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DANZHU SHI

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A capillary surface expressible as a graph over a domain containing a protruding corner can exhibit strikingly varying behavior, with discontinuous transitions, depending on local boundary conditions. Korevaar in 1980 showed that very different kinds of behavior must be expected when the corner opening exceeds π, and later Lancaster and Siegel extended that result to indicate a remarkable range in the kinds of behavior that can occur. This work characterizes all possible modes of behavior for this case, subject to a conjecture of Concus and Finn for the protruding angle case.

1. Introduction

A *capillary surface* S is the interface separating two immiscible fluids adjacent to each other. In this work we discuss interfaces that are ideally thin and can be represented as graphs $f(x, y)$ over a base domain Ω. We only consider equilibrium configurations. As shown initially by Laplace, for incompressible fluids in a vertical cylinder (a *capillary tube*), the shape of the surface is governed by the equation

$$\nabla \cdot Tf = \kappa f + \lambda \quad \text{in } \Omega, \tag{1}$$

$$Tf \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega, \tag{2}$$

where Tf is defined as $\nabla f/(\sqrt{1+|\nabla f|^2})$ and $\kappa = \rho g/\sigma$ is the *capillarity constant*, with ρ the density change across the surface, g the gravitational attraction, and σ the surface tension of the interface. The constant λ is a Lagrange parameter arising from a possible volume constraint, ν is the exterior normal vector on $\partial\Omega$, and $\gamma = \gamma(s)$ is a function of position on $\partial\Omega$, satisfying $0 \le \gamma(s) \le \pi$. The surface $z = f(x, y)$ describes the shape of the static liquid-gas interface in a vertical cylindrical tube of bounded cross-section Ω. In this paper, we assume $\kappa > 0$, corresponding to the case of a vertically downward gravity field, with the denser fluid below the surface.

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Keywords: capillary surfaces, reentrant corner, corner behavior, Concus–Finn conjecture.

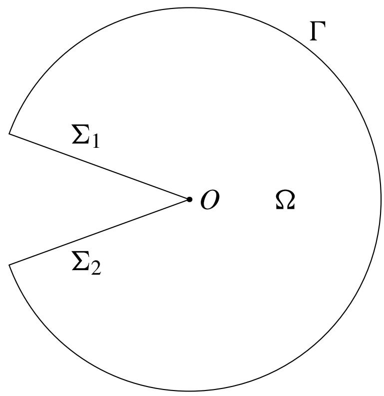


Figure 1. Reentrant wedge domain.

I will address the particular case in which Ω is a wedge domain with a reentrant corner. Specifically, I will assume that the corner of the wedge is the origin O of coordinates, and $\partial\Omega$ consists of three smooth portions:

$$\begin{aligned}\Gamma &= \{(\cos\varphi, \sin\varphi) : -\alpha \le \varphi \le \alpha\},\\ \Sigma_1 &= \{(\rho\cos\alpha, \rho\sin\alpha) : 0 < \rho < 1\},\\ \Sigma_2 &= \{(\rho\cos\alpha, -\rho\sin\alpha) : 0 < \rho < 1\}.\end{aligned}$$

See Figure 1. Σ_1 and Σ_2 can be relaxed to only asymptotic straight lines when approaching the origin. Let γ equal $\gamma_1(s)$, $\gamma_2(s)$ along the sides Σ_1, and Σ_2, respectively, where $s = 0$ corresponds to the point O.

Capillary surfaces can exhibit strikingly idiosyncratic behavior at corner points of the domain Ω of definition, as a consequence of the characteristic nonlinearities in Equation (1). This was initially observed by Concus and Finn [1974b; 1974a; 1974]. Later this behavior was further delineated in [Simon 1980; Tam 1986a; 1986b; Lieberman 1988; Miersemann 1985; Concus and Finn 1994; 1996; Finn 1986; Lancaster and Siegel 1996; Shi and Finn 2004].

We take [Lancaster and Siegel 1996] as the starting point for this work. There the authors considered the limiting values Rf of f along radial approaches within Ω to the vertex O. They showed that under very general hypotheses this limit always exists, and they delineated various possibilities for its behavior in terms of direction of approach to O. For protruding corners (opening angle $2\alpha < \pi$), their results can be considered close to definitive, subject to a conjecture of Concus and Finn. The present work addresses the complementary case of reentrant corners (opening angle $2\alpha > \pi$), for which the behavior can be very different.

We will proceed by indirect reasoning using methods of geometric measure theory. In Section 3 we will introduce the concept of a generalized solution of the

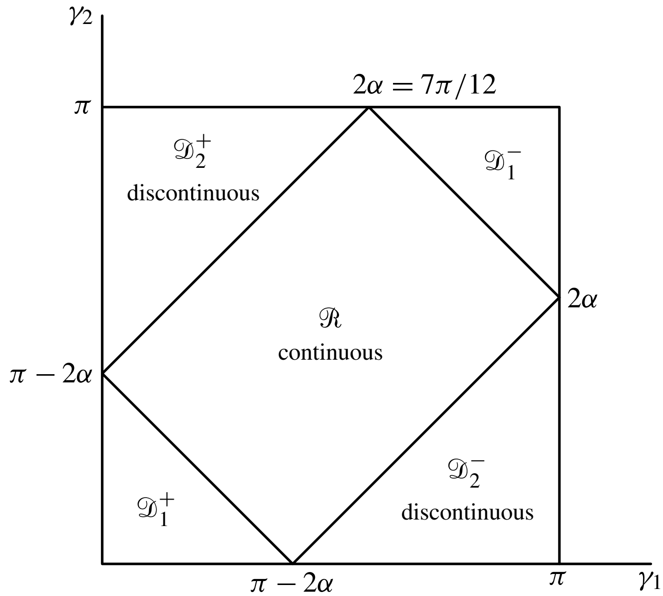


Figure 2. The Concus–Finn rectangle: regions in parameter space corresponding to different behaviors at a convex corner of angle 2α. In regions $\mathcal{D}_1^\pm$ there is no bounded graph.

minimal surface equations in the sense of Miranda [1964]. In Section 4 we adapt Giusti's minimal cone theory [1977]. These two sections, together with some other earlier results, will play an essential role later on in the proof of the main result.

At the end of this work, we distinguish various cases, according to the contact angles γ_1, γ_2 on the two sides of the wedge domain formed near O. These cases can be characterized geometrically, using a diagram analogous to one introduced by Concus and Finn [1996] for the protruding angle case. We show the Concus–Finn diagram in Figure 2. The central rectangle $\mathcal{R}$ is uniquely determined by the angle 2α. Its vertices along the coordinate axes have coordinate $\pi - 2\alpha$ (in the reentrant case the corresponding value will be $2\alpha - \pi$; see Figure 3). The four triangular regions are denoted $\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{D}_2^+, \mathcal{D}_2^-$.

A central new result of this paper is that, assuming the truth of the Concus–Finn conjecture for the protruding angle case, *any solution arising from data outside the rectangle $\mathcal{R}$ is necessarily discontinuous at O for the reentrant angle case as well, in the sense that different limit values must occur for different radial directions of approach.* We will completely characterize the ways in which the discontinuous behavior can be manifested. This can change according to the particular domain of the diagram where (γ_1, γ_2) lies.

For a reentrant corner, data within $\mathcal{R}$ can lead to a discontinuous solution; this contrasts with the protruding angle case, for which all solutions arising from such data are known to be not only continuous but Hölder differentiable up to O.

2. The Concus–Finn diagram

We start by reviewing results and notation from [Lancaster and Siegel 1996], and categorizing them using the Concus–Finn diagram.

Protruding wedge domains. Solutions can exist at corners for any data (γ_1, γ_2); but for any transition from $\mathcal{R}$ to $\mathcal{D}_1^{\pm}$ across a common boundary point (γ_1^*, γ_2^*), there is a discontinuous change in behavior, from uniform boundedness at O of the solution for all data up to and including (γ_1^*, γ_2^*), to unboundedness for all data in $\mathcal{D}_1^{\pm}$, with asymptotic behavior depending only on the local geometry.

The radial limits of f at the vertex of the corner will be denoted by $Rf(\theta) = \lim_{r\to 0+} f(r\cos\theta, r\sin\theta)$, $-\alpha < \theta < \alpha$ and $Rf(\pm\alpha) = \lim_{x,y\to 0} f(x, y)$ where $(x, y) \in \Sigma_{1,2}$, which are the limits of the boundary values of f on the two sides of the corner.

Theorem 2.1 [Lancaster and Siegel 1996]. *Let f be a bounded solution to* (1) *satisfying the boundary condition* (2) *on $\partial^{\pm}\Omega \setminus O$, discontinuous at O, with $0 < \gamma_0 \le \gamma^{\pm}(s) \le \gamma_1 < \pi$. If $\alpha \ge \pi/2$, then $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. If $\alpha < \pi/2$ and there exist constants $\underline{\gamma}^{\pm}$ and $\overline{\gamma}^{\pm}$ satisfying*

$$0 < \underline{\gamma}^{\pm} \le \pi/2, \quad \pi/2 \le \overline{\gamma}^{\pm} < \pi, \quad \underline{\gamma}^{+} + \underline{\gamma}^{-} > \pi - 2\alpha, \quad \overline{\gamma}^{\pm} + \overline{\gamma}^{-} < 2\alpha + \pi,$$

so that $\underline{\gamma}^{\pm} \le \gamma^{\pm}(s) \le \overline{\gamma}^{\pm}$ for all $s \in (0, s_0)$ for some s_0, then again $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. Furthermore, in either case $Rf(\theta)$ is a continuous function on $[-\alpha, \alpha]$ behaving in one of the following ways:

(i) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these cases* (I) and (D), respectively.

(ii) There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R \le \alpha, \alpha_R = \alpha_L + \pi$, and Rf is constant on $[-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and either increasing on $[\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2]$ or decreasing on $[\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2]$. Label these cases (ID) and (DI), respectively.

Corollary 2.2 [Lancaster and Siegel 1996]. *Let f be a bounded solution to* (1) *satisfying*

$$\begin{aligned} Tf \cdot \nu &= \cos\gamma_1 \quad \text{on} \quad \Sigma_1, \\ Tf \cdot \nu &= \cos\gamma_2 \quad \text{on} \quad \Sigma_2, \end{aligned} \tag{3}$$

with constant $\gamma_1, \gamma_2 \in (0, \pi)$. *For* $\alpha < \pi/2$, *assume in addition*

$$\pi - 2\alpha < \gamma_1 + \gamma_2 < \pi + 2\alpha.$$

Then

Case (I) cannot hold if $\gamma_1 - \gamma_2 \leq \pi - 2\alpha$.

Case (D) cannot hold if $\gamma_2 - \gamma_1 \leq \pi - 2\alpha$.

For $\alpha > \pi/2$, case (ID) cannot hold if $\gamma_1 + \gamma_2 \leq 3\pi - 2\alpha$.

For $\alpha > \pi/2$, case (DI) cannot hold if $\gamma_1 + \gamma_2 \geq 2\alpha - \pi$.

Corollary 2.3 [Lancaster and Siegel 1996]. *Let* f *be a bounded solution to* (1) *satisfying* (3), *with* $\gamma_1, \gamma_2 \in (0, \pi)$, $\alpha < \pi/2$, *and* $(\gamma_1, \gamma_2) \in \overline{\mathcal{R}}$, *then* f *must be continuous at* O.

Concus and Finn have conjectured that for data from $\mathcal{D}_2^{\pm}$, there could only be discontinuous solutions.

Conjecture 2.4 (Concus–Finn). *Suppose* $\alpha < \pi/2$, *and* $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{\pm}(2\alpha)$, *then the solution* f *has a discontinuity at the vertex* O.

In this paper, everything is based on the assumption that Conjecture 2.4 is true.

First, as indicated in Figure 2, we can summarize the situation for a protruding wedge domain in terms of the regions $\mathcal{R}$, $\mathcal{D}_1^{\pm}$, $\mathcal{D}_2^{\pm}$:

- The pair (γ_1, γ_2) lies in $\overline{\mathcal{R}}(2\alpha)$ if and only if both f and its outward unit normal are continuous up to O.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{+}(2\alpha)$, f must be a (D) case.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{-}(2\alpha)$, f must be an (I) case.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{\pm}(2\alpha)$, f is no longer a bounded graph near O.

Hence with this conjecture, we have completed categorizing the continuity of the solutions at the corner of the domain in terms of the Concus–Finn diagram.

Reentrant wedge domains. We now develop parallel results for reentrant wedge domains. When $2\alpha = \pi$, the boundary is smooth, which is not a case considered in this paper.

Theorem 2.5. *Assume* $\alpha > \pi/2$. *Let* f *be a solution to* (1) *satisfying* (3), *with* $\gamma_1, \gamma_2 \in (0, \pi)$.

(i) *For* $(\gamma_1, \gamma_2) \in \overline{\mathcal{R}}(2\alpha)$, f *can be continuous at* O *or in one of the cases* (I) or (D).

(ii) For $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{+}(2\alpha)$, f can be in one of the cases (DI), (I), or (D).

(iii) For $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{-}(2\alpha)$, f can be in one of the cases (ID), (I), or (D).

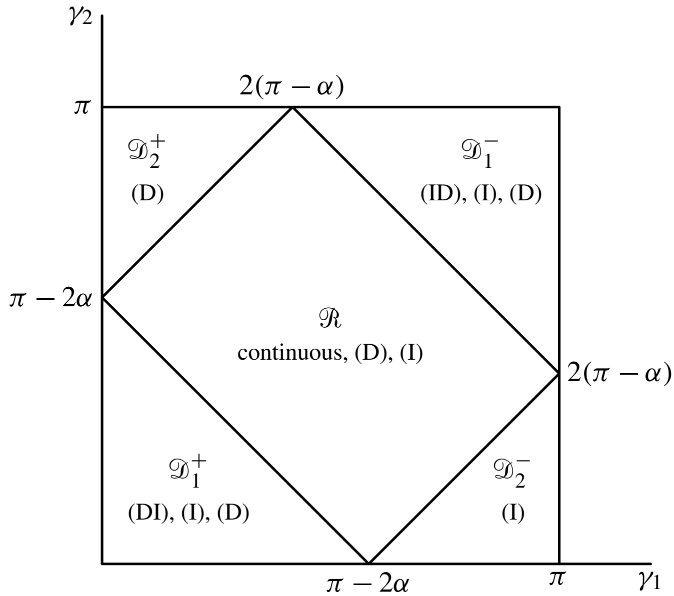


Figure 3. Analogue of the Concus–Finn rectangle for reentrant wedges.

(iv) For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^+(2\alpha)$, f must be in case (D).

(v) For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^-(2\alpha)$, f must be in case (I).

In Section 6 we construct examples showing that each of these cases actually occurs.

By Corollaries 2.2 and 2.3 we deduce:

(i′) For $\gamma_1, \gamma_2 \in \overline{\mathcal{R}}(2\alpha)$, f can only be continuous up to O, or be in case (I) or (D).

(ii′) For $\gamma_1, \gamma_2 \in \mathcal{D}_1^+(2\alpha)$, f cannot be in case (ID).

(iii′) For $\gamma_1, \gamma_2 \in \mathcal{D}_1^-(2\alpha)$, f cannot be in case (DI).

(iv′) For $\gamma_1, \gamma_2 \in \mathcal{D}_2^+(2\alpha)$, f can be continuous up to O, or be in case (D).

(v′) For $\gamma_1, \gamma_2 \in \mathcal{D}_2^-(2\alpha)$, f can be continuous up to O, or be in case (I).

In other words, all that is left to prove is that continuity is excluded from the $\mathcal{D}_1^\pm$, $\mathcal{D}_2^\pm$ regions, and that each of the cases can occur.

3. Generalized solutions

To discuss discontinuous capillary surfaces further, we introduce the definition of generalized (or weak) solution of the minimal surface equations in the sense of Miranda, and prove some results for capillary surfaces over a reentrant wedge domain.

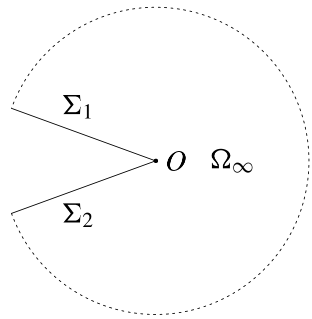


Figure 4. The infinite domain.

Set $\Omega_\infty = \{(\rho\cos\varphi, \rho\sin\varphi) : -\alpha < \varphi < \alpha,\ \rho > 0\}$. We redefine the symbols for the boundary pieces (see Figure 4):

$$\Sigma_1 = \{\rho > 0,\ \varphi = \alpha\},$$
$$\Sigma_2 = \{\rho > 0,\ \varphi = -\alpha\}$$

Definition 3.1. A function $u : \Omega_\infty \to [-\infty, \infty]$ is called a *generalized solution* of the equations

$$\begin{aligned} \nabla\cdot Tu &= 0 && \text{in } \Omega_\infty,\\ Tu\cdot\nu &= \cos\gamma_1 && \text{on } \Sigma_1,\\ Tu\cdot\nu &= \cos\gamma_2 && \text{on } \Sigma_2 \end{aligned} \tag{4}$$

if the subgraph of u defined by

$$U = \{(x, y, z) : (x, y) \in \Omega_\infty, z < u(x, y)\}$$

minimizes the functional

$$\int_{\Omega_\infty\times\mathbb{R}} |D\varphi_U| - \cos\gamma_1 \int_{\Sigma_1\times\mathbb{R}} \varphi_U\, dH_2 - \cos\gamma_2 \int_{\Sigma_2\times\mathbb{R}} \varphi_U\, dH_2,$$

where φ_U is the characteristic function of U and H_2 is 2-dimensional Hausdorff measure in $\mathbb{R}^3$. That means that for every Caccioppoli set (set of locally finite perimeter) $E \subset \Omega_\infty \times \mathbb{R}$ coinciding with U outside some compact set $K \subset \mathbb{R}^3$ we have

$$W(K, U) \le W(K, E), \tag{5}$$

where

$$W(K,U) = \int_{(\Omega_\infty\times\mathbb{R})\cap K} |D\varphi_U| - \cos\gamma_1 \int_{(\Sigma_1\times\mathbb{R})\cap K} \varphi_U\, dH_2 - \cos\gamma_2 \int_{(\Sigma_2\times\mathbb{R})\cap K} \varphi_U\, dH_2$$

and likewise for $W(K, E)$.

A sequence of functions v_n is said to *converge locally* to a function v in a domain Ω if the characteristic functions of the subgraphs of v_n converge almost everywhere to the characteristic function of the subgraph of v in $\Omega \times \mathbb{R}$.

The function u is allowed to take the values $\pm\infty$. It follows from [Miranda 1964] that every classical solution of equations (4) is a generalized solution; conversely, every locally bounded generalized solution is a classical solution of equations (4).

We introduce the sets

$$\begin{aligned} P(u) &= \{(x, y) \in \Omega_\infty : u(x, y) = +\infty\}, \\ N(u) &= \{(x, y) \in \Omega_\infty : u(x, y) = -\infty\}, \\ G(u) &= \Omega_\infty - (\bar{P}(u) \cup \bar{N}(u)). \end{aligned}$$

It follows that P minimizes the functional

$$\begin{aligned} \Phi(A) &= \iint_{\Omega_\infty} |D\chi_A| - \cos\gamma_1 \int_{\Sigma_1} \chi_A \, dH_1 - \cos\gamma_2 \int_{\Sigma_2} \chi_A \, dH_1 \\ &= H_1(\Omega_\infty \cap \partial A) - \cos\gamma_1 H_1(\Sigma_1 \cap \partial A) - \cos\gamma_2 H_1(\Sigma_2 \cap \partial A) \end{aligned}$$

in Ω_∞, where H_1 is 1-dimensional Hausdorff measure in $\mathbb{R}_2$.

Similarly, N minimizes the functional

$$\begin{aligned} \Psi(A) &= \iint_{\Omega_\infty} |D\chi_A| + \cos\gamma_1 \int_{\Sigma_1} \chi_A \, dH_1 + \cos\gamma_2 \int_{\Sigma_2} \chi_A \, dH_1 \\ &= H_1(\Omega_\infty \cap \partial A) + \cos\gamma_1 H_1(\Sigma_1 \cap \partial A) + \cos\gamma_2 H_1(\Sigma_2 \cap \partial A) \end{aligned}$$

After modification by a set of measure zero, the two sets

$$\partial P \cap \Omega_\infty \quad \text{and} \quad \partial N \cap \Omega_\infty$$

consist of straight lines that do not intersect inside of Ω_∞. Moreover:

Lemma 3.2. (i) *Let L be the portion of ∂P which lies inside Ω_∞. Suppose L is not empty, then L is a straight line which either meets Σ_i in an angle γ_i, or passes through the point O meeting the sides in angles β_i with $\beta_i \geq \gamma_i$, $i = 1, 2$. Here the angles β_i are measured inside P.*

(ii) *Let L' be the portion of ∂N which lies inside Ω_∞. Suppose L' is not empty, then L' is a straight line which either meets Σ_i in an angle $\pi - \gamma_i$ provided that $\Sigma_i \cap \partial N \neq \varnothing$, or passes point O in an angle ϑ_i with $\vartheta_i \geq \pi - \gamma_i$, $i = 1, 2$. Here the angles ϑ_i are measured inside N.*

From now on consider specifically the reentrant corner domain. For this case we can also say:

Theorem 3.3. *Suppose v is a generalized solution of* (4) *on Ω_∞ with $\alpha > \pi/2$. The P, N, G regions of v are as defined at the beginning of this section. Assume P and N are nonempty.*

(i) *Each component of $\partial P \cap \Omega_\infty$ and $\partial N \cap \Omega_\infty$ is infinite.*

(ii) *Suppose a component of P or N has two boundary lines inside Ω_∞. The lines either meet on $\partial\Omega_\infty$, or their extensions meet outside Ω_∞, with an angle η, measured from the side containing P, or N respectively. Then $\eta \geq \pi$.*

(iii) *There are at most two components of $\partial P \cap \Omega_\infty$, or $\partial N \cap \Omega_\infty$.*

Proof. (i) No line segment inside Ω_∞ can meet both sides of a reentrant corner domain.

(ii) We work by contradiction. Suppose there is a component of P which has two boundary lines inside Ω_∞. Their extension lines meet with an angle $\eta < \pi$. See Figure 5. Comparing P with $P - OACDB$, we get $AC - OA\cos\gamma_1 + BD - OB\cos\gamma_2 \leq CD$. Hence, successively,

$$AC + AE + BD + BE \leq CD + OA\cos\gamma_1 + OB\cos\gamma_2 + AE + BE,$$

$$CE + DE - CD \leq AE + BE + OA\cos\gamma_1 + OB\cos\gamma_2.$$

Move CD to infinity, parallel to itself. Then $CE + DE - CD \to \infty$, while $AE + BE + OA\cos\gamma_1 + OB\cos\gamma_2$ remains fixed. Contradiction.

(iii) If $\partial P \cap \Omega_\infty$ has a whole line as a component, there must be no other component. Because of (ii), it is easy to see $\partial P \cap \Omega_\infty$ cannot contain three or more half-lines. $\square$

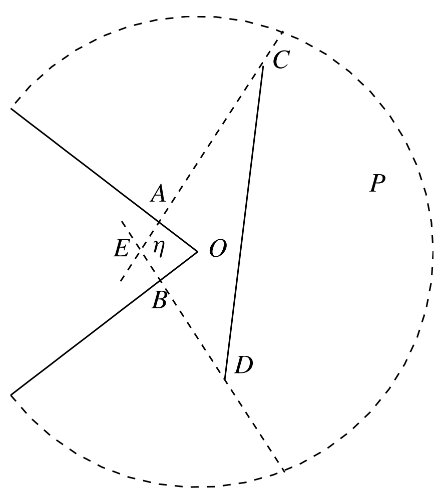


Figure 5. Proof of Theorem 3.3. Impossible case for P.

The following theorem describes the structure of the infinite sets P and N, the proof of the theorem is similar to [Tam 1986a, Theorem 2.4].

Corollary 3.4. *Under the assumptions of Theorem 3.3, the only possibilities for P other than $\varnothing$, Ω_∞ are the ones shown in Figure 6, namely:*

(i) *P consists of a single component, which is bounded between Σ_1 and a line L_1 with an opening angle β_1. L_1 meets Σ_1 at point A. Either $A \in \Sigma_1$ and $\beta_1 = \gamma_1$, or $A = O$ and $\beta_1 \geq \gamma_1$.*

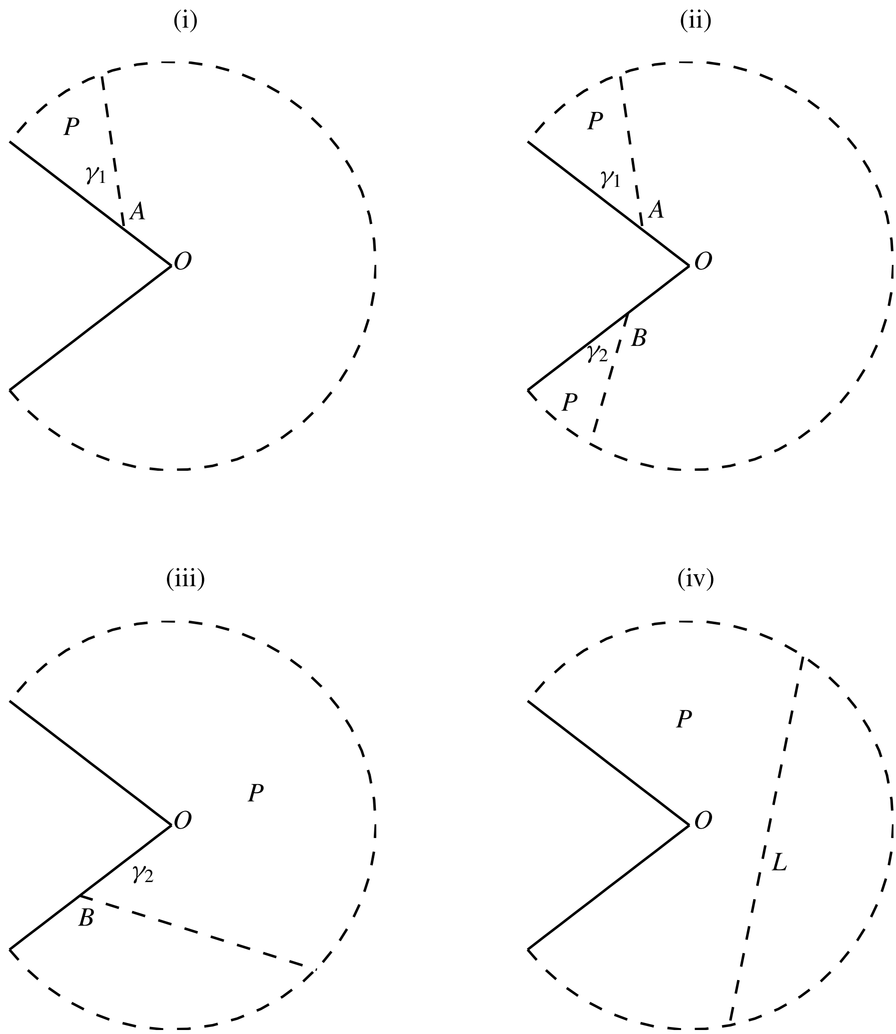


Figure 6. All possibilities for the region P.

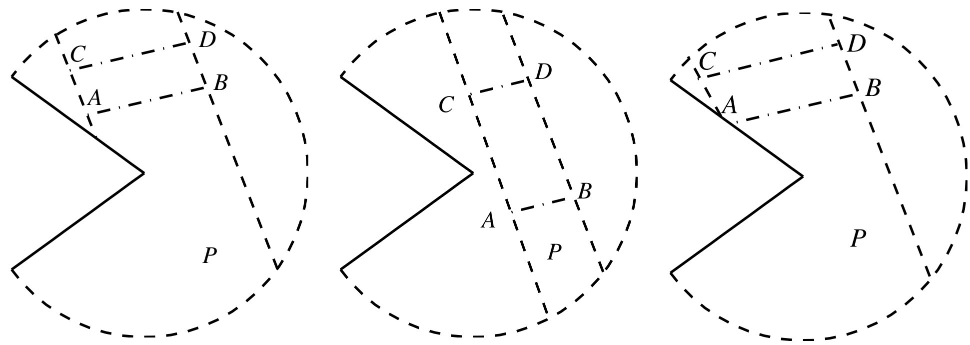


Figure 7. Impossible cases for the region P.

(ii) *P consists of two components. On is bounded between Σ_1 and a line L_1 with an opening angle β_1 and the other is bounded between Σ_2 and a line L_2 with an opening angle β_2. For $i = 1, 2$, suppose L_i meets Σ_i at point A_i. Either $A_i \in \Sigma_i$ and $\beta_i = \gamma_i$, or $A_i = O$ and $\beta_i \geq \gamma_i$.*

(iii) *P consists of a single component, which is bounded by Σ_1, $OB \subset \Sigma_2$, and line L_2 which meets Σ_2 at point B with an opening angle γ_2.*

(iv) *There is a whole line L which lies inside Ω_∞ and P is either the region bounded by $\partial\Omega_\infty$ and L, or the half plane bounded by L.*

Proof. Again we work by contradiction.

From Theorem 3.3 we obtain that P has at most two components. All the possibilities for the structure of P are indicated in Figures 6 and 7.

Next we prove that those cases in Figure 7 are impossible. Compare P with $P - ABDC$ in that figure. By definition,

$$\Phi(P) \leq \Phi(P - ABDC) \quad \text{and} \quad 2|AC| \leq 2|AB|.$$

Moving $\overline{CD}$ to infinity, parallel to itself, we get $\infty \leq |AB|$. Contradiction. $\square$

4. Radial linearity

In this section, we extend Giusti's work on minimal graphs [1977] to general H-graphs. We use the "blow-up" procedure to expand the capillary surface about 0 and show that the limit set C exists and that C is a minimal cone. Furthermore, this limit cone is unique.

The results in this section apply for any n-dimensional space. However, for our specific problem (1), we only need to consider $n = 3$.

Definition 4.1. Suppose Ω_∞ is the wedge domain in $\mathbb{R}^2$ defined as before. Define

$$Q = \Omega_\infty \times \mathbb{R}, \quad \delta Q = \partial\Omega_\infty \times \mathbb{R}.$$

Now suppose K is any open set in $\mathbb{R}^3$. Define

$$\delta K = \delta Q \cap K, \quad \delta^* K = Q \cap \partial K.$$

Denote by $\mathrm{BV}(K)$ the space of functions of bounded variation on K.

Definition 4.2. If $f \in \mathrm{BV}(K)$, set

$$\begin{aligned}
F_K(f) &= \int_{Q\cap K} |Df| - \int_{\delta K} \cos\gamma \cdot f \, dH_2 \\
\zeta(f, K) &= \inf\{F_K(g) : g \in \mathrm{BV}(K), \operatorname{supp}(g - f) \subset K\}, \\
\psi(f, K) &= F_K(f) - \zeta(f, K).
\end{aligned}$$

If f is the characteristic function of some set E with finite perimeter, we shall write $F_K(E)$, $\nu(E, K)$ and $\psi(E, K)$ instead of $F_K(\varphi_E)$, $\nu(\varphi_E, K)$ and $\psi(\varphi_E, K)$.

Definition 4.3. A set E is a *minimal* in K if $\psi(E, K) = 0$.

Definition 4.4. We call C a *cone* in $\mathbb{R}^3$ if

$$C = \{t(x, y, z) : t \geq 0, (x, y, z) \in A\},$$

for some set $A \in \mathbb{R}^3$.

This is the main theorem of this section.

Theorem 4.5. *Suppose that E minimizes the functional*

$$F_{B_1}(W) + \int_{Q\cap B_1} H\varphi_W,$$

for some H uniformly bounded on B_1, such that $0 \in \partial E$. *For* $t > 0$, *let*

$$E_t = \{x \in \mathbb{R}^3 : tx \in E\}.$$

Then as $t \to 0$, E_t converges locally in $\mathbb{R}^3$ to a set C. Moreover, C is a minimal cone.

In the rest of this section, we adopt the notation of trace in the sense of [Giusti 1977, Chapter 2]: f^- is the trace of f from above $z = 0$, and f^+ is the trace from below.

To prove Theorem 4.5 we need some lemmas. The first is an adaptation of [Giusti 1977, Lemma 5.3].

Lemma 4.6. *Let* $f \in \mathrm{BV}(B_R)$ *and let* $0 < \rho < r < R$. *Then*

$$\int_{\delta^* B_1} |f^-(rx) - f^-(\rho x)|\, dH_2 \le \int_{Q\cap(B_r - B_\rho)} \Big|\Big\langle \frac{x}{|x|^3}, Df\Big\rangle\Big|.$$

Proof. If $g \in C^1(A;\mathbb{R}^3)$, then $\int_A |\langle g, Df\rangle|$ is the total variation in A of the measure $\langle g, Df\rangle$; that is ,

$$\textstyle\int_A |\langle g, Df\rangle| = \sup\big\{\int f\nabla\cdot(\mu g)\,dx : \mu \in C_0^1(A),\ |\mu| \le 1\big\}.$$

Now set $g(x) = x/|x|^3$ and let h be any C^1 function. Define α by

$$\alpha(x) = h(x/|x|).$$

Then $(\nabla\cdot\alpha g) = 0$ in $\mathbb{R}^3 - \{0\}$, so from [Giusti 1977, 2.14] we have

$$\begin{aligned}\int_{Q\cap(B_r-\bar B_\rho)} \alpha\langle g, Df\rangle &= \int_{\delta^* B_r} \alpha f^-\Big\langle g, \frac{x}{|x|}\Big\rangle\,dH_2 - \int_{\delta^* B_\rho} \alpha f^+\Big\langle g, \frac{x}{|x|}\Big\rangle\,dH_2 \\ &\qquad + \int_{\delta(B_r-\bar B_\rho)} \alpha f^-\langle g, \vec n\rangle\,dH_2 \\ &= r^{-2}\int_{\delta^* B_r} \alpha f^-\,dH_2 - \rho^{-2}\int_{\delta^* B_\rho} \alpha f^+\,dH_2 \\ &= \int_{\delta^* B_1} h(x)\big(f(rx) - f(\rho x)\big)\,dH_2,\end{aligned}$$

where $\vec n$ is the outward unit normal of $\delta(B_r - \bar B_\rho)$. (Recall that $\langle x, \vec n\rangle = 0$ anywhere on δK.)

Next we restrict h so that $|h(x)| \le 1$ and hence $|\alpha(x)| \le 1$. By the definition of $\int_A |\langle g, Df\rangle|$, we have

$$\int_{\delta^* B_1} h(x)[f^-(rx) - f^+(\rho x)]\,dH_2 \le \int_{Q\cap(B_r-\bar B_\rho)} |\langle g, Df\rangle|$$

for any function h such that h is C^1 and $|h| \le 1$.

Now for almost all $\rho < r$ we have $\int_{\delta^* B_\rho} |Df| = 0$ and $f^+ = f^- = f$, by [Giusti 1977, Remark 2.13], so that

$$\int_{\delta^* B_1} h(x)[f^-(rx) - f^-(\rho x)]\,dH_2 \le \int_{Q\cap(B_r-B_\rho)} |\langle g, Df\rangle| \tag{6}$$

for almost all $\rho < r$. Thus if we take any $\rho < r$, we can choose a sequence $\{\rho_j\}$ such that $\rho_j \to \rho$, (6) holds for each ρ_j and $f^-(\rho_j x) \to f^-(\rho x)$ in $L^1(\partial B_1)$. Taking the limit as $j \to \infty$ we obtain (6) for every $\rho < r$. Now on taking the supremum over all h with $|h| \le 1$ we arrive at the desired inequality. □

The next three results correspond to [Giusti 1977, Lemmas 5.5, 5.6, 5.8].

Lemma 4.7. *Suppose $f \in \mathrm{BV}(B_R)$ and $\rho < R$. If $\{\rho_j\}$ is a sequence such that $\rho_j \le \rho$ and $\rho_j \to \rho$, then*

$$\lim_{j\to\infty} \zeta(f, B_{\rho_j}) = \zeta(f, B_\rho) \quad \textit{and} \quad \lim_{j\to\infty} \psi(f, B_{\rho_j}) = \psi(f, B_\rho).$$

Proof. Given $\varepsilon > 0$, by the definition of $\zeta(f, B_\rho)$ we can choose a function $g \in \mathrm{BV}(B_\rho)$ such that $\mathrm{supp}(g - f) \subset B_\rho$ and

$$F_{B_\rho}(g) \le \zeta(f, B_\rho) + \varepsilon.$$

For j large enough we have $\mathrm{supp}(g - f) \subset B_{\rho_j}$ and hence

$$F_{B_\rho}(g) \ge F_{B_{\rho_j}}(g) - \int_{\delta(B_\rho - B_{\rho_j})} \cos\gamma \cdot g\, dH_2 \ge \zeta(f, B_{\rho_j}) - \int_{\delta(B_\rho - B_{\rho_j})} \cos\gamma \cdot g\, dH_2.$$

Since $\varepsilon > 0$ is arbitrary,

$$\zeta(f, B_\rho) \ge \limsup_{j\to\infty} \zeta(f, B_{\rho_j}).$$

On the other hand, for $j \in N$, we can choose $g_j \in \mathrm{BV}(B_\rho)$ such that $g_j - f$ is supported in B_{ρ_j} and

$$\zeta(f, B_{\rho_j}) + \frac{1}{j} \ge F_{B_{\rho_j}}(g_j).$$

Also notice that

$$F_{B_{\rho_j}}(g_j) = F_{B_\rho}(g) - F_{B_\rho - B_{\rho_j}}(f),$$

so we have

$$\zeta(f, B_{\rho_j}) + \frac{1}{j} \ge \zeta(f, B_\rho) - F_{B_\rho - B_{\rho_j}}(f)$$

and therefore

$$\limsup_{j\to\infty} \zeta(f, B_{\rho_j}) \ge \zeta(f, B_\rho).$$

Thus we have proved the first equation. The second follows immediately from

$$\lim_{j\to\infty} F_{B_{\rho_j}}(f) = F_{B_\rho}(f). \qquad \square$$

Lemma 4.8. *Suppose $f, g \in \mathrm{BV}(B_R)$ and $\rho < R$. Then*

$$|\zeta(f, B_\rho) - \zeta(g, B_\rho)| \le \int_{\delta^* B_\rho} |f^- - g^-|\, dH_2.$$

Proof. Since the equality to be proved is symmetric in f and g, it is sufficient to show that

$$\zeta(f, B_\rho) - \zeta(g, B_\rho) \le \int_{\delta^* B_\rho} |f^- - g^-|\, dH_2.$$

Given $\varepsilon > 0$, we can choose $\varphi \in \mathrm{BV}(B_R)$ such that $\mathrm{supp}(\varphi - f) \subset B_\rho$ and

$$F_{B_\rho}(\varphi) \le \zeta(f, B_\rho) + \varepsilon.$$

Let $\{\rho_j\}$ be a sequence such that $\rho_j \le \rho$, $\rho_j \to \rho$,

$$\int_{\partial B_{\rho_j}} |Df| = \int_{\partial B_{\rho_j}} |Dg| = 0$$

and $\operatorname{supp}(f-\varphi) \subset B_{\rho_j}$. For every j, define

$$g_j = \begin{cases} \varphi & \text{in } B_{\rho_j}, \\ g & \text{in } B_R - \bar{B}_{\rho_j}. \end{cases}$$

Then, by [Giusti 1977, Proposition 2.8], we have $g_j \in \mathrm{BV}(B_R)$ and

$$\begin{aligned}
&\zeta(g, B_\rho)\\
&\quad \le F_{B_\rho}(g_j)\\
&\quad = \int_{Q\cap B_\rho} |Dg_j| - \int_{\delta B_\rho} \cos\gamma \cdot g_j \, dH_2\\
&\quad = \int_{Q\cap B_{\rho_j}} |D\varphi| + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |\varphi - g| \, dH_2 - \int_{\delta B_\rho} \cos\gamma \cdot g_j \, dH_2\\
&\quad \le \left(\int_{Q\cap B_\rho} |D\varphi| - \int_{\delta B_\rho} \cos\gamma \cdot \varphi dN_2\right) + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |\varphi - g| \, dH_2\\
&\qquad\qquad + \int_{\delta B_\rho} \cos\gamma(\varphi - g_j) \, dH_2\\
&\quad = F_{B_\rho}(\varphi) + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |f - g| \, dH_2 + \int_{\delta B_\rho} \cos\gamma(\varphi - g_j) \, dH_2\\
&\quad \le \zeta(f, B_\rho) + \varepsilon + \int_{Q\cap(B_\rho - B_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |f - g| \, dH_2 + \int_{\delta B_\rho} \cos\gamma(\varphi - g_j) \, dH_2.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary and the terms $\int_{Q\cap(B_\rho - B_{\rho_j})} |Dg|$ and $\int_{\delta B_\rho} \cos\gamma(\varphi - g_j)\, dH_2$ vanish as $j \to \infty$, the lemma follows by taking the limit. □

Lemma 4.9. *Suppose $f \in \mathrm{BV}(B_R)$ and $0 < \rho < r < R$. Then*

$$\left(\int_{Q_\rho^r} \left|\left\langle \frac{x}{|x|^3}, Df \right\rangle\right|\right)^2 \le \left(r^{-2} F_{B_r}(f) - \rho^{-2} F_{B_\rho}(f) + 2\int_\rho^r t^{-3} \psi(f, B_t)\, dt\right) \times 2\int_{Q_\rho^r} |x|^{-2} |Df|, \tag{7}$$

where $Q_\rho^r = Q \cap (B_r - B_\rho)$.

Proof. Suppose first that $f \in C^1(B_R)$ and then, for $0 < t < R$, define

$$f_t(x) = \begin{cases} f(x) & \text{for } t < |x| < R, \\ f(tx/|x|) & \text{for } |x| < t. \end{cases}$$

Then we have

$$\int_{Q\cap B_t} |Df_t|\,dx = \frac{t}{2}\int_{\delta^* B_t} |Df|\left(1-\frac{\langle x, Df\rangle^2}{|x|^2|Df|^2}\right)^{1/2} dH_2,$$

$$\int_{\delta B_t} \cos\gamma\cdot f_t\,dH_2 = \frac{t}{2}\int_{\delta(\partial B_t)} \cos\gamma\cdot f\,dH_1$$

which is to say

$$F_{B_t}(f_t) = \frac{t}{2}\left(\int_{\delta^* B_t} |Df|\left(1-\frac{\langle x, Df\rangle^2}{|x|^2|Df|^2}\right)^{1/2} dH_2 - \int_{\delta(\partial B_t)} \cos\gamma\cdot f dH_1\right),$$

and hence

$$\begin{aligned}\zeta(f, B_t) &= F_{B_t}(f) - \psi(f, B_t)\\ &\le F_{B_t}(f_t)\\ &\le \frac{t}{2}\left(\int_{\delta^* B_t} |Df|\,dH_2 - \frac{1}{2}\int_{\delta^* B_t} \frac{\langle x, Df\rangle^2}{|x|^2|Df|}\,dH_2 - \int_{\delta(\partial B_t)} \cos\gamma\cdot f dH_1\right)\end{aligned}$$

and

$$\tfrac{1}{2}t^{-2}\int_{\delta^* B_t} \frac{\langle x, Df\rangle^2}{|x|^2|Df|}\,dH_2 \le \frac{d}{dt}\big(t^{-2}F_{B_t}(f)\big) + 2t^{-3}\psi(f, B_t).$$

Now integrating with respect to t between ρ and r, we have

$$\frac{1}{2}\int_{Q\cap(B_r-B_\rho)} \frac{\langle x, Df\rangle^2}{|x|^4|Df|}\,dx \le r^{-2}F_{B_r}(f) - \rho^{-2}F_{B_\rho}(f) + 2\int_\rho^r t^{-3}\psi(f, B_t)\,dt.$$

On the other hand, from the Schwarz inequality we have

$$\left(\int_{Q\cap(B_r-B_\rho)} \left|\left\langle \frac{x}{|x|^3}, Df\right\rangle\right| dx\right)^2 \le \int_{Q\cap(B_r-B_\rho)} \frac{|Df|}{|x|^2}\,dx \int_{Q\cap(B_r-B_\rho)} \frac{\langle x, Df\rangle^2}{|x|^4|Df|} dx,$$

so Equation (7) holds for $f\in C^1(B_R)$.

Now suppose that $f\in \mathrm{BV}(B_R)$. By [Giusti 1977, Remarks 2.12 and 2.13] we can approximate f by C^1 functions f_k such that

$$\int_{Q\cap B_t} |Df_k| \to \int_{Q\cap B_t} |Df| \quad\text{and}\quad \int_{\partial(Q\cap B_t)} |f-f_k|\,dH_2 \to 0$$

for almost all t. If we write (7) for f_k and observe that, by Lemma 4.8, $\psi(f_k, B_t)\to \psi(f, B_t)$, we see that (7) holds for $f\in\mathrm{BV}(B_R)$ and almost all ρ, r. Finally we obtain (7) for every ρ and r by approximating with increasing sequences $\{\rho_j\}\to\rho$ and $\{r_j\}\to r$ for which (7) holds. □

Remark 4.10. By approximating at the final step with sequences decreasing to r and ρ we obtain (7) with $\overline{B}_r$, $\overline{B}_\rho$ instead of B_r and B_ρ.

Remark 4.11. From (7) it follows that, for every $\rho < r$,

$$\rho^{-2} F_{B_\rho}(f) \le r^{-2} F_{B_r}(f) + 2 \int_\rho^r t^{-3} \psi(f, B_t)\, dt. \tag{8}$$

In particular, $\psi(f, B_r) = 0$ implies $\rho^{-2} F_{B_\rho}(f) \le r^{-2} F_{B_r}(f)$. Hence $\rho^{-2} F_{B_\rho}$ is an increasing function of ρ.

The next result is adapted from [Giusti 1977, Lemma 9.1].

Lemma 4.12. *Let K be an open set in $\mathbb{R}^3$, and let $\{E_j\}$ be a sequence of Cacciop-poli sets such that*

$$\lim_{j\to\infty} \psi(E_j, A) = 0 \quad \text{for all } A \Subset K.$$

Suppose that there exists a set E such that

$$\varphi_{E_j} \to \varphi_E \text{ in } L^1_{\mathrm{loc}}(K).$$

Then E is a minimal set in K, that is,

$$\psi(E, A) = 0 \quad \text{for all } A \Subset K.$$

Moreover, if $L \Subset K$ is such that $\int_{\delta^* L} |D\varphi_E| = 0$, we have

$$\lim_{j\to\infty} F_L(E_j) = F_L(E).$$

Proof. Let $A \Subset K$. We may suppose that ∂A is smooth, so that for every j,

$$\begin{aligned} F_A(E_j) = \zeta(E_j, A) + \psi(E_j, A) &\le F_A(E_j - A^\circ) + \psi(E_j, A) \\ &\le H_2((\delta^* A) \cap E_j) + \psi(E_j, A) \le H_2(\delta^* A) + \psi(E_j, A). \end{aligned}$$

By [Giusti 1977, Theorem 9.1],

$$F_A(E) \le \liminf_{j\to\infty} F_A(E_j) \le H_2(\delta^* A).$$

For $t > 0$, set

$$A_t = \{x \in K : \mathrm{dist}(x, A) < t\}.$$

We have

$$\lim_{j\to\infty} \int_{Q\cap A_t} |\varphi_{E_j} - \varphi_E|\, dx = 0$$

so there exists a subsequence $\{E_{k_j}\}$ such that, for almost every t close to 0,

$$\lim_{j\to\infty} \int_{\delta^* A_t} |\varphi_{E_{k_j}} - \varphi_E|\, dH_2 = 0.$$

From Lemma 4.8 we have $\lim_{j\to\infty}\zeta(E_{k_j}, A_t)=\zeta(E, A_t)$ for these values of t, and therefore from [Giusti 1977, Theorem 1.9] we get

$$\psi(E, A_t)=0.$$

Now let $L \Subset K$ be such that

$$\int_{\delta^* L}|D\varphi_E|=0,$$

and let A be a smooth open set with $L \Subset A \Subset K$. Let $\{F_j\}$ be any subsequence of $\{E_j\}$. Reasoning as above we can find a set A_t and a subsequence $\{F_{k_j}\}$ such that

$$\lim_{j\to\infty}\zeta(F_{k_j}, A_t)=\zeta(E, A_t).$$

Since $\lim_{j\to\infty}\psi(F_{k_j}, A_t)=\psi(E, A_t)=0$ we have

$$\lim_{j\to\infty}F_{A_t}(F_{k_j})=F_{A_t}(E),$$

and hence from [Giusti 1977, Proposition 1.13]:

$$\lim_{j\to\infty}F_L(F_{k_j})=F_L(E). \qquad \square$$

Proof of Theorem 4.5. We first prove the conclusion for every sequence $\{t_j\}$ tending to zero; that is, for every sequence $\{t_j\}$ tending to zero there exists a subsequence $\{s_j\}$ such that E_{s_j} converges locally in $\mathbb{R}^3$ to a set C. Moreover C is a minimal cone.

Then we will prove that this limit cone C does not depend on the specific sequence $\{s_j\}$, and hence is the limit for E_t.

Suppose $t_j\to 0$. We show that for every $R>0$ there exists a subsequence $\{\sigma_j\}$ such that E_{σ_j} converges in B_R. We have

$$F_{B_R}(E_t)=\frac{1}{t^2}F_{B_{tR}}(E),$$

so choosing t sufficiently small (so that $tR<1$ and $t<1$), we guarantee that E_t minimizes

$$F_{B_R}(E_t)+t\int_{Q\cap B_R}H\varphi_{E_t}\le H_2(\delta^* B_R)+tM\cdot H_3(Q\cap B_R)=4\alpha R^2+tM\cdot\tfrac{4}{3}\alpha R^3,$$

where $|H|<M$.

Hence, by [Giusti 1977, Theorem 1.19] on compactness, a subsequence E_{σ_j} will converge to a set C_R in B_R. Taking a sequence $R_i\to\infty$ we obtain, by means of a diagonal process, a set $C\subset\mathbb{R}^3$ and a sequence $\{s_j\}$ such that $E_{s_j}\to C$ locally. Now applying Lemma 4.12, we see that C is minimal and it remains only to show that C is a cone.

Also by Lemma 4.12 we have

$$F_{B_R}(E_{s_j}) \to F_{B_R}(C) \quad \text{for almost all } R > 0.$$

Hence if we define

$$p(t) = \frac{1}{t^2} F_{B_t}(E) + \frac{16}{3}\alpha t M = F_{B_1}(E_t) + \frac{16}{3}\alpha t M,$$

we have, for almost all $R > 0$,

$$\lim_{j\to\infty} p(s_j R) = \frac{1}{R^2} F_{B_R}(C).$$

Moreover,

$$\psi(E, B_t) = F_{B_t}(E) - \zeta(E, B_t) = F_{B_t}(E) + \int_{Q\cap B_t} H\varphi_E - \int_{Q\cap B_t} H\varphi_E - \zeta(E, B_t).$$

By the definition of ζ, we can say for all $\varepsilon > 0$, there is a Caccioppoli set E_ε satisfying $\operatorname{supp}(\varphi_{E_\varepsilon} - \varphi_E) \subset B_t$ and $0 < F_{B_t}(E_\varepsilon) - \zeta(E, B_t) < \varepsilon$. Hence,

$$\begin{aligned}\psi(E, B_t) &\le F_{B_t}(E_\varepsilon) + \int_{Q\cap B_t} H\varphi_{E_\varepsilon} - \int_{Q\cap B_t} H\varphi_E - \zeta(E, B_t)\\ &\le \int_{Q\cap B_t} H(\varphi_{E_\varepsilon} - \varphi_E) + \varepsilon \le \frac{8}{3}\alpha M t^3 + \varepsilon.\end{aligned}$$

Then, by (8) and letting $\varepsilon \to 0$, we get

$$\rho^{-2} F_{B_\rho}(E) + \tfrac{16}{3}\alpha M\rho \le r^{-2} F_{B_r}(E) + \tfrac{16}{3}\alpha M r,$$

so that $p(t)$ is increasing in t.

If $\rho < R$, there exists for every j an $m_j > 0$ such that $s_j\rho > s_{j+m_j} R$. Thus $p(s_{j+m_j} R) \le p(s_j\rho) \le p(s_j R)$, so that

$$\lim_{j\to\infty} p(s_j\rho) = \lim_{j\to\infty} p(s_j R) = \frac{1}{R^2} F_{B_R}(C).$$

Thus we have proved that

$$\frac{1}{\rho^2} F_{B_\rho}(C)$$

is independent of ρ and so, from Lemmas 4.6, 4.8 and 4.9, we conclude that

$$\int_{\delta^* B_1} \left|\varphi_C(\rho x) - \varphi_C(r x)\right| dH_2 = 0$$

for almost all $\rho, r > 0$. Hence the set C differs only on a set of measure zero from a cone with vertex at the origin.

Now suppose we have two sequences $\{s_j\}$ and $\{s'_j\}$ that give us two minimal cones in the limit, C and C'. Recall that $p(t)$ is increasing in t. Therefore $R^{-2}F_{B_R}(C)$ is independent of both R and C; that is, for almost all R,

$$\lim_{t\to 0} p(t) = \frac{1}{R^2}F_{B_R}(C) = \frac{1}{R^2}F_{B_R}(C').$$

As on the previous page, we apply Lemmas 4.6, 4.8 and 4.9 to the set E and get, for $s_j > s'_k$,

$$\begin{aligned}
&\left(\int_{\delta^* B_1} |\varphi_E(s_j x) - \varphi_E(s'_k x)|\, dH_2\right)^2 \\
&\quad\le 2\int_{Q\cap(B_{s_j}-B_{s'_k})} \frac{|D\varphi_E|}{|x|^2}\left(s_j^{-2}F_{B_{s_j}}(E) - s_k'^{-2}F_{B_{s'_k}}(E) + 2\int_{s'_k}^{s_j} t^{-3}\psi(\varphi_E, B_t)\, dt\right) \\
&\quad\le 2\int_{Q\cap(B_{s_j}-B_{s'_k})} \frac{|D\varphi_E|}{|x|^2}\left(F_{B_1}(E_{s_j}) - F_{B_1}(E_{s'_k}) + \frac{16}{3}\alpha M(s_j - s'_k)\right).
\end{aligned}$$

Suppose $j, k \to \infty$. We have

$$\int_{\delta^* B_1} |\varphi_E(s_j x) - \varphi_E(s'_k x)|\, dH_2 \to 0;$$

that is,

$$\int_{\delta^* B_1} |\varphi_{E_{s_j}}(x) - \varphi_{E_{s'_k}}(x)|\, dH_2 \to 0,$$

which implies

$$\int_{\delta^* B_1} |\varphi_C(x) - \varphi_{C'}(x)|\, dH_2 = 0.$$

Hence C and C' are almost equal, completing the proof of the theorem. □

5. Continuity at a reentrant wedge

We now prove the well-definedness of a new boundary condition and we subdivide the reentrant wedge domain along the new boundary to get two protruding wedge domains, which enables us to apply Concus–Finn conjecture to prove the main theorem.

First we introduce a uniformity lemma for P, N:

Lemma 5.1 [Finn 1986, Lemma 7.1]. *Suppose we have a wedge domain Ω_∞ with $2\alpha \ge \pi$. A sequence of functions $\{f_j\}$ converges locally to a generalized solution of the corresponding minimal surface problem, f. Denote their subgraphs as V_j and V, respectively. Then for some point $(x_0, y_0) \in \Omega$, there exists $r_0 > 0$ and $C > 0$ not depending on j such that for all $t \in \mathbb{R}$, the following is true*:

If $\left|V'_{j,r}(x_0, y_0, t)\right| > 0$ *and* $\left|V_{j,r}(x_0, y_0, t)\right| > 0$ *for all* $r > 0$, *then*

$$|V_{j,r}(x_0, y_0, t)| \geq Cr^3 \quad \textit{and} \quad |V'_{j,r}(x_0, y_0, t)| \geq Cr^3$$

for all $0 < r \leq r_0$, *where*

$$C_r(x_0, y_0, t_0) = \left\{(x, y, t) \in \mathbb{R}^3 : \mathrm{dist}((x, y),\ (x_0, y_0)) < r,\ |t - t_0| < r\right\},$$
$$V'_{j,r}(x_0, y_0, t) = C_r(x_0, y_0, t) - V_j.$$

Lemma 5.2 [Chen et al. 1998, Lemma 6.1]. *If f is a classical minimal surface over $\mathbb{R}^2$ and is linear in every radial direction* (*that is, its restriction to each radial direction is a linear function*), *then f is either a plane or a helicoid.*

Lemma 5.3 [Chen et al. 1998, Lemma 6.2]. *If G is a nonempty domain, the only possibility that there is a classical minimal surface f defined on G which is linear in every radial direction is that $G = \Omega_\infty$ and $(\gamma_1, \gamma_2) \in \mathcal{R}$. Moreover, f is a plane.*

Proof. It follows from Lemma 5.2 that f is a plane or a helicoid defined on G, say, $f = a \tan^{-1}(y/x)$ for some constant a.

If P or $N \neq \varnothing$, then either $\partial P \cap \partial G \neq \varnothing$ or $\partial N \cap \partial G \neq \varnothing$. Without loss of generality, we may assume that $L = \partial P \cap \partial G \neq \varnothing$. It follows from Corollary 3.4 that L is either a line or a half-line in Ω_∞.

Let f, defined over G, be a helicoid or a plane and take $x_0 \in L$ distinct from O. Then $f \in C^1$ in $\overline{G} \cap B_\rho(x_0)$, where $B_\rho(x_0)$ is a small open disk with radius ρ and center x_0 such that the disk belongs to the sector Ω_∞. The subgraph F of f can not be a minimal surface in a small neighborhood of $(x_0, f(x_0))$ since F will violate the inequality (5) if K is small enough so that $K \cap (\partial\Omega_\infty \times \mathbb{R}) = \varnothing$; hence $P = N = \varnothing$. To see that F violates (5) we will construct a "better" comparison set as follows.

Let T_0 be the tangential plane to the surface $S : z = f(x)$, $x \in G$, at the point $(x_0, f(x_0))$. Take a plane E_0 parallel to the edge $T_0 \cap (L \times \mathbb{R})$ which intersects the vertical plane $L \times \mathbb{R}$ and the plane T_0 in a distance h from the edge $T_0 \cap (L \times \mathbb{R})$.

Choose the plane that lies above of $T_0 \cap (L \times \mathbb{R})$. Take two further planes perpendicular to the edge $T_0 \cap (L \times \mathbb{R})$ and with distances $\pm a$ from $(x_0, f(x_0))$.

This construction defines a prismatic set W. Set $V = F \cup W$ and $h = a^2$, and let ω be the opening angle of the edge $T_0 \cap (L \times \mathbb{R})$ as indicated in the figure. Then

$$\int_{(\Omega_\infty \times \mathbb{R}) \cap K} |D\varphi_F| - \int_{(\Omega_\infty \times \mathbb{R}) \cap K} |D\varphi_V| = 4a^3(1 - \sin \tfrac{1}{2}\omega) + o(a^3) \quad \text{as } a \to 0.$$

Thus $G = \Omega_\infty$ and f satisfies equations (4). Since a helicoid $f = a \tan^{-1}(y/x)$ cannot make constant contact angles on Σ_1 and Σ_2, the only possibility is that f is a plane. The conclusion that $\pi + (\gamma_2 - \gamma_1) < \theta < \pi + (\gamma_2 + \gamma_1)$ then follows. $\square$

Lemma 5.4. *Suppose $f \in C^2(\Omega_\infty)$ solves the problem*

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \quad && \text{in } \Omega_\infty, \\ Tf \cdot \nu_1 &= \cos\gamma_1 && \text{on } \Sigma_1, \\ Tf \cdot \nu_2 &= \cos\gamma_2 && \text{on } \Sigma_2, \end{aligned}$$

where $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$, $\kappa \geq 0$ and f is bounded on Ω_∞. Then there is a radial line L and some curve $\mathscr{C}$, such that $\mathscr{C}$ is tangent to L when approaching the vertex and

$$\lim_{\substack{(x,y)\in\mathscr{C} \\ (x,y)\to(0,0)}} Tf(x, y) = \nu_L,$$

where ν_L is the unit normal vector of L pointing toward P.

Proof. Denote by E the subgraph of f, and set

$$E_t = \{x \in \mathbb{R}^3 : tx \in E\}.$$

By Theorem 4.5 we see that as $t \to 0$, E_t converges locally in $\mathbb{R}^3$ to a minimal cone C.

It is well known that the only minimal cone in $\mathbb{R}^3$ is a plane. Thus C is a plane.

By the assumption that $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$ we see that C has to be a vertical plane passing through z-axis. Let the line L be the projection of C onto the xy-plane.

To complete the proof of the lemma we just need to show the following:

Claim. There is a level curve $\mathscr{C}$ of $f(0, 0)$ that reaches $(0, 0)$ and is tangent to L.

Proof. The level set of $f(0, 0)$ must lie within a cusp region with $(0, 0)$ as its tip. For assume to the contrary that there exists a sequence $\{(x_k, y_k)\}$ in the level set with the property that

$$(x^*, y^*) := \lim_{k\to\infty} \left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}} \right) \notin L. \tag{9}$$

Take the blow-up sequence

$$f_k(x, y) = \frac{1}{\varepsilon_k} \big(f(\varepsilon_k x, \varepsilon_k y) - f(0, 0) \big),$$

where $\varepsilon_k = \sqrt{x_k^2 + y_k^2}$. Then

$$f_k\left(\frac{x_k}{\varepsilon_k}, \frac{y_k}{\varepsilon_k} \right) = 0 \quad \text{for all } k.$$

By Lemma 5.1, then, the limit minimal cone must go through $(x^*, y^*, 0)$, which is impossible.

Next we check that, in some neighborhood of the origin, there is no point of the level set of $(0, 0)$ where the gradient of f vanishes. For suppose to the contrary that there exists a sequence $\{(x_k, y_k)\} \to (0, 0)$ of points in the level set satisfying

$$\nabla f(x_k, y_k) = 0 \quad \text{for all } k.$$

(Recall that f is C^2 in Ω_∞ and satisfies the problem in Lemma 5.4.) Construct another blow-up sequence as above and let

$$f_k(x, y) = \frac{1}{\varepsilon_k}\big(f(\varepsilon_k x, \varepsilon_k y) - f(0, 0)\big),$$

where $\varepsilon_k = \sqrt{x_k^2 + y_k^2}$. Then $\nabla f_k(x_k/\varepsilon_k, y_k/\varepsilon_k) = \nabla f(x_k, y_k) = 0$ for all k. In the notation of (9), we have

$$(x^*, y^*) = \lim_{k\to\infty}\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}\right) \in L;$$

therefore [Massari and Pepe 1975, Theorem 3] yields

$$\vec{\nu}_k\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, 0\right) \to \vec{\nu}_C(x^*, y^*, 0) = \vec{\nu}_L,$$

contradicting the equality

$$\vec{\nu}_k\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, 0\right) = \vec{\nu}_f(x_k, y_k, 0) = (0, 0, 1).$$

Thus we see that the level set of $f(0, 0)$ is locally a union of *unbranched* level curves, which do not stop at any interior point of Ω_∞; moreover $(0, 0)$ is an accumulation point of this level set. To conclude the proof of the claim, assume to the contrary that no one level curve approaches $(0, 0)$; in other words, there is a sequence of distinct level curves approaching the vertex. On each of them, choose a point (x_k, y_k) nearest to $(0, 0)$. Then $\vec{\nu}_f(x_k, y_k)\cdot\vec{\nu}_L \to 0$, which again contradicts Massari's theorem, which says that $\vec{\nu}_f(x_k, y_k)\cdot\vec{\nu}_L \to 1$. □

Now we exclude continuity from $\mathcal{D}$-regions for reentrant corner domains, assuming Conjecture 2.4.

Theorem 5.5. *Let f be a bounded solution to* (1) *satisfying* (3), *with* $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$ *and $\alpha > \pi/2$. Under the assumption that Conjecture 2.4 is true, f must have discontinuous radial limits at O.*

Proof. Assume to the contrary that $\gamma_1, \gamma_2 \notin \overline{\mathcal{R}}$ and f has continuous radial limits at $(0, 0)$.

Denote by Σ_1', Σ_2' the lines extending Σ_1', Σ_2'. Consider, for $t \to 0$, the blow-up functions

$$f_t(x, y) = \frac{1}{t}\big(f(tx, ty) - f(0, 0)\big)$$

which converge locally to a generalized solution $v(x, y)$ to the corresponding minimal surface problem defined in Ω_∞.

By Lemma 5.3, the G region of $v(x, y)$ must be empty. Hence v defines a vertical plane, whose projection L onto the xy-plane equals $\partial P \cap \partial N$. By the conclusion of the previous section, L must be either a half-line or a whole line passing through the origin.

Assume L is a half-line; we claim that it must lie between Σ_1' and Σ_2', inclusive (Figure 8, left). Otherwise either the P or the N region will cover a subdomain which is again a reentrant wedge. This leads to a contradiction with Lemma 5.1 on uniformity.

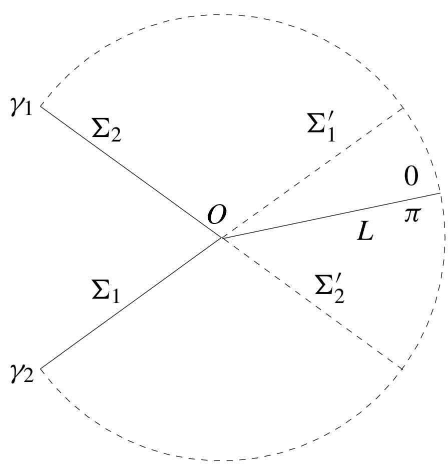
By Lemma 5.4 we know that along any radial half-line approaching the origin, $\lim Tf \cdot \nu$ is well defined. Therefore we can split the domain Ω_∞ from L, and get two subproblems:

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda && \text{between } \Sigma_1 \text{ and } L, \\ Tf \cdot \nu &= \cos \gamma_1 && \text{on } \Sigma_1, \\ Tf \cdot \nu &= \pm 1 && \text{on } L \end{aligned}$$

and

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda && \text{between } L \text{ and } \Sigma_2, \\ Tf \cdot \nu &= \mp 1 && \text{on } L, \\ Tf \cdot \nu &= \cos \gamma_2 && \text{on } \Sigma_2. \end{aligned}$$

Both of them admit a continuous solution.



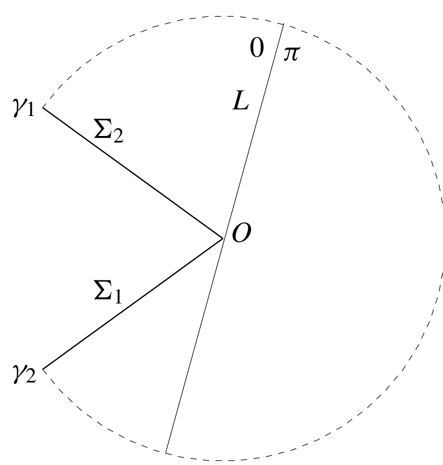


Figure 8. Proof of Theorem 5.5. Left: case where L is a half-line. Right: case where L is a whole line.

By the Concus–Finn conjecture, the angle between Σ_1 and L is either $\pi - \gamma_1$ or γ_1, while the angle between L and Σ_2 is correspondingly either γ_2 or $\pi - \gamma_2$. This leads to $|\gamma_1 - \gamma_2| = 2\alpha - \pi$, which is impossible.

Now assume instead that L is a whole line passing through the origin. Then the region between Σ_1' and Σ_2' must lie on one side of L (see Figure 8, right). We again split into two subproblems:

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \text{between } \Sigma_1 \text{ and } L_1, \\ Tf \cdot \nu &= \cos\gamma_1 \quad \text{on } \Sigma_1, \\ Tf \cdot \nu &= \pm 1 \qquad \text{on } L_1 \end{aligned}$$

and

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \text{between } L_2 \text{ and } \Sigma_2, \\ Tf \cdot \nu &= \pm 1 \qquad\qquad\qquad \text{on } L_2, \\ Tf \cdot \nu &= \cos\gamma_2 \qquad\qquad \text{on } \Sigma_2. \end{aligned}$$

Using a similar reasoning as in the previous case, we see that

$$\gamma_1 + \gamma_2 = 2\alpha - \pi, \ \text{ or } \gamma_1 + \gamma_2 = 3\pi - 2\alpha,$$

which is again impossible. □

6. Examples

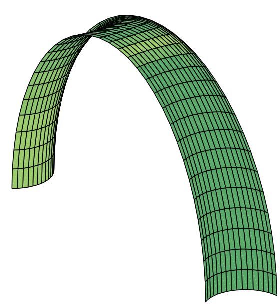
We now construct explicit examples for some of the discontinuous cases given in Theorem 2.5. The notation (D), (I), (DI) and (ID) is defined in Section 3.

Example 6.1. For any $(\gamma_1, \gamma_2) \in \mathcal{R}(2\alpha) \cup \mathcal{D}_2^+ \cup \mathcal{D}_1^-$, we have case (D).

Referring to Figures 9–11, consider the region $\Omega_1 \in \Omega$ bounded by two (close) parallel sides Σ_1, $\tilde{\Sigma}_1$ and two circular arcs C, $\tilde{C}$ which are symmetric about a line orthogonal to Σ_1 $\tilde{\Sigma}_1$.

The arcs are constrained to meet Σ_1 in the angle γ_1, and $\tilde{\Sigma}_1$ in the (fixed) angle $\tilde{\gamma}_1$, which is chosen in the interval $(0, \pi - \gamma_1)$. Thus each arc is part of a circle, with radius ε/a_1, and with $a_1 = \cos\gamma_1 + \cos\tilde{\gamma}_1$.

We distinguish two cases, according as $\gamma_1 < 2\alpha - \pi$ or $\gamma_1 > 2\alpha - \pi$; they are indicated in Figures 9 and 10, respectively. In the former case, we can position a disk of radius δ (independent of ε) in Ω and tangent to Σ_2 at O. Following a construction initiated by Korevaar [1980], we next construct an upper half of the inner side of a torus containing the (horizontal) arcs C, $\tilde{C}$ (see figure on the right). This has the appearance of a Japanese footbridge. It can be represented as a function $g(x, y)$ over Ω_1, with



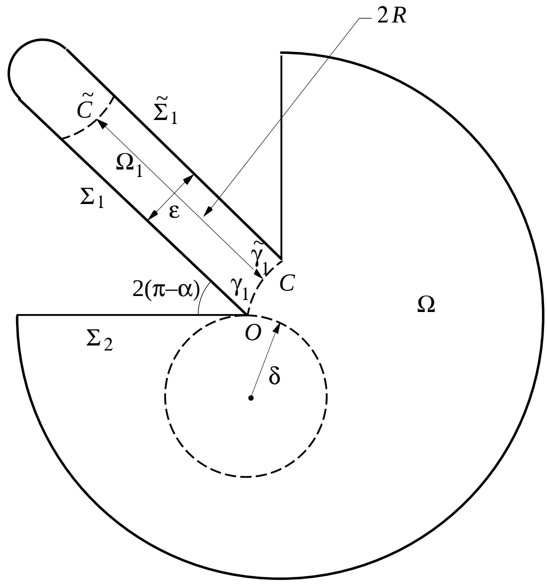


Figure 9. Construction of example of case (D), with $(\gamma_1, \gamma_2) \in \mathcal{R} \cup \mathcal{D}_2^+ \cup \mathcal{D}_1^-$ and $\gamma_1 < 2\alpha - \pi$.

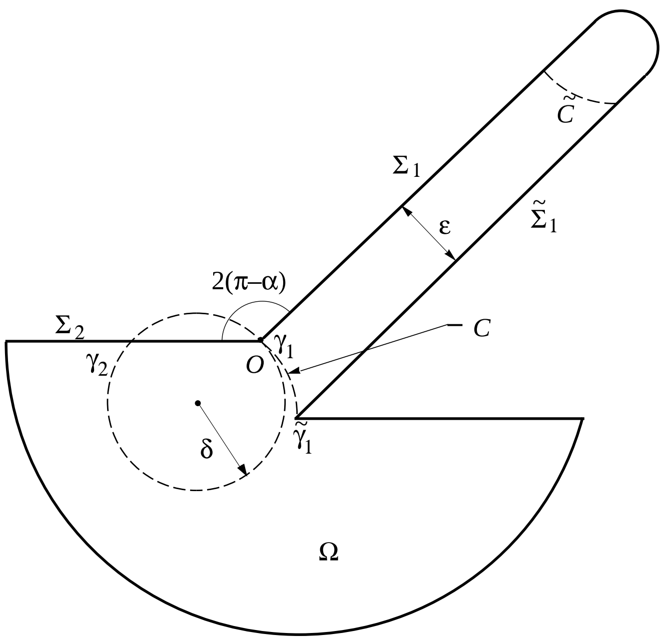


Figure 10. Construction of example of case (D), with $(\gamma_1, \gamma_2) \in \mathcal{R} \cup \mathcal{D}_2^+ \cup \mathcal{D}_1^-$ and $\gamma_1 > 2\alpha - \pi$.

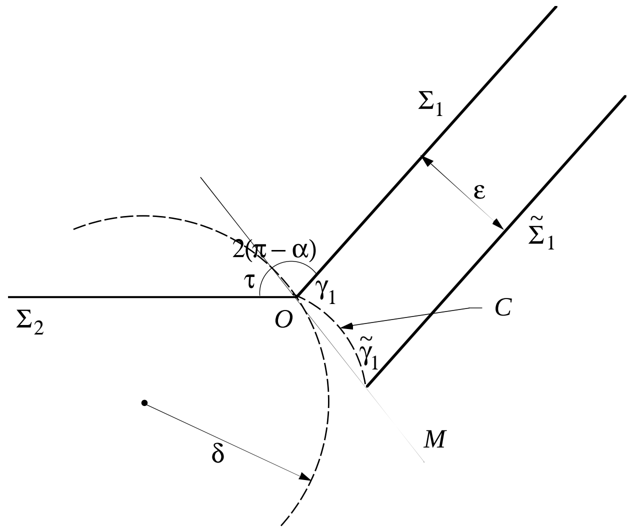


Figure 11. Detail of construction.

$\nu \cdot Tg = -1$ on C and $\tilde{C}$, ν being unit exterior normal.On Σ_1 and $\tilde{\Sigma}_1$, the torus meets vertical walls over the sides in the constant angles γ_1, $\tilde{\gamma}_1$, so $\nu \cdot Tg = \cos\gamma_1$ on Σ_1, $\nu \cdot Tg = \cos\tilde{\gamma}_1$ on $\tilde{\Sigma}_1$.

We extend γ_1, $\tilde{\gamma}_1$ smoothly to the remaining boundary of Ω. By theorems of Emmer [1973] and of Finn and Gerhardt [1977], a (unique) solution $f(x, y)$ of (1)+(3) exists in Ω, with $\nu \cdot Tf = \cos\gamma_1$ on Σ_1, $\nu \cdot Tf = \cos\tilde{\gamma}_1$ on $\tilde{\Sigma}_1$. On C, $\tilde{C}$ there holds $\nu \cdot Tf > -1$, since $|Tf| < 1$ for any function f.

We adjust f by an additive constant so that $\max g = (1/K)(a_1/\varepsilon - 1/R - \lambda)$, where R (the inner radius of the torus) is half the distance between C and $\tilde{C}$. The mean curvature of the torus is given by $\frac{1}{2}\nabla \cdot (Tg)$, and is minimized at the upper symmetry point, at which $\nabla \cdot (Tg) = (a_1/\varepsilon - 1/R)$. We thus find $\nabla \cdot (Tg) \geq \kappa g + \lambda$. By the comparison principle of Concus and Finn [1996, Theorem 5.1], we then obtain $f \geq g > \max g - (R + \varepsilon/a_1)$ in Ω_1, and thus $\lim_{\varepsilon\to 0} f = \infty$, uniformly over Ω_1, and in particular for any radial approach to O from within Ω_1. On the other hand, we find using [Concus and Finn 1996, Theorem 5.2] that $f < 2/(\kappa\delta) + \delta + \lambda/\kappa$ throughout the interior of the disk of radius δ, and hence also for any radial approach to O within that disk. We conclude that for small enough ε, the behavior at O must be either (D) or else (DI). However, (DI) is excluded by [Lancaster and Siegel 1996, Corollary 3].

If $\gamma_1 > 2\alpha - \pi$ the construction above does not work, because with decreasing ε the segment $\tilde{\Sigma}_1$ would enter the δ-disk. We start instead by constructing that disk to be tangent to C at O (Figure 10). We then have $\tau = \pi - 2\alpha + \gamma_1$. But

$\gamma_1 - \gamma_2 < 2\alpha - \pi$ by assumption, and thus $\tau < \gamma_2$. We now rotate the disk slightly about O, so that its circumference meets C at a positive angle, but still maintaining the condition $\tau < \gamma_2$ (see Figure 11). From [Concus and Finn 1996, Theorem 5.3] we now derive that again $f < 2/(\kappa\delta) + \delta + \lambda/\kappa$ in the intersection of the disk with Ω. It remains to narrow the Ω_1 region in a way ensuring that $\tilde{\Sigma}_1$ does not enter the δ-disk; but that again elevates the solution height unboundedly within that Ω_1.

We can do that simply by introducing a tangent line M to the δ-disk at O. For each choice of ε (tending to zero) we choose C to be the unique circular arc meeting Σ_1 at O in the angle γ_1 and meeting $\tilde{\Sigma}_1$ at the intersection of $\tilde{\Sigma}_1$ with M. The angle $\tilde{\gamma}_1$ will not remain constant in this construction, but the radius of C tends to zero with ε, and thus the comparison principle can again be applied to show that the solutions then uniformly to infinity throughout the domains $\Omega_1(\varepsilon)$. We are done.

Example 6.2. For any $(\gamma_1, \gamma_2) \in \mathcal{R}(2\alpha) \cup \mathcal{D}_1^+ \cup \mathcal{D}_2^-$, we are in case (I).

Use the same construction of Example 6.1, interchanging the sides Σ_1 and Σ_2.

Example 6.3. For any $(\gamma_1, \gamma_2) \in \mathcal{R}(2\alpha)$, we have a capillary surface continuous up to the vertex.

First we introduce a lemma on the general existence of a continuous solution of the capillary equation

$$\nabla \cdot Tf = \kappa f + \lambda \quad \text{in } \Omega, \tag{10}$$

which, for rotationally symmetric solutions, becomes

$$(r \sin\psi)_r = \kappa r u, \; u_r = \tan\psi, \tag{11}$$

where ψ is the inclination angle of the vertical surface section $u(r)$ with the r-axis.

Lemma 6.4 [Johnson and Perko 1968]. *Given $u_0 > 0$, there exists a unique $R_0(u_0) > 0$ and solution $u(r; u_0)$ of (11) in $0 < r < R_0$ such that*

(1) *u and ψ extend continuously to the closed interval $0 \le r \le R_0$,*

(2) *$u(0; u_0) = u_0$, $\psi(0; u_0) = 0$, $\psi(R_0; \psi) = \pi$, and*

(3) *the functions u and ψ both are monotone increasing on $0 \le r \le R_0$.*

This guarantees the existence of a convex rotation surface S with a single "bottom" point, that become vertical on a horizontal circular ring, and for which the height $u(x, y)$ is a solution of (10).

Lemma 6.5 [Finn 1986, pp. 67–69]. *Let $u_0 > 0$ and let Π be any nonvertical plane. There is a (unique) plane parallel to Π and tangent to the solution surface of Lemma 6.4.*

Proof. First move Π parallel to itself in a direction orthogonal to itself until it doesn't meet S. Then move Π parallel to itself toward S. If the first point of

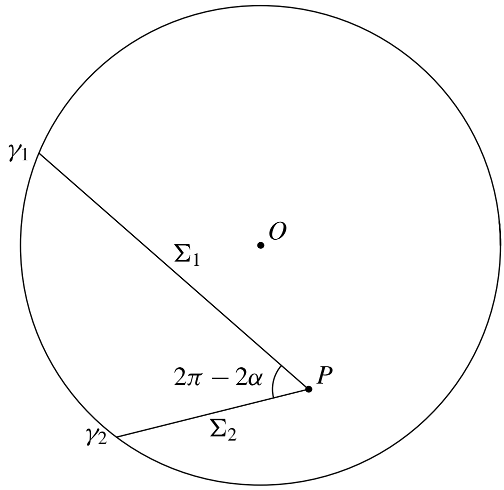


Figure 12. Projection of the continuous capillary surface on the xy-plane.

contact with S is an interior point of the surface, that gives us the plane we want. If the first contact point is with the ring on which S is vertical, continue moving Π until there is a last point of contact with S. That is then the plane we want. $\diamond$

So any nonvertical plane can be realized up to rigid motion preserving the normal, as a tangent plane of S. But [Concus and Finn 1996, Theorem 1] characterizes the interior of $\mathcal{R}$ as exactly the set of intersection angles (γ_1, γ_2) with the wedge planes, that arise from all possible nonvertical planes. Since we can get any nonvertical plane as a tangent to S, we can get any point of $\mathcal{R}$.

Now, by [Concus and Finn 1996, Theorem 1], starting from any point (γ_1, γ_2) in $\mathcal{R}$, a unique nonvertical plane is determined. There is a unique point p of the plane that lies above the vertex of the wedge. Position S to be tangent to the plane at p.

Then consider a neighborhood of p on S. Recall that $(\gamma_1, \gamma_2) \in \mathcal{R}$ guarantees that $2\alpha - \pi < \gamma_1 + \gamma_2 < 3\pi - 2\alpha$ and $|\gamma_1 - \gamma_2| < 2\alpha - \pi$. Therefore we can always find suitable positions for sides Σ_1, Σ_2; see Figure 12 for the projection of the surface onto the xy-plane. Now we have constructed a continuous solution to the equation that also satisfies the boundary condition for the given $(\gamma_1, \gamma_2) \in \mathcal{R}$. $\square$

Example 6.6. For any given $(\gamma_1, \gamma_2) \in \mathcal{D}_1^+(2\alpha)$, we are in case (DI).

On each of the regions Ω_i, $i = 1, 2$, consider two circles which is symmetric about the dashed line. Each circle has radius ε_i / a_i, $a_i = \cos\gamma_i + \cos\tilde{\gamma}_i$, where $0 < \tilde{\gamma}_i < \pi - \gamma_i$, and meets side Σ_i with an angle γ_i and side $\tilde{\Sigma}_i$ with an angle

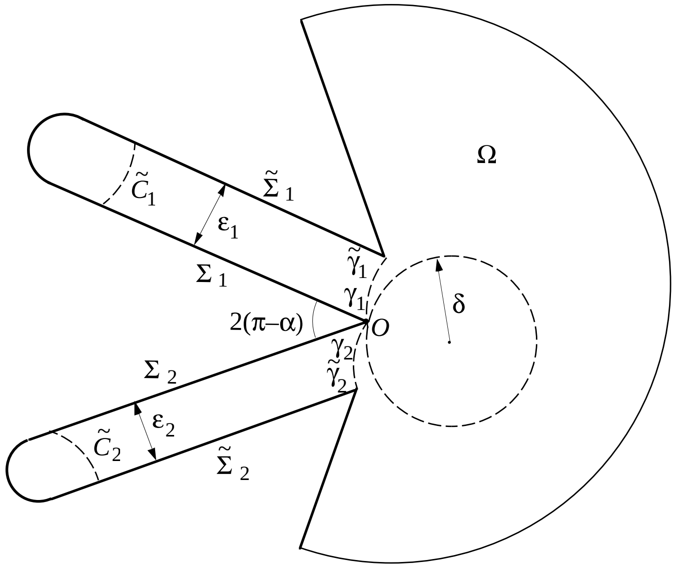


Figure 13. Construction for case (DI) (Example 6.6): $\mathscr{D}_1^+$ and O can be joined within Ω by a circle of fixed radius δ, independent of ε_1 and ε_2.

$\tilde{\gamma}_i$. Let g_1, g_2 both be the portion of a torus obtained by rotating one of the circular arcs above the xy-plane, like g_1 in Example 6.1.

Again it follows from the comparison principle that

$$f \geq g_i \geq \frac{1}{\kappa}\Big(\frac{a_1}{\varepsilon_i} - \frac{1}{R-\varepsilon_i}\Big) - R \quad \text{in } \Omega_i,\ i = 1, 2.$$

Since now $(\gamma_1, \gamma_2) \in \mathscr{D}_1^+$, which says that $\gamma_1 + \gamma_2 < 2\alpha - \pi$, we can construct a ball Ω_0 of radius δ which is also contained in Ω not overlapping with Ω_1 or Ω_2. Thus

$$f \leq \frac{2}{\kappa\delta} + \delta \quad \text{in } \Omega_0.$$

By making ε_i sufficiently small we can make

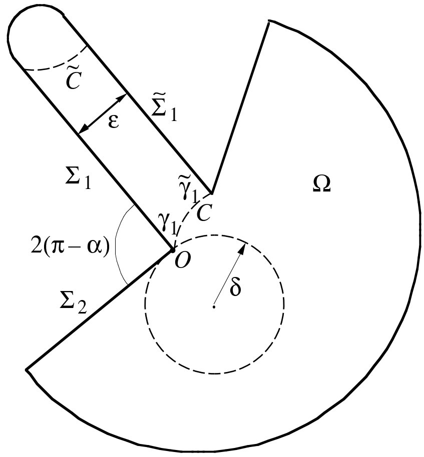
$$\frac{2}{\kappa\delta} + \delta \leq \frac{1}{\kappa}\Big(\frac{a_i}{\varepsilon_i} - \frac{1}{R-\varepsilon_i}\Big) - R$$

for $i = 1, 2$, forcing f to be in case (DI) at O.

Example 6.7. For any $(\gamma_1, \gamma_2) \in \mathscr{D}_1^-(2\alpha)$, we are in case (ID).

Similar to Example 6.6. Consider $-f$.

Example 6.8. For any $(\gamma_1, \gamma_2) \in \mathscr{D}_1^+(2\alpha)$, we are in case (D).



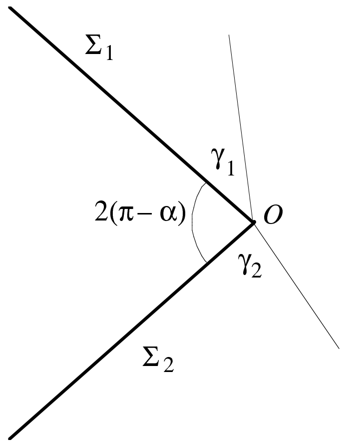


Figure 14. Left: Construction for case (D) in region $\mathscr{D}_1^+$. Right: Exclusion of central fan.

When (γ_1, γ_2) is in this region there are more possibilities of discontinuous solutions. So when constructing a (D) case there is an additional complexity: differ case (D) from case (DI).

We use a construction very similar to that given in Example 6.1.

Referring to Figure 14, left, consider the region Ω_1 bounded by two parallel straight sides, Σ_1, $\tilde{\Sigma}_1$, and two circular arcs which are symmetric about the dashed line.

First choose an angle θ_1 such that $\gamma_1 \le \theta_1 < 2\alpha - \pi$ and $\theta_1 + \gamma_2 > 2\alpha - \pi$. Then let each arc meet side Σ_1 with an angle θ_1 and side $\tilde{\Sigma}_1$ with an angle $\tilde{\theta}_1$. Here $\tilde{\theta}_1$is any value in $(0, \pi - \theta_1)$. Hence each arc is a part of a circle with radius ε_1/a_1, where $a_1 = \cos\theta_1 + \cos\tilde{\theta}_1$.

Region Ω_2 contains a disk B with some radius δ. Σ_2 is a part of the boundary $\partial\Omega_2$. We fix δ, and make Ω_1 and Ω_2 not overlap each other for any $\varepsilon_1 > 0$. Notice that when $(\gamma_1, \gamma_2) \in \mathscr{D}_1^+$ this is always possible.

Construct the torus in region Ω_1 and the lower hemisphere in region Ω_2 the same way as in Example 6.1. By making ε_1, and ε_2 sufficiently small we can force f to have a jump discontinuity at O.

Now the only thing to do before we can say for sure this is a (D) case is to eliminate the possibility of a (DI) case.

Lancaster and Siegel [1996, Theorem 2] proved that in a (DI) case, there exist fans of constant radial limits adjacent to Σ_1 and Σ_2. And the size of the fan on side Σ_i is no less than γ_i for $i = 1, 2$. This indicates a jump discontinuity of the radial limits happens in radial directions away from Σ_2 by an angle at least γ_2. On the other hand, by the above construction we know another jump discontinuity happens in radial directions away from Σ_1 by an angle at least θ_1. Since $\theta_1 + \gamma_2 > 2\alpha - \pi$, between the two discontinuities there is no enough space for a half plane constant radial limits to happen. Therefore a (DI) case is impossible.

Finally we proved that this construction gives us a (D) case.

Example 6.9. For any given $(\gamma_1, \gamma_2) \in \mathcal{D}_1^-(2\alpha)$, this is an example of case (I).

Similar configuration as in Example 6.8. Reverse the sides Σ_1 and Σ_2.

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DANZHU SHI
sdanzhu@Stanfordalumni.org