{\rho(h)} + \int_{p_0}^{p(h)} \frac{dp}{\rho}. \tag{5}$$

We consider a thin tube of sectional area $\delta\Omega$ extending from the base level to the surface $u(x)$. At the height h we focus attention on an element of the tube of height δh. If this element is to be in equilibrium, the pressure change from the bottom to the top must be

$$\delta p = -\rho g\, \delta h,$$

and thus

$$\int_{p_0}^{p(h)} \frac{dp}{\rho} = -gh. \tag{6}$$

We assume a relation $\rho = \phi(p; p_0) > 0$. We can then solve (6) for $p = P(h; p_0)$. From this we obtain $\rho = \phi(P(h; p_0); p_0) = \Phi(h; p_0)$. Note that the expansion energy doesn't enter here.

Returning to (5) and using (6), we find that the energy released by the indicated element of mass $\rho\,\delta h\,\delta\Omega$ is

$$\delta \boldsymbol{E}_e = \rho\,\delta h\,\delta\Omega\,\delta^1 \boldsymbol{E}_e = \Big(\frac{p_0}{\rho_0}\Phi - P - \Phi g h\Big)\,\delta h\,\delta\Omega,$$

and thus

$$\boldsymbol{E}_e = \int_\Omega d\Omega \int_0^u \Big(\frac{p_0}{\rho_0}\Phi - P - \Phi g h\Big)\,dh.$$

We add this energy to those previously introduced in [Finn 2001]. From established procedures of the calculus of variations, we obtain the equation

$$\text{(7)} \qquad \operatorname{div} Tu = \frac{\Phi_u}{\Phi}\cos\omega - \frac{P}{\Phi}\frac{\rho_0}{\sigma} + \lambda\frac{\rho_0}{\sigma} + \frac{p_0}{\sigma}$$

in Ω, with the boundary condition (2) unchanged.

In the special case $\rho = \rho_0 + \chi(p - p_0)$, (6) yields $\rho = \rho_0 e^{-\chi g u}$, and (7) becomes

$$\text{(8)} \qquad \operatorname{div} Tu = \frac{\rho_0 - \chi p_0}{\chi\sigma}\big(e^{\chi g u} - 1\big) - \chi g\cos\omega + \lambda\frac{\rho_0}{\sigma}.$$

We address here the classical problem of a cylindrical tube open at both ends, dipped into an unbounded reservoir of liquid. In this case, $\lambda = 0$, and (8) becomes

$$\text{(9)} \qquad \operatorname{div} Tu = \frac{\rho_0 - \chi p_0}{\chi\sigma}\big(e^{\chi g u} - 1\big) - \chi g\cos\omega$$

in Ω. We seek conditions under which there will be a solution of (9) in Ω subject to (2) on Σ. In the interest of obtaining well behaved solutions, we are driven to the further hypothesis

$$\rho_0 - \chi p_0 > 0.$$

In the limit as $\chi \to 0$, we obtain the classical Young–Laplace–Gauss equation (1), as is to be expected. However, the limiting procedure is not uniform in the height u. Note that despite the absence of mass constraint, (9) is not satisfied by the function $u \equiv 0$ when $\gamma = \pi/2$. That is a consequence of the imposed variation of density with height. The fluid rises in the tube as consequence of the decreasing density, until the effect is compensated by the weight of the lifted fluid. The rest level for this trivial solution is the constant height

$$\text{(10)} \qquad u_c = \frac{\chi\sigma}{\rho_0}$$

for (3) with $\lambda = 0$, or

$$u_c = \frac{1}{\chi g} \ln\left(1 + \frac{\chi^2 \sigma g}{\rho_0 - \chi p_0}\right) \tag{11}$$

for (9). This reference value will appear in Theorem 2.8 as a universal upper bound when $\pi/2 \le \gamma \le \pi$, and also implicitly in other contexts.

We will establish varying existence and uniqueness properties, for solutions of (3)+(2) or of (8)+(2), in domains of general shape; additionally we will establish a priori bounds on solutions of (3) or (8), irrespective of boundary conditions. Some of these bounds are idiosyncratic to the particular kinds of nonlinearities considered, and have no counterparts in classical theory of elliptic equations. In configurations for which uniqueness cannot be established by methods at our disposal, we obtain instead comparison theorems, estimating a priori the difference between possible solutions. We will establish growth and comparison properties and discontinuous behavior of solutions in particular domains, depending on inequalities for boundary data. The remainder of the paper is organized as follows:

In Section 2 we present a priori estimates on solution heights, in a somewhat more general context than the particular cases (3) and (8).

In Section 3 we give the gradient estimates up to the boundary for $C^{2,\mu}$ domains Ω, adapting a procedure introduced by Ural'tseva [1973; 1975].

In Section 4 we provide the existence and uniqueness assertions, for $C^{2,\mu}$ domains.

In Section 5 we adapt a procedure used in [Finn and Gerhardt 1977] to prove the existence of "variational solutions" in piecewise smooth domains. Limited knowledge of boundary behavior at corner points is available for such solutions; however, boundedness or growth properties can be established, depending on local geometry, and "near-uniqueness" properties are obtained.

Finally, we note that the height estimates obtained by comparison to hemispheres trivially extend to hold for domains in any dimension, $\Omega \subset \mathbb{R}^n$. We prove the gradient estimates in n dimensions. Our results for domains with corners are formulated for $n = 2$.

2. A priori height estimates

We consider generally solutions $u(x)$ of

$$\operatorname{div} Tu = -\frac{a^2}{\sqrt{1+|Du|^2}} + \mathcal{F}(u), \qquad Tu = \frac{Du}{\sqrt{1+|Du|^2}}, \tag{12}$$

in a bounded, piecewise smooth domain Ω. It is assumed that $\mathcal{F}(u)$ is monotone increasing, with $\mathcal{F}(0) = 0$, and that a is a constant.

For the following definition, let us note that every $f \in H^{1,1}(\Omega)$ has a trace $f^t \in L^1(\partial\Omega)$, which we will denote by f. We call $u(x)$ a *variational solution* of (12) in Ω, corresponding to a boundary contact angle γ, if $u \in C^2(\Omega)$, if $\mathcal{F}(u)$ is integrable over Ω, and if

$$\int_\Omega [D\eta \cdot Tu + \eta\,\mathcal{F}(u)]\,dx = \int_\Omega \eta \frac{a^2}{\sqrt{1+|Du|^2}}\,dx + \int_{\partial\Omega} \eta \cos\gamma\,ds, \tag{13}$$

for every $\eta \in \mathcal{Q}(\Omega) := L^\infty \cap H^{1,1}(\Omega)$. We note in (13) that even though the nominal boundary condition involves derivatives of u, neither the derivatives nor the function itself occurs in the boundary integral. We assume γ to be piecewise continuous on $\partial\Omega$, with $0 \le \gamma \le \pi$.

The following lemma extends slightly Lemma 3 in [Finn and Gerhardt 1977].

Lemma 2.1. *Let $\mathcal{F}(u)$ be nondecreasing. Let Ω be a piecewise smooth domain exhausted by smooth domains $\Omega^j \subset \Omega$. Let u, v be functions in $H^{1,1}_{\mathrm{loc}}(\Omega)$, such that*

$$\limsup_{j\to\infty} \int_{\Omega^j} \big(D\eta\cdot(Tv - Tu) + \eta\,(\mathcal{F}(v) - \mathcal{F}(u))\big)\,dx \ge 0 \tag{14}$$

for every $\eta \in \mathcal{Q}_{\mathrm{loc}}(\Omega) := L^\infty \cap H^{1,1}_{\mathrm{loc}}(\Omega)$ with $\eta \ge 0$. If $\mathcal{F}(u)$ is strictly increasing, there follows $v \ge u$ almost everywhere in Ω. Otherwise either $v \ge u$ in Ω or else $v \equiv u + c$, c constant, throughout Ω. If strict inequality holds in (14), then the inequalities $v \ge u$ can be replaced by $v > u$.

We will apply this lemma in varying contexts to the particular cases

$$\mathcal{F}(u) = \frac{\rho_0 g}{\sigma} u, \qquad a^2 = \chi g, \tag{15}$$

$$\mathcal{F}(u) = \frac{\rho_0 - \chi p_0}{\chi\sigma}(e^{\chi g u} - 1), \quad a^2 = \chi g, \quad 0 < \chi < \frac{\rho_0}{p_0}. \tag{16}$$

The first case corresponds to the situation studied in [Finn 2001], with unconstrained total mass, with $\kappa = \rho_0 g/\sigma$, and with $a^2 = \chi g$; the case of prescribed mass, subject to the (necessary) condition $\chi g M < \rho_0 |\Omega|$, is retrieved by adding a constant to u, see the discussion in [Finn 2001, p. 147]. The second case yields the more exact equation introduced in this paper, again with unconstrained mass. The same necessary condition applies; however a prescribed mass can no longer be achieved by a rigid vertical translation of the surface.

In view of the first term on the right in (12), it is not immediately clear whether the solutions are unique or satisfy a maximum principle. We do obtain that the difference of two solutions satisfies an elliptic equation for which a maximum principle holds, and we can use that information for the following result:

Theorem 2.2. *Suppose that Ω is a C^1 domain, satisfying an internal sphere condition, and that u, v are C^2 solutions of (12) in Ω, both C^1 on $\overline{\Omega}$. If $Tu \cdot \nu \le Tv \cdot \nu$ on $\Sigma = \partial\Omega$, then either $u < v$ in Ω, or else $u \equiv v$ in Ω.*

By an internal sphere condition (ISC) we mean that every boundary point can be contacted from within Ω by a disk contained in $\overline{\Omega}$.

Proof. Let $w = u - v$ denote the difference of the two solutions, and assume w has a maximum M at a point $p \in \overline{\Omega}$. If $p \in \Sigma$, then at p the tangential derivative along Σ vanishes, $w_s = 0$. Thus the exterior normal derivative $w_\nu = \partial w / \partial \nu$ satisfies $w_\nu \ge 0$. In view of the internal sphere condition, we may apply the boundary point lemma, obtaining that either $w \equiv M$ or else $w_\nu > 0$. In the former case we conclude $M = 0$ since $\mathcal{F}(u)$ is strictly increasing; the latter case conflicts with the hypothesis, and we may thus assume that $p \in \Omega$.

We can exclude an interior positive maximum for w by using the maximum principle as noted; however, we present here a geometric argument.

Since w attains a maximum at p, we remark that the values of the angle ω are equal for both surfaces at the point. Were $u(p) > v(p)$, we would have $\operatorname{div} Tu(p) > \operatorname{div} Tv(p)$ by (12). Since these expressions are twice the mean curvature of the respective surfaces, we conclude that at least one of the principal curvatures of the surface $S_u = \operatorname{graph} u$ would exceed that for the surface $S_v = \operatorname{graph} v$, contradicting that w has a maximum at p. □

If less smoothness is known for Ω or for the solution, one nevertheless has:

Theorem 2.3. *Let u^1, u^2 be variational solutions in a piecewise smooth Ω of (12)+(15), corresponding to data $\beta^1 = \cos\gamma^1 \le \beta^2 = \cos\gamma^2$ on $\Sigma = \partial\Omega$. Then*

$$u^1 < u^2 + \frac{\chi\sigma}{\rho_0}.$$

If instead u^1, u^2 are variational solutions of (12)+(16), for which $u^1, u^2 > -A > -\infty$, then

$$u^1 < u^2 + \frac{\chi\sigma}{\rho_0 - \chi p_0} e^{\chi g A}.$$

Proof. To prove the first assertion, we observe that in view of (13) we have, for positive η,

$$\begin{aligned}\int_{\Omega^j} \Big(D\eta \cdot (Tu^2 - Tu^1) + \eta \, \frac{\rho_0 g}{\sigma} (u^2 - u^1) \Big) \, dx \\ > \chi g \int_\Omega \eta \left(\frac{1}{\sqrt{1+|Du^2|^2}} - \frac{1}{\sqrt{1+|Du^1|^2}} \right) dx \\ > -\chi g \int_\Omega \eta \, dx.\end{aligned}$$

Writing $u^1 = w^1 + \chi\sigma/\rho_0$, we find

$$\int_\Omega \Big(D\eta \cdot (Tu^2 - Tw^1) + \eta\, \frac{\rho_0 g}{\sigma}\, (u^2 - w^1) \Big)\, dx \;>\; 0,$$

for all $\eta \in \mathcal{Q}_{\mathrm{loc}}(\Omega)$, $\eta \geq 0$. By Lemma 2.1 we have $u^2 > w^1 = u^1 - \chi\sigma/\rho_0$, which completes the proof of the initial assertion. The second assertion follows similarly, using the estimate

$$e^{\chi g w^1 + c} - e^{\chi g w^1} = \chi g \int_{w^1}^{w^1+c} e^{\chi g t}\, dt > \chi g e^{\chi g w^1} c \quad \text{if } c > 0. \qquad \square$$

We may apply a variant of the method to obtain universal bounds, above and below, on solutions of (12)+(15) interior to a given domain Ω; with regard to (12)+(16) we find a universal bound above, and a universal bound below for solutions over a sufficiently large disk:

Theorem 2.4. *Let u be a variational solution of (12)+(15) interior to a ball B_δ. Then*

$$-\frac{2\sigma}{\rho_0 g \delta} - \delta < u < \frac{\chi\sigma}{\rho_0} + \frac{2\sigma}{\rho_0 g \delta} + \delta \tag{17}$$

throughout B_δ. If u is a variational solution of (12)+(16) in B_δ, then

$$u < \delta + \frac{1}{\chi g} \ln \left(1 + \frac{\chi\sigma}{\rho_0 - \chi p_0} \Big(\chi g + \frac{2}{\delta} \Big) \right) \tag{18}$$

throughout B_δ. In this case, if in addition $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$, then

$$u > \frac{1}{\chi g} \ln \left(1 - \frac{2\chi\sigma}{(\rho_0 - \chi p_0)\delta} \right) - \delta. \tag{19}$$

Proof. We compare the given solution u of (12) with a lower hemisphere $v(x)$ of radius δ and projecting into B_δ. This function has constant mean curvature $1/\delta$ and thus satisfies the auxiliary equation

$$\operatorname{div} Tv = 2/\delta$$

over B_δ. We verify the relation

$$\begin{aligned} &\int_{B_\delta} \big(D\eta \cdot (Tv - Tu) + \eta\, (\mathcal{F}(v) - \mathcal{F}(u)) \big)\, dx \\ &\qquad = \int_{\partial B_\delta} \eta (1 - \cos\gamma_\delta)\, ds + \int_{B_\delta} \eta \left(\mathcal{F}(v) - \frac{\chi g}{\sqrt{1 + |Du|^2}} - \frac{2}{\delta} \right) dx. \end{aligned} \tag{20}$$

Here $\cos\gamma_\delta = \nu \cdot Tu$ evaluated on ∂B_δ, and we have used that $\nu \cdot Tv = 1$ on ∂B_δ, since the hemisphere is vertical on that arc.

We position the hemisphere so that $\mathcal{F}(v) = \chi g + 2/\delta$ at its lowest point. By the monotonicity of $\mathcal{F}$, the right side of (20) will then be positive for any positive $\eta \in \mathcal{Q}_{\mathrm{loc}}(B_\delta)$. Thus, the left side will also be positive, and we conclude from Lemma 2.1 that $u < v$ in B_δ. Since the total height change of v from the center to the edge of B_δ is δ, this inequality establishes (18) and the right-hand side of (17).

The left side of (17) and also (19) follow similarly, using an upper hemisphere as comparison surface. The restriction $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ must be imposed, as the inverse function for $\mathcal{F}(u)$ in (16) is not defined for $\mathcal{F} < -(\rho_0 - \chi p_0)/\sigma\chi$. □

This result can be sharpened significantly in the particular case where B_δ is the definition domain Ω, with a constant contact angle γ_δ achieved in the variational sense on ∂B_δ. Then we may choose v to be a spherical cap meeting the cylinder wall $r = \delta$ in the angle γ_δ. The boundary integral in (20) then vanishes, and we find

$$\tag{21} \begin{aligned} \int_{B_\delta} \big(D\eta \cdot (Tv - Tu) + \eta\,(\mathcal{F}(v) - \mathcal{F}(u))\big)\,dx = \\ \int_{B_\delta} \eta\left(\mathcal{F}(v) - \frac{\chi g}{\sqrt{1+|Du|^2}} - \frac{2}{\delta}\cos\gamma_\delta\right)dx. \end{aligned}$$

We distinguish four cases, according to whether $\gamma_\delta < \pi/2$ or $\gamma_\delta > \pi/2$, and whether we seek upper or lower bounds. If $\gamma_\delta < \pi/2$ and we seek an upper bound, we position the cap so that $\mathcal{F}(v) = \chi g + 2(\cos\gamma_\delta)/\delta$ at the point of symmetry. Then both sides of (21) will be positive for all positive η, and we conclude $u < v$. If we seek a lower bound, we position the cap so that $\mathcal{F}(v) = 2(\cos\gamma_\delta)/\delta$ at the point $r = \delta$. Then both sides of (21) will be negative for positive η, from which follows $u > v$. Analogous reasoning applies when $\gamma_\delta > \pi/2$. We are led to:

Corollary 2.5. *Suppose $\Omega = B_\delta$ and $0 \le \gamma \le \pi$.*

(a) *If $u(x)$ is a variational solution of (12)+(15) in Ω, there holds*

$$\tag{22} \frac{2\sigma}{\rho_0 g}\frac{\cos\gamma}{\delta} - \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta < u < \frac{2\sigma}{\rho_0 g}\frac{\cos\gamma}{\delta} + \frac{\chi\sigma}{\rho_0} + \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta.$$

(b) *If $u(x)$ is a variational solution of (12)+(16) in Ω then*

$$\tag{23} \begin{aligned} \frac{1}{\chi g}\ln\left(1 + \frac{2\chi\sigma}{(\rho_0 - \chi p_0)}\frac{\cos\gamma}{\delta}\right) - \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta \\ < u < \frac{1}{\chi g}\ln\left(1 + \frac{\chi\sigma}{(\rho_0 - \chi p_0)}\left(\frac{2\cos\gamma}{\delta} + \chi g\right)\right) + \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta. \end{aligned}$$

In these last relations, the logarithmic terms must be replaced by $-\infty$ if the arguments are nonpositive. This is not an accident of the method; we return to this point below, where we will show that if the argument on the right side is nonpositive, then no solution of (12)+(16) can exist in the disk. (See also the remark on nonexistence on page 221.)

These bounds can be improved in some respects by relaxing the boundary condition for v; compare the proof of Theorem 2.8.

The hypotheses of Theorem 2.4 clearly apply to configurations in which $u(x)$ is defined as a solution in a domain Ω containing B_δ; more generally if B_δ does not lie entirely interior to Ω but if u assumes (in a variational sense) data γ on $B_\delta \cap \Sigma$, it suffices to focus attention on a component of $B_\delta \cap \Omega$ for which the hemispheres introduced in the proof meet the vertical walls of Σ in angles majorizing γ. Specifically, we obtain:

Theorem 2.6. *Let u be a variational solution of (12)+(15) or of (12)+(16) interior to a component Z_δ of $B_\delta \cap \Omega$. If on $Z_\delta \cap \Sigma$ the lower hemisphere under B_δ meets the vertical walls under Σ in angles $\gamma^\delta \le \gamma$, then the right side of (22) holds in Z_δ for the system (12)+(15) and the right side of (23) holds for (12)+(16). If $\gamma^\delta \ge \gamma$, then the remaining inequalities apply in the respective cases.*

Further, using the definition for an internal sphere condition $\mathrm{ISC}_{\delta,\gamma^\delta}$ as given in [Finn and Gerhardt 1977, pp. 15–16], we may state:

Corollary 2.7. *If Ω can be covered by disks of radius δ for some fixed $\delta > 0$, then (17) and (18) hold throughout Ω. If that can be done with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$, then (19) also holds throughout Ω. More generally, if Ω satisfies an internal sphere condition $\mathrm{ISC}_{\delta,\gamma^\delta}$, with $\gamma^\delta \le \gamma$, then the right sides of (22) and (23) hold in the respective cases. If a condition $\mathrm{ISC}_{\delta,\pi-\gamma^\delta}$ holds, with $\pi - \gamma^\delta \ge \pi - \gamma$, then the remaining statements of Corollary 2.5 apply.*

In general, if some a priori information is known on boundary behavior of the solution u, then the bounds in (17) and (18) can to some extent be sharpened. Assume first that $\gamma < \gamma_0 < \pi/2$. We take as comparison surface v a lower hemisphere whose center projects to a point of Ω, and of radius R large enough that the projection covers Ω and such that the contact angle $\gamma^v \ge \gamma_0$. We obtain now the relation

$$\begin{aligned}\int_\Omega \big(D\eta\cdot(Tu - Tv) + \eta\,(\mathcal{F}(u) - \mathcal{F}(v))\big)\,dx \\ = \int_{\partial\Omega} \eta(\cos\gamma - \cos\gamma^v)\,ds + \int_\Omega \eta\left(\frac{2}{R} + \frac{\chi g}{\sqrt{1+|Du|^2}} - \mathcal{F}(v)\right)dx,\end{aligned}$$

and it thus suffices to choose v such that $\mathcal{F}(v) < 2/R$. R will in general not be known explicitly, however a universal choice, suitable both for (15) and for (16), is provided by the function $v = 0$; that yields $\mathcal{F}(v) = 0$ in both cases, from which $u > 0$ follows by Lemma 2.1. In the other direction, we introduce for v an upper hemisphere, and are led to the relation

$$\int_\Omega \big(D\eta\cdot(Tv-Tu)+\eta\,(\mathcal{F}(v)-\mathcal{F}(u))\big)\,dx \\ =\int_{\partial\Omega}\eta(\cos\gamma^v-\cos\gamma)\,ds+\int_\Omega \eta\left(\frac{2}{R}-\frac{\chi g}{\sqrt{1+|Du|^2}}+\mathcal{F}(v)\right)dx,$$

and we see that it suffices in general to have $\mathcal{F}(v)>\chi g$. Again we may let $R\to\infty$, leading to the choice

$$v\equiv\frac{\chi\sigma}{\rho_0}\quad\text{for (15)}\qquad\text{and}\qquad v\equiv\text{const}=\frac{1}{\chi g}\ln\left(1+\frac{\chi^2\sigma g}{\rho_0-\chi p_0}\right)\quad\text{for (16)}.$$

We have proved:

Theorem 2.8. *Let u be a variational solution of either (12)+(15) or (12)+(16) in a piecewise smooth domain Ω. If $0\le\gamma<\pi/2$, there holds $u>0$ in Ω in both cases (15) and (16). If $\pi/2<\gamma\le\pi$, there holds in Ω*

$$u<\frac{\chi\sigma}{\rho_0}\quad\textit{in case (15)}\qquad\textit{and}\qquad u<\frac{1}{\chi g}\ln\left(1+\frac{\chi^2\sigma g}{\rho_0-\chi p_0}\right)\quad\textit{in case (16)}.$$

The material above provides global estimates for solutions over a prescribed domain Ω. We turn our attention now to behavior near corner points of Ω. For simplicity, we assume that the boundary consists locally at the corner P of two line segments, intersecting in an angle $2\alpha<\pi$, measured interior to Ω. We assume that $|\gamma-\pi/2|>\alpha$ – and thus that $|\cos\gamma|>\sin\alpha$ – in a neighborhood of P on $\partial\Omega$. (If $|\gamma-\pi/2|\le\alpha$ in such a neighborhood, the bounds indicated in Theorem 2.6 apply.) We assume first that $0\le\gamma<\pi/2$, and observe that then any point $p\in\Omega$ of (sufficiently small) distance r from P lies in a disk of radius $r\sin\alpha/\cos\gamma$ that meets the boundary segments Σ at an angle γ, as in the figure:

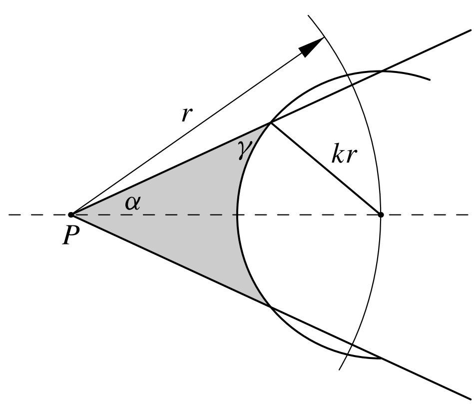

Figure 2. Construction for bounding solution below, in a wedge domain.

The lower hemisphere $v(x)$ with Σ as equatorial circle meets the vertical walls through Σ in that same angle γ. By Theorem 2.6, we find in the case of (12)+(15), setting $k = \sin\alpha/\cos\gamma$,

$$u(x) < \frac{2\sigma}{\rho_0 g}\frac{1}{kr} + \frac{\chi\sigma}{\rho_0} + kr \tag{24}$$

and in the case (12)+(16)

$$u(x) < \frac{1}{\chi g}\ln\left(1 + \frac{2\chi\sigma}{\rho_0 - \chi p_0}\frac{1}{kr} + \frac{\chi^2\sigma g}{\rho_0 - \chi p_0}\right) + kr. \tag{25}$$

To obtain appropriate lower bounds, we adapt a procedure introduced by Korevaar [1980], and use the upper inner side of a torus as a comparison surface. Corresponding to points at distance not exceeding r from the vertex, we consider the torus $v(\boldsymbol{x})$, $\boldsymbol{x} = (x, y, z)$ defined in terms of parameters ϕ, ψ relative to the vertex as coordinate origin by

$$x = (A - a\cos\psi)\cos\phi, \quad y = a\sin\psi, \quad z = (A - a\cos\psi)\sin\phi,$$

with $a = r\sin\alpha/(\cos\gamma - \sin\alpha)$. Here $A > a$, and the parameters satisfy $-\psi_0 < \psi < \psi_0$, $0 < \phi < \phi_0$, with ϕ_0, $\psi_0 < \pi/2$ fixed but arbitrary. The general appearance is that of a Japanese footbridge, drawn here in perspective:

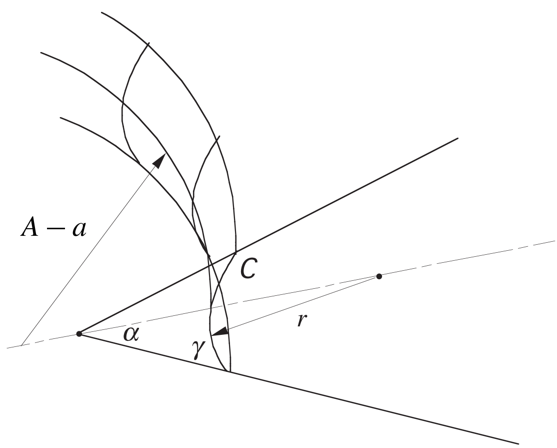

Figure 3. Construction for bounding solution above at a corner point.

The crucial observation is that $\nu \cdot Tv = -1$ on the curve $\mathcal{C} = \{\phi = 0\}$, ν being the exterior unit normal, and thus the boundary condition on that curve minorizes that of any solution u in a common domain of definition.

For small a, the torus cuts off a small piece of the corner, as indicated in the figure, with the curve $\mathcal{C}$ meeting the bounding segments at an angle γ. We observe that v satisfies

$$\operatorname{div} Tv = 2H(x) > \frac{1}{a} - \frac{1}{A-a} = \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right).$$

If r is small enough, this expression will be positive. Since the unit normal to the torus is continuous and is directed horizontally toward the vertex at the symmetry point of $\mathcal{C}$, there will hold for small enough r that $\nu \cdot Tv < \sin\alpha + \varepsilon$ on both the segments cut off at the corner, with $\sin\alpha + \varepsilon < \cos\gamma$ on these segments.

Following the procedure of Theorem 2.6, we find for the case (12)+(15) that $u > v$ in the domain cut off at the vertex, provided that v can be chosen so that

$$\frac{\rho_0 g}{\sigma} v < \chi g + \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right).$$

We may translate v vertically so that this inequality holds at a particular point of the domain; we then find on the basis of the construction that

$$\frac{\rho_0 g}{\sigma} v > \chi g + \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right) - \omega(\varepsilon)$$

with $\lim_{\varepsilon\to 0} \omega(\varepsilon) = 0$.

We now wish to let $r \to 0$. A convenient way to do that is by a similarity transformation, which leaves all boundary angles and the geometric configuration unchanged. We obtain the result that for sufficiently small r, there holds at all points (x, y) of distance r from the vertex the inequality

$$u(x, y) > \frac{\sigma}{\rho_0 g}\frac{1-k}{kr} + C, \tag{26}$$

for a fixed constant C independent of r. Together with (24), this result implies that every solution of (12)+(15) in a wedge domain with $\alpha + \gamma < \pi/2$ is unbounded at the corner, with a growth rate $O(1/r)$.

In the case (12)+(16) an analogous reasoning yields, observing that the choice of $A > a$ is arbitrary,

$$u(x, y) > \frac{1}{\chi g}\ln\left(1 + \frac{\chi\sigma}{\rho_0 - \chi p_0}\frac{1-k}{k}\frac{1-\varepsilon}{r} - C(\varepsilon)\right) \tag{27}$$

asymptotically as $r \to 0$, for any $\varepsilon > 0$ and fixed $C(\varepsilon)$ independent of r.

We turn our attention now to the case $\pi/2 < \gamma \le \pi$. A procedure analogous to that yielding (24) (and resuming the notation $x \in \Omega$) leads now, for solutions of (12)+(15), to

$$u(x) > -\frac{2\sigma}{\rho_0 g}\frac{1}{kr} - kr \tag{28}$$

and a procedure analogous to that yielding (26) now yields

$$u(x) < -\frac{\sigma}{\rho_0 g}\frac{1-k}{kr} + C \tag{29}$$

in the case (12)+(15).

With regard to solutions of (12)+(16) the situation is now simpler. We investigate (12) over a wedge triangle:

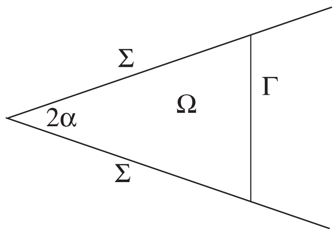

Figure 4. Wedge domain.

In view of the boundary condition, we obtain

$$2|\Sigma|\cos\gamma + \int_\Gamma \nu\cdot Tu\,ds = \int_\Omega\left(-\frac{\chi g}{\sqrt{1+|Du|^2}} + \frac{\rho_0-\chi p_0}{\chi\sigma}(e^{2\chi u}-1)\right)dx,$$

from which, since $|\nu\cdot Tu| < 1$, we conclude that

$$2|\Sigma|\cos\gamma + |\Gamma| > -\left(\chi g + \frac{\rho_0-\chi p_0}{\chi\sigma}\right)|\Omega|.$$

Thus, since $\gamma > \pi/2$ and $|\gamma-\pi/2| > \alpha$ so that $|\cos\gamma| > \sin\alpha$, we find

$$0 < 2(|\cos\gamma| - \sin\alpha) < \left(\chi g + \frac{\rho_0-\chi p_0}{\chi\sigma}\right)|\Sigma|\cos\alpha\sin\alpha, \tag{30}$$

and we obtain a contradiction by letting Γ move in parallel translation toward the vertex.

Gathering the material above, we have proved:

Theorem 2.9. *Suppose that γ is constant in a neighborhood of a corner point of opening 2α. If $|\gamma-\pi/2| \le \alpha$ then the estimates of Theorem 2.6 apply. If $\alpha+\gamma < \pi/2$ then the estimates (24) and (26) hold for the case (12)+(15), and the estimates (25) and (27) hold for the case (12)+(16). If $\gamma > \alpha+\pi/2$ then (28) and (29) apply for the case (12)+(15); however for the case of (12)+(16) no solution can exist in such a wedge.*

If the boundary of Ω is not rectilinear at the corner point, we still obtain the same results as above, but under the stronger condition $|\gamma-\pi/2| < \alpha$.

Finally, we remark an immediate consequence of Theorem 2.4:

Theorem 2.10. *Any solution of (12)+(15) is bounded at any isolated singular point. Any solution of (12)+(16) is bounded above at an isolated singular point.*

In [Finn 1963] it is proved that the "classical" capillary equation

$$\operatorname{div} Tu = \frac{\rho g}{\sigma} u$$

admits only removable isolated singularities. We do not know to what extent that theorem extends to the more general configurations considered in this paper.

3. Gradient estimates

We study the case of equation (8). We derive the gradient estimate following techniques introduced by Ural'tseva [1973; 1975] and used in [Gerhardt 1976; Huisken 1985].

We follow closely the procedure in [Huisken 1985]. For the convenience of the reader we state here the results of that paper which we use.

We consider the equation

$$\begin{aligned} &\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}} && \text{in } \Omega, \\ &Tu \cdot \nu = \beta && \text{on } \Sigma, \end{aligned} \tag{31}$$

with $a^2 = \chi\sigma/\rho_0$ and the function $\mathcal{F}(u)$ defined either as in (15) or as in (16). The main assumption on $\mathcal{F}$ needed for the gradient estimate is that $\mathcal{F}' > 0$. For the present considerations we assume $\beta \in C^{0,1}(\Sigma)$ to satisfy

$$|\beta| \le 1 - \tilde{\alpha}, \qquad \tilde{\alpha} > 0. \tag{32}$$

As above, we denote by T the operator defined by

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}.$$

We also introduce the notations

$$a^i(p) = \frac{p^i}{\sqrt{1+|p|^2}}, \qquad a^{ij} = \frac{\partial a^i}{\partial p_j},$$

for $p \in \mathbb{R}^n$, and denote by $H(x, u, Du)$ the right-hand side of (31):

$$H(x, u, Du) = \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}}. \tag{33}$$

Given $\Sigma \in C^{2,\mu}$, we can extend β and ν to the interior of Ω, in such a way that $\beta \in C^{0,1}(\Omega)$ still satisfies (32) and ν is uniformly Lipschitz continuous in Ω, with $|\nu| < 1$.

We denote by $S = \operatorname{graph} u$ the liquid-air interface and by $\nabla^S f$ the tangential gradient on S of a function $f \in C^1(\Omega)$:

$$\nabla^S f = Df - (Df \cdot \nu_S)\, \nu_S,$$

with ν_S the unit normal to the interface S.

The main idea is to work with the function

$$v = \sqrt{1 + |Du|^2} + \beta\, (Du \cdot \nu) \equiv W + \beta\, (Du \cdot \nu)$$

as in [Ural'tseva 1973; Gerhardt 1976], and to prove that v is uniformly bounded in Ω. This in turn gives the gradient estimate, since

$$|Du| \le \sqrt{1 + |Du|^2} \;=\; W \;\le \frac{1}{\tilde{\alpha}} v.$$

We will bound the function

$$w = \log v$$

instead of v; we can follow all the steps as in [Huisken 1985, (2.12)–(2.30)], with the first real difference being the derivative $D_k H$ needed in (2.25) of that paper, which is computed in (2.31). In our case, we find

$$\begin{aligned}(34)\quad &\int_\Omega a^{ij} \left(D_j v - D_j(\beta \nu^k)\, D_k u\right)\, D_i \eta + \frac{1}{2n}\, |H|^2\, \eta\, dx \\ &\le - \int_\Omega D_k H\, (a^k + \beta \nu^k)\, \eta\, dx + c_\varepsilon \int_\Omega \left(1 + \frac{\nabla^S v}{W}\right) \eta\, dx + c_3 \int_\Sigma \eta\, d\mathcal{H}^{n-1}.\end{aligned}$$

Inequality (34) is almost identical with [Huisken 1985, (2.32)], except that we want to explicitly calculate the first term on the right-hand side, since our problem only differs in the form of the prescribed mean curvature function H. In view of (33) we have

$$\begin{aligned}(35)\quad & - \int_\Omega D_k H (a^k + \beta \nu^k)\, \eta\, dx \\ &= - \int_\Omega \left(\mathcal{F}'(u) D_k u + \frac{\mu}{W^3} D_l u D_k D_l u\right)\, (a^k + \beta \nu^k)\, \eta\, dx.\end{aligned}$$

The first term on the right-hand side of (35) is negative, since $\mathcal{F}'(u) > 0$ and

$$D_k u\, (a^k + \beta \nu^k) = v - W^{-1} > 0.$$

Therefore it can be ignored. For the second term on the right in (35), we can use the equality

$$D_l u \left(D_k D_l u\, (a^k + \beta \nu^k)\right) = D_l u \left(D_l v - \, D_l(\beta \nu^k)\, D_k u\right),$$

which follows from [Huisken 1985, (2.26)]. In view of this equality, (35) becomes

$$-\int_\Omega D_k H\,(a^k+\beta\nu^k)\,\eta\,dx \le -a^2\int_\Omega \frac{1}{W^3} D_l u\, D_l v\,\eta\,dx + a^2\int_\Omega \frac{1}{W^3} D_l(\beta\nu^k)\, D_l u\, D_k u\,\eta\,dx.$$

Denote the integrals on the right-hand side by I_1 and I_2. They can be estimated by

$$I_2 \le a^2 c_4 \int_\Omega \frac{|Du|^2}{W^3}\,\eta\,dx \le \mu\, c_4 \int_\Omega \eta\,dx, \tag{36}$$

$$\begin{aligned} I_1 &\le \frac{1}{2\tilde\varepsilon}\, a^4 \int_\Omega \frac{|Du|^2}{W^3}\,\eta\,dx + \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|Dv|^2}{W^3}\,\eta\,dx \\ &\le \frac{1}{2\tilde\varepsilon}\, a^4 \int_\Omega \eta\,dx + \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx. \end{aligned} \tag{37}$$

Here c_4 depends on the Lipschitz constant of $\beta\nu$ and we have used the inequalities $|Du|^2/W^2 \le 1$ and $|Dv|^2/W^2 \le |\nabla^S v|^2$, the latter being proved as follows:

$$\begin{aligned} |Dv|^2 &\le |\nabla^{n+1}|^2 = |\nabla^S|^2 + |(\nabla^{n+1}\cdot\nu_S)\,\nu_S|^2 \\ &= |\nabla^S v|^2 + \left|\frac{(Dv,0)\cdot(-Du,1)}{W}\right|^2 |\nu_S|^2 = |\nabla^S v|^2 + \frac{|Dv|^2\,|Du|^2}{W^2}. \end{aligned}$$

Both the first term on the right-hand side of (37) and the estimate (36) for I_2 are of the same form and can be incorporated into the second term on the right-hand side of (34) with a new constant $c_5 = c_\varepsilon + \mu c_4 + a^4/\tilde\varepsilon$ replacing c_ε. Using the above considerations, we conclude that (34) gives

$$\begin{aligned} \int_\Omega a^{ij}\big(D_j v - D_j(\beta\nu^k)\,D_k u\big)\, D_i\eta + \frac{1}{2n}|H|^2\,\eta\,dx \\ \le \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx + c_5\int_\Omega \left(1+\frac{\nabla^S v}{W}\right)\eta\,dx + c_3\int_\Sigma \eta\,d\mathcal{H}^{n-1}. \end{aligned} \tag{38}$$

As a test function η we choose

$$\eta = v\max(w-k,0) \equiv v z$$

for positive k, and define

$$A(k) = \big\{p=(x,u(x))\in S : w(x) > k\big\}, \quad |A(k)| = \mathcal{H}^n(A(k)).$$

For the first term on the right-hand side of (38) we note that $\eta\, W^{-1} \le 2z$, since $v \le 2W$ and we have

$$\frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx \le \tilde\varepsilon\int_\Omega |\nabla^S v|^2\, z\,dx. \tag{39}$$

This term will then be taken to the left-hand side of the inequality (38).

We next show that (38) is equivalent to

$$\int_{A(k)} |\nabla^S v|^2\, d\mathcal{H}^n + \frac{1}{n}\int_{A(k)} |H|^2 z\, d\mathcal{H}^n \le c\,|A(k)| + c\int_{A(k)} z\, d\mathcal{H}^n, \tag{40}$$

where $c = c\big(\tilde\alpha, n, |D\nu|_\Omega, |D\beta|_\Omega\big)$. For this, we estimate each term separately, starting with the first term on the left; we use [Huisken 1985, (2.27)–(2.30)] and the equalities $w = \log v$, $D_i\eta = (z+1)\, D_i v$. Setting $\Omega_\eta = \Omega \cap \operatorname{supp}\eta$, we get

$$\begin{aligned}
&\int_{\Omega_\eta} a^{ij}\big(D_j v - D_j(\beta\nu^k)\, D_k u\big)\, D_i\eta\, dx \\
&\quad\ge \int_{\Omega_\eta} \Big(a^{ij}\, D_j v\, D_i v - \big|a^{ij}\, (D_j(\beta\nu^k)\, D_k u)\, D_i v\big|\Big)\, (z+1)\, dx \\
&\quad\ge \int_{\Omega_\eta} W^{-1}|\nabla^S v|^2\, (z+1)\, dx \\
&\qquad\qquad - \int_{\Omega_\eta} \left|\frac{1}{2\varepsilon} a^{ij}(D_j(\beta\nu^k)\, D_k u)(D_i(\beta\nu^k)\, D_k u) + \frac{\varepsilon}{2} a^{ij}\, D_i v\, D_j v\right| (z{+}1)\, dx \\
&\quad\ge \Big(1 - \frac{\varepsilon}{2}\Big)\int_{\Omega_\eta} W^{-1}|\nabla^S v|^2 (z+1)\, dx - \frac{1}{2\varepsilon}\int_{\Omega_\eta} W^{-1}|\nabla^S(\beta\nu)|^2 |Du|^2 (z+1)\, dx \\
&\quad\ge \Big(1 - \frac{\varepsilon}{2}\Big)\int_{\Omega_\eta} W^{-1}\, v^2|\nabla^S z|^2\, dx - \frac{1}{2\varepsilon}|D(\beta\nu)|^2_\Omega \int_{\Omega_\eta} (z+1) W\, dx \\
&\quad\ge \tilde\alpha^2\Big(1 - \frac{\varepsilon}{2}\Big)\int_{A(k)} |\nabla^S z|^2\, d\mathcal{H}^n - \frac{1}{2\varepsilon}|D(\beta\nu)|^2_\Omega \int_{A(k)} (z+1)\, d\mathcal{H}^n.
\end{aligned}$$

For the second term on the left-hand side of (38), we find using [Huisken 1985, (2.29)] that

$$\frac{1}{2n}\int_{\Omega_\eta} |H|^2\, \eta\, dx \ge \frac{\tilde\alpha}{2n}\int_{A(k)} |H|^2 z\, d\mathcal{H}^n.$$

For the second term on the right-hand side of (38), and again by [Huisken 1985, (2.29)], we estimate

$$\begin{aligned}
\int_{\Omega_\eta} \Big(1 + \frac{|\nabla^S v|}{W}\Big)\, \eta\, dx &\le \int_{A(k)} vz\, W^{-1}\, d\mathcal{H}^n + \int_{\Omega_\eta} \frac{|\nabla^S v|}{W}\, vz\, dx \\
&\le 2\int_{A(k)} z\, d\mathcal{H}^n + \frac{\tilde{\tilde\varepsilon}}{2}\int_{\Omega_\eta} \frac{|\nabla^S v|^2}{W}\, z\, dx + \frac{1}{2\tilde{\tilde\varepsilon}}\int_{\Omega_\eta} \frac{v^2}{W}\, z\, dx \\
&\le 2\int_{A(k)} z\, d\mathcal{H}^n + \frac{\tilde{\tilde\varepsilon}}{2}\int_{\Omega_\eta} \frac{|\nabla^S v|^2}{W}\, z\, dx + \frac{1}{2\tilde{\tilde\varepsilon}}\int_{A(k)} z\, d\mathcal{H}^n.
\end{aligned}$$

For the third term on the right-hand side of (38), and in view of [Huisken 1985, (2.20)], we have

$$\begin{aligned}\int_\Sigma \eta\, d\mathcal{H}^{n-1} &= \int_\Sigma vz\, d\mathcal{H}^{n-1}\\ &\le \int_{A(k)} |\nabla^S z|\, d\mathcal{H}^n + \int_{A(k)} (|H| + |\nabla^S \nu|)\, z\, d\mathcal{H}^n\\ &\le \frac{\tilde\varepsilon}{2}\int_{A(k)} |\nabla^S z|^2\, d\mathcal{H}^n + \frac{1}{2\tilde\varepsilon}\int_{A(k)} d\mathcal{H}^n + c_6\int_{A(k)} z\, d\mathcal{H}^n,\end{aligned}$$

with $c_6 = c_6(|H|_\Omega, |D\nu|_\Omega)$.

Taking into consideration all the estimates following (40), we can easiliy obtain (40) from (38).

Inequality (40) is exactly of the same form as [Huisken 1985, (2.34)], and the subsequent procedure in that paper is independent of the choice of the function H prescribing the mean curvature of the surface S. Therefore, we can conclude in the same manner that

$$w = \log v \le k_0 + c|A(k_0)|,$$

where $k_0 = k_0(\tilde\alpha, n)$ and $c = c\big(n, \tilde\alpha, \Omega, |D\beta|_\Omega, |D\nu|_\Omega\big)$.

This concludes the gradient estimate in a neighborhood of the boundary $\Sigma = \partial\Omega$, which we state in Theorem 3.1 below.

Definition. We call a domain *admissible* if it is open, bounded, simply connected, and of class $C^{2,\mu}$.

This definition is such that we are able to obtain uniform height bounds as in Section 2. The following theorem would still be true if we just assumed these uniform bounds instead. (For the notation $\mathrm{ISC}_{\delta,\pi-\gamma}$ see [Finn and Gerhardt 1977, pp. 15–16].)

Theorem 3.1. *Let Ω be an admissible domain. Assume u to be a $C^2(\Omega)$ solution of (31), with the function $\mathcal{F}(u)$ defined either as in (15), or as in (16), in which case we also require an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ when $\gamma > \pi/2$ (for the uniform height estimates to hold). We denote by β, ν the Lipschitz extensions into the interior of Ω of β and ν_Σ, and assume β to satisfy (32); that is, $|\beta| \le 1 - \tilde\alpha$ with $\tilde\alpha > 0$, and $|\nu| \le 1$. Then there exists a constant $C = C\big(n, \tilde\alpha, \Omega, |D\beta|_\Omega, |D\nu|_\Omega\big)$ such that*

$$|Du| \le C$$

in a neighborhood of the boundary Σ.

The interior gradient estimate in admissible domains can be obtained by means of the maximum principle:

Theorem 3.2. *Assume Ω and u satisfy the assumptions of Theorem 3.1. Then*

$$|Du| \leq C$$

in Ω, with C the constant of Theorem 3.1.

Proof. We rearrange the equation (31), satisfied by u in Ω, to find

$$a_{ij} D_i D_j u - \mathcal{F}(u)\sqrt{1+|Du|^2} + a^2 = 0, \tag{41}$$

where

$$a_{ij} = \delta_{ij} - \frac{D_i u \, D_j u}{1+|Du|^2}.$$

By general elliptic theory we can assume the local existence of derivatives of all orders for u. We differentiate (41) with respect to x_k, for any $k \in \{1, \ldots, n\}$, and set $D_k u = v$ to obtain

$$a_{ij} D_i D_j v + b_i D_i v + cv = 0$$

in Ω, with $c = -\mathcal{F}'(u)\sqrt{1+|Du|^2} \leq 0$. The equation satisfied by v is elliptic, and we can apply the maximum or minimum principle to deduce the claimed interior gradient bound. □

With Theorems 3.1 and 3.2, we have the main result of this section:

Theorem 3.3. *Under the assumptions of Theorem 3.1, there is a constant $M > 0$, such that for any solution u of (31) we have*

$$|Du| \leq M.$$

4. Existence in smooth domains, uniqueness of solutions and nonexistence results

The following result is contained in Theorem 2.2.

Theorem 4.1 (Uniqueness). *Suppose $\mathcal{F}' > 0$, and let $u(x)$, $v(x)$ be solutions of (12) in a domain Ω with boundary $\Sigma = \partial\Omega$ of class C^1, which satisfies an internal sphere condition. We suppose $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We suppose further that on Σ there holds $\nu \cdot Tu = \nu \cdot Tv$. Then $u(x) \equiv v(x)$ in Ω.*

For the case of domains with corner points, we refer to Theorem 2.9 above.

The gradient estimates enable us now to prove existence in domains with $C^{2,\mu}$ boundary using a continuity method.

Theorem 4.2. *Assume Ω to be an admissible domain, and consider the problem (31), with H defined as in (33), $\mathcal{F}$ given by either (15) or (16), $0 < \gamma < \pi$ and β taken to be $C^{1,\mu}$ in its arguments. If $\mathcal{F}$ is as in (16), and if $\gamma > \pi/2$, we assume in addition an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$.*

Then the problem (31) has a unique solution $u \in C^{2,\mu}(\bar{\Omega})$, where the exponent μ, $0 < \mu < 1$ depends on the above quantities.

Remark. If $0 < \gamma < \pi/2$ we have uniform height estimates from above and below for both cases (15) and (16), as shown by Corollary 2.7 and Theorem 2.8. For $\pi/2 < \gamma < \pi$ and for $\mathcal{F}$ is as in (15), the height estimates also hold, but for case (16) an additional internal sphere condition is needed in the statement of the theorem in order for the uniform lower height estimate to hold. The condition on δ is optimal as discussed in the remark on nonexistence following the proof.

The cases $\gamma = 0$ and $\gamma = \pi$ are not considered due to assumption (32), which is essential for the gradient estimate.

Proof of Theorem 4.2. The proof follows exactly the steps in [Gerhardt 1976, proof of Theorem 2.1]; we only outline it here.

For $\tau \in \mathbb{R}$, $0 < \tau < 1$, consider the problem

$$\begin{aligned} -\operatorname{div} \frac{Du_\tau}{\sqrt{1 + |Du_\tau|^2}} + \tau H(x, u_\tau, Du_\tau) &= 0 \qquad \text{in } \Omega, \\ Tu_\tau \cdot \nu &= \tau\beta \qquad \text{on } \Sigma. \end{aligned} \tag{42}$$

One then proves that the set

$$T = \{\tau : \text{there exists a solution } u_\tau \in C^2(\bar{\Omega})\}$$

is open and closed.

The idea is to look at a uniformly elliptic operator that coincides with the given one in (42) whenever $|Du_\tau|_\Omega \le K$ for some constant K. This allows us to apply [Ladyzhenskaya and Ural'tseva 1968, Chapter 10, Theorem 2.2]; the change from the equation considered in [Gerhardt 1976], namely that the H term is different, does not interfere. Everything else follows verbatim. □

Remark on nonexistence. If $\gamma > \pi/2$, then in the case (16) existence can fail if $\delta < 2\chi\sigma/\rho_0 - \chi p_0$. To see that, we integrate (12) over Ω, obtaining

$$\int_\Omega \left(\frac{\rho_0 - \chi p_0}{\chi\sigma} (e^{\chi g u} - 1) + \chi g(1 - \cos\omega) \right) dx = 2\pi |\Sigma| \cos\gamma$$

from which it follows that

$$\frac{\rho_0 - \chi p_0}{\chi\sigma} |\Omega| > -|\Sigma| \cos\gamma,$$

which leads to a contradiction if the domain is scaled to be small enough. For the special case of a disk B_δ, we obtain

$$\delta > -\frac{2\chi\sigma}{\rho_0 - \chi p_0}\cos\gamma$$

providing a slight improvement over the criterion yielded by Theorem 2.3.

This last result applies to the "unconstrained" case of an open circular tube dipped into an infinite reservoir of fluid. Physically, it signifies that if the tube is too narrow, the fluid will disappear down the tube to negative infinity. Finn and Luli [≥ 2007] studied the "constrained" case of a circular tube closed at the bottom and filled with a prescribed mass of fluid. For that problem they were able to show that for any γ with $0 \le \gamma < \pi$, and for any prescribed total mass $\mathcal{M}$, there is at least one symmetric solution of the problem, and that the height for this solution will lie over any prescribed level if $\mathcal{M}$ is sufficiently large. If $\gamma \le \pi/2$, the solution is unique among symmetric solutions with the prescribed mass. From Theorem 2.2 then follows that the solution is unique among all solutions with the same Lagrange parameter.

5. Existence of solutions in domains with corners

For this section we need Theorem 4 of [Ladyzhenskaya and Ural'tseva 1970], which adapted to our situation yields:

Theorem 5.1. *Let u be a classical solution of*

$$\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}} = \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}},$$

in a bounded domain Ω, with $\mathcal{F}$ as defined in (15) or (16). Assume $\sup_\Omega |u| \le M$. Then for any strictly interior subdomain Ω' of Ω with $d := \operatorname{dist}(\Omega', \partial\Omega)$,

$$\max_{\Omega'} |Du(x)| \le C,$$

with $C = C(n, M, d)$. (Compare also [Simon 1977, Theorem 2″].)

For the existence result in this section, we assume the domain Ω to be open and bounded, with piecewise $C^{1,\mu}$ boundary Σ and to have a finite number of "well-behaved" corners. By this we mean that if a corner is located at the point O, we can parametrize the arcs on either side of O by smooth functions $c_i(s)$, $0 < s < s_0$, $i = 1, 2$, such that $\lim c_i(s) = O$, as $s \to 0$, with an angle $0 < 2\alpha < \pi$ formed by $\lim c_i'(s)$ as $s \to 0$.

Ω can be exhausted by an expanding sequence of admissible domains $\Omega^j \subset \Omega$, whose boundaries Σ^j converge uniformly in C^1 in any neighborhood U_{x_0} of a boundary point $x_0 \in \Sigma$, whose closure $\overline{U}_{x_0} \cap \Sigma$ lies in the smooth portion of Σ.

With similar arguments as in [Finn and Gerhardt 1977, Theorem 1], we can prove the following existence result:

Theorem 5.2. *Let Ω be as described above. Let γ be constant and $0 < \gamma < \pi$. In the case of $\mathcal{F}$ being given by (16) and if $\gamma > \pi/2$ we also require an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ to hold for Ω^j.*

Then there exists a variational solution u of (13). If $|\gamma - \pi/2| < \alpha$ then $u \in \mathcal{Q}(\Omega)$.

If u, v are two variational solutions of (12)+(15), there holds

$$|u - v| < \frac{\chi\sigma}{\rho_0};$$

for variational solutions of (12)+(16) such that $u, v > -A > -\infty$, there holds

$$|u - v| < \frac{\chi\sigma}{\rho_0 - \chi p_0} e^{\chi g A}.$$

Proof. In view of the conditions on Ω^j, we can obtain a solution $u^j \in C^{2,\alpha}(\Omega^j)$ of

$$\operatorname{div} T u^j = \mathcal{F}(u^j) - \frac{a^2}{W^j} \tag{43}$$

in each Ω^j, with fixed boundary data γ on Σ^j, as in Theorem 4.2 above. They will satisfy the corresponding weak form; i.e., they will be variational solutions, each u^j satisfying (13) in Ω^j:

$$\int_{\Sigma^j} \eta \cos\gamma \, ds = \int_{\Omega^j} \left(D\eta \cdot T u^j + \eta\, \mathcal{F}(u^j) - \eta \frac{a^2}{\sqrt{1 + |Du^j|^2}} \right) dx, \tag{44}$$

for every $\eta \in \mathcal{Q}(\Omega^j)$.

In view of the assumption on the contact angle γ and the additional $\mathrm{ISC}_{\delta,\pi-\gamma}$ condition on Ω^j in case (16), the height and gradient estimates (Theorems 2.6 and 3.3) and the existence results hold for u^j in Ω^j without any additional restrictions being needed. As the height estimates depend on the distance of Σ^j to a corner, and the gradient estimates depend on the Lipschitz extension of the normal to the boundary, these estimates are not uniform in j. To overcome this obstacle, for any fixed j_0 we consider a fixed j_1, and solutions u^j in Ω^j, where $j \geq N(j_1) > j_1 > j_0$, such that the distance from Ω^{j_0} to $\partial\Omega^{j_1}$ and from Ω^{j_1} to $\partial\Omega^j$ is strictly positive. These u^j will satisfy (43) in Ω^{j_1} and Ω^{j_0}. The height bounds in Ω^{j_1} are uniform, as shown in Theorems 2.6 and 2.9, and by Theorem 5.1 we obtain uniform gradient bounds in Ω^{j_0}. Therefore, in Ω^{j_0} we have uniform height and gradient bounds.

Using general results on elliptic equations [Ladyzhenskaya and Ural'tseva 1968, Chapter 10, Theorem 2.2], we can extend the uniform height and gradient estimates to higher regularity of the solutions u^j $(j \geq N(j_0))$ of (43) in Ω^{j_0}, for every j_0.

Using the Arzelà–Ascoli theorem we can find a subsequence (not relabeled), converging uniformly together with all its derivatives in any Ω^{j_0}, to a solution $u(x)$ of (43).

We choose $\eta \in \mathcal{Q}(\Omega)$, so that in particular $\eta \in \mathcal{Q}(\Omega^j)$. We remark that $\eta \in H^{1,1}(\Omega)$ has a well-defined trace function in $L^1(\partial\Omega)$, which we denote again by η. We also note that $\eta \in H^{1,1}(\Omega)$ can be approximated in the $H^{1,1}$ norm by uniformly continuous functions in Ω. Their boundary values approximate the trace of η on $\partial\Omega$ in the $L^1(\partial\Omega)$ norm; see [Giusti 1984, Theorem 2.11].

We consider (44). Regarding the convergence of the right-hand side of (44), we again fix j_0, and note that $|Tu^j| < 1$ and $a^2/\sqrt{1+|Du^j|^2} \le a^2$ in Ω. We have

$$\left|\int_{\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}} \right) dx \right| \le c,$$

with c depending on $|\eta|_{L^1(\Omega)}$, $|D\eta|_{L^1(\Omega)}$, and the size of Ω, but independent of j_0. Also, given the uniform convergence of u^j and Du^j in Ω^{j_0}, we have

$$\lim_{j\to\infty} \int_{\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}} \right) dx = \int_{\Omega^{j_0}} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx. \tag{45}$$

Now we can let j_0 vary, and conclude that the first and third terms on the right-hand side of (44) converge to

$$\int_{\Omega} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx.$$

To see this we consider

$$\begin{aligned}
&\left| \int_{\Omega^j} \left(D\eta\cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}} \right) dx - \int_{\Omega} \left(D\eta\cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx \right| \\
&\le \left| \int_{\Omega^{j_0}} \left(D\eta\cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}} \right) dx - \int_{\Omega^{j_0}} \left(D\eta\cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx \right| \\
&\qquad + \left| \int_{\Omega^j - \Omega^{j_0}} \left(D\eta\cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}} \right) dx \right| \\
&\qquad + \left| \int_{\Omega - \Omega^{j_0}} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx \right|.
\end{aligned}$$

By (45) the first term on the right-hand side is less than $\varepsilon/3$ for $j > J$, for large enough J. The second and third terms on the right-hand side can be estimated by $c\big(|\eta|_{L^1(\Omega)}, |D\eta|_{L^1(\Omega)}\big)|\Omega^j - \Omega^{j_0}| < \varepsilon/3$ and $c\big(|\eta|_{L^1(\Omega)}, |D\eta|_{L^1(\Omega)}\big)|\Omega - \Omega^{j_0}| < \varepsilon/3$, respectively, due to the convergence of Ω^j to Ω, for $j > J$ and appropriately large j_0.

For the left-hand side of (44), we remark that $|\eta \cos\gamma|$ is bounded. Therefore

$$\int_{\Sigma^j \cap B_r(O)} \eta \cos\gamma \, ds \to 0 \ \text{ for } r \to 0, \ \text{ uniformly in } j, \tag{46}$$

where $B_r(O)$ denotes a ball of small radius r centered at a corner O. We also have

$$\int_{\Sigma \cap B_r(O)} \eta \cos\gamma \, ds \to 0 \ \text{ for } r \to 0. \tag{47}$$

In what follows it suffices to assume Ω to have only one corner, O.

In the following estimate we split integrals into their parts over $B_r(O)$ and its complement, $B_r^c(O)$.

$$\begin{aligned} \left| \int_{\Sigma^j \cap B_R(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_R(O)} \eta \cos\gamma \, ds \right| & \\ \le \left| \int_{\Sigma^j \cap B_r(O)} \eta \cos\gamma \, ds \right| + \left| \int_{\Sigma \cap B_r(O)} \eta \cos\gamma \, ds \right| & \\ + \left| \int_{\Sigma^j \cap B_r^c(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_r^c(O)} \eta \cos\gamma \, ds \right|. & \end{aligned} \tag{48}$$

We choose r sufficiently small to ensure that the first and second summands on the right are each less than $\varepsilon/3$, by (46) and (47) respectively.

By the assumptions on the convergence of Σ^j in neighborhoods of Σ not containing corners, the last summand $\left|\int_{\Sigma^j \cap B_r^c(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_r^c(O)} \eta \cos\gamma \, ds\right|$ is also less than $\varepsilon/3$ for any $j > J$, with J large enough. We thus obtain from (48)

$$\left| \int_{\Sigma^j \cap B_R(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_R(O)} \eta \cos\gamma \, ds \right| < \varepsilon,$$

and the convergence of the boundary integral in (44) to $\int_\Sigma \eta \cos\gamma \, ds$ is proved.

Having shown the convergence of all terms of (44) except $\int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx$, we conclude that this term converges too. We will show it converges to $\int_\Omega \eta \, \mathcal{F}(u) \, dx$.

We know that u^j satisfies (44) in Ω^j, which we rewrite as

$$\int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx = \int_{\Omega^j} [D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1 + |Du^j|^2}}] \, dx - \int_{\Sigma^j} \eta \cos\gamma \, ds.$$

The right-hand side here converges; therefore

$$\left| \int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx \right| \le c,$$

with a constant c depending on η, but independent of j.

We consider two cases:

(i) *The angle γ satisfies $|\gamma - \pi/2| < \alpha$.* In view of Theorem 2.9 we have uniform height bounds on u^j, independent of j, since they are independent of the distance of Σ^j to the corner O, and the same bounds hold for the limit function u. We fix j_0 as before, and with the continuity of $\mathcal{F}$, we have $\mathcal{F}(u^j)$ converging to $\mathcal{F}(u)$ in any Ω^{j_0}, and the corresponding uniform bounds for $\mathcal{F}(u)$. So

$$\lim_{j\to\infty} \int_{\Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx = \int_{\Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx$$

and, using the uniform bounds on $\mathcal{F}(u)$ over all of Ω, we can let $j_0 \to \infty$ and obtain the result as in the previous considerations.

(ii) *The angle γ satisfies $|\gamma - \pi/2| > \alpha$.* In this case the height estimates will depend on the distance of Σ^j to the corner, u^j becoming unbounded as we approach O. However the growth of $|u^j|$ means that $|\mathcal{F}(u^j)|$ is proportional to r^{-1}, as proved in Theorem 2.9.

Again, it suffices to assume Ω to have only one corner, O. We estimate, after adding and subtracting the terms $\int_{\Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx$ and $\int_{\Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx$,

$$\begin{aligned} \left| \int_{\Omega^j} \eta\, \mathcal{F}(u^j)\, dx - \int_{\Omega} \eta\, \mathcal{F}(u)\, dx \right| & \\ \le \left| \int_{\Omega^j - \Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx \right| + \left| \int_{\Omega - \Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx \right| & \\ + \left| \int_{\Omega^{j_0}} \big(\eta\, \mathcal{F}(u^j) - \eta\, \mathcal{F}(u)\big)\, dx \right| &. \end{aligned} \tag{49}$$

The last summand on the right is less than $\varepsilon/4$ for all $j > J$, with J large enough; to see this, use the continuity of $\mathcal{F}$ and the uniform bounds in Ω^{j_0}.

The second summand on the right can be estimated by

$$\begin{aligned} \int_{\Omega - \Omega^{j_0}} |\eta \mathcal{F}(u)|\, dx & \\ \le \sup|\eta| \int_{\Omega - \Omega^{j_0}} |\mathcal{F}(u)|\, dx & \\ \le \sup|\eta| \bigg(\int_{(\Omega - \Omega^{j_0}) \cap B_r(O)} |\mathcal{F}(u)|\, dx + \int_{(\Omega - \Omega^{j_0}) \cap B_r^c(O)} |\mathcal{F}(u)|\, dx \bigg)&, \end{aligned}$$

where $B_r^c(O)$ denotes the complement in $\mathbb{R}^2$ of the disk $B_r(O)$.

In $(\Omega - \Omega^{j_0}) \cap B_r^c(O)$, we are at a positive distance from O, and have bounds for $\mathcal{F}(u)$, so

$$\sup|\eta| \int_{(\Omega - \Omega^{j_0}) \cap B_r^c(O)} |\mathcal{F}(u)|\, dx < \varepsilon/4,$$

due to the convergence of Ω^j to Ω, for large j_0, after possibly adjusting the previous choice of J.

For the integral over $(\Omega-\Omega^{j_0})\cap B_r(O)$ we introduce polar coordinates and can show, using the inequality $|\mathcal{F}(u)| < Cr^{-1}$, that

$$\sup|\eta|\int_{(\Omega-\Omega^{j_0})\cap B_r(O)}|\mathcal{F}(u)|\,dx \le \sup|\eta|\int_{B_r(O)}|\mathcal{F}(u)|\,dx$$
$$\le \sup|\eta|\int_0^r\int_0^{2\pi} C\cos\theta\,d\theta\,dr \ \le \varepsilon/4,$$

after choosing r appropriately small.

The first summand on the right in (49) can be dealt with in a similar way, but more easily, since $\Omega^j\cap B_r(O)=\varnothing$ for small r.

Returning to (49), we have shown that $\int_{\Omega^j}\eta\,\mathcal{F}(u^j)\,dx$ converges to $\int_\Omega \eta\,\mathcal{F}(u)\,dx$ as $j\to\infty$.

We had approximated $\eta\in H^{1,1}(\Omega)$ and worked with uniformly continuous functions. We have shown that u satisfies

$$\int_\Omega\left(D\eta\cdot Tu+\eta\,\mathcal{F}(u)-\eta\frac{a^2}{\sqrt{1+|Du|^2}}\right)dx=\int_\Sigma \eta\cos\gamma\,ds$$

for such η. Going over to $\eta\in\mathcal{Q}(\Omega)$, we conclude in both cases that u is a variational solution in Ω.

By Theorem 2.9, u is bounded if $|\gamma-\pi/2|<\alpha$ and therefore $u\in\mathcal{Q}(\Omega)$.

We also remark that if we have different limits u and v obtained by two different subsequences, we still know that they are not "too far apart", in the sense of the estimate given in Theorem 2.3. We emphasize that this is true even though the solution might become unbounded when approaching a corner. □

Remark 1. Theorem 5.2 is the best possible result one can obtain for this problem. As observed in Section 4, existence fails in small domains in the case (16) and $\gamma>\pi/2$. This is taken care of by the internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta>2\chi\sigma/(\rho_0-\chi p_0)$ for Ω^j.

Remark 2. We obtain a variational solution for our problem in both cases (15) and (16) after imposing the additional $\mathrm{ISC}_{\delta,\pi-\gamma}$, for general $0<\gamma<\pi$, despite the fact that the values for any solution become unbounded in a narrow corner, when $|\gamma-\pi/2|>\alpha$, as stated in Theorem 2.9.

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References

[Bashforth and Adams 1883] F. Bashforth and J. G. Adams, *An attempt to test the theories of capillary action by comparing the theoretical and measured forms of drops of fluids*, Cambridge Univ. Press, Cambridge, 1883.

[Concus et al. 1999] P. Concus, R. Finn, and M. Weislogel, "Capillary surfaces in an exotic container: results from space experiments", *J. Fluid Mech.* **394** (1999), 119–135. MR 2000e:76001 Zbl 0944.76519

[Concus et al. 2000] P. Concus, R. Finn, and M. Weislogel, "Measurement of critical contact angle in a microgravity space experiment", *Exps. Fluids* **28** (2000), 197–205.

[Emmer 1973] M. Emmer, "Esistenza, unicità e regolarità nelle superfici de equilibrio nei capillari", *Ann. Univ. Ferrara Sez. VII* (*N.S.*) **18** (1973), 79–94. MR 49 #1281 Zbl 0275.49005

[Finn 1963] R. Finn, "On partial differential equations (whose solutions admit no isolated singularities)", *Scripta Math.* **26** (1963), 107–115. MR 28 #4225 Zbl 0114.30401

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Math. Wiss. **284**, Springer, New York, 1986. MR 816345 (88f:49001) Zbl 0583.35002

[Finn 1999] R. Finn, "Capillary surface interfaces", *Notices Amer. Math. Soc.* **46**:7 (1999), 770–781. MR 2000g:76033

[Finn 2001] R. Finn, "On the equations of capillarity", *J. Math. Fluid Mech.* **3**:2 (2001), 139–151. MR 2002c:76026 Zbl 0999.76025

[Finn 2006] R. Finn, "The contact angle in capillarity", *Phys. Fluids* **18** (2006), #047102.

[Finn and Gerhardt 1977] R. Finn and C. Gerhardt, "The internal sphere condition and the capillary problem", *Ann. Mat. Pura Appl.* (4) **112** (1977), 13–31. MR 55 #7106 Zbl 0349.49019

[Finn and Luli $\geq$ 2007] R. Finn and G. K. Luli, "On the capillarity problem for compressible fluids", *J. Math. Fluid Mech.*. To appear.

[Gauss 1830] C. F. Gauss, "Principia generalia theoriae figurae fluidorum", *Comment. Soc. Regiae Scient. Gottingensis Rec.* **7** (1830). German translation: *Grundlagen einer Theorie der Gestalt von Flüssigkeiten im Zustand des Gleichgewichts*, Ostwald's Klassiker der exakten Wissenschaften **135**, W. Engelmann, Leipzig, 1903.

[Gerhardt 1976] C. Gerhardt, "Global regularity of the solutions to the capillarity problem", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **3**:1 (1976), 157–175. MR 58 #29199 Zbl 0338.49008

[Giusti 1984] E. Giusti, *Minimal surfaces and functions of bounded variation*, Monographs in Mathematics **80**, Birkhäuser, Basel, 1984. MR 87a:58041 Zbl 0545.49018

[Huisken 1985] G. Huisken, "Capillary surfaces over obstacles", *Pacific J. Math.* **117**:1 (1985), 121–141. MR 87c:49047 Zbl 0561.49009

[Korevaar 1980] N. J. Korevaar, "On the behavior of a capillary surface at a re-entrant corner", *Pacific J. Math.* **88**:2 (1980), 379–385. MR 82c:49033 Zbl 0475.49034

[Ladyzhenskaya and Ural'tseva 1968] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, Mathematics in Science and Engineering **46**, Academic Press, New York, 1968. MR 39 #5941 Zbl 0164.13002

[Ladyzhenskaya and Ural'tseva 1970] O. A. Ladyzhenskaya and N. N. Ural'tseva, "Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations", *Comm. Pure Appl. Math.* **23** (1970), 677–703. MR 42 #654 Zbl 0193.07202

[Laplace 1805–1806] P. S. de Laplace, *Traité de mécanique céleste* (suppléments au Livre X), 1805–1806. Reprinted in vol. 4 of his *Œuvres complètes*, Gaulthier-Villars, Paris, 1882. See also the annotated translation by N. Bowditch, Boston, 1839; reprinted by Chelsea, New York, 1966.

[Simon 1977] L. Simon, "Equations of mean curvature type in 2 independent variables", *Pacific J. Math.* **69**:1 (1977), 245–268. MR 56 #13099 Zbl 0354.35040

[Ural'tseva 1973] N. N. Ural'tseva, "The solvability of the capillarity problem", *Vestnik Leningrad. Univ.* issue 19 (= Mat. Mekh. Astronom. 4) (1973), 54–64. In Russian; translation in *Vestnik Leningrad Univ. Math.* **6** (1979), 363–375. MR 58 #30682

[Ural'tseva 1975] N. N. Ural'tseva, "The solvability of the capillarity problem, II", *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* issue 1 (= Mat. Mekh. Astronom. 1) (1975), 143–149. In Russian; translation in *Vestnik Leningrad Univ. Math.* **8** (1980), 151–158. MR 58 #30683

[Young 1805] T. Young, "An essay on the cohesion of fluids", *Philos. Trans. Roy. Soc. London* **95** (1805), 65–87.

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