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SPECIAL ISSUE ON

THE MATHEMATICAL THEORY OF

CAPILLARITY

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PREFACE

The present issue of the Pacific Journal consists of invited research articles on mathematical problems of capillarity.

A *capillary surface* is the interface separating two fluids that lie adjacent to each other and do not mix. In conjunction with boundary conditions imposed by rigid "supporting walls", such interfaces can exhibit remarkable geometric properties and seemingly strange behavior, occasionally confounding intuition. The earliest known writing on the topic, due to Aristoteles, comains basic misconceptions that apparently went unchallenged for almost 2000 years, when Galileo addressed them in his *Discorsi*. Quantitative progress had to await the later discovery of the Calculus. The characterization of rise height in a circular cylindrical glass "capillary tube" dipped into a reservoir of liquid became a major scientific challenge of the eighteenth century, and was not achieved during that period. Initial breakthroughs came in 1805 and 1806 with insights of Thomas Young and Pierre-Simon de Laplace. Young professed to scorn the mathematical method but nevertheless introduced the mathematical concept of mean curvature that now underlies the entire theory. The framework for the theory achieved a clear definitive form with the 1830 paper of Gauss, who gained conceptual advantage by basing his study on an energy principle, in preference to the force balance conceived by his predecessors. Even so, the Gauss framework still leaves room for more inclusive discussion, as is pointed out in the initial article of the present volume.

During almost a century and a half following the Gauss paper interest for the topic declined, although the physical foundations continued to be studied on a molecular level by van der Waals and by his successors. With regard to global macroscopic problems, those studies led to no changes in the equations or boundary conditions, which present nonlinearities that initially defied analysis. Achievements during that time period were limited to some isolated striking insights due to Kelvin, Rayleigh and a few others, and some of the explicit unsolved problems of the time served as an impetus toward development of modern numerical methods. For the equations that apply in a gravity field, only a single nontrivial closed form solution has as yet been discovered, and classical linearizing procedures have provided little substantive information.

Inspired perhaps by the needs of space technology and of medicine, and utilizing new insights appearing in geometric measure theory, an explosion of activity has occurred during the past thirty-five years, in many directions. New problems have been attacked, new methods introduced, and discoveries of basically new nature have appeared. Already during the initial ten years of that explosion, enough substantive new material had appeared to justify an entire issue (88:2) of this journal devoted exclusively to capillarity theory and related problems. The present issue, about a quarter century later and somewhat more restrictive as to topics addressed, is intended as a sequel to that initial one. It will be apparent to those familiar with the earlier collection that some perspectives and also some participants have changed, but that the level of activity and the interest in the problems and in the methods have not lessened. Nor have the individual results and the new insights become less striking. That point is of course best made by the papers themselves, which present their own messages. Unfortunately space and time limitations have forced us to restrict the number of papers included here; the present collection should be regarded as an effort to make accessible in a single location a representative section of the (considerable) current activity, in the context of its varying methods and perspectives.

Much of the impetus for this volume developed at the First International Summer School on Capillarity, held at the Max-Planck-Institut für Mathematik in den Naturwissenschaften, in Leipzig, Germany, 2003. A number of the papers that follow had their origins in intense discussions held during that gathering, as did early scientific training for several students who have since continued to successful graduate degrees.

The reader will perceive that the fortress guarding the inner mysteries of capillarity is under heavy siege but has not yet succumbed. We trust that the materials joined together here will serve as a stimulus leading ultimately to completion of the conquest.

Robert Finn
V. S. Varadarajan

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COMPRESSIBLE FLUIDS IN A CAPILLARY TUBE

MARIA ATHANASSENAS AND ROBERT FINN

We study a mathematical model for a compressible liquid in a capillary tube. We establish necessary and sufficient conditions for existence and for uniqueness or near uniqueness of solutions, and we provide general height estimates for solutions, depending on the geometrical structure of the definition domain. We show that solutions exhibit discontinuous dependence properties in domains with corners, analogous to those that are known for the classical capillarity equation.

1. Introduction

The mathematical theory of capillary surfaces was founded by Young [1805], by Laplace [1805–1806] and by Gauss [1830]. The profound investigations of these authors led to the equation

$$\operatorname{div} Tu = \kappa u + \lambda, \qquad Tu = \frac{Du}{\sqrt{1+|Du|^2}} \tag{1}$$

for the rise height $u(x, y)$ in a vertical cylindrical capillary tube of general section $\Omega \subset \mathbb{R}^2$. Here $\kappa = \rho g/\sigma$, with ρ the density change across the surface, g the gravitational acceleration, σ the interfacial tension, and λ a constant to be determined by an eventual volume constraint. On the boundary $\Sigma = \partial\Omega$, and with ν the outer unit normal to Σ, the condition

$$\nu \cdot Tu = \cos\gamma \tag{2}$$

is imposed, which asserts that the free surface S meets the bounding cylinder surface in the (prescribed) angle γ. These relations were established by Young and by Laplace using force balance reasoning that was not clearly defined and in some respects incorrect (see [Finn 2006]), then later obtained independently by Gauss using Johann Bernoulli's "principle of virtual work", under the hypothesis that position variations internal to the bulk fluid do not affect the mechanical energy of the system. That was certainly reasonable to suppose at the time, but nevertheless may now be appropriate to question.

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Keywords: capillary surfaces, capillary tube, mean curvature, compressible fluid, elliptic nonlinear second-order PDE.

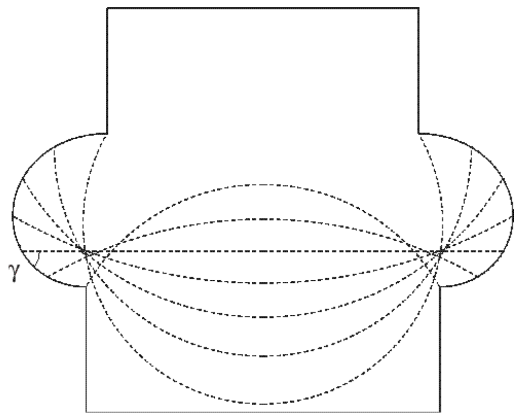

Figure 1. Profiles of static equilibrium surfaces in an "exotic container".

The equations (1)+(2) have served for two centuries, though perhaps not as well as might initially have been hoped, in view of their seemingly intractable nonlinearities. During the initial century, some isolated particular solutions were found by essentially numerical procedures [Bashforth and Adams 1883], and many attempts were made to obtain general information via linearization procedures; these latter attempts led to little information of substantive interest, and in fact to some misconceptions as to the behavior of the solutions. (See also [Finn 1986; 1999] for an overview.)

During the past half century, the problems were attacked anew on the basis of the full nonlinearity of the equations, yielding unexpected predictions of discontinuous behavior; some of these predictions were since verified by experiment; see, for example, [Concus et al. 2000; 1999; Finn 1999, p. 773]. The first existence proofs for (1)+(2) appeared in [Emmer 1973; Ural'tseva 1973; 1975], followed by a number of others under varying conditions.

In this sense, the qualitative validity of (1)+(2) as descriptions of reality was clearly established. Nevertheless, there remain significant questions as to their correctness in quantitative detail. Figure 1 displays profile curves of seven of the continuum of rotationally symmetric equilibrium surfaces in an "exotic container" [Concus et al. 1999]; all of these surfaces bound the same volume of fluid below them, all provide identical mechanical energies in the sense of Gauss, and all of them meet the boundary walls in the same contact angle γ.

Neither the system (1)+(2) nor the variational procedure of Gauss can distinguish among these formal solutions. Nevertheless, there are significant distinctions among the surfaces relative to the physical criteria that underlie those procedures. According to a discovery of Young, there is a pressure jump across each surface

S of magnitude $\delta p = 2\sigma H$, where H is the scalar mean curvature of S. We may assume vacuum ($p = 0$) above each surface in the family. Since H varies widely among the surfaces, so will the fluid pressures, and one must expect corresponding changes in the internal energy of the fluid.

Finn [2001] took an initial step to account for such energy changes, by assuming a slightly compressible fluid, with a phenomenological pressure/density relation $\rho = \rho_0 + \chi(p - p_0)$. By taking account of the thus induced effects of gravity on density, he was led to the equation

$$\operatorname{div} Tu = \frac{\rho_0 g}{\sigma} u - \chi g \cos\omega + \lambda, \tag{3}$$

where ω is the angle between the upward directed surface normal and the vertical, and λ is a Lagrange parameter, depending on an eventual mass constraint. For the problem of a prescribed mass $\mathcal{M}$ in a tube closed at the bottom, Finn found a necessary condition

$$\mathcal{M} < \rho_0 |\Omega| / \chi g \tag{4}$$

on $\mathcal{M}$ for existence of a solution, and he showed that for a circular tube (4) also suffices for existence of a uniquely determined solution.

In the present work we study (3) for domains Ω of general shape in the absence of a mass constraint, and we also consider the equation that arises on taking account of the expansion energy in fluid elements, resulting from density changes. In both cases, although mass is not prescribed, (4) will appear as a general bound for the mass lifted above the rest level $u \equiv 0$; see the discussion in [Finn 2001], which applies to all cases considered here.

The energy released in the expansion of a unit mass of compressible liquid on being raised from the base level 0 to level h is

$$\delta^1 \boldsymbol{E}_e = -\int_{p_0}^{p(h)} p\, d(1/\rho) = \frac{p_0}{\rho_0} - \frac{p(h)}{\rho(h)} + \int_{p_0}^{p(h)} \frac{dp}{\rho}. \tag{5}$$

We consider a thin tube of sectional area $\delta\Omega$ extending from the base level to the surface $u(x)$. At the height h we focus attention on an element of the tube of height δh. If this element is to be in equilibrium, the pressure change from the bottom to the top must be

$$\delta p = -\rho g\, \delta h,$$

and thus

$$\int_{p_0}^{p(h)} \frac{dp}{\rho} = -gh. \tag{6}$$

We assume a relation $\rho=\phi(p;\,p_0)>0$. We can then solve (6) for $p=P(h;\,p_0)$. From this we obtain $\rho=\phi(P(h;\,p_0);\,p_0)=\Phi(h;\,p_0)$. Note that the expansion energy doesn't enter here.

Returning to (5) and using (6), we find that the energy released by the indicated element of mass $\rho\,\delta h\,\delta\Omega$ is

$$\delta \boldsymbol{E}_e=\rho\,\delta h\,\delta\Omega\,\delta^1\boldsymbol{E}_e=\Big(\frac{p_0}{\rho_0}\Phi-P-\Phi g h\Big)\,\delta h\,\delta\Omega,$$

and thus

$$\boldsymbol{E}_e=\int_\Omega d\Omega\int_0^u\Big(\frac{p_0}{\rho_0}\Phi-P-\Phi g h\Big)\,dh.$$

We add this energy to those previously introduced in [Finn 2001]. From established procedures of the calculus of variations, we obtain the equation

$$\operatorname{div} Tu=\frac{\Phi_u}{\Phi}\cos\omega-\frac{P}{\Phi}\frac{\rho_0}{\sigma}+\lambda\frac{\rho_0}{\sigma}+\frac{p_0}{\sigma} \tag{7}$$

in Ω, with the boundary condition (2) unchanged.

In the special case $\rho=\rho_0+\chi(p-p_0)$, (6) yields $\rho=\rho_0 e^{-\chi g u}$, and (7) becomes

$$\operatorname{div} Tu=\frac{\rho_0-\chi p_0}{\chi\sigma}\big(e^{\chi g u}-1\big)-\chi g\cos\omega+\lambda\frac{\rho_0}{\sigma}. \tag{8}$$

We address here the classical problem of a cylindrical tube open at both ends, dipped into an unbounded reservoir of liquid. In this case, $\lambda=0$, and (8) becomes

$$\operatorname{div} Tu=\frac{\rho_0-\chi p_0}{\chi\sigma}\big(e^{\chi g u}-1\big)-\chi g\cos\omega \tag{9}$$

in Ω. We seek conditions under which there will be a solution of (9) in Ω subject to (2) on Σ. In the interest of obtaining well behaved solutions, we are driven to the further hypothesis

$$\rho_0-\chi p_0>0.$$

In the limit as $\chi\to 0$, we obtain the classical Young–Laplace–Gauss equation (1), as is to be expected. However, the limiting procedure is not uniform in the height u. Note that despite the absence of mass constraint, (9) is not satisfied by the function $u\equiv 0$ when $\gamma=\pi/2$. That is a consequence of the imposed variation of density with height. The fluid rises in the tube as consequence of the decreasing density, until the effect is compensated by the weight of the lifted fluid. The rest level for this trivial solution is the constant height

$$u_c=\frac{\chi\sigma}{\rho_0} \tag{10}$$

for (3) with $\lambda = 0$, or

$$u_c = \frac{1}{\chi g} \ln\left(1 + \frac{\chi^2 \sigma g}{\rho_0 - \chi p_0}\right) \tag{11}$$

for (9). This reference value will appear in Theorem 2.8 as a universal upper bound when $\pi/2 \le \gamma \le \pi$, and also implicitly in other contexts.

We will establish varying existence and uniqueness properties, for solutions of (3)+(2) or of (8)+(2), in domains of general shape; additionally we will establish a priori bounds on solutions of (3) or (8), irrespective of boundary conditions. Some of these bounds are idiosyncratic to the particular kinds of nonlinearities considered, and have no counterparts in classical theory of elliptic equations. In configurations for which uniqueness cannot be established by methods at our disposal, we obtain instead comparison theorems, estimating a priori the difference between possible solutions. We will establish growth and comparison properties and discontinuous behavior of solutions in particular domains, depending on inequalities for boundary data. The remainder of the paper is organized as follows:

In Section 2 we present a priori estimates on solution heights, in a somewhat more general context than the particular cases (3) and (8).

In Section 3 we give the gradient estimates up to the boundary for $C^{2,\mu}$ domains Ω, adapting a procedure introduced by Ural'tseva [1973; 1975].

In Section 4 we provide the existence and uniqueness assertions, for $C^{2,\mu}$ domains.

In Section 5 we adapt a procedure used in [Finn and Gerhardt 1977] to prove the existence of "variational solutions" in piecewise smooth domains. Limited knowledge of boundary behavior at corner points is available for such solutions; however, boundedness or growth properties can be established, depending on local geometry, and "near-uniqueness" properties are obtained.

Finally, we note that the height estimates obtained by comparison to hemispheres trivially extend to hold for domains in any dimension, $\Omega \subset \mathbb{R}^n$. We prove the gradient estimates in n dimensions. Our results for domains with corners are formulated for $n = 2$.

2. A priori height estimates

We consider generally solutions $u(x)$ of

$$\operatorname{div} Tu = -\frac{a^2}{\sqrt{1+|Du|^2}} + \mathcal{F}(u), \qquad Tu = \frac{Du}{\sqrt{1+|Du|^2}}, \tag{12}$$

in a bounded, piecewise smooth domain Ω. It is assumed that $\mathcal{F}(u)$ is monotone increasing, with $\mathcal{F}(0) = 0$, and that a is a constant.

For the following definition, let us note that every $f \in H^{1,1}(\Omega)$ has a trace $f^t \in L^1(\partial\Omega)$, which we will denote by f. We call $u(x)$ a *variational solution* of (12) in Ω, corresponding to a boundary contact angle γ, if $u \in C^2(\Omega)$, if $\mathcal{F}(u)$ is integrable over Ω, and if

$$\int_\Omega [D\eta \cdot Tu + \eta\,\mathcal{F}(u)]\,dx = \int_\Omega \eta \frac{a^2}{\sqrt{1+|Du|^2}}\,dx + \int_{\partial\Omega} \eta \cos\gamma\,ds, \tag{13}$$

for every $\eta \in \mathcal{Q}(\Omega) := L^\infty \cap H^{1,1}(\Omega)$. We note in (13) that even though the nominal boundary condition involves derivatives of u, neither the derivatives nor the function itself occurs in the boundary integral. We assume γ to be piecewise continuous on $\partial\Omega$, with $0 \le \gamma \le \pi$.

The following lemma extends slightly Lemma 3 in [Finn and Gerhardt 1977].

Lemma 2.1. *Let $\mathcal{F}(u)$ be nondecreasing. Let Ω be a piecewise smooth domain exhausted by smooth domains $\Omega^j \subset \Omega$. Let u, v be functions in $H^{1,1}_{\mathrm{loc}}(\Omega)$, such that*

$$\limsup_{j\to\infty} \int_{\Omega^j} \big(D\eta \cdot (Tv - Tu) + \eta\,(\mathcal{F}(v) - \mathcal{F}(u))\big)\,dx \ge 0 \tag{14}$$

for every $\eta \in \mathcal{Q}_{\mathrm{loc}}(\Omega) := L^\infty \cap H^{1,1}_{\mathrm{loc}}(\Omega)$ with $\eta \ge 0$. If $\mathcal{F}(u)$ is strictly increasing, there follows $v \ge u$ almost everywhere in Ω. Otherwise either $v \ge u$ in Ω or else $v \equiv u + c$, c constant, throughout Ω. If strict inequality holds in (14), then the inequalities $v \ge u$ can be replaced by $v > u$.

We will apply this lemma in varying contexts to the particular cases

$$\mathcal{F}(u) = \frac{\rho_0 g}{\sigma} u, \qquad\qquad a^2 = \chi g, \tag{15}$$

$$\mathcal{F}(u) = \frac{\rho_0 - \chi p_0}{\chi\sigma}(e^{\chi g u} - 1), \quad a^2 = \chi g, \quad 0 < \chi < \frac{\rho_0}{p_0}. \tag{16}$$

The first case corresponds to the situation studied in [Finn 2001], with unconstrained total mass, with $\kappa = \rho_0 g/\sigma$, and with $a^2 = \chi g$; the case of prescribed mass, subject to the (necessary) condition $\chi g M < \rho_0|\Omega|$, is retrieved by adding a constant to u, see the discussion in [Finn 2001, p. 147]. The second case yields the more exact equation introduced in this paper, again with unconstrained mass. The same necessary condition applies; however a prescribed mass can no longer be achieved by a rigid vertical translation of the surface.

In view of the first term on the right in (12), it is not immediately clear whether the solutions are unique or satisfy a maximum principle. We do obtain that the difference of two solutions satisfies an elliptic equation for which a maximum principle holds, and we can use that information for the following result:

Theorem 2.2. *Suppose that Ω is a C^1 domain, satisfying an internal sphere condition, and that u, v are C^2 solutions of (12) in Ω, both C^1 on $\overline{\Omega}$. If $Tu \cdot \nu \leq Tv \cdot \nu$ on $\Sigma = \partial\Omega$, then either $u < v$ in Ω, or else $u \equiv v$ in Ω.*

By an internal sphere condition (ISC) we mean that every boundary point can be contacted from within Ω by a disk contained in $\overline{\Omega}$.

Proof. Let $w = u - v$ denote the difference of the two solutions, and assume w has a maximum M at a point $p \in \overline{\Omega}$. If $p \in \Sigma$, then at p the tangential derivative along Σ vanishes, $w_s = 0$. Thus the exterior normal derivative $w_\nu = \partial w/\partial\nu$ satisfies $w_\nu \geq 0$. In view of the internal sphere condition, we may apply the boundary point lemma, obtaining that either $w \equiv M$ or else $w_\nu > 0$. In the former case we conclude $M = 0$ since $\mathcal{F}(u)$ is strictly increasing; the latter case conflicts with the hypothesis, and we may thus assume that $p \in \Omega$.

We can exclude an interior positive maximum for w by using the maximum principle as noted; however, we present here a geometric argument.

Since w attains a maximum at p, we remark that the values of the angle ω are equal for both surfaces at the point. Were $u(p) > v(p)$, we would have $\operatorname{div} Tu(p) > \operatorname{div} Tv(p)$ by (12). Since these expressions are twice the mean curvature of the respective surfaces, we conclude that at least one of the principal curvatures of the surface $S_u = \operatorname{graph} u$ would exceed that for the surface $S_v = \operatorname{graph} v$, contradicting that w has a maximum at p. $\square$

If less smoothness is known for Ω or for the solution, one nevertheless has:

Theorem 2.3. *Let u^1, u^2 be variational solutions in a piecewise smooth Ω of (12)+(15), corresponding to data $\beta^1 = \cos\gamma^1 \leq \beta^2 = \cos\gamma^2$ on $\Sigma = \partial\Omega$. Then*

$$u^1 < u^2 + \frac{\chi\sigma}{\rho_0}.$$

If instead u^1, u^2 are variational solutions of (12)+(16), for which $u^1, u^2 > -A > -\infty$, then

$$u^1 < u^2 + \frac{\chi\sigma}{\rho_0 - \chi p_0} e^{\chi g A}.$$

Proof. To prove the first assertion, we observe that in view of (13) we have, for positive η,

$$\begin{aligned}\int_{\Omega^j} \Big(D\eta \cdot (Tu^2 - Tu^1) + \eta\, \frac{\rho_0 g}{\sigma}(u^2 - u^1) \Big)\, dx \\ > \chi g \int_\Omega \eta \left(\frac{1}{\sqrt{1+|Du^2|^2}} - \frac{1}{\sqrt{1+|Du^1|^2}} \right) dx \\ > -\chi g \int_\Omega \eta\, dx.\end{aligned}$$

Writing $u^1 = w^1 + \chi\sigma/\rho_0$, we find

$$\int_\Omega \Big(D\eta \cdot (Tu^2 - Tw^1) + \eta \, \frac{\rho_0 g}{\sigma} \, (u^2 - w^1) \Big) \, dx \; > \; 0,$$

for all $\eta \in \mathcal{Q}_{\text{loc}}(\Omega)$, $\eta \geq 0$. By Lemma 2.1 we have $u^2 > w^1 = u^1 - \chi\sigma/\rho_0$, which completes the proof of the initial assertion. The second assertion follows similarly, using the estimate

$$e^{\chi g w^1 + c} - e^{\chi g w^1} = \chi g \int_{w^1}^{w^1+c} e^{\chi g t} \, dt > \chi g e^{\chi g w^1} c \quad \text{if } c > 0. \qquad \square$$

We may apply a variant of the method to obtain universal bounds, above and below, on solutions of (12)+(15) interior to a given domain Ω; with regard to (12)+(16) we find a universal bound above, and a universal bound below for solutions over a sufficiently large disk:

Theorem 2.4. *Let u be a variational solution of (12)+(15) interior to a ball B_δ. Then*

$$-\frac{2\sigma}{\rho_0 g \delta} - \delta < u < \frac{\chi\sigma}{\rho_0} + \frac{2\sigma}{\rho_0 g \delta} + \delta \tag{17}$$

throughout B_δ. If u is a variational solution of (12)+(16) in B_δ, then

$$u < \delta + \frac{1}{\chi g} \ln \left(1 + \frac{\chi\sigma}{\rho_0 - \chi p_0} \Big(\chi g + \frac{2}{\delta} \Big) \right) \tag{18}$$

throughout B_δ. In this case, if in addition $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$, then

$$u > \frac{1}{\chi g} \ln \left(1 - \frac{2\chi\sigma}{(\rho_0 - \chi p_0)\delta} \right) - \delta. \tag{19}$$

Proof. We compare the given solution u of (12) with a lower hemisphere $v(x)$ of radius δ and projecting into B_δ. This function has constant mean curvature $1/\delta$ and thus satisfies the auxiliary equation

$$\operatorname{div} Tv = 2/\delta$$

over B_δ. We verify the relation

$$\begin{aligned} &\int_{B_\delta} \big(D\eta \cdot (Tv - Tu) + \eta \, (\mathcal{F}(v) - \mathcal{F}(u)) \big) \, dx \\ &\qquad = \int_{\partial B_\delta} \eta (1 - \cos\gamma_\delta) \, ds + \int_{B_\delta} \eta \left(\mathcal{F}(v) - \frac{\chi g}{\sqrt{1 + |Du|^2}} - \frac{2}{\delta} \right) dx. \end{aligned} \tag{20}$$

Here $\cos\gamma_\delta = \nu \cdot Tu$ evaluated on ∂B_δ, and we have used that $\nu \cdot Tv = 1$ on ∂B_δ, since the hemisphere is vertical on that arc.

We position the hemisphere so that $\mathcal{F}(v) = \chi g + 2/\delta$ at its lowest point. By the monotonicity of $\mathcal{F}$, the right side of (20) will then be positive for any positive $\eta \in \mathcal{Q}_{\text{loc}}(B_\delta)$. Thus, the left side will also be positive, and we conclude from Lemma 2.1 that $u < v$ in B_δ. Since the total height change of v from the center to the edge of B_δ is δ, this inequality establishes (18) and the right-hand side of (17).

The left side of (17) and also (19) follow similarly, using an upper hemisphere as comparison surface. The restriction $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ must be imposed, as the inverse function for $\mathcal{F}(u)$ in (16) is not defined for $\mathcal{F} < -(\rho_0 - \chi p_0)/\sigma\chi$. □

This result can be sharpened significantly in the particular case where B_δ is the definition domain Ω, with a constant contact angle γ_δ achieved in the variational sense on ∂B_δ. Then we may choose v to be a spherical cap meeting the cylinder wall $r = \delta$ in the angle γ_δ. The boundary integral in (20) then vanishes, and we find

$$\text{(21)} \quad \int_{B_\delta} \big(D\eta \cdot (Tv - Tu) + \eta\,(\mathcal{F}(v) - \mathcal{F}(u))\big)\,dx = \int_{B_\delta} \eta \left(\mathcal{F}(v) - \frac{\chi g}{\sqrt{1+|Du|^2}} - \frac{2}{\delta}\cos\gamma_\delta\right) dx.$$

We distinguish four cases, according to whether $\gamma_\delta < \pi/2$ or $\gamma_\delta > \pi/2$, and whether we seek upper or lower bounds. If $\gamma_\delta < \pi/2$ and we seek an upper bound, we position the cap so that $\mathcal{F}(v) = \chi g + 2(\cos\gamma_\delta)/\delta$ at the point of symmetry. Then both sides of (21) will be positive for all positive η, and we conclude $u < v$. If we seek a lower bound, we position the cap so that $\mathcal{F}(v) = 2(\cos\gamma_\delta)/\delta$ at the point $r = \delta$. Then both sides of (21) will be negative for positive η, from which follows $u > v$. Analogous reasoning applies when $\gamma_\delta > \pi/2$. We are led to:

Corollary 2.5. *Suppose* $\Omega = B_\delta$ *and* $0 \le \gamma \le \pi$.

(a) *If* $u(x)$ *is a variational solution of* (12)+(15) *in* Ω, *there holds*

$$\text{(22)} \quad \frac{2\sigma}{\rho_0 g}\frac{\cos\gamma}{\delta} - \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta < u < \frac{2\sigma}{\rho_0 g}\frac{\cos\gamma}{\delta} + \frac{\chi\sigma}{\rho_0} + \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta.$$

(b) *If* $u(x)$ *is a variational solution of* (12)+(16) *in* Ω *then*

$$\text{(23)} \quad \frac{1}{\chi g}\ln\left(1 + \frac{2\chi\sigma}{(\rho_0 - \chi p_0)}\frac{\cos\gamma}{\delta}\right) - \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta < u < \frac{1}{\chi g}\ln\left(1 + \frac{\chi\sigma}{(\rho_0-\chi p_0)}\left(\frac{2\cos\gamma}{\delta} + \chi g\right)\right) + \left|\frac{1-\sin\gamma}{\cos\gamma}\right|\delta.$$

In these last relations, the logarithmic terms must be replaced by $-\infty$ if the arguments are nonpositive. This is not an accident of the method; we return to this point below, where we will show that if the argument on the right side is nonpositive, then no solution of (12)+(16) can exist in the disk. (See also the remark on nonexistence on page 221.)

These bounds can be improved in some respects by relaxing the boundary condition for v; compare the proof of Theorem 2.8.

The hypotheses of Theorem 2.4 clearly apply to configurations in which $u(x)$ is defined as a solution in a domain Ω containing B_δ; more generally if B_δ does not lie entirely interior to Ω but if u assumes (in a variational sense) data γ on $B_\delta \cap \Sigma$, it suffices to focus attention on a component of $B_\delta \cap \Omega$ for which the hemispheres introduced in the proof meet the vertical walls of Σ in angles majorizing γ. Specifically, we obtain:

Theorem 2.6. *Let u be a variational solution of (12)+(15) or of (12)+(16) interior to a component Z_δ of $B_\delta \cap \Omega$. If on $Z_\delta \cap \Sigma$ the lower hemisphere under B_δ meets the vertical walls under Σ in angles $\gamma^\delta \le \gamma$, then the right side of (22) holds in Z_δ for the system (12)+(15) and the right side of (23) holds for (12)+(16). If $\gamma^\delta \ge \gamma$, then the remaining inequalities apply in the respective cases.*

Further, using the definition for an internal sphere condition $\mathrm{ISC}_{\delta,\gamma^\delta}$ as given in [Finn and Gerhardt 1977, pp. 15–16], we may state:

Corollary 2.7. *If Ω can be covered by disks of radius δ for some fixed $\delta > 0$, then (17) and (18) hold throughout Ω. If that can be done with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$, then (19) also holds throughout Ω. More generally, if Ω satisfies an internal sphere condition $\mathrm{ISC}_{\delta,\gamma^\delta}$, with $\gamma^\delta \le \gamma$, then the right sides of (22) and (23) hold in the respective cases. If a condition $\mathrm{ISC}_{\delta,\pi-\gamma^\delta}$ holds, with $\pi - \gamma^\delta \ge \pi - \gamma$, then the remaining statements of Corollary 2.5 apply.*

In general, if some a priori information is known on boundary behavior of the solution u, then the bounds in (17) and (18) can to some extent be sharpened. Assume first that $\gamma < \gamma_0 < \pi/2$. We take as comparison surface v a lower hemisphere whose center projects to a point of Ω, and of radius R large enough that the projection covers Ω and such that the contact angle $\gamma^v \ge \gamma_0$. We obtain now the relation

$$\int_\Omega \big(D\eta\cdot(Tu - Tv) + \eta\,(\mathcal{F}(u) - \mathcal{F}(v))\big)\,dx \\ = \int_{\partial\Omega} \eta(\cos\gamma - \cos\gamma^v)\,ds + \int_\Omega \eta\left(\frac{2}{R} + \frac{\chi g}{\sqrt{1+|Du|^2}} - \mathcal{F}(v)\right)dx,$$

and it thus suffices to choose v such that $\mathcal{F}(v) < 2/R$. R will in general not be known explicitly, however a universal choice, suitable both for (15) and for (16), is provided by the function $v = 0$; that yields $\mathcal{F}(v) = 0$ in both cases, from which $u > 0$ follows by Lemma 2.1. In the other direction, we introduce for v an upper hemisphere, and are led to the relation

$$\int_\Omega \big(D\eta\cdot(Tv-Tu)+\eta\,(\mathcal{F}(v)-\mathcal{F}(u))\big)\,dx \\ = \int_{\partial\Omega}\eta(\cos\gamma^v-\cos\gamma)\,ds+\int_\Omega \eta\left(\frac{2}{R}-\frac{\chi g}{\sqrt{1+|Du|^2}}+\mathcal{F}(v)\right)dx,$$

and we see that it suffices in general to have $\mathcal{F}(v)>\chi g$. Again we may let $R\to\infty$, leading to the choice

$$v\equiv\frac{\chi\sigma}{\rho_0}\quad\text{for (15)}\qquad\text{and}\qquad v\equiv\text{const}=\frac{1}{\chi g}\ln\left(1+\frac{\chi^2\sigma g}{\rho_0-\chi p_0}\right)\quad\text{for (16)}.$$

We have proved:

Theorem 2.8. *Let u be a variational solution of either (12)+(15) or (12)+(16) in a piecewise smooth domain Ω. If $0\le\gamma<\pi/2$, there holds $u>0$ in Ω in both cases (15) and (16). If $\pi/2<\gamma\le\pi$, there holds in Ω*

$$u<\frac{\chi\sigma}{\rho_0}\quad\textit{in case (15)}\qquad\textit{and}\qquad u<\frac{1}{\chi g}\ln\left(1+\frac{\chi^2\sigma g}{\rho_0-\chi p_0}\right)\quad\textit{in case (16)}.$$

The material above provides global estimates for solutions over a prescribed domain Ω. We turn our attention now to behavior near corner points of Ω. For simplicity, we assume that the boundary consists locally at the corner P of two line segments, intersecting in an angle $2\alpha<\pi$, measured interior to Ω. We assume that $|\gamma-\pi/2|>\alpha$ – and thus that $|\cos\gamma|>\sin\alpha$ – in a neighborhood of P on $\partial\Omega$. (If $|\gamma-\pi/2|\le\alpha$ in such a neighborhood, the bounds indicated in Theorem 2.6 apply.) We assume first that $0\le\gamma<\pi/2$, and observe that then any point $p\in\Omega$ of (sufficiently small) distance r from P lies in a disk of radius $r\sin\alpha/\cos\gamma$ that meets the boundary segments Σ at an angle γ, as in the figure:

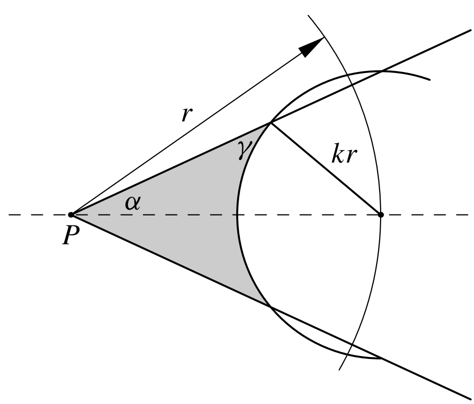

Figure 2. Construction for bounding solution below, in a wedge domain.

The lower hemisphere $v(x)$ with Σ as equatorial circle meets the vertical walls through Σ in that same angle γ. By Theorem 2.6, we find in the case of (12)+(15), setting $k = \sin\alpha/\cos\gamma$,

$$u(x) < \frac{2\sigma}{\rho_0 g}\frac{1}{kr} + \frac{\chi\sigma}{\rho_0} + kr \tag{24}$$

and in the case (12)+(16)

$$u(x) < \frac{1}{\chi g}\ln\left(1 + \frac{2\chi\sigma}{\rho_0 - \chi p_0}\frac{1}{kr} + \frac{\chi^2\sigma g}{\rho_0 - \chi p_0}\right) + kr. \tag{25}$$

To obtain appropriate lower bounds, we adapt a procedure introduced by Korevaar [1980], and use the upper inner side of a torus as a comparison surface. Corresponding to points at distance not exceeding r from the vertex, we consider the torus $v(\boldsymbol{x})$, $\boldsymbol{x} = (x, y, z)$ defined in terms of parameters ϕ, ψ relative to the vertex as coordinate origin by

$$x = (A - a\cos\psi)\,\cos\phi, \quad y = a\sin\psi, \quad z = (A - a\cos\psi)\,\sin\phi,$$

with $a = r\sin\alpha\,/(\cos\gamma - \sin\alpha)$. Here $A > a$, and the parameters satisfy $-\psi_0 < \psi < \psi_0$, $0 < \phi < \phi_0$, with ϕ_0, $\psi_0 < \pi/2$ fixed but arbitrary. The general appearance is that of a Japanese footbridge, drawn here in perspective:

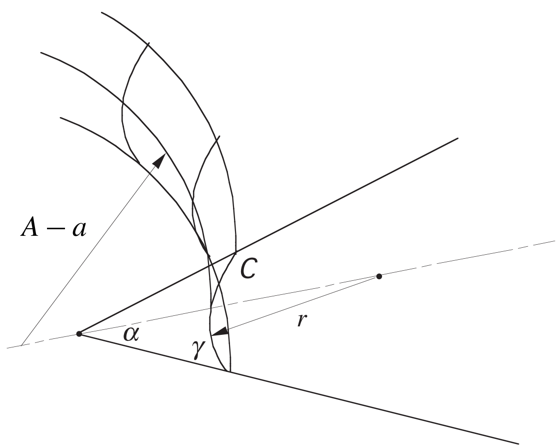

Figure 3. Construction for bounding solution above at a corner point.

The crucial observation is that $\nu \cdot Tv = -1$ on the curve $\mathcal{C} = \{\phi = 0\}$, ν being the exterior unit normal, and thus the boundary condition on that curve minorizes that of any solution u in a common domain of definition.

For small a, the torus cuts off a small piece of the corner, as indicated in the figure, with the curve $\mathcal{C}$ meeting the bounding segments at an angle γ. We observe that v satisfies

$$\operatorname{div} Tv = 2H(x) > \frac{1}{a} - \frac{1}{A-a} = \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right).$$

If r is small enough, this expression will be positive. Since the unit normal to the torus is continuous and is directed horizontally toward the vertex at the symmetry point of $\mathcal{C}$, there will hold for small enough r that $\nu \cdot Tv < \sin\alpha + \varepsilon$ on both the segments cut off at the corner, with $\sin\alpha + \varepsilon < \cos\gamma$ on these segments.

Following the procedure of Theorem 2.6, we find for the case (12)+(15) that $u > v$ in the domain cut off at the vertex, provided that v can be chosen so that

$$\frac{\rho_0 g}{\sigma} v < \chi g + \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right).$$

We may translate v vertically so that this inequality holds at a particular point of the domain; we then find on the basis of the construction that

$$\frac{\rho_0 g}{\sigma} v > \chi g + \frac{1-k}{k}\left(\frac{1}{r} - \frac{k}{(1-k)A - kr}\right) - \omega(\varepsilon)$$

with $\lim_{\varepsilon\to 0}\omega(\varepsilon) = 0$.

We now wish to let $r \to 0$. A convenient way to do that is by a similarity transformation, which leaves all boundary angles and the geometric configuration unchanged. We obtain the result that for sufficiently small r, there holds at all points (x, y) of distance r from the vertex the inequality

$$u(x, y) > \frac{\sigma}{\rho_0 g}\frac{1-k}{kr} + C, \tag{26}$$

for a fixed constant C independent of r. Together with (24), this result implies that every solution of (12)+(15) in a wedge domain with $\alpha + \gamma < \pi/2$ is unbounded at the corner, with a growth rate $O(1/r)$.

In the case (12)+(16) an analogous reasoning yields, observing that the choice of $A > a$ is arbitrary,

$$u(x, y) > \frac{1}{\chi g}\ln\left(1 + \frac{\chi\sigma}{\rho_0 - \chi p_0}\frac{1-k}{k}\frac{1-\varepsilon}{r} - C(\varepsilon)\right) \tag{27}$$

asymptotically as $r \to 0$, for any $\varepsilon > 0$ and fixed $C(\varepsilon)$ independent of r.

We turn our attention now to the case $\pi/2 < \gamma \le \pi$. A procedure analogous to that yielding (24) (and resuming the notation $x \in \Omega$) leads now, for solutions of (12)+(15), to

$$u(x) > -\frac{2\sigma}{\rho_0 g}\frac{1}{kr} - kr \tag{28}$$

and a procedure analogous to that yielding (26) now yields

$$u(x) < -\frac{\sigma}{\rho_0 g}\frac{1-k}{kr} + C \tag{29}$$

in the case (12)+(15).

With regard to solutions of (12)+(16) the situation is now simpler. We investigate (12) over a wedge triangle:

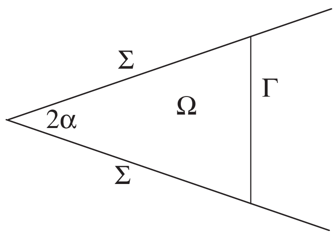

Figure 4. Wedge domain.

In view of the boundary condition, we obtain

$$2|\Sigma|\cos\gamma + \int_\Gamma \nu\cdot Tu\,ds = \int_\Omega \left(-\frac{\chi g}{\sqrt{1+|Du|^2}} + \frac{\rho_0 - \chi p_0}{\chi\sigma}(e^{2\chi u}-1) \right) dx,$$

from which, since $|\nu\cdot Tu| < 1$, we conclude that

$$2|\Sigma|\cos\gamma + |\Gamma| > -\left(\chi g + \frac{\rho_0-\chi p_0}{\chi\sigma}\right)|\Omega|.$$

Thus, since $\gamma > \pi/2$ and $|\gamma - \pi/2| > \alpha$ so that $|\cos\gamma| > \sin\alpha$, we find

$$0 < 2(|\cos\gamma| - \sin\alpha) < \left(\chi g + \frac{\rho_0-\chi p_0}{\chi\sigma}\right)|\Sigma|\cos\alpha\sin\alpha, \tag{30}$$

and we obtain a contradiction by letting Γ move in parallel translation toward the vertex.

Gathering the material above, we have proved:

Theorem 2.9. *Suppose that γ is constant in a neighborhood of a corner point of opening 2α. If $|\gamma-\pi/2| \le \alpha$ then the estimates of Theorem 2.6 apply. If $\alpha+\gamma < \pi/2$ then the estimates (24) and (26) hold for the case (12)+(15), and the estimates (25) and (27) hold for the case (12)+(16). If $\gamma > \alpha + \pi/2$ then (28) and (29) apply for the case (12)+(15); however for the case of (12)+(16) no solution can exist in such a wedge.*

If the boundary of Ω is not rectilinear at the corner point, we still obtain the same results as above, but under the stronger condition $|\gamma-\pi/2| < \alpha$.

Finally, we remark an immediate consequence of Theorem 2.4:

Theorem 2.10. *Any solution of (12)+(15) is bounded at any isolated singular point. Any solution of (12)+(16) is bounded above at an isolated singular point.*

In [Finn 1963] it is proved that the "classical" capillary equation

$$\operatorname{div} Tu = \frac{\rho g}{\sigma} u$$

admits only removable isolated singularities. We do not know to what extent that theorem extends to the more general configurations considered in this paper.

3. Gradient estimates

We study the case of equation (8). We derive the gradient estimate following techniques introduced by Ural'tseva [1973; 1975] and used in [Gerhardt 1976; Huisken 1985].

We follow closely the procedure in [Huisken 1985]. For the convenience of the reader we state here the results of that paper which we use.

We consider the equation

$$\begin{aligned} \operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} &= \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}} \quad && \text{in } \Omega, \\ Tu \cdot \nu &= \beta && \text{on } \Sigma, \end{aligned} \tag{31}$$

with $a^2 = \chi\sigma/\rho_0$ and the function $\mathcal{F}(u)$ defined either as in (15) or as in (16). The main assumption on $\mathcal{F}$ needed for the gradient estimate is that $\mathcal{F}' > 0$. For the present considerations we assume $\beta \in C^{0,1}(\Sigma)$ to satisfy

$$|\beta| \le 1 - \tilde{\alpha}, \qquad \tilde{\alpha} > 0. \tag{32}$$

As above, we denote by T the operator defined by

$$Tu = \frac{Du}{\sqrt{1+|Du|^2}}.$$

We also introduce the notations

$$a^i(p) = \frac{p^i}{\sqrt{1+|p|^2}}, \qquad a^{ij} = \frac{\partial a^i}{\partial p_j},$$

for $p \in \mathbb{R}^n$, and denote by $H(x, u, Du)$ the right-hand side of (31):

$$H(x, u, Du) = \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}}. \tag{33}$$

Given $\Sigma \in C^{2,\mu}$, we can extend β and ν to the interior of Ω, in such a way that $\beta \in C^{0,1}(\Omega)$ still satisfies (32) and ν is uniformly Lipschitz continuous in Ω, with $|\nu| < 1$.

We denote by $S = \operatorname{graph} u$ the liquid-air interface and by $\nabla^S f$ the tangential gradient on S of a function $f \in C^1(\Omega)$:

$$\nabla^S f = Df - (Df \cdot \nu_S)\, \nu_S,$$

with ν_S the unit normal to the interface S.

The main idea is to work with the function

$$v = \sqrt{1+|Du|^2} + \beta\,(Du \cdot \nu) \equiv W + \beta\,(Du \cdot \nu)$$

as in [Ural'tseva 1973; Gerhardt 1976], and to prove that v is uniformly bounded in Ω. This in turn gives the gradient estimate, since

$$|Du| \le \sqrt{1+|Du|^2} \;=\; W \;\le \frac{1}{\tilde{\alpha}} v.$$

We will bound the function

$$w = \log v$$

instead of v; we can follow all the steps as in [Huisken 1985, (2.12)–(2.30)], with the first real difference being the derivative $D_k H$ needed in (2.25) of that paper, which is computed in (2.31). In our case, we find

$$\begin{aligned}(34)\quad &\int_\Omega a^{ij}\left(D_j v - D_j(\beta\nu^k)\, D_k u\right)\, D_i\eta + \frac{1}{2n}\,|H|^2\,\eta\,dx \\ &\quad\le -\int_\Omega D_k H\,(a^k + \beta\nu^k)\,\eta\,dx + c_\varepsilon \int_\Omega \left(1 + \frac{\nabla^S v}{W}\right)\eta\,dx + c_3 \int_\Sigma \eta\, d\mathcal{H}^{n-1}.\end{aligned}$$

Inequality (34) is almost identical with [Huisken 1985, (2.32)], except that we want to explicitly calculate the first term on the right-hand side, since our problem only differs in the form of the prescribed mean curvature function H. In view of (33) we have

$$\begin{aligned}(35)\quad &-\int_\Omega D_k H (a^k + \beta\nu^k)\,\eta\,dx \\ &\qquad = -\int_\Omega \left(\mathcal{F}'(u) D_k u + \frac{\mu}{W^3} D_l u D_k D_l u\right)\,(a^k + \beta\nu^k)\,\eta\,dx.\end{aligned}$$

The first term on the right-hand side of (35) is negative, since $\mathcal{F}'(u) > 0$ and

$$D_k u\,(a^k + \beta\nu^k) = v - W^{-1} > 0.$$

Therefore it can be ignored. For the second term on the right in (35), we can use the equality

$$D_l u\left(D_k D_l u\,(a^k + \beta\nu^k)\right) = D_l u\left(D_l v - \,D_l(\beta\nu^k)\, D_k u\right),$$

which follows from [Huisken 1985, (2.26)]. In view of this equality, (35) becomes

$$-\int_\Omega D_k H\,(a^k+\beta\nu^k)\,\eta\,dx \le -a^2\int_\Omega \frac{1}{W^3} D_l u\, D_l v\,\eta\,dx + a^2\int_\Omega \frac{1}{W^3} D_l(\beta\nu^k)\, D_l u\, D_k u\,\eta\,dx.$$

Denote the integrals on the right-hand side by I_1 and I_2. They can be estimated by

$$I_2 \le a^2 c_4 \int_\Omega \frac{|Du|^2}{W^3}\,\eta\,dx \le \mu\, c_4 \int_\Omega \eta\,dx, \tag{36}$$

$$\begin{aligned} I_1 &\le \frac{1}{2\tilde\varepsilon}\,a^4\int_\Omega \frac{|Du|^2}{W^3}\,\eta\,dx + \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|Dv|^2}{W^3}\,\eta\,dx \\ &\le \frac{1}{2\tilde\varepsilon}\,a^4\int_\Omega \eta\,dx + \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx. \end{aligned} \tag{37}$$

Here c_4 depends on the Lipschitz constant of $\beta\nu$ and we have used the inequalities $|Du|^2/W^2 \le 1$ and $|Dv|^2/W^2 \le |\nabla^S v|^2$, the latter being proved as follows:

$$\begin{aligned} |Dv|^2 &\le |\nabla^{n+1}|^2 = |\nabla^S|^2 + |(\nabla^{n+1}\cdot\nu_S)\,\nu_S|^2 \\ &= |\nabla^S v|^2 + \left|\frac{(Dv,0)\cdot(-Du,1)}{W}\right|^2 |\nu_S|^2 = |\nabla^S v|^2 + \frac{|Dv|^2\,|Du|^2}{W^2}. \end{aligned}$$

Both the first term on the right-hand side of (37) and the estimate (36) for I_2 are of the same form and can be incorporated into the second term on the right-hand side of (34) with a new constant $c_5 = c_\varepsilon + \mu c_4 + a^4/\tilde\varepsilon$ replacing c_ε. Using the above considerations, we conclude that (34) gives

$$\begin{aligned} \int_\Omega a^{ij}\big(D_j v - D_j(\beta\nu^k)\,D_k u\big)\,D_i\eta + \frac{1}{2n}\,|H|^2\,\eta\,dx \\ \le \frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx + c_5\int_\Omega\left(1+\frac{\nabla^S v}{W}\right)\eta\,dx + c_3\int_\Sigma \eta\,d\mathcal{H}^{n-1}. \end{aligned} \tag{38}$$

As a test function η we choose

$$\eta = v\max(w-k,0) \equiv v z$$

for positive k, and define

$$A(k) = \big\{p=(x,u(x))\in S : w(x)>k\big\}, \quad |A(k)| = \mathcal{H}^n(A(k)).$$

For the first term on the right-hand side of (38) we note that $\eta\,W^{-1} \le 2z$, since $v \le 2W$ and we have

$$\frac{\tilde\varepsilon}{2}\int_\Omega \frac{|\nabla^S v|^2}{W}\,\eta\,dx \le \tilde\varepsilon\int_\Omega |\nabla^S v|^2\,z\,dx. \tag{39}$$

This term will then be taken to the left-hand side of the inequality (38).

We next show that (38) is equivalent to

$$\int_{A(k)} |\nabla^S v|^2 \, d\mathcal{H}^n + \frac{1}{n} \int_{A(k)} |H|^2 z \, d\mathcal{H}^n \le c\,|A(k)| + c \int_{A(k)} z \, d\mathcal{H}^n, \tag{40}$$

where $c = c\big(\tilde{\alpha}, n, |D\nu|_\Omega, |D\beta|_\Omega\big)$. For this, we estimate each term separately, starting with the first term on the left; we use [Huisken 1985, (2.27)–(2.30)] and the equalities $w = \log v$, $D_i \eta = (z+1)\, D_i v$. Setting $\Omega_\eta = \Omega \cap \operatorname{supp} \eta$, we get

$$\begin{aligned}
&\int_{\Omega_\eta} a^{ij} \big(D_j v - D_j(\beta \nu^k)\, D_k u\big)\, D_i \eta \, dx \\
&\quad \ge \int_{\Omega_\eta} \Big(a^{ij} D_j v \, D_i v - \big|a^{ij}\,(D_j(\beta\nu^k)\, D_k u)\, D_i v\big|\Big)\,(z+1)\, dx \\
&\quad \ge \int_{\Omega_\eta} W^{-1} |\nabla^S v|^2 \,(z+1)\, dx \\
&\qquad\qquad - \int_{\Omega_\eta} \left| \frac{1}{2\varepsilon} a^{ij} (D_j(\beta\nu^k)\, D_k u)\,(D_i(\beta\nu^k)\, D_k u) + \frac{\varepsilon}{2} a^{ij}\, D_i v\, D_j v \right| (z{+}1)\, dx \\
&\quad \ge \Big(1 - \frac{\varepsilon}{2}\Big) \int_{\Omega_\eta} W^{-1} |\nabla^S v|^2 (z+1)\, dx - \frac{1}{2\varepsilon} \int_{\Omega_\eta} W^{-1} |\nabla^S(\beta\nu)|^2 |Du|^2 (z+1)\, dx \\
&\quad \ge \Big(1 - \frac{\varepsilon}{2}\Big) \int_{\Omega_\eta} W^{-1}\, v^2 |\nabla^S z|^2 \, dx - \frac{1}{2\varepsilon} |D(\beta\nu)|^2_\Omega \int_{\Omega_\eta} (z+1) W \, dx \\
&\quad \ge \tilde{\alpha}^2 \Big(1 - \frac{\varepsilon}{2}\Big) \int_{A(k)} |\nabla^S z|^2 \, d\mathcal{H}^n - \frac{1}{2\varepsilon} |D(\beta\nu)|^2_\Omega \int_{A(k)} (z+1)\, d\mathcal{H}^n.
\end{aligned}$$

For the second term on the left-hand side of (38), we find using [Huisken 1985, (2.29)] that

$$\frac{1}{2n} \int_{\Omega_\eta} |H|^2 \eta \, dx \ge \frac{\tilde{\alpha}}{2n} \int_{A(k)} |H|^2 z \, d\mathcal{H}^n.$$

For the second term on the right-hand side of (38), and again by [Huisken 1985, (2.29)], we estimate

$$\begin{aligned}
\int_{\Omega_\eta} \Big(1 + \frac{|\nabla^S v|}{W}\Big)\, \eta \, dx &\le \int_{A(k)} v z \, W^{-1} \, d\mathcal{H}^n + \int_{\Omega_\eta} \frac{|\nabla^S v|}{W}\, v z \, dx \\
&\le 2 \int_{A(k)} z \, d\mathcal{H}^n + \frac{\tilde{\tilde{\varepsilon}}}{2} \int_{\Omega_\eta} \frac{|\nabla^S v|^2}{W} z \, dx + \frac{1}{2\tilde{\tilde{\varepsilon}}} \int_{\Omega_\eta} \frac{v^2}{W} z \, dx \\
&\le 2 \int_{A(k)} z \, d\mathcal{H}^n + \frac{\tilde{\tilde{\varepsilon}}}{2} \int_{\Omega_\eta} \frac{|\nabla^S v|^2}{W} z \, dx + \frac{1}{2\tilde{\tilde{\varepsilon}}} \int_{A(k)} z \, d\mathcal{H}^n.
\end{aligned}$$

For the third term on the right-hand side of (38), and in view of [Huisken 1985, (2.20)], we have

$$\begin{aligned}\int_\Sigma \eta\, d\mathcal{H}^{n-1} &= \int_\Sigma vz\, d\mathcal{H}^{n-1}\\ &\le \int_{A(k)} |\nabla^S z|\, d\mathcal{H}^n + \int_{A(k)} (|H| + |\nabla^S \nu|)\, z\, d\mathcal{H}^n\\ &\le \frac{\tilde\varepsilon}{2}\int_{A(k)} |\nabla^S z|^2\, d\mathcal{H}^n + \frac{1}{2\tilde\varepsilon}\int_{A(k)} d\mathcal{H}^n + c_6 \int_{A(k)} z\, d\mathcal{H}^n,\end{aligned}$$

with $c_6 = c_6(|H|_\Omega, |D\nu|_\Omega)$.

Taking into consideration all the estimates following (40), we can easiliy obtain (40) from (38).

Inequality (40) is exactly of the same form as [Huisken 1985, (2.34)], and the subsequent procedure in that paper is independent of the choice of the function H prescribing the mean curvature of the surface S. Therefore, we can conclude in the same manner that

$$w = \log v \le k_0 + c\,|A(k_0)|,$$

where $k_0 = k_0(\tilde\alpha, n)$ and $c = c\big(n, \tilde\alpha, \Omega, |D\beta|_\Omega, |D\nu|_\Omega\big)$.

This concludes the gradient estimate in a neighborhood of the boundary $\Sigma = \partial\Omega$, which we state in Theorem 3.1 below.

Definition. We call a domain *admissible* if it is open, bounded, simply connected, and of class $C^{2,\mu}$.

This definition is such that we are able to obtain uniform height bounds as in Section 2. The following theorem would still be true if we just assumed these uniform bounds instead. (For the notation $\mathrm{ISC}_{\delta,\pi-\gamma}$ see [Finn and Gerhardt 1977, pp. 15–16].)

Theorem 3.1. *Let Ω be an admissible domain. Assume u to be a $C^2(\Omega)$ solution of (31), with the function $\mathcal{F}(u)$ defined either as in (15), or as in (16), in which case we also require an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ when $\gamma > \pi/2$ (for the uniform height estimates to hold). We denote by β, ν the Lipschitz extensions into the interior of Ω of β and ν_Σ, and assume β to satisfy (32); that is, $|\beta| \le 1 - \tilde\alpha$ with $\tilde\alpha > 0$, and $|\nu| \le 1$. Then there exists a constant $C = C\big(n, \tilde\alpha, \Omega, |D\beta|_\Omega, |D\nu|_\Omega\big)$ such that*

$$|Du| \le C$$

in a neighborhood of the boundary Σ.

The interior gradient estimate in admissible domains can be obtained by means of the maximum principle:

Theorem 3.2. *Assume Ω and u satisfy the assumptions of Theorem 3.1. Then*

$$|Du| \le C$$

in Ω, with C the constant of Theorem 3.1.

Proof. We rearrange the equation (31), satisfied by u in Ω, to find

$$a_{ij} D_i D_j u - \mathcal{F}(u)\sqrt{1+|Du|^2} + a^2 = 0, \tag{41}$$

where

$$a_{ij} = \delta_{ij} - \frac{D_i u\, D_j u}{1+|Du|^2}.$$

By general elliptic theory we can assume the local existence of derivatives of all orders for u. We differentiate (41) with respect to x_k, for any $k \in \{1, \dots, n\}$, and set $D_k u = v$ to obtain

$$a_{ij} D_i D_j v + b_i D_i v + cv = 0$$

in Ω, with $c = -\mathcal{F}'(u)\sqrt{1+|Du|^2} \le 0$. The equation satisfied by v is elliptic, and we can apply the maximum or minimum principle to deduce the claimed interior gradient bound. □

With Theorems 3.1 and 3.2, we have the main result of this section:

Theorem 3.3. *Under the assumptions of Theorem 3.1, there is a constant $M > 0$, such that for any solution u of (31) we have*

$$|Du| \le M.$$

4. Existence in smooth domains, uniqueness of solutions and nonexistence results

The following result is contained in Theorem 2.2.

Theorem 4.1 (Uniqueness). *Suppose $\mathcal{F}' > 0$, and let $u(x)$, $v(x)$ be solutions of (12) in a domain Ω with boundary $\Sigma = \partial\Omega$ of class C^1, which satisfies an internal sphere condition. We suppose $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We suppose further that on Σ there holds $\nu \cdot Tu = \nu \cdot Tv$. Then $u(x) \equiv v(x)$ in Ω.*

For the case of domains with corner points, we refer to Theorem 2.9 above.

The gradient estimates enable us now to prove existence in domains with $C^{2,\mu}$ boundary using a continuity method.

Theorem 4.2. *Assume Ω to be an admissible domain, and consider the problem (31), with H defined as in (33), $\mathcal{F}$ given by either (15) or (16), $0 < \gamma < \pi$ and β taken to be $C^{1,\mu}$ in its arguments. If $\mathcal{F}$ is as in (16), and if $\gamma > \pi/2$, we assume in addition an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$.*

Then the problem (31) has a unique solution $u \in C^{2,\mu}(\overline{\Omega})$, where the exponent μ, $0 < \mu < 1$ depends on the above quantities.

Remark. If $0 < \gamma < \pi/2$ we have uniform height estimates from above and below for both cases (15) and (16), as shown by Corollary 2.7 and Theorem 2.8. For $\pi/2 < \gamma < \pi$ and for $\mathcal{F}$ is as in (15), the height estimates also hold, but for case (16) an additional internal sphere condition is needed in the statement of the theorem in order for the uniform lower height estimate to hold. The condition on δ is optimal as discussed in the remark on nonexistence following the proof.

The cases $\gamma = 0$ and $\gamma = \pi$ are not considered due to assumption (32), which is essential for the gradient estimate.

Proof of Theorem 4.2. The proof follows exactly the steps in [Gerhardt 1976, proof of Theorem 2.1]; we only outline it here.

For $\tau \in \mathbb{R}$, $0 < \tau < 1$, consider the problem

$$-\operatorname{div}\frac{Du_\tau}{\sqrt{1+|Du_\tau|^2}} + \tau H(x, u_\tau, Du_\tau) = 0 \qquad \text{in } \Omega, \tag{42}$$
$$Tu_\tau \cdot \nu = \tau\beta \qquad \text{on } \Sigma.$$

One then proves that the set

$$T = \{\tau : \text{there exists a solution } u_\tau \in C^2(\overline{\Omega})\}$$

is open and closed.

The idea is to look at a uniformly elliptic operator that coincides with the given one in (42) whenever $|Du_\tau|_\Omega \le K$ for some constant K. This allows us to apply [Ladyzhenskaya and Ural'tseva 1968, Chapter 10, Theorem 2.2]; the change from the equation considered in [Gerhardt 1976], namely that the H term is different, does not interfere. Everything else follows verbatim. □

Remark on nonexistence. If $\gamma > \pi/2$, then in the case (16) existence can fail if $\delta < 2\chi\sigma/\rho_0 - \chi p_0$. To see that, we integrate (12) over Ω, obtaining

$$\int_\Omega \left(\frac{\rho_0 - \chi p_0}{\chi\sigma}(e^{\chi g u} - 1) + \chi g(1 - \cos\omega) \right) dx = 2\pi|\Sigma|\cos\gamma$$

from which it follows that

$$\frac{\rho_0 - \chi p_0}{\chi\sigma}|\Omega| > -|\Sigma|\cos\gamma,$$

which leads to a contradiction if the domain is scaled to be small enough. For the special case of a disk B_δ, we obtain

$$\delta > -\frac{2\chi\sigma}{\rho_0 - \chi p_0}\cos\gamma$$

providing a slight improvement over the criterion yielded by Theorem 2.3.

This last result applies to the "unconstrained" case of an open circular tube dipped into an infinite reservoir of fluid. Physically, it signifies that if the tube is too narrow, the fluid will disappear down the tube to negative infinity. Finn and Luli [$\geq$ 2007] studied the "constrained" case of a circular tube closed at the bottom and filled with a prescribed mass of fluid. For that problem they were able to show that for any γ with $0 \leq \gamma < \pi$, and for any prescribed total mass $\mathcal{M}$, there is at least one symmetric solution of the problem, and that the height for this solution will lie over any prescribed level if $\mathcal{M}$ is sufficiently large. If $\gamma \leq \pi/2$, the solution is unique among symmetric solutions with the prescribed mass. From Theorem 2.2 then follows that the solution is unique among all solutions with the same Lagrange parameter.

5. Existence of solutions in domains with corners

For this section we need Theorem 4 of [Ladyzhenskaya and Ural'tseva 1970], which adapted to our situation yields:

Theorem 5.1. *Let u be a classical solution of*

$$\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}} = \mathcal{F}(u) - \frac{a^2}{\sqrt{1+|Du|^2}},$$

in a bounded domain Ω, with $\mathcal{F}$ as defined in (15) or (16). Assume $\sup_\Omega |u| \leq M$. *Then for any strictly interior subdomain Ω' of Ω with $d :=$* $\operatorname{dist}(\Omega', \partial\Omega)$,

$$\max_{\Omega'} |Du(x)| \leq C,$$

with $C = C(n, M, d)$. (Compare also [Simon 1977, Theorem 2$''$].)

For the existence result in this section, we assume the domain Ω to be open and bounded, with piecewise $C^{1,\mu}$ boundary Σ and to have a finite number of "well-behaved" corners. By this we mean that if a corner is located at the point O, we can parametrize the arcs on either side of O by smooth functions $c_i(s)$, $0 < s < s_0$, $i = 1, 2$, such that $\lim c_i(s) = O$, as $s \to 0$, with an angle $0 < 2\alpha < \pi$ formed by $\lim c_i'(s)$ as $s \to 0$.

Ω can be exhausted by an expanding sequence of admissible domains $\Omega^j \subset \Omega$, whose boundaries Σ^j converge uniformly in C^1 in any neighborhood U_{x_0} of a boundary point $x_0 \in \Sigma$, whose closure $\overline{U}_{x_0} \cap \Sigma$ lies in the smooth portion of Σ.

With similar arguments as in [Finn and Gerhardt 1977, Theorem 1], we can prove the following existence result:

Theorem 5.2. *Let Ω be as described above. Let γ be constant and $0 < \gamma < \pi$. In the case of $\mathcal{F}$ being given by (16) and if $\gamma > \pi/2$ we also require an internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ to hold for Ω^j.*

Then there exists a variational solution u of (13). If $|\gamma - \pi/2| < \alpha$ then $u \in \mathcal{Q}(\Omega)$.

If u, v are two variational solutions of (12)+(15), there holds

$$|u - v| < \frac{\chi\sigma}{\rho_0};$$

for variational solutions of (12)+(16) such that $u, v > -A > -\infty$, there holds

$$|u - v| < \frac{\chi\sigma}{\rho_0 - \chi p_0} e^{\chi g A}.$$

Proof. In view of the conditions on Ω^j, we can obtain a solution $u^j \in C^{2,\alpha}(\Omega^j)$ of

$$\operatorname{div} T u^j = \mathcal{F}(u^j) - \frac{a^2}{W^j} \tag{43}$$

in each Ω^j, with fixed boundary data γ on Σ^j, as in Theorem 4.2 above. They will satisfy the corresponding weak form; i.e., they will be variational solutions, each u^j satisfying (13) in Ω^j:

$$\int_{\Sigma^j} \eta \cos\gamma \, ds = \int_{\Omega^j} \left(D\eta \cdot T u^j + \eta\, \mathcal{F}(u^j) - \eta \frac{a^2}{\sqrt{1 + |Du^j|^2}} \right) dx, \tag{44}$$

for every $\eta \in \mathcal{Q}(\Omega^j)$.

In view of the assumption on the contact angle γ and the additional $\mathrm{ISC}_{\delta,\pi-\gamma}$ condition on Ω^j in case (16), the height and gradient estimates (Theorems 2.6 and 3.3) and the existence results hold for u^j in Ω^j without any additional restrictions being needed. As the height estimates depend on the distance of Σ^j to a corner, and the gradient estimates depend on the Lipschitz extension of the normal to the boundary, these estimates are not uniform in j. To overcome this obstacle, for any fixed j_0 we consider a fixed j_1, and solutions u^j in Ω^j, where $j \geq N(j_1) > j_1 > j_0$, such that the distance from Ω^{j_0} to $\partial\Omega^{j_1}$ and from Ω^{j_1} to $\partial\Omega^j$ is strictly positive. These u^j will satisfy (43) in Ω^{j_1} and Ω^{j_0}. The height bounds in Ω^{j_1} are uniform, as shown in Theorems 2.6 and 2.9, and by Theorem 5.1 we obtain uniform gradient bounds in Ω^{j_0}. Therefore, in Ω^{j_0} we have uniform height and gradient bounds.

Using general results on elliptic equations [Ladyzhenskaya and Ural'tseva 1968, Chapter 10, Theorem 2.2], we can extend the uniform height and gradient estimates to higher regularity of the solutions u^j ($j \geq N(j_0)$) of (43) in Ω^{j_0}, for every j_0.

Using the Arzelà–Ascoli theorem we can find a subsequence (not relabeled), converging uniformly together with all its derivatives in any Ω^{j_0}, to a solution $u(x)$ of (43).

We choose $\eta \in \mathcal{Q}(\Omega)$, so that in particular $\eta \in \mathcal{Q}(\Omega^j)$. We remark that $\eta \in H^{1,1}(\Omega)$ has a well-defined trace function in $L^1(\partial\Omega)$, which we denote again by η. We also note that $\eta \in H^{1,1}(\Omega)$ can be approximated in the $H^{1,1}$ norm by uniformly continuous functions in Ω. Their boundary values approximate the trace of η on $\partial\Omega$ in the $L^1(\partial\Omega)$ norm; see [Giusti 1984, Theorem 2.11].

We consider (44). Regarding the convergence of the right-hand side of (44), we again fix j_0, and note that $|Tu^j| < 1$ and $a^2/\sqrt{1+|Du^j|^2} \le a^2$ in Ω. We have

$$\left|\int_{\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}\right) dx\right| \le c,$$

with c depending on $|\eta|_{L^1(\Omega)}$, $|D\eta|_{L^1(\Omega)}$, and the size of Ω, but independent of j_0. Also, given the uniform convergence of u^j and Du^j in Ω^{j_0}, we have

$$\lim_{j\to\infty} \int_{\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}\right) dx = \int_{\Omega^{j_0}} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}}\right) dx. \tag{45}$$

Now we can let j_0 vary, and conclude that the first and third terms on the right-hand side of (44) converge to

$$\int_{\Omega} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}}\right) dx.$$

To see this we consider

$$\begin{aligned}
&\left|\int_{\Omega^{j}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}\right) dx - \int_{\Omega} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}}\right) dx\right| \\
&\quad\le \left|\int_{\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}\right) dx - \int_{\Omega^{j_0}} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}}\right) dx\right| \\
&\qquad + \left|\int_{\Omega^{j}-\Omega^{j_0}} \left(D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}\right) dx\right| \\
&\qquad + \left|\int_{\Omega-\Omega^{j_0}} \left(D\eta \cdot Tu - \eta \frac{a^2}{\sqrt{1+|Du|^2}}\right) dx\right|.
\end{aligned}$$

By (45) the first term on the right-hand side is less than $\varepsilon/3$ for $j > J$, for large enough J. The second and third terms on the right-hand side can be estimated by $c\big(|\eta|_{L^1(\Omega)}, |D\eta|_{L^1(\Omega)}\big)|\Omega^j - \Omega^{j_0}| < \varepsilon/3$ and $c\big(|\eta|_{L^1(\Omega)}, |D\eta|_{L^1(\Omega)}\big)|\Omega - \Omega^{j_0}| < \varepsilon/3$, respectively, due to the convergence of Ω^j to Ω, for $j > J$ and appropriately large j_0.

For the left-hand side of (44), we remark that $|\eta \cos\gamma|$ is bounded. Therefore

$$\int_{\Sigma^j \cap B_r(O)} \eta \cos\gamma \, ds \to 0 \ \text{ for } r \to 0, \quad \text{uniformly in } j, \tag{46}$$

where $B_r(O)$ denotes a ball of small radius r centered at a corner O. We also have

$$\int_{\Sigma \cap B_r(O)} \eta \cos\gamma \, ds \to 0 \ \text{ for } r \to 0. \tag{47}$$

In what follows it suffices to assume Ω to have only one corner, O.

In the following estimate we split integrals into their parts over $B_r(O)$ and its complement, $B_r^c(O)$.

$$\begin{aligned} \Bigl| \int_{\Sigma^j \cap B_R(O)} \eta \cos\gamma \, ds &- \int_{\Sigma \cap B_R(O)} \eta \cos\gamma \, ds \Bigr| \\ &\le \Bigl| \int_{\Sigma^j \cap B_r(O)} \eta \cos\gamma \, ds \Bigr| + \Bigl| \int_{\Sigma \cap B_r(O)} \eta \cos\gamma \, ds \Bigr| \\ &\quad + \Bigl| \int_{\Sigma^j \cap B_r^c(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_r^c(O)} \eta \cos\gamma \, ds \Bigr|. \end{aligned} \tag{48}$$

We choose r sufficiently small to ensure that the first and second summands on the right are each less than $\varepsilon/3$, by (46) and (47) respectively.

By the assumptions on the convergence of Σ^j in neighborhoods of Σ not containing corners, the last summand $\bigl|\int_{\Sigma^j \cap B_r^c(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_r^c(O)} \eta \cos\gamma \, ds\bigr|$ is also less than $\varepsilon/3$ for any $j > J$, with J large enough. We thus obtain from (48)

$$\Bigl| \int_{\Sigma^j \cap B_R(O)} \eta \cos\gamma \, ds - \int_{\Sigma \cap B_R(O)} \eta \cos\gamma \, ds \Bigr| < \varepsilon,$$

and the convergence of the boundary integral in (44) to $\int_\Sigma \eta \cos\gamma \, ds$ is proved.

Having shown the convergence of all terms of (44) except $\int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx$, we conclude that this term converges too. We will show it converges to $\int_\Omega \eta \, \mathcal{F}(u) \, dx$.

We know that u^j satisfies (44) in Ω^j, which we rewrite as

$$\int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx = \int_{\Omega^j} [D\eta \cdot Tu^j - \eta \frac{a^2}{\sqrt{1+|Du^j|^2}}] \, dx - \int_{\Sigma^j} \eta \cos\gamma \, ds.$$

The right-hand side here converges; therefore

$$\Bigl| \int_{\Omega^j} \eta \, \mathcal{F}(u^j) \, dx \Bigr| \le c,$$

with a constant c depending on η, but independent of j.

We consider two cases:

(i) *The angle γ satisfies $|\gamma - \pi/2| < \alpha$.* In view of Theorem 2.9 we have uniform height bounds on u^j, independent of j, since they are independent of the distance of Σ^j to the corner O, and the same bounds hold for the limit function u. We fix j_0 as before, and with the continuity of $\mathcal{F}$, we have $\mathcal{F}(u^j)$ converging to $\mathcal{F}(u)$ in any Ω^{j_0}, and the corresponding uniform bounds for $\mathcal{F}(u)$. So

$$\lim_{j\to\infty} \int_{\Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx = \int_{\Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx$$

and, using the uniform bounds on $\mathcal{F}(u)$ over all of Ω, we can let $j_0 \to \infty$ and obtain the result as in the previous considerations.

(ii) *The angle γ satisfies $|\gamma - \pi/2| > \alpha$.* In this case the height estimates will depend on the distance of Σ^j to the corner, u^j becoming unbounded as we approach O. However the growth of $|u^j|$ means that $|\mathcal{F}(u^j)|$ is proportional to r^{-1}, as proved in Theorem 2.9.

Again, it suffices to assume Ω to have only one corner, O. We estimate, after adding and subtracting the terms $\int_{\Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx$ and $\int_{\Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx$,

$$\begin{aligned} \left| \int_{\Omega^j} \eta\, \mathcal{F}(u^j)\, dx - \int_{\Omega} \eta\, \mathcal{F}(u)\, dx \right| &\le \left| \int_{\Omega^j - \Omega^{j_0}} \eta\, \mathcal{F}(u^j)\, dx \right| + \left| \int_{\Omega - \Omega^{j_0}} \eta\, \mathcal{F}(u)\, dx \right| \\ &\quad + \left| \int_{\Omega^{j_0}} \bigl(\eta\, \mathcal{F}(u^j) - \eta\, \mathcal{F}(u)\bigr)\, dx \right|. \end{aligned} \tag{49}$$

The last summand on the right is less than $\varepsilon/4$ for all $j > J$, with J large enough; to see this, use the continuity of $\mathcal{F}$ and the uniform bounds in Ω^{j_0}.

The second summand on the right can be estimated by

$$\begin{aligned} \int_{\Omega - \Omega^{j_0}} |\eta \mathcal{F}(u)|\, dx &\le \sup|\eta| \int_{\Omega - \Omega^{j_0}} |\mathcal{F}(u)|\, dx \\ &\le \sup|\eta| \biggl(\int_{(\Omega - \Omega^{j_0}) \cap B_r(O)} |\mathcal{F}(u)|\, dx + \int_{(\Omega - \Omega^{j_0}) \cap B_r^c(O)} |\mathcal{F}(u)|\, dx \biggr), \end{aligned}$$

where $B_r^c(O)$ denotes the complement in $\mathbb{R}^2$ of the disk $B_r(O)$.

In $(\Omega - \Omega^{j_0}) \cap B_r^c(O)$, we are at a positive distance from O, and have bounds for $\mathcal{F}(u)$, so

$$\sup|\eta| \int_{(\Omega - \Omega^{j_0}) \cap B_r^c(O)} |\mathcal{F}(u)|\, dx < \varepsilon/4,$$

due to the convergence of Ω^j to Ω, for large j_0, after possibly adjusting the previous choice of J.

For the integral over $(\Omega - \Omega^{j_0}) \cap B_r(O)$ we introduce polar coordinates and can show, using the inequality $|\mathcal{F}(u)| < Cr^{-1}$, that

$$\begin{aligned}\sup|\eta| \int_{(\Omega-\Omega^{j_0})\cap B_r(O)} |\mathcal{F}(u)|\ dx &\le \sup|\eta| \int_{B_r(O)} |\mathcal{F}(u)|\ dx \\ &\le \sup|\eta| \int_0^r \int_0^{2\pi} C\cos\theta\, d\theta\, dr\ \le \varepsilon/4,\end{aligned}$$

after choosing r appropriately small.

The first summand on the right in (49) can be dealt with in a similar way, but more easily, since $\Omega^j \cap B_r(O) = \varnothing$ for small r.

Returning to (49), we have shown that $\int_{\Omega^j} \eta\, \mathcal{F}(u^j)\, dx$ converges to $\int_\Omega \eta\, \mathcal{F}(u)\, dx$ as $j \to \infty$.

We had approximated $\eta \in H^{1,1}(\Omega)$ and worked with uniformly continuous functions. We have shown that u satisfies

$$\int_\Omega \left(D\eta \cdot Tu + \eta\, \mathcal{F}(u) - \eta \frac{a^2}{\sqrt{1+|Du|^2}} \right) dx = \int_\Sigma \eta \cos\gamma\, ds$$

for such η. Going over to $\eta \in \mathcal{Q}(\Omega)$, we conclude in both cases that u is a variational solution in Ω.

By Theorem 2.9, u is bounded if $|\gamma - \pi/2| < \alpha$ and therefore $u \in \mathcal{Q}(\Omega)$.

We also remark that if we have different limits u and v obtained by two different subsequences, we still know that they are not "too far apart", in the sense of the estimate given in Theorem 2.3. We emphasize that this is true even though the solution might become unbounded when approaching a corner. □

Remark 1. Theorem 5.2 is the best possible result one can obtain for this problem. As observed in Section 4, existence fails in small domains in the case (16) and $\gamma > \pi/2$. This is taken care of by the internal sphere condition $\mathrm{ISC}_{\delta,\pi-\gamma}$ with $\delta > 2\chi\sigma/(\rho_0 - \chi p_0)$ for Ω^j.

Remark 2. We obtain a variational solution for our problem in both cases (15) and (16) after imposing the additional $\mathrm{ISC}_{\delta,\pi-\gamma}$, for general $0 < \gamma < \pi$, despite the fact that the values for any solution become unbounded in a narrow corner, when $|\gamma - \pi/2| > \alpha$, as stated in Theorem 2.9.

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BEHAVIOR OF SOME CMC CAPILLARY SURFACES AT CONVEX CORNERS

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We construct examples of nonparametric surfaces $z = h(x, y)$ of zero mean curvature which satisfy contact angle boundary conditions in a cylinder in $\mathbb{R}^3$ over a convex domain Ω with corners. When the contact angles for two adjacent walls of the cylinder differ by more than $\pi - 2\alpha$, where 2α is the opening angle between the walls, the (bounded) solution h is shown to be discontinuous at the corresponding corner. This is exactly the behavior predicted by the Concus–Finn conjecture. These examples currently constitute the largest collection of capillary surfaces for which the validity of the Concus–Finn conjecture is known and, in particular, provide examples for all contact angle data satisfying the condition above for opening angles $2\alpha \in (\pi/2, \pi)$.

1. Introduction

Let Ω be an open set in the plane whose boundary is smooth except at a number of corner points. Assume that near each such corner point $P \in \partial\Omega$, the boundary consists of two curves, ω^+ and ω^-, meeting at P at an angle $2\alpha \in (0, \pi)$; this condition characterizes a *convex corner*. Let $\gamma : \partial\Omega \to [0, \pi]$ be continuous on each smooth piece of $\partial\Omega$, and assume that at each corner the limits

$$\lim_{\substack{Q \in \omega^+ \\ Q \to P}} \gamma(Q) =: \gamma_1 \quad \text{and} \quad \lim_{\substack{Q \in \omega^- \\ Q \to P}} \gamma(Q) =: \gamma_2$$

both exist. Also let $\Lambda = \Omega \times \mathbb{R}$ be the cylinder over Ω. We ask about the existence of a *capillary graph* over Ω with contact angle data γ; that is, does there exist a surface $z = h(x, y)$ defined over $\overline{\Omega} \setminus \{\text{corners}\}$, satisfying the physical conditions that characterize a liquid interface for prescribed values of gravity and density, and meeting the walls of Λ at the prescribed angle γ? (See Equation (2) for a formal statement.)

This question has received considerable interest. The *local* question of the existence and boundedness of a capillary graph near a corner P has been solved by

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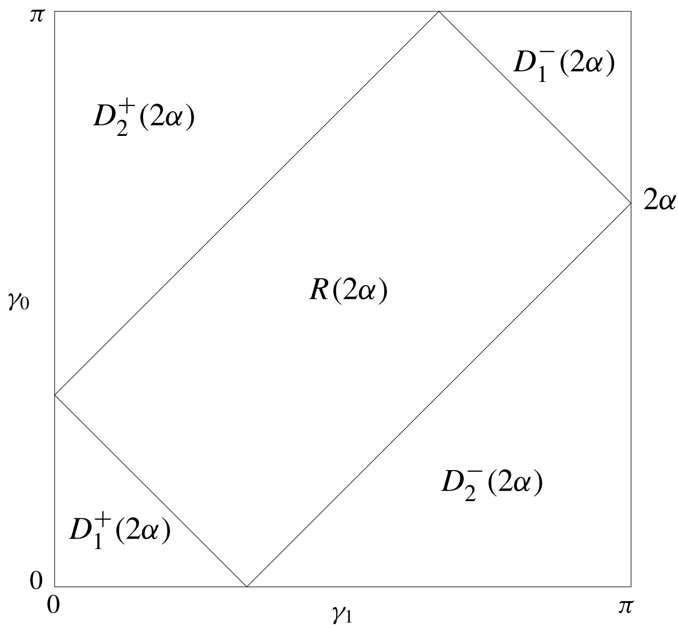

Figure 1. The Concus–Finn rectangle

Paul Concus and Robert Finn in all but one case. The current state of knowledge [Concus and Finn 1991; 1994; 1996; Finn 1986; 1996; Simon 1980; Tam 1986] is summarized by referring to Figure 1, in which the horizontal variable is γ_1, the vertical variable is γ_2 and the corner opening angle is 2α:

(i) A solution $z = h(x, y)$ will be continuous at P if $(\gamma_1, \gamma_2) \in R(2\alpha)$.

(ii) There is no solution if $\kappa = 0$ and $(\gamma_1, \gamma_2) \in D_1^{\pm}(2\alpha)$.

(iii) There is no solution which is bounded at P if $(\gamma_1, \gamma_2) \in D_1^{\pm}(2\alpha)$.

(iv) There can exist a bounded solution $z = h(x, y)$ if $(\gamma_1, \gamma_2) \in D_2^{\pm}(2\alpha)$.

In case (iv), the continuity of the solution at P is unknown, but we have:

Conjecture [Concus and Finn 1996; Finn 1996]. *A local capillary graph at a corner P with data from a $D_2(2\alpha)$ domain has a jump discontinuity at P, whether in zero gravity or not.*

Fix $\delta \in (0, \pi/4)$ and consider the diamond-shaped region $\Omega \subset \mathbb{R}^2$ symmetric with respect to the coordinate axes and having vertices $(0, \pm 1)$ and $(\pm \tan\delta, 0)$. Label the vertices as in Figure 2, so the convex angle OAB has measure δ and the convex angle ABC has measure $2\alpha = \pi - 2\delta$. As before, set $\Lambda = \Omega \times \mathbb{R}$.

Let $\gamma_1, \gamma_2 \in (0, \pi)$ satisfy

(1) $$|\gamma_1 + \gamma_2 - \pi| \le 2\alpha \quad \text{and} \quad |\gamma_1 - \gamma_2| > \pi - 2\alpha;$$

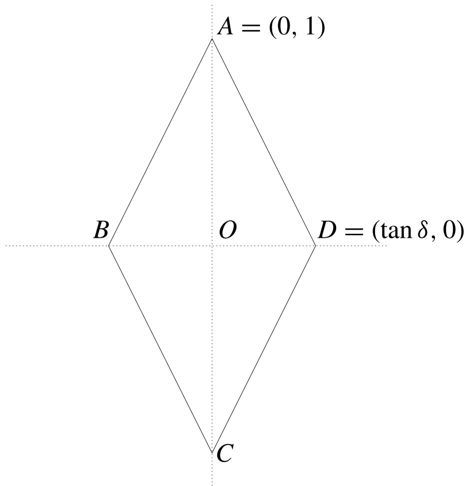

Figure 2. The quadrilateral domain Ω with $OAB = \delta$.

this is equivalent to saying that $(\gamma_1, \gamma_2) \in D_2^+(2\alpha) \cup D_2^-(2\alpha)$. Define the function $\gamma : \partial\Omega \to \mathbb{R}$ by

$$\gamma(x, y) = \begin{cases} \gamma_1 & \text{if } (x, y) \in AB, \\ \gamma_2 & \text{if } (x, y) \in BC, \\ \pi - \gamma_2 & \text{if } (x, y) \in CD, \\ \pi - \gamma_1 & \text{if } (x, y) \in DA. \end{cases}$$

We now formally define the capillary problem in the cylinder Λ with contact angle boundary data γ, gravitational constant $\kappa \geq 0$ and Lagrange multiplier λ. By a solution of this problem, we mean a function $h : \Omega \to \mathbb{R}$ with

$$h \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{A, B, C, D\})$$

which satisfies

$$\begin{aligned} \operatorname{div}(Th) &= \kappa h + \lambda \quad \text{in } \Omega\,, \\ Th \cdot \boldsymbol{\nu} &= \cos\gamma \qquad \text{on } \partial\Omega \setminus \{A, B, C, D\}, \end{aligned} \tag{2}$$

where $\boldsymbol{\nu}$ is the outer unit normal to $\partial\Omega$ and

$$Th = \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}},$$

as in [Finn 1986]. We are interested in the behavior of the solution in zero gravity, $\kappa = 0$. In this case the divergence theorem together with (2) implies

$$\lambda|\Omega| = \int_\Omega \operatorname{div}(Th)\, dA = \int_{\partial\Omega} \cos\gamma\, ds.$$

Since $\cos\gamma(x, y)$ is an odd function of x, we see that $\lambda = 0$. This means a solution h will be a minimal surface. The contact angles from each side at B are γ_1 and γ_2 and at D are $\pi-\gamma_2$ and $\pi-\gamma_1$. Our principal interest is in the behavior of solutions at B and D and our proof will focus on the behavior of solutions at B.

Theorem 1.1. *Suppose $\gamma_1, \gamma_2 \in (0, \pi)$ satisfy (1). Let Ω and γ be as defined above. There exists a unique solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{A, B, C, D\})$ of the boundary value problem (2) with $\kappa = \lambda = 0$ which satisfies $h(0, 0) = 0$. This solution is discontinuous at B and D, continuous at A if and only if $|\gamma_1 - \pi/2| \le \pi/2 - \delta$, and continuous at C if and only if $|\gamma_2 - \pi/2| \le \pi/2 - \delta$.*

To prove this theorem, we first isolate and prove the most difficult case:

Lemma 1.2. *Suppose $\gamma_1 \in [\delta, \pi/2]$ and $\gamma_2 \in [\pi/2, \pi-\delta]$ satisfy (1). Let Ω and γ be as above. There exists a unique solution*

$$h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{B, D\})$$

of the boundary value problem (2) with $\kappa = \lambda = 0$ which satisfies $h(0, 0) = 0$. This solution h is discontinuous at B and D.

One accomplishment in this paper is that it provides an example of a capillary surface for each corner angle $2\alpha = \pi - 2\delta \in (\pi/2, \pi)$ and each pair of contact angles in the regions $D_2^{\pm}(\pi-2\delta)$ in which the validity of the Concus–Finn conjecture is unknown. The Concus–Finn conjecture is the principal outstanding open problem in the mathematical theory of capillarity. To the best of our knowledge, there is only one paper [Huff and McCuan 2006] that provides examples of capillary surfaces with data in the $D_2^{\pm}$ regions and in which the continuity of the nonparametric capillary surface at the corner is determined; it considers contact angle data only along the line $\gamma_1 + \gamma_2 = \pi$. We give here, then, the first collection of examples corresponding to all of the contact angle pairs in $D_2^{\pm}(\pi-2\delta)$ in which the continuity at the corner is determined, and in these examples the Concus–Finn conjecture correctly predicts the behavior at B and D of these capillary surfaces.

2. Proof of Lemma 1.2

Assume the hypotheses of Lemma 1.2 hold. We then know from (1) that $\gamma_2 - \gamma_1 > 2\delta$. We begin by assuming

$$\gamma_1 < \pi/2 < \gamma_2, \quad \text{that is,} \quad \gamma_1, \gamma_2 \neq \pi/2.$$

Let Ω_0 be the portion of Ω in $\{x < 0\}$, so that $\partial\Omega_0$ is the triangle with successive vertices A, B and C. Let $B_1 = \{w \in \mathbb{C} : |w| < 1\}$ and set

$$E_0 = \{w \in B_1 : \operatorname{Im} w > 0, |w - w_1| > \tan\gamma_1, |w - w_3| > \tan(\pi-\gamma_2)\},$$

where

$$w_1 = u_1 + i v_1 = -\cos\delta \sec\gamma_1 + i \sin\delta \sec\gamma_1,$$
$$w_3 = u_3 + i v_3 = \cos\delta \sec(\pi-\gamma_2) + i \sin\delta \sec(\pi-\gamma_2).$$

Also set $r_1 = \tan\gamma_1$ and $r_3 = \tan(\pi-\gamma_2)$. Let $E = \bar{E}_0$ (Figure 3). We remark that E will eventually be shown to be the image of the stereographic projection of the Gauss map to the (closure of the) graph of the nonparametric solution h over $\overline{\Omega}_0$ when this graph is given a downward orientation. Now E is a connected, simply connected subset of the closed unit disk which is star-like with respect to the origin. The boundary of E consists of portions of the circles

$$C_1 = \{w : |w - w_1| = \tan\gamma_1\} \quad \text{and} \quad C_3 = \{w : |w - w_3| = \tan(\pi-\gamma_2)\}$$

(which are orthogonal to the unit circle ∂B_1), the real axis ($v = 0$) and the unit half-circle $\{w \in \partial B_1 : \operatorname{Im} w \geq 0\}$. The condition $\gamma_2 - \gamma_1 > 2\delta$ implies (and is actually equivalent to) $C_1 \cap C_3 = \varnothing$.

Write

$$\sigma_1 = \partial E \cap C_1, \quad \sigma_2 = \{w \in \partial E : \operatorname{Im}(w) = 0\},$$
$$\sigma_3 = \partial E \cap C_3, \quad \sigma_4 = \partial E \cap \partial B_1.$$

We denote the corners of ∂E (in counterclockwise order) as t_1, t_2, t_3 and t_4, with $t_1, t_2 \in \sigma_1$, $t_3, t_4 \in \sigma_3$, $t_1, t_4 \in \partial B_1$ and $\operatorname{Im} t_2 = \operatorname{Im} t_3 = 0$. Notice that $t_1 = e^{(\pi-\delta-\gamma_1)i}$ and $t_4 = e^{(\delta+\pi-\gamma_2)i}$.

Our numbering scheme, associating center w_3 and circle C_3 with the cylinder side BC whose prescribed contact angle is γ_2, is chosen because it provides a clearer and more consistent notation for the Riemann–Hilbert problem we will consider later. There is no "second" circle C_2 (unless one wishes to consider the

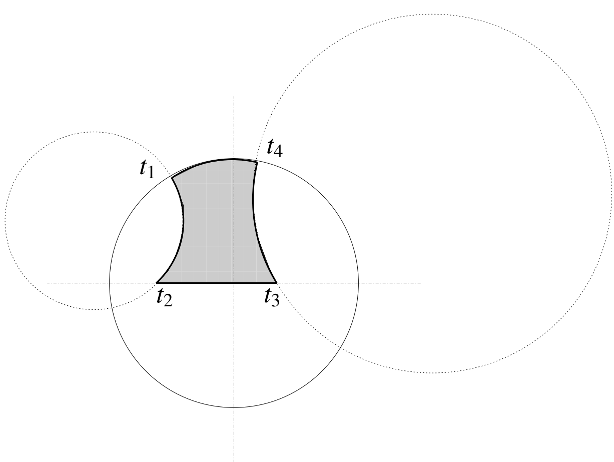

Figure 3. The region E.

line $v = 0$ to be a circle with infinite radius) and no "second" center w_2. If we wished to introduce a fourth circle, we would set $C_4 = \partial B_1$ and $w_4 = 0$.

Define $g : E \to E$ by $g(w) = w$. Our goal is to find $f \in C^0(E \setminus \{t_1, \dots, t_4\})$ with, at worst, integrable singularities at t_1, t_2, t_3, and t_4 which is analytic in E_0 and to define $X \in C^0(E : \mathbb{R}^3) \cap C^2(E_0 : \mathbb{R}^3)$ with

$$X(u + iv) = (x(u, v), y(u, v), z(u, v)) \tag{3}$$

and $K(u + iv) = (x(u, v), y(u, v))$ for $u + iv \in E$, satisfying certain conditions:

(i) The analytic functions (f, g) form the Weierstrass representation of X (see [Osserman 1969], for instance, or [Huff 2006] in this volume).

(ii) K is a homeomorphism between σ_1 and the line segment AB, between σ_2 and AC, and between σ_3 and BC.

(iii) K is constant on σ_4.

Here we say f has an integrable singularity at t_k if and only if $|f(w)| \le C|w - t_k|^s$ with $-1 < s < 0$ and $C \ge 0$ for w near t_k. Notice that $f \equiv 0$ corresponds to a "surface" consisting of a single point, as in (5) below, and therefore does not yield a solution of (2).

We now formulate the Riemann–Hilbert problem which we will solve by temporarily assuming the existence of a suitable function f.

The boundary requirements (ii) imply

$$\begin{cases} y(u, v) = \cot\delta\, x(u, v) + 1 & \text{for } u + iv \in \sigma_1, \\ x(u, v) = 0 \text{ and } 0 \le y(u, v) \le 1 & \text{for } u + iv \in \sigma_2, \\ y(u, v) = -\cot\delta\, x(u, v) - 1 & \text{for } u + iv \in \sigma_3. \end{cases} \tag{4}$$

Write $f(u + iv) = f_1(u, v) + i f_2(u, v)$, where f_1 and f_2 are real-valued.

Now (i) implies

$$\begin{cases} x_w = f(w)(1 - w^2)/2, \\ y_w = i f(w)(1 + w^2)/2, \\ z_w = w f(w); \end{cases} \tag{5}$$

for $w \in E$; see [Elcrat and Lancaster 1989, p. 1061], for example. Since $d/dw = \frac{1}{2}(\partial/\partial u - i\partial/\partial v)$, the equations above yield

$$\begin{aligned} x_u(u, v) &= \quad \operatorname{Re}(f(w)(1 - w^2)) = (1 - u^2 + v^2) f_1 + 2uv f_2, \\ x_v(u, v) &= -\operatorname{Im}(f(w)(1 - w^2)) = 2uv f_1 - (1 - u^2 + v^2) f_2, \end{aligned} \tag{6}$$

$$\begin{aligned} y_u(u,v) &= \quad \mathrm{Re}(if(w)(1+w^2)) = -2uvf_1 - (1+u^2-v^2)f_2, \\ y_v(u,v) &= -\mathrm{Im}(if(w)(1+w^2)) = -(1+u^2-v^2)f_1 + 2uvf_2, \end{aligned} \tag{7}$$

and

$$\begin{aligned} z_u(u,v) &= \quad \mathrm{Re}(2wf(w)) = 2(uf_1 - vf_2), \\ z_v(u,v) &= -\mathrm{Im}(2wf(w)) = 2(-vf_1 - uf_2), \end{aligned} \tag{8}$$

where we use the notation $w = u + iv$. If we parametrize σ_k, $k = 1, \dots, 4$, as

$$\sigma_k = \{w_k(t) = u_k(t) + iv_k(t)\},$$

we find that the equalities (4) imply, respectively,

$$\begin{aligned} &y_u u_1' + y_v v_1' = \cot\delta\,(x_u u_1' + x_v v_1'), \\ &x_u(u_2(t), 0) = 0, \\ &y_u u_3' + y_v v_3' = -\cot\delta\,(x_u u_3' + x_v v_3'), \end{aligned}$$

and that condition (iii) of the previous page implies

$$x_u u_4' + x_v v_4' = 0.$$

Now $(u_1(t) - u_1)^2 + (v_1(t) - v_1)^2 = r_1^2$ implies

$$\frac{u_1'(t)}{v_1'(t)} = -\frac{v_1(t) - v_1}{u_1(t) - u_1} = -\frac{\mathrm{Im}(w_1(t) - w_1)}{\mathrm{Re}(w_1(t) - w_1)}$$

and similarly for $u_3'(t)/v_3'(t)$. Recall that

$$\cot\delta = -\frac{\mathrm{Re}\,w_1}{\mathrm{Im}\,w_1} = -\frac{u_1}{v_1} \quad \text{and} \cot\delta = \frac{\mathrm{Re}\,w_3}{\mathrm{Im}\,w_3} = \frac{u_3}{v_3}.$$

If we rewrite $x_u, \dots, z_v$ in terms of f, u and v, we obtain

$$\mathrm{Re}\big((a_k(u,v) + ib_k(u,v))f(u+iv)\big) = 0 \tag{9}$$

when $(u,v) \in \sigma_k$, which we could also write as

$$a_k(u,v)f_1(u,v) - b_k(u,v)f_2(u,v) = 0,$$

for $k = 1, \dots, 4$, where

$$\begin{aligned} a_1(u,v) + ib_1(u,v) &= ie^{i\delta}(w - w_1)(e^{-2\delta i} - w^2) && \text{if } w = u + iv \in \sigma_1, \\ a_2(u,v) + ib_2(u,v) &= -1 && \text{if } u + iv \in \sigma_2, \\ a_3(u,v) + ib_3(u,v) &= ie^{-i\delta}(w - w_3)(e^{2\delta i} - w^2) && \text{if } w = u + iv \in \sigma_3, \\ a_4(u,v) + ib_4(u,v) &= (u+iv)^2 && \text{if } w = u + iv \in \sigma_4. \end{aligned}$$

We now define $a, b : \partial E \to \mathbb{R}$ by $a(u+iv) = a_k(u, v)$ and $b(u+iv) = b_k(u, v)$ if $u+iv \in \sigma_k$, for $k \in \{1, \dots, 4\}$, and define $G : \partial E \to \mathbb{C}$ by

$$G(w) = a(w) + ib(w).$$

We wish to find a function $f \in C^0(E \setminus \{t_1, \dots, t_4\})$ which is analytic in E_0 and satisfies

$$\operatorname{Re}(G(w) f(w)) = 0 \quad \text{for } \ w \in \partial E \setminus \{t_1, \dots, t_4\}.$$

This is a Riemann–Hilbert problem with discontinuous coefficients G; in the notation of [Monakhov 1983, Chapter 1, §4], this is a "Hilbert problem with piecewise Hölder coefficients" (see also [Athanassenas and Lancaster 2004]). In order to use the results in [Monakhov 1983], we need to compute the index of this Hilbert problem in an appropriate function class $O(m) = O(t_{k_1}, \dots, t_{k_m})$ for some $m \in \{0, \dots, 4\}$. Define $G_1 : \partial E \to \mathbb{C}$ by

$$G_1(w) = -\frac{\overline{G(w)}}{G(w)}.$$

Notice that $G_1(w) = -1$ for $w \in \sigma_2$ and $G_1(w) = -\bigl(\overline{w}/|w|\bigr)^4 = -\overline{w}^4$ for $w \in \sigma_4$. Set $\omega = e^{i\delta}$. Moreover

$$\begin{aligned}
G_1(w) &= |G(w)|^{-2} e^{-2\delta i} (\overline{w} - \overline{w}_1)^2 \bigl(\omega^2 - \overline{w}^2\bigr)^2 \\
&= |G(w)|^{-2} e^{-2\delta i} (\overline{w} - \overline{w}_1)^2 \bigl(\omega - \overline{w}\bigr)^2 \bigl(\omega + \overline{w}\bigr)^2 \quad \text{for } w \in \sigma_1, \\
G_1(w) &= |G(w)|^{-2} e^{2\delta i} (\overline{w} - \overline{w}_3)^2 \bigl(\overline{\omega} - \overline{w}\bigr)^2 \bigl(\overline{\omega} + \overline{w}\bigr)^2 \qquad \text{for } w \in \sigma_3.
\end{aligned}$$

For $k \in \{1, \dots, 4\}$, set

$$\theta_k = \frac{1}{2\pi} \bigl(\arg G_1(t_k - 0) - \arg G_1(t_k + 0) \bigr),$$

where $\arg G_1(t_k - 0)$ means the limit at t_k of the argument of G_1 along the arc σ_{k-1} (with σ_0 here being σ_4) and $\arg G_1(t_k + 0)$ means the limit at t_k of the argument of G_1 along the arc σ_k. The argument is taken to be continuous along each component of each set σ_k. We have

$$\begin{aligned}
&\arg G_1(t_1 - 0) = -\pi + 4\gamma_1 + 4\delta, && \arg G_1(t_1 + 0) = 4\gamma_1 + 4\delta - 2\pi, \\
&\arg G_1(t_2 - 0) = -2\bigl(\delta + \tau_{1B} + \lambda_{1B}\bigr), && \arg G_1(t_2 + 0) = \pi, \\
&\arg G_1(t_3 - 0) = \pi, && \arg G_1(t_3 + 0) = 2\delta - 2\bigl(\tau_{2B} + \lambda_{2B}\bigr), \\
&\arg G_1(t_4 - 0) = 4\pi - 4\gamma_2 - 4\delta, && \arg G_1(t_4 + 0) = 3\pi - 4\gamma_2 - 4\delta.
\end{aligned}$$

Here $t_2 = w_1 + r_1 e^{i\tau_{1B}}$ for some $\tau_{1B} \in [-\pi/2, -\delta)$, $t_3 = w_2 + r_2 e^{i\tau_{2B}}$ for some $\tau_{2B} \in (\delta - \pi, -\pi/2]$, $\overline{\omega}^2 - t_2^2 = \bigl|\overline{\omega}^2 - t_2^2\bigr| \, e^{i\lambda_{1B}}$ for some $\lambda_{1B} \in [-\pi/2 - \delta, -2\delta)$

and $\omega^2+t_3^2=|\omega^2+t_3^2|\, e^{i\lambda_{2B}}$ for some $\lambda_{2B}\in(2\delta,\pi/2+\delta]$. This implies

$$\theta_1=\frac{1}{2},\quad \theta_4=\frac{1}{2},\quad \frac{2\delta}{\pi}-\frac{1}{2}<\theta_2\le\frac{1}{2}\quad\text{and}\quad \frac{2\delta}{\pi}-\frac{1}{2}<\theta_3\le\frac{1}{2}.$$

We set $\nu_1=\nu_2=\nu_3=\nu_4=0$. Consider $k\in\{2,3\}$. If $\theta_k\in(-1,0)$, the solution f of our Hilbert problem will be unbounded and have an integrable singularity at t_k while if $\theta_k\in(0,1)$, f will be continuous and vanish at t_k; if $\theta_k=0$, f will be continuous and nonzero at t_k. Let $n\in\{0,1,2\}$ be the sum of the greatest integer function of θ_2 and of θ_3 and set $m=4-n$. A function in the function class $O(m)$ is an analytic function in E_0 which is continuous at each point of $\bar{E}$ except possibly at the corners $\{t_1,\dots,t_4\}$ of ∂E, does not vanish on $E\setminus\{t_1,\dots,t_4\}$, is continuous at m of the corners and vanishes at some or all of these corners and has an integrable singularity at the remaining $4-m$ corners. The index of our Hilbert problem in $O(m)$ is $\kappa=\nu_1+\cdots+\nu_4=0$ [Monakhov 1983, page 49] and our problem has a "canonical" solution F in $O(m)$ which is continuous at t_1 and t_4 and possibly at t_2 or t_3 [Monakhov 1983, pp. 42–53]. The general form of any solution (in $O(m)$) is $c_0F(w)$ for any $c_0\in\mathbb{R}$. Equation (9) with $k=2$ implies $\operatorname{Re}F=0$ on σ_2; since F is nonvanishing on $E\setminus\{t_1,t_2,t_3,t_4\}$, $\operatorname{Im}F$ is either strictly positive or strictly negative on the entire open interval $\sigma_2\setminus\{t_2,t_3\}$. Let us select c_1 by requiring

$$c_1\int_{t_2}^{t_3}(1+u^2)\operatorname{Im}F(u)\,du=-2 \tag{10}$$

(recall that $\operatorname{Im}t_2=\operatorname{Im}t_3=0$). We now define $f(u+iv)=f_1(u,v)+if_2(u,v)$ to be $c_1F(u+iv)$.

Any two (complex) functions analytic on and without common zeros in the same simply connected domain in $\mathbb{C}$ can be used to form a (parametric) minimal surface whose components will satisfy (5). Let $X\in C^0(E)\cap C^2(E_0)$ be the minimal surface with Weierstrass representation (f,g) which satisfies $X(0)=(0,y_0,0)$ for some y_0 to be determined. Let us use the notation in (3) and define $K(u+iv)=(x(u,v),y(u,v))$. Recall that the image E of g is star-like with respect to the origin. Using, for example, [Nitsche 1989], we see that X is strictly monotonic on ∂E.

If $u+iv\in\sigma_2$, then $v=0$ and $u\in[t_2,t_3]$. For $u\in[t_2,t_3]$, (7) implies

$$y(u,0)-y_0=\int_0^u(1+s^2)f_2(s)\,ds$$

and (10) yields $y(t_3,0)-y(t_2,0)=-2$. Now set $y_0=-1-\int_0^{t_3}(1+s^2)f_2(s)\,ds$, so that $y(t_3,0)=-1$ and therefore $y(t_2,0)=1$. From (9) with $k=2$, we have $(-1)f_1(u,v)+(0)f_2(u,v)=0$ and so $f_1(u,v)=0$ for $u+iv\in\sigma_2$. Now (6) and (8) imply $x_u(u,0)=0$ and $z_u(u,0)=0$, so x and z are constant on σ_2. Since $x(0,0)=z(0,0)=0$, we see that X and K map σ_2 strictly monotonically onto AC.

If $u+iv\in\sigma_4$, then $u+iv=e^{i\theta}$ for some $\theta\in(0,\pi)$. Writing $u=u(\theta)=\cos\theta$ and $v=v(\theta)=\cos\theta$, we have

$$\begin{aligned}\frac{d}{d\theta}\big(x(u(\theta),v(\theta))\big)&=-vx_u(u,v)+ux_v(u,v)\\&=2v\big((u^2-v^2)f_1(u,v)-2uvf_2(u,v)\big)=0,\\\frac{d}{d\theta}\big(y(u(\theta),v(\theta))\big)&=-vy_u(u,v)+uy_v(u,v)\\&=-2u\big((u^2-v^2)f_1(u,v)-2uvf_2(u,v)\big)=0,\end{aligned}$$

since $a_4f_1-b_4f_2=0$ on σ_4. This implies x and y are constant on σ_4. Thus K is constant on σ_4.

Consider the behavior of K on σ_1. Writing $u=u_1(t)$ and $v=v_1(t)$, we have

$$\begin{aligned}&\frac{d}{dt}\big(y(u_1(t),v_1(t))-\cot\delta\,x(u_1(t),v_1(t))\big)\\&\qquad=y_uu_1'(t)+y_vv_1'(t)-\cot\delta\,\big(x_uu_1'(t)+x_vv_1'(t)\big)=\frac{v_1'(t)}{u-u_1}\big(a_1f_1-b_1f_2\big)=0,\end{aligned}$$

so $y-\cot\delta\,x$ is constant on σ_1. Since $y(t_2,0)=1$ and $x(t_2,0)=0$, we see that $y-\cot\delta\,x=1$ on σ_1 and $K(t_2)=A$.

Now consider the behavior of K on σ_3. Writing $u=u_3(t)$ and $v=v_3(t)$, we get

$$\begin{aligned}&\frac{d}{dt}\big(y(u_3(t),v_3(t))+\cot\delta\,x(u_3(t),v_3(t))\big)\\&\qquad=y_uu_3'(t)+y_vv_3'(t)-\cot\delta\,\big(x_uu_3'(t)+x_vv_3'(t)\big)=\frac{v_3'(t)}{u-u_3}\big(a_3f_1-b_3f_2\big)=0,\end{aligned}$$

so $y+\cot\delta\,x$ is constant on σ_3. Since $y(t_3,0)=-1$ and $x(t_3,0)=0$, we see that $y+\cot\delta\,x=-1$ on σ_3 and $K(t_3)=C$.

Since $K\in C^0(\partial E)$ and K is constant on σ_4, $K(t_1)=K(t_4)$. Now $K(t_1)$ lies on the line $y=\cot\delta\,x+1$ and $K(t_4)$ lies on the line $y=-\cot\delta\,x-1$; hence $K(t_1)=K(t_4)$ must lie on the intersection of these lines, which is the point B. Therefore $K(t_1)=K(t_4)=B$, K maps σ_1 onto AB, and K maps σ_3 onto BC. Set

$$\Gamma=\big\{(x(u,v),y(u,v),z(u,v))\;:\;u+iv\in\partial E\big\}.$$

Since Γ projects onto the convex triangle ABC and this projection is a bijection from $X(\partial E\setminus\sigma_4)$ onto $\partial\Omega_0\setminus\{B\}$, $X(E\setminus\sigma_4)$ is the graph of a function

$$h\in C^2(\Omega_0)\cap C^0(\overline{\Omega}_0\setminus\{B\}),$$

K maps E onto $\overline{\Omega}_0$ and $h(x(u,v),y(u,v))=z(u,v)$ for $u+iv\in E\setminus\sigma_4$; in fact, $h\in C^1(\overline{\Omega}_0\setminus\{B\})$; see, for example, [Nitsche 1989, §400, p. 349; Finn 1986]. Since $z(u,v)=0$ if $u+iv\in\sigma_2$, $h=0$ on AC.

We wish to demonstrate that the contact angle condition $Th \cdot \boldsymbol{\nu} = \cos\gamma$ from (2) is satisfied on AB; the demonstration that it is satisfied on BC is similar. Now the exterior unit normal $\nu \in \mathbb{R}^2$ to AB is $(-\cos\delta, \sin\delta)$ and the corresponding horizontal unit normal in $\mathbb{R}^3$ is $\eta = (-\cos\delta, \sin\delta, 0)$. Set $S = X(E)$. The downward unit normal (in $\mathbb{R}^3$) to S at $(x, y, h(x, y))$ is

$$\vec{N}(x, y) = \frac{(h_x(x, y), h_y(x, y), -1)}{\sqrt{1 + |\nabla h(x, y)|^2}}$$

and the Gauss map $\vec{G} : E \to S$ of X is given by $\vec{G}(u + iv) = \vec{N}(x(u, v), y(u, v))$ for $u + iv \in E \setminus \sigma_4$ and $\vec{G}(e^{i\theta}) = (\cos\theta, \sin\theta, 0)$ for $e^{i\theta} \in \sigma_4$. Recall that g is the stereographic projection of $\vec{G}$ and C_1 is the stereographic projection of the circle $\{(u, v, t) \in S^2 \; : \; (u - u_1)^2 + (v - v_1)^2 + t^2 = r_1^2\}$, which can also be described as the intersection of the unit sphere with the cone $\{Y \in \mathbb{R}^3 \; : \; Y \cdot \eta = |Y| \cos\gamma_1\}$. We see therefore that $\vec{G}(w) \cdot \eta = \cos\gamma_1$ for $w \in \sigma_1$ and so

$$Th(x, y) \cdot \nu = \vec{N}(x, y) \cdot \eta = \vec{G}(w) \cdot \eta = \cos\gamma_1,$$

where $w = u + iv \in \sigma_1$ satisfies $(x(u, v), y(u, v)) = (x, y)$. Thus the contact angle condition is satisfied on the (open) interval AB.

We claim that h is discontinuous at B and, in fact, has a jump discontinuity at B. Using either [Lancaster and Siegel 1996] or the general maximum principle for minimal surfaces together with standard comparison surfaces, such as planes, we see that

$$\min\{z(t_1), z(t_4)\} \le \liminf_{(x,y)\to B} h(x, y) \le \limsup_{(x,y)\to B} h(x, y) \le \max\{z(t_1), z(t_4)\},$$

where we have abused notation by, for example, writing $z(t_1)$ for $z(\operatorname{Re} t_1, \operatorname{Im} t_1)$. Since

$$\lim_{(x,y)\to B^+} h(x, y) = z(t_1) \quad \text{and} \quad \lim_{(x,y)\to B^-} h(x, y) = z(t_4),$$

where the first limit means approaching B along AB and the second limit means approaching B along BC, establishing this claim only requires us to prove that $z(t_1) \neq z(t_4)$. Now

$$\frac{d}{d\theta} z(\cos\theta, \sin\theta) = -2 \operatorname{Im}(w^2 f(w)),$$

and, from (9) with $k = 4$, we have $\operatorname{Re}\big(w^2 f(w)\big) = 0$, where $u = \cos\theta$, $v = \sin\theta$ and $w = u + iv$. Since f does not vanish on $\sigma_4 \setminus \{t_1, t_4\}$ and w does not vanish on σ_4, the derivative $(d/d\theta) z(\cos\theta, \sin\theta)$ cannot vanish for any $\theta \in (\delta + \gamma_2, \pi - \delta - \gamma_1)$. Therefore $z(\cos\theta, \sin\theta)$ is either strictly increasing or strictly decreasing in θ for $\theta \in [\delta + \gamma_2, \; \pi - \delta - \gamma_1]$, so $z(t_1) \neq z(t_4)$.

We now define h on $\overline{\Omega}$ by extending the minimal surface $z = h(x, y)$ by reflection across the line segment $\{(0, y, 0) : |y| \le 1\}$, so that $h \in C^0(\overline{\Omega} \setminus \{B, D\})$ and $h(x, y)$ is an odd function of x. Then the condition $Th \cdot \boldsymbol{\nu} = \cos\gamma$ is satisfied at each point of $\partial\Omega \setminus \{A, B, C, D\}$. Since h is discontinuous at B, it is also discontinuous at D.

Suppose that $z = h_1(x, y)$ is any solution of the capillary problem with $h_1(0, 0) = 0$. Using the comparison principle for capillary surfaces [Finn 1986; Finn and Hwang 1989], we see that $h_1 = h + C$ for some constant C. Since $h_1(0, 0) = 0 = h(0, 0)$, we see that $h_1 = h$. Notice that h has the boundary behavior described in Theorem 1.1.

Suppose that $\gamma_1 < \pi/2$ and $\gamma_2 = \pi/2$. The arguments above continue to hold, but now $E = \overline{E}_0$, with

$$E_0 = \{w \in B_1 : \operatorname{Im} w > 0,\ |w - w_1| > \tan\gamma_1,\ \operatorname{Re} \bar{\omega}\, w < 0\},$$

as in Figure 4; recall that $\omega = e^{\delta i}$. The case in which $\gamma_1 = \pi/2$ and $\gamma_2 > \pi/2$ is similar.

3. Proof of Theorem 1.1

Consider $\gamma_1 \in (0, \pi/2]$ and $\gamma_2 \in [\pi/2, \pi)$ satisfying (1) and such that one of the following cases holds:

$$\gamma_1 \in (0, \delta) \quad \text{and} \quad \gamma_2 \in [\pi/2, \pi - \delta]; \tag{11}$$

$$\gamma_1 \in [\delta, \pi/2] \quad \text{and} \quad \gamma_2 \in (\pi - \delta, \pi); \tag{12}$$

$$\gamma_1 \in (0, \delta) \quad \text{and} \quad \gamma_2 \in (\pi - \delta, \pi). \tag{13}$$

Together with the results of Lemma 1.2, the proof that our stated conclusions hold in these three cases will complete the proof of Theorem 1.1 when $\gamma_1 \in (0, \pi/2]$

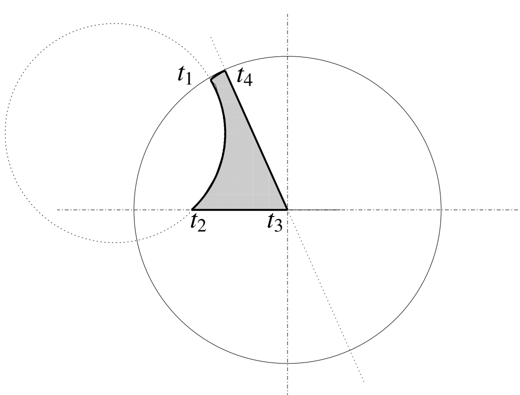

Figure 4. The domain E in the case $\gamma_2 = \pi/2$.

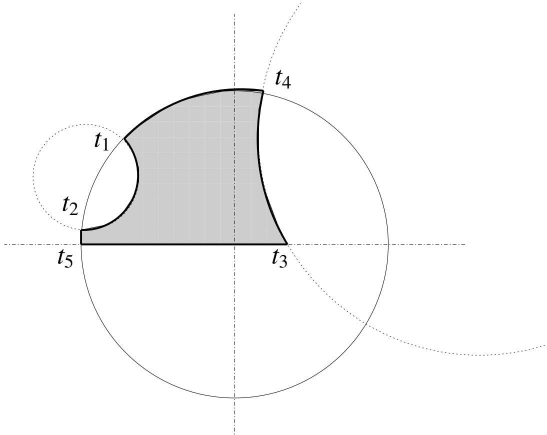

Figure 5. The domain E in the case $\gamma_1 \in (0, \delta)$, $\gamma_2 \in [\pi/2, \pi - \delta]$.

and $\gamma_2 \in [\pi/2, \pi)$. By reflecting Ω about the $y-$axis (or by considering the contact angles near D), we see that the situation where $\gamma_2 \in (0, \pi/2]$ and $\gamma_1 \in [\pi/2, \pi)$ will also be covered.

We begin by assuming that (11) holds. Then σ_4 has two components and E, which is illustrated in Figure 5, has an extra corner at $t_5 = -1$. We will continue to use the notation introduced in the proof of Lemma 1.2. Then

$$t_1 = e^{(\pi-\delta-\gamma_1)i}, \quad t_2 = e^{(\pi-\delta+\gamma_1)i},$$

and t_3 and t_4 are the same as in the previous section (see Figure 3). That we know t_2 explicitly makes our work here easier. The functions $G(w)$ and $G_1(w)$ remain the same and we wish to find $f \in C^0(E \setminus \{t_1, \dots, t_5\})$ analytic in E_0 and satisfying $\mathrm{Re}(G(w) f(w)) = 0$ for $w \in \{t_1, \dots, t_5\}$. A little work shows that θ_1, θ_3 and θ_4 remain as before and

$$\begin{aligned} &\arg G_1(t_2 - 0) = 2\pi + 4\delta - 4\gamma_1, &&\arg G_1(t_2 + 0) = -3\pi + 4\delta - 4\gamma_1, \\ &\arg G_1(t_5 - 0) = -3\pi, &&\arg G_1(t_5 + 0) = \pi, \end{aligned}$$

so $\theta_2 = \frac{5}{2}$ and $\theta_5 = -2$. Set $\nu_1 = 0$, $\nu_2 = 2$, $\nu_3 = 0$, $\nu_4 = 0$ $\nu_5 = -2$ and $\alpha_k = \theta_k - \nu_k$, $1 \le k \le 5$. If we select $c_1 \in \mathbb{R}$ such that

$$c_1 \int_{-1}^{t_3} (1 + u^2) \,\mathrm{Im}\, F(u)\, du = -2,$$

where $F(w)$ is a "canonical solution" as in Section 2, the argument used there shows that there is a unique solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{A, B, D\})$ of (2) and this solution is discontinuous at A, B and D. If $\gamma_2 = \pi/2$, then E is modified as in the previous section (see Figure 4) and this conclusion continues to be valid.

Case (12) is similar to case (11), with corners at t_1, t_2, t_3, t_4 and $t_6 = 1$. Here θ_1, θ_2 and θ_4 are as in Section 2 and $\theta_3 = \frac{1}{2}$, $\theta_6 = \frac{1}{2}$. We leave this case as an exercise for the reader.

Suppose case (13) holds. Then E has six corners (with $t_5 = -1$ and $t_6 = 1$) and

$$\theta_1 = \tfrac{1}{2}, \quad \theta_2 = \tfrac{5}{2}, \quad \theta_3 = \tfrac{1}{2}, \quad \theta_4 = \tfrac{1}{2}, \quad \theta_5 = -2, \quad \theta_6 = \tfrac{1}{2}.$$

If we set $\nu_1 = 0$, $\nu_2 = 2$, $\nu_3 = 0$, $\nu_4 = 0$, $\nu_5 = -2$ and $\nu_6 = 0$, then our conclusions follow as in case (a).

It remains to show that our claims are true when (1) hold and γ_1 and γ_2 are both in $(0, \pi/2)$ (or both are in $(\pi/2, \pi)$). Let us assume $\gamma_1, \gamma_2 \in (0, \pi/2)$ satisfy (1) with $\gamma_1 - \gamma_2 > 2\delta$. We redefine w_3 and r_3 by

$$w_3 = u_3 + i v_3 = -\cos\delta \sec\gamma_2 - i \sin\delta \sec\gamma_2 \quad \text{and} \quad r_3 = \tan\gamma_3.$$

We set $E = \bar{E}_0$, where

$$E_0 = \bigl\{ w \in B_1 : \operatorname{Im} w < 0,\ |w - w_1| < r_1,\ |w - w_3| < r_3 \bigr\}.$$

Let C_3 denote the circle $|w - w_3| = r_3$ and set

$$\sigma_1 = \partial E \cap C_1, \qquad \sigma_2 = \{ w \in \partial E : \operatorname{Im} w = 0 \},$$
$$\sigma_3 = \partial E \cap C_2, \qquad \sigma_4 = \partial E \cap \partial B_1.$$

We have two cases to consider: $\gamma_2 < \delta$ and $\gamma_2 \geq \delta$. The situation can then be taken to be as in the left and right panels, respectively, of Figure 6. For if we can obtain our desired conclusions in these two situations, we will have proved that Theorem 1.1 is valid in one of the four triangular regions remaining where (1) is

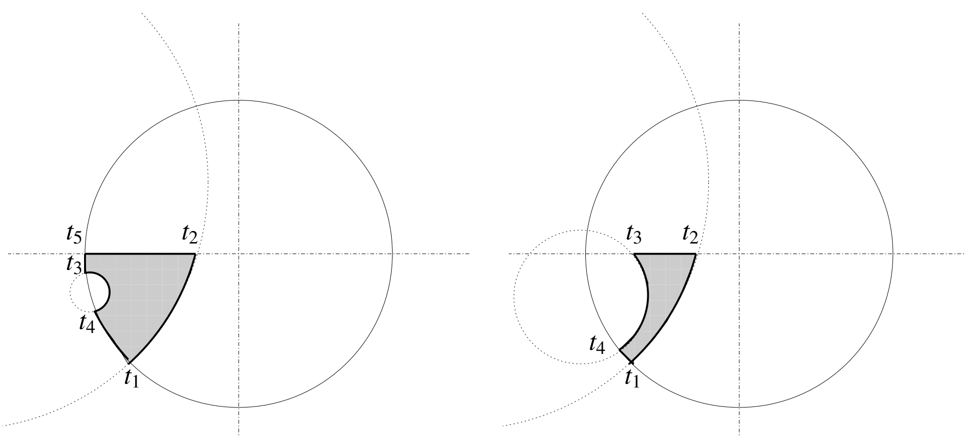

Figure 6. The domain E in the case $\gamma_1 \in (0, \delta)$, $\gamma_2 \in (\pi - \delta, \pi)$. Left: $\gamma_2 < \delta$; right: $\gamma_2 \geq \delta$.

satisfied. The validity of Theorem 1.1 in the other three regions will then follow by symmetry and/or the interchange of γ_1 and γ_2.

Suppose $\gamma_2 < \delta$, and refer to Figure 6, left. Denote the corners of E by $t_1, \ldots, t_5$, where

$$t_1 = e^{(\pi-\delta+\gamma_1)i}, \quad t_3 = e^{(\pi+\delta-\gamma_2)i}, \quad t_4 = e^{(\pi+\delta+\gamma_2)i}, \quad t_5 = -1$$

and $t_2 \in \sigma_1$ with $\operatorname{Im} t_2 = 0$. The functions $G(w)$ and $G_1(w)$ remain the same as in Section 2 (using our redefined w_3 and r_3). Then

$$\begin{aligned}
&\arg G_1(t_1-0) = -3\pi + 4(\delta-\gamma_1), && \arg G_1(t_1+0) = 3\pi + 4(\delta-\gamma_1),\\
&\arg G_1(t_2-0) = -2(\delta+\tau_{1B}+\lambda_{1B}), && \arg G_1(t_2+0) = \pi,\\
&\arg G_1(t_3-0) = -3\pi - 4(\delta-\gamma_2), && \arg G_1(t_3+0) = 4(\gamma_2-\delta),\\
&\arg G_1(t_4-0) = 2\pi - 4(\gamma_2+\delta), && \arg G_1(t_4+0) = -3\pi - 4(\gamma_2+\delta),\\
&\arg G_1(t_5-0) = \pi, && \arg G_1(t_5+0) = -3\pi,
\end{aligned}$$

where $t_2 = w_1 + r_1 e^{i\tau_{1B}}$ for some $\tau_{1B} \in [-\pi/2, -\delta)$ and $\omega^2 - t_2^2 = |\omega^2 - t_2^2| e^{i\lambda_{1B}}$ for some $\lambda_{1B} \in [-\pi/2-\delta, -2\delta)$ as in Section 2. Then

$$\theta_1 = -3, \quad \theta_2 = -\frac{1}{2} - \frac{\delta+\tau_{1B}+\lambda_{1B}}{\pi}, \quad \theta_3 = -\frac{3}{2}, \quad \theta_4 = \frac{5}{2}, \quad \theta_4 = 2.$$

Set $\nu_1 = -3$, $\nu_2 = 0$, $\nu_3 = -1$, $\nu_4 = 2$, $\nu_5 = 2$ and $\alpha_k = \theta_k - \nu_k$, $1 \le k \le 5$. Since $\nu_1+\nu_2+\nu_3+\nu_4+\nu_5 = 0$, we may argue as before and obtain a unique solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{B, C, D\})$ of (2); this solution is discontinuous at B, C, D.

Suppose $\gamma_2 \ge \delta$, and refer to Figure 6, right. Let the corners of E be denoted by $t_1, \ldots, t_4$, where t_1, t_2 and t_4 are as in the previous case and $t_3 \in \sigma_3$ with $\operatorname{Im} t_3 = 0$. If we write $t_3 = w_3 + r_3 e^{i\tau_{2B}}$ for some $\tau_{2B} \in (\delta, \pi/2]$ and $\omega^2 - t_3^2 = |\omega^2 - t_3^2| e^{i\lambda_{2B}}$ for some $\lambda_{2B} \in (2\delta, \pi/2+\delta]$, then we find

$$\theta_1 = -3, \quad \theta_2 = -\frac{1}{2} - \frac{\delta+\tau_{1B}+\lambda_{1B}}{\pi}, \quad \theta_3 = \frac{1}{2} + \frac{\tau_{2B}+\lambda_{2B}-\delta}{\pi}, \quad \theta_4 = \frac{5}{2}.$$

Set $\nu_1 = -3$, $\nu_2 = 0$, $\nu_3 = 1$, $\nu_4 = 2$ and $\alpha_k = \theta_k - \nu_k$, $1 \le k \le 4$. Then there is a unique solution $h \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{B, D\})$ of (2) and this solution is discontinuous at B and D.

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MINIMAL SOLUTIONS TO THE LOCAL CAPILLARY WEDGE PROBLEM

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We give sufficient conditions for the existence of minimal capillary graphs over quadrilaterals symmetric with respect to a diagonal. The proof is constructive, making use of the Weierstrass representation theorem for minimal surfaces. In the process, we construct minimal solutions to the local capillary wedge problem for any wedge angle $0 < \phi < \pi$ and contact angles $\gamma_1, \gamma_2 \in (0, \pi)$ such that $|\gamma_1 - \gamma_2| \le \pi - \phi$. When $|\gamma_1 - \gamma_2| < \pi - \phi$, the solution presented here has a jump discontinuity at the wedge corner.

1. Introduction and statement of results

Given a convex quadrilateral Q (each interior angle strictly less than π) with edges $\{s_k\}_{k=1}^4$, and given *contact angles* $\{\gamma_k\}_{k=1}^4$, we ask under what conditions there exists a corresponding *capillary graph*, that is, a minimal surface that is a graph over Q (except perhaps at the vertices) and that meets each wall $s_k \times \mathbb{R}$ at a constant angle γ_k. We will give sufficient conditions in the case where Q is symmetric with respect to a diagonal.

Physically, a capillary graph models the behavior of a surface formed by a liquid in a container, which in our case is a cylinder with quadrilateral cross section. In the absence of gravity any such graph given by u satisfies

$$\operatorname{div} \frac{\operatorname{grad} u}{\sqrt{1+|\operatorname{grad} u|^2}} = H \qquad \text{in } Q,$$

$$\left\langle \frac{\operatorname{grad} u}{\sqrt{1+|\operatorname{grad} u|^2}}, \nu_k \right\rangle = \cos \gamma_k \quad \text{along } s_k,$$

where ν_k is the outward pointing unit normal to s_k and H is a constant. The first equation means the graph has constant mean curvature H, and the second is just the contact angle condition along the edges.

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The wedge problem. A necessary condition for the existence of a capillary graph over a convex quadrilateral comes from the local capillary wedge problem, which deals with the existence of capillary graphs defined locally in a neighborhood of a wedge vertex. In this setting, Concus and Finn [1996] have shown it is not possible for a capillary graph with constant mean curvature to exist if the contact angles (γ_1, γ_2) along the two sides of the wedge are such that

$$|\gamma_1 + \gamma_2 - \pi| > \phi.$$

where $\phi \in (0, \pi)$ is the wedge angle. The forbidden region thus defined is the union $\mathcal{D}_1^+ \cup \mathcal{D}_1^-$ in Figure 1. Thus, a minimal capillary graph over a convex quadrilateral can exist only if

$$(\gamma_k, \gamma_{k+1}) \notin \mathcal{D}_1^+ \cup \mathcal{D}_1^- \text{ with respect to } \alpha_k, \quad k = 1, 2, 3, 4,$$

where α_k is the interior angle between s_k and s_{k+1} and $k = 4$ implies $k + 1 = 1$.

Also labeled in Figure 1 are the regions

$$\mathcal{R} = \{(\gamma_1, \gamma_2) : |\gamma_1 + \gamma_2 - \pi| < \phi \text{ and } |\gamma_1 - \gamma_2| < \pi - \phi\},$$
$$\mathcal{D}_2^+ \cup \mathcal{D}_2^- = \{(\gamma_1, \gamma_2) : |\gamma_1 - \gamma_2| > \pi - \phi\}.$$

By considering portions of planes and spheres over linear wedges, one can see that a solution to the local capillary problem (in zero gravity) exists for any $(\gamma_1, \gamma_2) \in \mathcal{R}$ and any mean curvature value H. According to a conjecture by Concus and Finn [1996], existence for any H should also hold in the closure of $\mathcal{D}_2^\pm$. Here, we make progress towards this conjecture by proving the following theorem, establishing

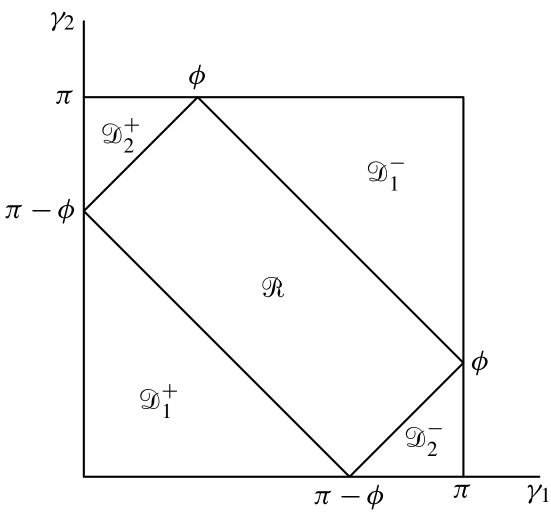

Figure 1. Contact angle diagram.

existence in the closure of $\mathfrak{D}_2^\pm$ minus the points where either γ_1 or γ_2 equals 0 or π. Because of the geometric nature of our construction, we are also able to determine the behavior of the solutions at the wedge corner.

Theorem 1. *There is a minimal solution to the local capillary wedge problem for any wedge angle* $0 < \phi < \pi$ *and contact angle pair*

$$(\gamma_1, \gamma_2) \in \overline{\mathfrak{D}}_2^\pm$$

with respect to ϕ *such that* $0 < \gamma_1, \gamma_2 < \pi$. *Moreover, if* (γ_1, γ_2) *lies in the interior* $\mathring{\mathfrak{D}}_2^\pm$ *of* $\mathfrak{D}_2^\pm$, *a solution exists with a finite jump discontinuity.*

Finn [1996] showed existence for any $H \neq 0$ in $\overline{\mathfrak{D}}_2^\pm - \overline{\mathfrak{D}}_1^\pm$ so long as the wedge angle is less than 31.5°. Combining this with the theorem, we obtain:

Corollary. *There is a solution to the local capillary wedge problem for any mean curvature value* H *and any* $(\gamma_1, \gamma_2) \in \overline{\mathfrak{D}}_2^\pm$ *such that* $0 < \gamma_1, \gamma_2 < \pi$ *so long as the wedge angle is less than* 31.5°.

Note. Crenshaw and Lancaster [2006] have proved Theorem 1 for wedge angles $\pi/2 < \phi < \pi$ and contact angles $(\gamma_1, \gamma_2) \in \mathring{\mathfrak{D}}_2^\pm$, by solving an appropriate Riemann–Hilbert problem.

Statement of the global existence theorem. Theorem 1 will arise as a corollary of Theorem 2 below, concerning the global existence of minimal capillary graphs over convex quadrilaterals. Unfortunately, it is too ambitious at this point to consider general convex quadrilaterals; instead, we restrict our attention to those that are symmetric with respect to a diagonal. To prove Theorem 2, we use the Weierstrass representation theorem for minimal surfaces to construct the graph; the sufficient conditions we derive for global existence result from studying the Gauss map on the graph.

Let Q be a convex quadrilateral that is symmetric with respect to a diagonal. Orient Q in the xy-plane so that the line of symmetry is the x-axis, and label the edges s_1, s_2, s_3, s_4 along with the wedge angle ϕ between s_1 and s_2 as shown in Figure 2. Next, assume the existence of a minimal capillary graph over Q having contact angle γ_k along the edge s_k. Furthermore, assume that the portion of the x-axis contained in Q, labeled b in Figure 2, is contained in the graph. By the Schwarz reflection principle for minimal surfaces, this last assumption implies the graph is symmetric with respect to 180° rotation around b, and this symmetry of the graph results in a symmetry of the contact angles:

$$\gamma_3 = \pi - \gamma_1 \quad \text{and} \quad \gamma_4 = \pi - \gamma_2. \tag{1}$$

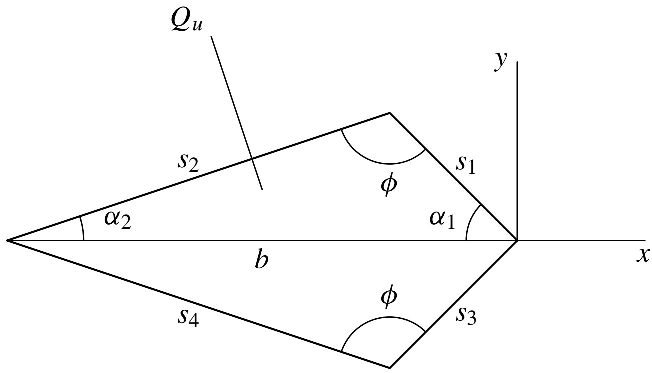

Figure 2. The quadrilateral Q oriented in the xy-plane.

Theorem 2. *Let Q, oriented and labeled as in Figure 2, be any convex quadrilateral symmetric with respect to the x-axis. Let $0 < \gamma_1, \gamma_2 < \pi$ and set $\gamma_3 = \pi - \gamma_1$, $\gamma_4 = \pi - \gamma_2$. Suppose further that*

$$(\gamma_1, \gamma_2) \in \overline{\mathfrak{D}}_2^{\pm}$$

with respect to the wedge angle ϕ between s_1 and s_2. Then there exists a minimal capillary graph over Q with contact angle γ_k on the edge s_k, $k = 1, 2, 3, 4$. Furthermore, if

$$(\gamma_1, \gamma_2) \in \mathring{\mathfrak{D}}_2^{\pm},$$

a solution exists with a finite jump discontinuity at the vertices $s_1 \cap s_2$ and $s_3 \cap s_4$.

Note that under the hypotheses of the theorem, we also have $(\gamma_3, \gamma_4) \in \overline{\mathfrak{D}}_2^{\pm}$ with respect to ϕ, by symmetry.

Before proving Theorem 2, we recall the Weierstrass representation theorem for minimal surfaces.

2. Background

The Weierstrass representation. Given a domain $\Omega \subset \mathbb{C}$, the Weierstrass representation theorem says that any (orientation preserving) conformal minimal immersion

$$X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$$

can be expressed, up to translation, in terms of a meromorphic function g and a holomorphic one-form dh by the formula

$$X(z) = \operatorname{Re} \int_{\cdot}^{z} \left(\frac{1}{2}(g^{-1} - g)\,dh,\ \frac{i}{2}(g^{-1} + g)\,dh,\ dh \right), \tag{2}$$

where g is the stereographic projection of the Gauss map and

$$dh = \left(\frac{\partial X_3}{\partial x} - i\frac{\partial X_3}{\partial y}\right) dz$$

is called the complex height differential (note that $\operatorname{Re} dh = dX_3$). Conversely, the theorem states that if g is a meromorphic function and dh a holomorphic one-form on Ω such that dh has a zero of order n at z if and only if g has a zero or pole of order n at z, then (2) gives an (orientation preserving) conformal minimal immersion on Ω that is well-defined provided that

$$\operatorname{Re} \int_c \left(\frac{1}{2}(g^{-1} - g)\,dh,\ \frac{i}{2}(g^{-1} + g)\,dh,\ dh\right) = 0$$

for every simple closed curve $c \subset \Omega$. (This condition is satisfied automatically if Ω is simply connected.)

Determining dh via the second fundamental form. For a minimal surface given by Weierstrass data g and dh, we have, for tangent vectors v and w,

$$\frac{dg(v)\,dh(w)}{g} = \mathrm{II}(v, w) - i\,\mathrm{II}(v, iw),$$

where II is the second fundamental form on the surface (see [Hoffman and Karcher 1997] for details). Therefore:

$$c \text{ is a principal curve} \iff \frac{dg(\dot{c})\,dh(\dot{c})}{g} \in \mathbb{R}; \tag{3}$$

$$c \text{ is an asymptotic curve} \iff \frac{dg(\dot{c})\,dh(\dot{c})}{g} \in i\mathbb{R}. \tag{4}$$

We see from (3) and (4) that the function ζ given by

$$\zeta(z) = \int_{\cdot}^{z} \sqrt{\frac{dg\,dh}{g}} \tag{5}$$

maps principal curves into vertical or horizontal lines and asymptotic curves into lines of slope ± 1. The map ζ is called the *developing map* of the one-form $\sqrt{dg\,dh/g}$. It is a local isometry between the minimal surface equipped with the conformal cone metric $|dg\,dh/g|$ and the Euclidean plane.

Each surface considered will have boundary consisting of principal and asymptotic curves, which will allow us to determine the function ζ. Once this is done, we can use (5) to conclude that

$$dh = \frac{g(d\zeta)^2}{dg}.$$

3. Proof of Theorem 2

Determining the image of the Gauss map. To construct the desired capillary graph in $\mathbb{R}^3$, we will find a parametrization of its image Ω under the Gauss map. Because of symmetry it is sufficient to consider the graph over the triangle Q_u with base b and base angles α_1 and α_2 (see Figure 2). From now on we assume without loss of generality, thanks to (1), that $\gamma_1 \leq \gamma_2$ and $\gamma_1 \leq \pi/2$. It will be convenient to distinguish two cases:

$$0 < \gamma_1 \leq \pi/2 \leq \gamma_2 < \pi, \tag{C1}$$

$$0 < \gamma_1 \leq \gamma_2 \leq \pi/2. \tag{C2}$$

We can also exclude the situation $\gamma_1 = \gamma_2 = \pi/2$, since then the desired graph is just part of a horizontal plane.

Because of the contact angle conditions, the Gauss map takes the interior of an edge s_k into (part of) a circle C_k on the sphere. Under stereographic projection, C_k is described as follows:

If $\gamma_k \neq \pi/2$, the circle C_k is the boundary of the disk

$$D_k = D(\sec\gamma_k\, e^{i\theta_k}, |\tan\gamma_k|) \tag{7}$$

of radius $|\tan\gamma_k|$ and center $\sec\gamma_k\, e^{i\theta_k}$, where

$$\theta_1 = \pi/2 - \alpha_1, \quad \theta_2 = \pi/2 + \alpha_2. \tag{8}$$

If $\gamma_k = \pi/2$, then C_k is the line through the origin in the direction of s_k and we define D_k as one of the half-planes bounded by this line: in case (C1),

$$D_1 = \bigcup_{0<\gamma<\pi/2} D(\sec\gamma\, e^{i\theta_1}, |\tan\gamma|)$$

or

$$D_2 = \bigcup_{\pi/2<\gamma<\pi} D(\sec\gamma\, e^{i\theta_2}, |\tan\gamma|);$$

in case (C2),

$$D_2 = \bigcup_{0<\gamma<\pi/2} D(\sec\gamma\, e^{i\theta_2}, |\tan\gamma|).$$

Finally, we note that the Gauss map takes the edge b into a segment of the imaginary axis, which we again label b.

In case (C1) we take the Gauss image Ω to be the region common to $\mathbb{C} - \bar{D}_1$, $\mathbb{C} - \bar{D}_2$, the half-plane $\{x > 0\}$, and the unit disk $D(0, 1)$, while in case (C2) we take Ω to be the region common to the exterior of the smaller disk D_1, the interior of the larger disk D_2, the half-plane $\{x > 0\}$, and the unit disk. We now show that these descriptions make sense under the hypotheses of Theorem 2. The circle C_k,

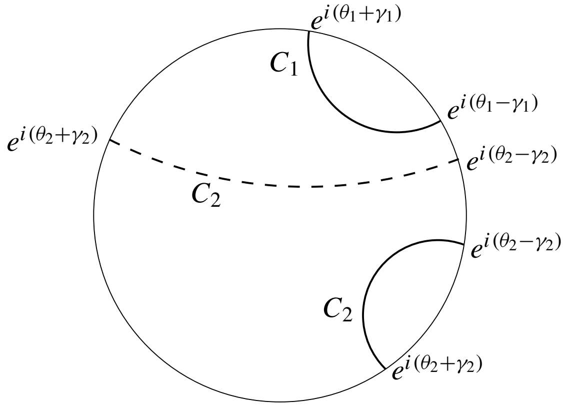

Figure 3. Intersections of C_1 and C_2 with the unit circle. The solid-line C_2 corresponds to case (C1), while the dashed-line C_2 corresponds to case (C2).

$k = 1, 2$, intersects the unit circle at $e^{i(\theta_k \pm \gamma_k)}$. Referring to Figure 3, we see that

$$\begin{aligned}(\gamma_1, \gamma_2) \in \bar{\mathfrak{D}}_2^{\pm} \text{ with respect to } \phi &\iff \gamma_2 - \gamma_1 \geq \alpha_1 + \alpha_2 = \pi - \phi \\ &\iff \pi/2 + \alpha_2 - \gamma_2 \leq \pi/2 - \alpha_1 - \gamma_1 \\ &\iff \theta_2 - \gamma_2 \leq \theta_1 - \gamma_1 \\ &\iff \begin{cases} D_1 \cap D_2 = \varnothing & \text{in case (C1)}, \\ C_1 \subset \bar{D}_2 & \text{in case (C2)}, \end{cases}\end{aligned}$$

as required. Thus the Gauss image Ω is the stated intersection. More explicitly, Ω is the region bounded by a curvilinear polygon consisting of the contact-angle arcs C_1, C_2, the segment b of the imaginary axis, and up to three arcs of the unit circle, labeled a_1, a_2, a_3 in the order shown in Figure 4, left. Each a_k present on $\partial\Omega$ comes from a finite jump discontinuity (a vertical line segment) over a vertex. Arc a_2 is present if and only if $(\gamma_1, \gamma_2) \in \mathring{\mathfrak{D}}_2^{\pm}$ with respect to ϕ; the arc (a_1 or a_3) connecting C_k to b is present if and only if $(\gamma_k, \pi - \gamma_k) \in \mathring{\mathfrak{D}}_2^{\pm}$ with respect to $2\alpha_k$. Note that from now on we use C_1 and C_2 to refer to arcs on the boundary of Ω, rather than whole circles.

Since Ω only depends on the contact angles and the interior angles of the triangle Q_u, we will construct graphs over one triangle per similarity class. To deal with other triangles in a congruence class, we simply apply a homothety of $\mathbb{R}^3$ to the graph, which changes the edge lengths of the triangle but preserves minimality and contact angles.

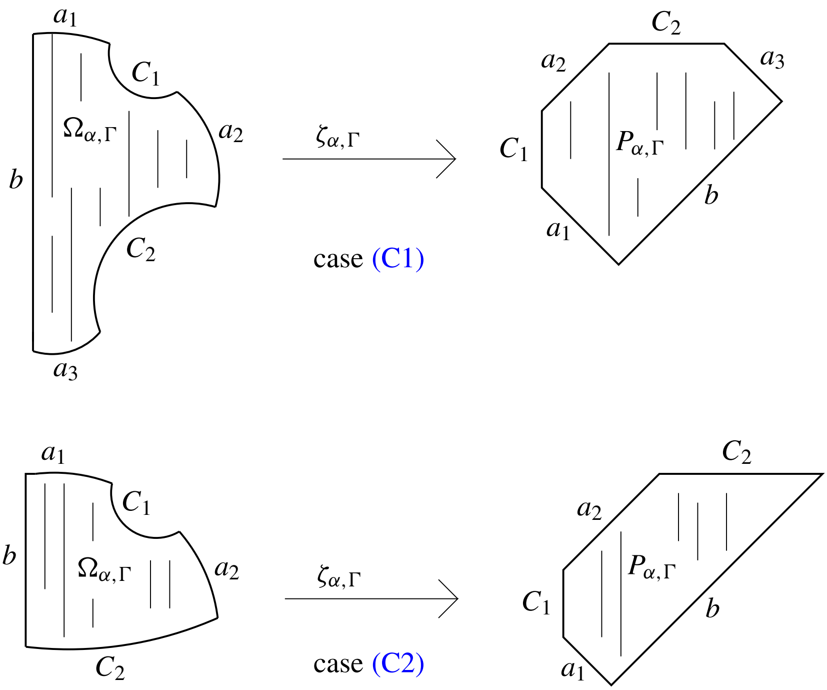

Figure 4. The developing map $\zeta_{\alpha,\Gamma}$.

We now seek to parametrize the capillary graph via the inverse of its Gauss map. The stereographic projection of the Gauss map of a minimal surface is conformal and orientation preserving, so its inverse can be expressed in terms of Weierstrass data g and dh using formula (2) above, where in our case $g(z) = z$ by construction. Hence, it remains to determine dh, which we will do in terms of the developing map of the complexified second fundamental form. Thus we need to investigate what properties this map should satisfy. It will sometimes be convenient to write $\Omega_{\alpha,\Gamma}$ instead of Ω, where we have set

$$\Gamma = (\gamma_1, \gamma_2) \quad \text{and} \quad \alpha = (\alpha_1, \alpha_2).$$

Existence of the developing map. Consider the function $\zeta = \zeta_{\alpha,\Gamma}$ on $\Omega_{\alpha,\Gamma}$ given by (5). Each edge a_k corresponds to an asymptotic curve, because ζ maps it into a vertical line over a vertex of Q_u. It follows from (4) that the image of each such edge under ζ is a segment of slope ± 1. Edges C_1 and C_2 correspond to the contact curves, which are planar curves along which the graph meets the plane of the curve at a constant angle. By Joachimstahl's Theorem, these are principal curves, so it follows from (3) that they are mapped by ζ into horizontal or vertical lines. We conclude that ζ maps $\Omega_{\alpha,\Gamma}$ conformally onto a Euclidean polygon $P_{\alpha,\Gamma}$. Diagrams of this map when $\Omega_{\alpha,\Gamma}$ contains the maximum number of edges — six in case (C1) and five in case (C2) — are shown in Figure 4. Thus we have reduced our task to proving that such a map always exists, for an appropriate choice of the Euclidean polygon $P_{\alpha,\Gamma}$.

We do this using a continuity argument and certain properties of extremal length, which we record here in the context of interest; for more generality and proofs, see [Ahlfors 1973]. Given a curvilinear polygon Δ, a Borel-measurable function $\rho > 0$ on Δ defines a conformal metric $\rho(dx^2 + dy^2)$. The *extremal length* between two edges A and B of Δ, or (A, B)-extremal length, is defined as

$$\mathrm{Ext}_{A,B}(\Delta) := \sup_{\rho} \frac{(\inf_{\gamma} \rho\text{-length of } \gamma)^2}{\rho\text{-area of } \Delta},$$

where the infimum is over all curves $\gamma : [0, 1] \to \Delta$ such that $\gamma(0) \in A,\ \gamma(1) \in B$, and $\gamma(t) \subset \mathring{\Delta}$ for $t \in (0, 1)$. Extremal length is invariant under biholomorphisms and has the following properties:

(i) If A and B are adjacent, $\mathrm{Ext}_{A,B}(\Delta) = 0$.

(ii) If B is degenerate (a point) and $\mathrm{dist}(A, B) > 0$, then $\mathrm{Ext}_{A,B}(\Delta) = \infty$.

(iii) If Δ is a Euclidean rectangle with edges $\{B_k\}$, $k = 1, 2, 3, 4$,

$$\mathrm{Ext}_{B_1,B_3}(\Delta) = \frac{1}{\mathrm{Ext}_{B_2,B_4}(\Delta)} = \frac{|B_2|}{|B_1|},$$

where the bars denote Euclidean length.

(iv) If $\Delta_1 \subset \Delta_2$ are such that edges $A_k, B_k \subset \Delta_k$, $k = 1, 2$, satisfy $A_1 \subset A_2$ and $B_1 \subset B_2$, then

$$\mathrm{Ext}_{A_2,B_2}(\Delta_2) \le \mathrm{Ext}_{A_1,B_1}(\Delta_1),$$

and the inequality is strict if $A_1 \ne A_2$ or $B_1 \ne B_2$.

(v) $\mathrm{Ext}_{A,B}(\Delta)$ depends continuously on the edge lengths of Δ.

We will prove the existence of $P_{\alpha,\Gamma}$ and the required biholomorphic map ζ in case (C1), assuming that a_1, a_2, a_3 are nondegenerate, as in Figure 4 (top). The proofs of the remaining cases are similar and simpler.

Consider the space $\mathcal{P}_6$ of Euclidean hexagons P as in Figure 4, normalized so that $C_1 \cap a_1 = 0 \in \mathbb{C}$ and $|C_1| = 1$. Any polygon $P \in \mathcal{P}_6$ is uniquely determined by the three (Euclidean) edge lengths $|a_1|$, $|a_2|$, $|C_2|$. This allows us to parametrize $\mathcal{P}_6$ using $(|a_1|, |a_2|, |C_2|)$ as coordinates:

$$P = P(|a_1|,\ |a_2|,\ |C_2|).$$

Choose any $|a_1|$, $|a_2|$ that are the first two coordinates of some $P \in \mathcal{P}_6$. Allowing $|C_2|$ to vary, we see that as $|C_2|$ approaches its lower limit (which is zero), the edges a_3 and a_2 become adjacent. By property (i) above, the (a_2, a_3)-extremal length tends to 0. Inversely, as $|C_2|$ approaches its upper limit, a_3 degenerates to a point and property (ii) says that $\mathrm{Ext}_{a_2,a_3}(P) \to \infty$. By the continuity property (v), there exists an intermediate $|\hat{C}_2| = f_1(|a_1|, |a_2|)$ such that

$$\mathrm{Ext}_{a_2,a_3}(\Omega_{\alpha,\Gamma}) = \mathrm{Ext}_{a_2,a_3}\big(P(|a_1|, |a_2|, f_1(|a_1|, |a_2|))\big). \tag{9}$$

Claim 1. *The function f_1 is continuous.*

Proof. Suppose f_1 is not continuous at some point $a = (|a_1|, |a_2|)$. Then we can find a subsequence $a_k \to a$ such that $f_1(a_k)$ converges to some $|C_2'| \neq f_1(a)$. Since extremal length depends continuously on edge lengths (property (v) above), it follows that $\mathrm{Ext}_{a_2,a_3}\big(P(a_k, f_1(a_k))\big)$ converges to $\mathrm{Ext}_{a_2,a_3}\big(P(a, |C_2'|)\big)$, and equation (9) tells us that $\mathrm{Ext}_{a_2,a_3}\big(P(a, |C_2'|)\big) = \mathrm{Ext}_{a_2,a_3}(\Omega_{a,\Gamma}) = \mathrm{Ext}_{a_2,a_3}\big(P(a, f_1(a))\big)$. However, since $|C_2'| \neq f_1(a)$, property (iv) implies that $\mathrm{Ext}_{a_2,a_3}\big(P(a, |C_2'|)\big) \neq \mathrm{Ext}_{a_2,a_3}\big(P(a, f_1(a))\big)$, a contradiction. □

Continuing, fix a length $|a_1|$ and consider $P = P(|a_1|, |a_2|, f_1(|a_1|, |a_2|))$. As $|a_2|$ approaches its lower limit of zero, property (ii) on the previous page says that

$$\mathrm{Ext}_{a_2,b}(P) \to \infty.$$

As $|a_2|$ approaches its upper limit of infinity, it is also true that $|b|$ approaches infinity. Therefore, consider a rectangle $R = R(|a_1|, |a_2|)$ with opposite sides $e_1 \subset a_2$ and $e_2 \subset b$ such that $|e_k| \to \infty$ $(k = 1, 2)$ as $|a_2| \to \infty$. Then property (iii) implies that $\mathrm{Ext}_{e_1,e_2}(R) \to 0$ as $|a_2|$ approaches infinity, and property (iv) shows that

$$\mathrm{Ext}_{a_2,b}(P) \to 0$$

as $|a_2|$ approaches infinity. By the continuity of f_1, there exists an intermediate $|\hat{a}_2| = f_2(|a_1|)$ such that

$$\mathrm{Ext}_{a_2,b}(\Omega_{\alpha,\Gamma}) = \mathrm{Ext}_{a_2,b}\big(P(|a_1|,\ f_2(|a_1|),\ f_1(|a_1|, f_2(|a_1|)))\big). \tag{10}$$

The continuity of f_2 is crucial to the remainder of the proof, and we prove it now.

Claim 2. *The function f_2 is continuous.*

Proof. As in the proof of Claim 1, we assume f_2 is discontinuous at some point $|a_1|$. We can find a subsequence $|a_1^k| \to |a_1|$ such that $f_2(|a_1^k|)$ converges to some $|a_2'| \neq f_2(|a_1|)$.

Let P and P' be the hexagons corresponding to $f_2(|a_1|)$ and $|a_2'|$, respectively. If the jump from $f_2(|a_1|)$ to $|a_2'|$ is a decrease, there are two possibilities: Either P' is strictly contained in P, or P' is such that there is a jump increase in $|b|$.

In the first case, it follows from property (iv) that $\mathrm{Ext}_{a_2,b}(P') > \mathrm{Ext}_{a_2,b}(P)$, so that equation (10) does not hold at P', a contradiction.

In the second case, we first decrease P' to a hexagon P'' by shortening the edges C_2 and b so that the edge a_3 of P'' is contained in the edge a_3 of P. By property (iv), we have $\mathrm{Ext}_{a_2,a_3}(P') > \mathrm{Ext}_{a_2,a_3}(P'')$. Another application of (iv) shows that $\mathrm{Ext}_{a_2,a_3}(P'') > \mathrm{Ext}_{a_2,a_3}(P)$, and hence that $\mathrm{Ext}_{a_2,a_3}(P') > \mathrm{Ext}_{a_2,a_3}(P)$. This implies that (9) is not satisfied at P', a contradiction.

Similarly, we reach contradictions if the jump from $f_2(|a_1|)$ to a_2' is an increase. Thus, we have shown that f_2 is continuous. □

Finally, we let $|a_1|$ vary within the family of polygons

$$P = P\big(|a_1|,\ f_2(|a_1|),\ f_1(|a_1|, f_2(|a_1|))\big).$$

As $|a_1|$ approaches its lower limit of zero, it follows from (9) and (10) that P approaches a pentagon with $|a_1| = 0$ and all other lengths nonzero. Thus,

$$\mathrm{Ext}_{b,C_1}(P) \to 0$$

as $|a_1|$ approaches zero.

As $|a_1|$ approaches infinity, consider the renormalized hexagon

$$P' = |a_1|^{-1} P\big(|a_1|, f_2(|a_1|), f_1(|a_1|, f_2(|a_1|))\big),$$

which has the properties that $|a_1| = 1$ and $|C_1|$ approaches zero as $|a_1|$ approaches infinity. Since extremal length is a conformal invariant, equations (9) and (10) also hold in P'. Now, if $|b|$ in P' approaches infinity as $|a_1|$ approaches infinity, it follows from the geometry of the hexagons that $|a_2|$ must also approach infinity. In such a case, we can apply properties (iii) and (iv) from page 255 to show that $\mathrm{Ext}_{a_2,b}(P')$ approaches zero as $|a_1|$ approaches infinity, violating the condition that equation (10) be satisfied. Thus, $|b|$ must be bounded in the family $\{P'\}$, and hence

$$\mathrm{Ext}_{b,C_1}(P') = \mathrm{Ext}_{b,C_1}\big(P(|a_1|, f_2(|a_1|), f_1(|a_1|, f_2(|a_1|)))\big) \to \infty$$

as $|a_1|$ approaches infinity. By the continuity of f_1 and f_2, there is an intermediate $|\hat{a}_1|$ such that $\hat{P} := P(|\hat{a}_1|, f_2(|\hat{a}_1|),\ f_1(|\hat{a}_1|, f_2(|\hat{a}_1|)))$ satisfies

$$\mathrm{Ext}_{b,C_1}(\Omega_{\alpha,\Gamma}) = \mathrm{Ext}_{b,C_1}(\hat{P})\,. \tag{11}$$

From the Riemann mapping theorem and the fact that $\partial\Omega_{\alpha,\Gamma}$ and $\partial\hat{P}$ are simple closed curves, it follows that there is a biholomorphic map ζ between $\Omega_{\alpha,\Gamma}$ and $\hat{P}$, and we can normalize so that

$$\zeta(a_2 \cap C_1) = a_2 \cap C_1, \quad \zeta(a_2 \cap C_2) = a_2 \cap C_2 \ \text{ and } \ \zeta(a_3 \cap C_2) = a_3 \cap C_2. \tag{12}$$

Since (9) is satisfied and extremal length is invariant under biholomorphisms, property (iv) implies that

$$\zeta(a_3 \cap b) = a_3 \cap b.$$

Given this and the equality $\mathrm{Ext}_{a_2,b}(\Omega_{\alpha,\Gamma}) = \mathrm{Ext}_{a_2,b}(\hat{P})$ arising from (10), we get

$$\zeta(a_1 \cap b) = a_1 \cap b,$$

which in turn, together with $\mathrm{Ext}_{a_2,a_3}(\Omega_{\alpha,\Gamma}) = \mathrm{Ext}_{a_2,a_3}(\hat{P})$ from (11), shows that

$$\zeta(a_1 \cap C_1) = a_1 \cap C_1.$$

Thus, the function ζ is the desired $\zeta_{\alpha,\Gamma}$ and the polygon $\hat{P}$ is the desired $P_{\alpha,\Gamma}$.

Verification of the parametrizations. With the existence of the map $\zeta = \zeta_{\alpha,\Gamma}$, we obtain a parametrization on $\Omega = \Omega_{\alpha,\Gamma}$ of a minimal surface given by Weierstrass data

$$g(z) = z \quad \text{and} \quad dh = \frac{g(d\zeta)^2}{dg}.$$

It now needs to be checked this surface is indeed a graph with the desired properties.

Choosing a base point $z_0 \in \Omega$ and using the formulas immediately above, the parametrization (2) takes the form

$$X(z) = \mathrm{Re} \int_{z_0}^{z} (1 - z^2, i(1 + z^2), 2z) \frac{(d\zeta)^2}{2\,dz}.$$

So that the resulting quadrilateral will be oriented and labeled as in Figure 2, we choose $z_0 = a_1 \cap b$ or $z_0 = C_1 \cap b$ if a_1 is not present in Ω.

Now, the map X is continuous on $\overline{\Omega}$. To see this, take a vertex v of Ω and denote the angle at v, $\zeta(v)$, by φ, ψ, respectively. Then, near v we have

$$\zeta(z) = \zeta(v) + (z - v)^{\psi/\varphi} \zeta_0(z),$$

where ζ_0 is holomorphic and nonzero at v. Thus, $\zeta'(z)^2 = (z - v)^{2(\psi/\varphi - 1)} \zeta_1(z)$, where ζ_1 is holomorphic and nonzero at v. It is easily verified that $\psi/\varphi > \frac{1}{2}$. Hence,

$$2\Big(\frac{\psi}{\varphi} - 1\Big) > -1,$$

and it follows that

$$\frac{(d\zeta)^2}{dz} = (\zeta')^2 dz$$

is integrable on $\overline{\Omega}_\Gamma$. Therefore,

X is continuous on $\overline{\Omega}$.

To analyze X on $\partial\Omega$, we parametrize C_k counterclockwise by

$$z_k(t) = \sec(\gamma_k) e^{i\theta_k} + |\tan \gamma_k| e^{it} \quad \text{if } \gamma_k \neq \pi/2,$$

and by

$$z_k(t) = \pm e^{i(\theta_k - \pi/2)} t \quad \text{if } \gamma_k \neq \pi/2,$$

where the sign factor is negative if $k = 2$ in case (C1), and positive otherwise (compare the definition of D_k on page 252). Since ζ maps C_1 into a vertical line and C_2 into a horizontal line, we have

$$(-1)^k d\zeta(\dot{z}_k)^2 > 0. \tag{13}$$

If $\gamma_k \neq \pi/2$, then $dz(\dot{z}_k) = i\,|\tan\gamma_k|e^{it}$ and we have

$$|\tan\gamma_k| = (-1)^{k-1}\tan\gamma_k \text{in case (C1)}.$$

Thus, in this case we compute

(14)

$$\begin{aligned}
dX_1(\dot{z}_k) &= \operatorname{Re}\left((1-z_k^2)\frac{d\zeta(\dot{z}_k)^2}{2\,dz(\dot{z}_k)}\right)\\
&= \frac{d\zeta(\dot{z}_k)^2}{2}\operatorname{Re}\left(\frac{1-\sec^2(\gamma_k)e^{i2\theta_k}-2\sec\gamma_k\,|\tan\gamma_k|e^{i(\theta_k+t)}-\tan^2\gamma_k\,e^{i2t}}{i\,|\tan\gamma_k|e^{it}}\right)\\
&= -\frac{d\zeta(\dot{z}_k)^2}{2\,|\tan\gamma_k|}\operatorname{Re}(ie^{-it}-i\sec^2\gamma_k\,e^{i(2\theta_k-t)}-i2\sec\gamma_k|\tan\gamma_k|e^{i\theta_k}-i\tan^2\gamma_k\,e^{it})\\
&= -\frac{d\zeta(\dot{z}_k)^2}{2\,|\tan\gamma_k|}(\sin t+\sec^2\gamma_k\sin(2\theta_k-t)+2\sec\gamma_k|\tan\gamma_k|\sin\theta_k+\tan^2\gamma_k\sin t)\\
&= \frac{(-1)^k d\zeta(\dot{z}_k)^2}{2\sin\gamma_k\cos\gamma_k}\left(\sin t+\sin(2\theta_k-t)+(-1)^{k-1}2\sin\gamma_k\sin\theta_k\right)\\
&= \frac{(-1)^k d\zeta(\dot{z}_k)^2}{\sin\gamma_k\cos\gamma_k}\sin\theta_k\left(\cos(\theta_k-t)+(-1)^{k-1}\cos\left(\gamma_k-\frac{\pi}{2}\right)\right).
\end{aligned}$$

Similarly, we have

$$dX_2(\dot{z}_k) = \frac{(-1)^{k-1}d\zeta(\dot{z}_k)^2}{\sin\gamma_k\cos\gamma_k}\cos\theta_k\left(\cos(\theta_k-t)+(-1)^{k-1}\cos\left(\gamma_k-\frac{\pi}{2}\right)\right).$$

In case (C2), we have

$$|\tan\gamma_k| = \tan\gamma_k.$$

Hence, we compute as above to obtain

$$\begin{aligned}
dX_1(\dot{z}_k) &= -\frac{d\zeta(\dot{z}_k)^2}{\sin\gamma_k\cos\gamma_k}\sin\theta_k\left(\cos(\theta_k-t)+\cos\left(\gamma_k-\frac{\pi}{2}\right)\right),\\
dX_2(\dot{z}_k) &= \frac{d\zeta(\dot{z}_k)^2}{\sin\gamma_k\cos\gamma_k}\cos\theta_k\left(\cos(\theta_k-t)+\cos\left(\gamma_k-\frac{\pi}{2}\right)\right).
\end{aligned} \tag{15}$$

Thus, in both cases we have

$$\frac{dX_2(\dot z_k)}{dX_1(\dot z_k)} = -\cot\theta_k. \tag{16}$$

Hence, the curve $X(C_k)$ is contained in a plane parallel to the vertical plane $V_k = \{(x_1, x_2, x_3) \mid x_2 = -x_1 \cot\theta_k\}$. That $X(C_k)$ is a contact curve of contact angle γ_k follows immediately from the fact that $g(z) = z$.

Moreover, we will show that

$$\begin{aligned} &dX_1(\dot z_1) < 0 \text{ on } \mathring{C}_1 \text{ in both cases,}\\ &dX_1(\dot z_2) < 0 \text{ on } \mathring{C}_2 \text{ in case (C1),}\\ &dX_1(\dot z_2) > 0 \text{ on } \mathring{C}_2 \text{ in case (C2).}\end{aligned} \tag{17}$$

To see this, consider again the points $e^{i(\theta_k+\gamma_k)}$ and $e^{i(\theta_k-\gamma_k)}$ on the unit circle (see Figure 3), assume for the moment that $\gamma_1, \gamma_2 \neq \pi/2$, and define $\epsilon_k = +1$ or -1 according to whether $\tan\gamma_k$ is positive or negative — explicitly, $\epsilon_k = 1$ except for $k = 2$ in case (C1). Then

$$e^{i(\theta_k\pm\gamma_k)} = \sec\gamma_k\, e^{i\theta_k} + |\tan\gamma_k| e^{i(\theta_k\pm\gamma_k\pm\epsilon_k\pi/2)},$$

so that z_k is defined for t in an interval $[a_k, b_k]$, where

$$a_k \geq \theta_k + \epsilon_k\gamma_k + \pi/2, \quad b_k \leq \theta_k - \epsilon_k\gamma_k + 3\pi/2.$$

Hence, for z_k we have $\pi/2 + \epsilon_k\gamma_k \leq t - \theta_k \leq 3\pi/2 - \epsilon_k\gamma_k$, so that

$$\cos(\theta_k - t) + \epsilon_k \cos(\pi/2 - \gamma_k) < 0 \quad \text{on } \mathring{C}_k. \tag{18}$$

The inequalities (17) follow from (13), (14), (15), and (18). The computations when $\gamma_k = \pi/2$ for some k are similar to those above and are therefore omitted.

If some a_k is present as an edge of Ω, we parametrize it counterclockwise by $w_k(t) = e^{it}$, so that $dz(\dot w_k) = ie^{it}$. Recall that ζ maps a_k into a line of slope -1 for $k = 1, 3$ and slope 1 for $k = 2$ (see Figure 4). Thus, $d\zeta(\dot w_k)^2 = (-1)^k i\, |d\zeta(\dot w_k)|^2$, so that

$$\begin{aligned} dX_1(\dot w_k) &= (-1)^k |d\zeta(\dot w_k)|^2\, \tfrac{1}{2}\,\mathrm{Re}(e^{-it} - e^{it}) = 0,\\ dX_2(\dot w_k) &= (-1)^k |d\zeta(\dot w_k)|^2\, \tfrac{1}{2}\,\mathrm{Re}(i(e^{-it} + e^{it})) = 0,\\ dX_3(\dot w_k) &= (-1)^k |d\zeta(\dot w_k)|^2.\end{aligned}$$

Thus X maps a_k monotonically onto a vertical line segment in $\mathbb{R}^3$.

Finally, parametrize b from bottom to top by $z_b(t) = it$. Then $dz(\dot z_b) \equiv i$, and since ζ maps b into a line of slope 1, we have $d\zeta(\dot z_b)^2 = i\, |d\zeta(\dot z_b)|^2$. Computing, we have

$$dX_1(\dot{z}_b) = |d\zeta(\dot{z}_b)|^2 \tfrac{1}{2}(1+t^2) > 0,$$
$$dX_2(\dot{z}_b) = |d\zeta(\dot{z}_b)|^2 \tfrac{1}{2}(1-t^2)\,\mathrm{Re}(i) = 0,$$
$$dX_3(\dot{z}_b) = |d\zeta(\dot{z}_b)|^2\, t\,\mathrm{Re}(i) = 0.$$

Thus, b is mapped monotonically by ζ onto a line segment in the x-direction.

Summing up, we see that $X(\partial\Omega)$ projects onto the boundary of a triangle with interior angles α_1, α_2, $\phi = \pi - \alpha_1 - \alpha_2$ and edges s_1, s_2, b such that α_k is the interior angle of the triangle between s_k and b. The projection is one-to-one except for vertical line segments that may lie over the vertices. A sharpened version of Radó's Theorem (see [Dierkes et al. 1992]) then implies that $X(\mathring{\Omega})$ is a graph over the interior of the triangle, and we have finished the proof of Theorem 2.

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MINIMAL CAPILLARY GRAPHS OVER REGULAR $2n$-GONS

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This paper follows previous work by Huff and McCuan, who provided for $0 < \gamma < \pi$ a geometric construction of minimal capillary graphs over a square with constant contact angles on the edges alternating between γ and $\pi - \gamma$. Here the result is extended to regular $2n$-gons. Regularity results are obtained for these graphs, and explicit, conformal parametrizations are given for the Jenkins–Serrin graphs corresponding to $\gamma \in \{0, \pi\}$.

Introduction

In this paper, we prove the following theorem, where the uniqueness statement follows from [Finn and Lu 1998, Theorem 3.1].

Theorem 1. *Let Q_n be a regular $2n$-gon and $0 \le \gamma \le \pi$. There is a unique* (*up to vertical translation*) *minimal graph over Q_n with constant contact angles on the edges alternating between γ and $\pi - \gamma$. If*

$$0 < \gamma < \frac{(n-1)\pi}{2n} \quad or \quad \frac{(n+1)\pi}{2n} < \gamma < \pi,$$

then there is a finite jump discontinuity over each vertex. If $\gamma \in \{0, \pi\}$, then the corresponding graph is a Jenkins–Serrin graph.

The case $n = 2$ and $0 < \gamma < \pi$ has previously been studied in [Huff and McCuan 2006], and by Concus, Finn, and McCuan in [Concus et al. 2001]. Existence was proved in the latter paper, while regularity and existence of the jump discontinuity was shown in the former. To prove existence here, we assume symmetries and then determine the image under the Gauss map, which is conformal on a minimal surface, of our fundamental piece. Next, we determine the image of the conformal map developing the (square root of) the complexified second fundamental form on the graph. As a result, we obtain conformal parametrizations of the graphs, and those corresponding to $\gamma \in \{0, \pi\}$ (Jenkins–Serrin graphs [1966]) and $(n-1)\pi/(2n) \le \gamma \le (n+1)\pi/(2n)$ can be made explicit. Another consequence of the construction is that Sobolev embedding theorems can be used to compute appropriate regularity properties of the graphs.

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1. Background

The Weierstrass representation. Given a domain $\Omega \subset \mathbb{C}$, the Weierstrass representation theorem says that any orientation-preserving conformal minimal immersion

$$X = (X_1, X_2, X_3) : \Omega \to \mathbb{R}^3$$

can be expressed, up to translation, in terms of a meromorphic function g and a holomorphic one-form dh by the formula

$$X(z) = \operatorname{Re} \int_{\cdot}^{z} \Big(\frac{1}{2}(g^{-1} - g)\, dh,\ \frac{i}{2}(g^{-1} + g)\, dh,\ dh \Big), \tag{1}$$

where g is the stereographic projection of the Gauss map and

$$dh = \left(\frac{\partial X_3}{\partial x} - i \frac{\partial X_3}{\partial y} \right) dz$$

is called the complex height differential (note that $\operatorname{Re} dh = dX_3$). Conversely, the theorem states that if g is a meromorphic function and dh a holomorphic one-form on Ω such that dh has a zero of order n at z if and only if g has a zero or pole of order n at z, then (1) gives an orientation-preserving conformal minimal immersion on Ω that is well-defined, provided that

$$\operatorname{Re} \int_{c} \Big(\frac{1}{2}(g^{-1} - g)\, dh,\ \frac{i}{2}(g^{-1} + g)\, dh,\ dh \Big) = 0$$

for every simple closed curve $c \subset \Omega$; this condition is satisfied automatically if Ω is simply connected.

Determining dh via the second fundamental form. For a minimal surface given by Weierstrass data g and dh, we have, for tangent vectors v and w,

$$\frac{dg(v)\, dh(w)}{g} = II(v, w) - i\, II(v, iw),$$

where II is the second fundamental form on the surface (for details, see [Hoffman and Karcher 1997]). It follows that

$$c \text{ is a principal curve} \iff \frac{dg(\dot{c})\, dh(\dot{c})}{g} \in \mathbb{R} \tag{2}$$

and

$$c \text{ is an asymptotic curve} \iff \frac{dg(\dot{c})\, dh(\dot{c})}{g} \in i\mathbb{R}. \tag{3}$$

We see from these two equivalences that the function ζ given by

$$\zeta(z) = \int_{\cdot}^{z} \sqrt{\frac{dg\,dh}{g}} \tag{4}$$

maps principal curves into vertical or horizontal lines and asymptotic curves into lines in one of the directions $e^{\pm i\pi/4}$.

The map ζ is called the *developing map* of the one-form $\sqrt{dg\,dh/g}$. It is a local isometry between the minimal surface equipped with the conformal cone metric $|dg\,dh/g|$ and $\mathbb{R}^2$ equipped with the Euclidean metric.

Each surface considered in this paper will have boundary consisting of principal and asymptotic curves, which will allow us to determine the function ζ. Once this is done, we can use (4) to conclude that

$$dh = \frac{g(d\zeta)^2}{dg}. \tag{5}$$

Extremal length. To prove the existence of an appropriate ζ, we will need to show the existence of a biholomorphic, edge-preserving map between two curvilinear polygons (polygons whose edges are arcs of circles or Euclidean line segments). To do this, we will need some properties of the conformal invariant extremal length. We will restrict our attention to curvilinear polygons, although in general extremal length is defined on arbitrary domains.

Given a curvilinear polygon Δ, a Borel measurable function $\rho > 0$ on Δ defines a conformal metric $\rho(dx^2 + dy^2)$. The length of a curve $\gamma \subset \Delta$ with respect to ρ is denoted $\ell_\rho(\gamma)$ (with $|\gamma|$ denoting Euclidean length), and the ρ-area of Δ is denoted by A_ρ. With this notation, we define the extremal length between edges A and B by

$$\mathrm{Ext}_\Delta(A, B) = \sup_\rho \frac{\inf_\gamma \ell_\rho^2(\gamma)}{A_\rho},$$

where the infimum is taken over all curves $\gamma : [0, 1] \to \Delta$ such that $\gamma(0) \in A$, $\gamma(1) \in B$, and $\gamma(t) \subset \mathring{\Delta}$ for $t \in (0, 1)$. Extremal length is invariant under biholomorphisms and has the following properties, which we record here (see [Ahlfors 1973] for details).

Proposition. (i) *If A and B are adjacent, then* $\mathrm{Ext}_\Delta(A, B) = 0$.

(ii) *If B is degenerate* (*a point*) *and* $\mathrm{dist}(A, B) > 0$, *then* $\mathrm{Ext}_\Delta(A, B) = \infty$.

(iii) *If $\Delta_1 \subset \Delta_2$ are such that edges $A_k, B_k \subset \Delta_k$, $k = 1, 2$, satisfy $A_1 \subset A_2$ and $B_1 \subset B_2$, then*

$$\mathrm{Ext}_{\Delta_2}(A_2, B_2) \le \mathrm{Ext}_{\Delta_1}(A_1, B_1),$$

where the inequality is strict if $A_1 \neq A_2$ or $B_1 \neq B_2$.

2. Construction

Determining the image of the Gauss map. Given $0 \leq \gamma \leq \pi$ and a regular $2n$-gon Q_n centered at the origin, let's assume the existence of a minimal graph $M_{\gamma,n}$ over Q_n with constant contact angles on the edges alternating between γ and $\pi - \gamma$. Such a graph, should it exist, is unique up to vertical translation, and so we normalize so that $0 \in M_{\gamma,n}$.

By the symmetry of the contact angle condition, it is sufficient to consider only $0 \leq \gamma < \pi/2$ (Note that $M_{\pi/2,n} = Q_n$), and we can simplify the problem further if we assume the following additional symmetries:

(S1) $M_{\gamma,n}$ is symmetric with respect to reflection through any vertical plane containing a bisector of two opposite edges of Q_n.

(S2) $M_{\gamma,n}$ is symmetric with respect to 180 degree rotation around any line connecting two opposite vertices of Q_n.

If we take the quotient by the symmetries (S1) and (S2), we are left with a fundamental piece $\hat{M}_{\gamma,n}$ that is a graph over a triangle T_n (see Figure 1) which is the quotient of Q_n by its symmetry group. For computational purposes, we rotate Q_n if necessary so that the edge s_1 of T_n connecting the center of Q_n to the midpoint of one of its edges lies on the positive x_1-axis (again, see Figure 1).

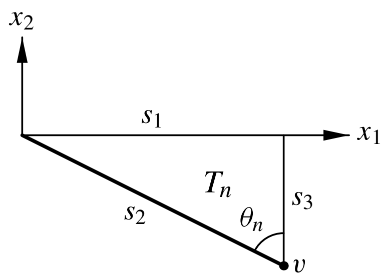


Figure 1. The fundamental triangle T_n.

We now wish to determine the image of the (downward pointing) Gauss map N on $\partial \hat{M}_{\gamma,n}$ under the stereographic projection σ that takes the south pole $(0, 0, -1)$ of S^2 to $0 \in \mathbb{C}$, the north pole to ∞, and the equator to the unit circle. Beginning with s_1, we assume the corresponding curve of $\partial \hat{M}_{\gamma,n}$ given by $f(x_1)$ is such that $f'' > 0$. Then it follows from the symmetries (S1) that the image of $\sigma \circ N$ along this curve is contained in the positive x-axis. Continuing, from the symmetries (S2) we have

$$s_2 \subset \partial \hat{M}_{\gamma,n},$$

and hence it follows that $\sigma \circ N(s_2)$ is contained in the line

$$L_n = \mathbb{R}e^{i\theta_n},$$

where

$$\theta_n = \frac{(n-1)\pi}{2n}.$$

For s_3, we have from the contact angle condition that $\sigma \circ N$ is contained in the circle

$$C_\gamma = \partial B(\sec\gamma, \tan\gamma),$$

where $B(\sec\gamma, \tan\gamma)$ is the disk centered at $\sec\gamma$ with radius $\tan\gamma$. Note that if $\gamma = 0$, then C_γ is just a point. In this case, as we will see below, the (Jenkins–Serrin) graph $M_{0,n}$ is infinite over the edges of Q_n.

To conclude our analysis of N on $\partial\hat{M}_{\gamma,n}$, we consider the behavior of the graph at the vertex v labeled in Figure 1. This behavior, which depends on the relation of the contact angle γ to the wedge angle $2\theta_n$, falls into one of the three cases below, as illustrated by Figure 2. (In the first two cases, we denote both the vertex and the jump discontinuity over the vertex by v.)

(C1) $\gamma = 0$: We assume there is an infinite jump discontinuity at v. That is, the vertical line in $\mathbb{R}^3$ passing through v is contained in $\partial M_{0,n}$. Since $\sigma \circ N$ along a vertical line is contained in the unit circle S^1, we conclude $\Omega_{0,n}$ is the curvilinear triangle shown in Figure 2 bounded by a segment of the positive real axis, a segment of L_n, and an arc of S^1.

(C2) $0 < \gamma < \theta_n$: We assume there is a finite jump discontinuity at v. That is, a vertical line segment passing through v is contained in $\partial M_{\gamma,n}$. Here $\Omega_{\gamma,n}$ is a curvilinear quadrilateral, as shown in Figure 2, bounded by a segment of the positive real axis, a segment of L_n, an arc of C_γ, and an arc of S^1.

(C3) $\theta_n \le \gamma < \pi/2$: In this case, Concus and Finn [Concus and Finn 1996] have shown $u_{\gamma,n}$ must be continuous at v if $\gamma \ne \theta_n$, where $\mathrm{Graph}(u_{\gamma,n}) = M_{\gamma,n}$, and we assume continuity for the case $\gamma = \theta_n$. Thus, we conclude $\Omega_{\gamma,n}$ is a curvilinear triangle as shown in Figure 2 bounded by a segment of the positive real axis, a segment of L_n, and an arc of C_γ.

Determining the developed image of $\sqrt{dg\,dh/g}$. We wish to parametrize $\hat{M}_{\gamma,n}$ on $\Omega_{\gamma,n}$ by finding the appropriate Weierstrass data g and dh. Since $\Omega_{\gamma,n}$ is the image of $\hat{M}_{\gamma,n}$ under stereographic projection of the Gauss map, we take

$$g(z) = z$$

for our first piece of data. For the second piece of data, we determine the conformal map $\zeta = \zeta_{\gamma,n}$ on $\Omega_{\gamma,n}$ given by (4). Then we solve for dh in terms of $\zeta_{\gamma,n}$ and obtain equation (5).

To determine $\zeta_{\gamma,n}$, we first note that each curve in $\partial\hat{M}_{\gamma,n}$ is either an asymptotic curve or a principal curve. Indeed, since s_2 and v are straight lines or straight

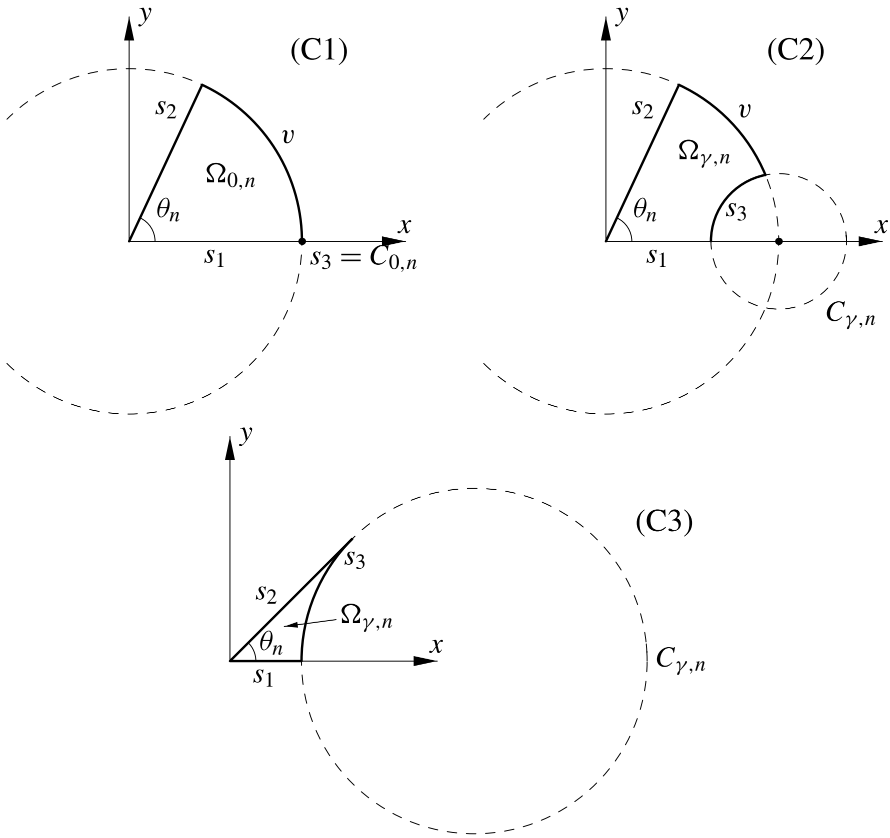
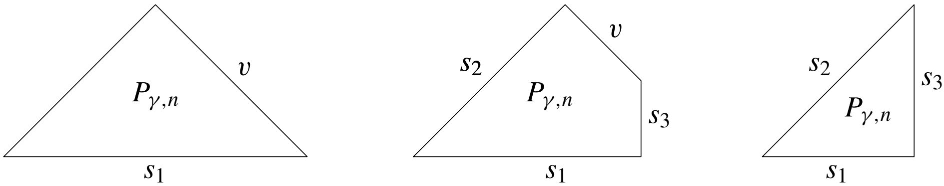


Figure 2. The image $\Omega_{\gamma,n}$ corresponding to cases (C1)–(C3).

segments, it follows immediately that they are asymptotic. For s_1 and s_3, we have that each is a planar curve along which the surface meets the plane of the curve at a constant angle. By Joachimstahl's theorem, such curves are principal. Thus, by (3), the curves s_2 and v are mapped by ζ into lines in one of the directions $e^{\pm i\pi/4}$, while s_1 and s_3 are mapped into horizontal or vertical lines. Based on this information, we conclude the image of ζ is a Euclidean polygon $P_{\gamma,n}$ with edges oriented and labeled as in Figure 3, where the number of edges and the labeling of the edges depend on the cases (C1), (C2), and (C3). Now, scaling $P_{\gamma,n}$ by a real number $\lambda > 0$ results in scaling dh, and thus $\hat{M}_{\gamma,n}$, by λ^2. Therefore, we can select one graph from each homothety class by normalizing $P_{\gamma,n}$ so that $|s_1| = 1$. Note that with this normalization, there is only one $P_{\gamma,n}$ corresponding to case (C1) and only one $P_{\gamma,n}$ corresponding to case (C3). In case (C2), the space $\{P_{\gamma,n}\}$ is one-dimensional. This space can be parametrized by the length of the edge s_3, where $0 < |s_3| < 1$.



(C1) (C2) (C3)

Figure 3. The developed image of ζ corresponding to cases (C1)–(C3).

Therefore, the map $\zeta_{\gamma,n}$ is an edge-preserving biholomorphism between $\Omega_{\gamma,n}$ and some $P_{\gamma,n}$. Since each of the two domains is simply connected and bounded by a simple closed curve, it follows that there exists a biholomorphism between them. Furthermore, we are allowed to specify the images of three points on the boundary. Thus, if in cases (C1) and (C3) we specify that the vertices of $\Omega_{\gamma,n}$ are mapped to the corresponding vertices of $P_{\gamma,n}$, then the edge-preserving property follows immediately.

For case (C2), the two domains are quadrilaterals, and so the result is not immediate. Here we normalize by specifying the images of three vertices of $\Omega_{\gamma,n}$ so that the edges s_1 and s_2 are preserved. What remains is a one-parameter family of biholomorphisms, and we aim to show there exists a map within this family that preserves all four edges. To prove this, we consider the quantity

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3).$$

First of all, it follows from part (i) of the Proposition (page 265) that

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3) \to 0 \ \text{ as } \ |s_3| \to 1.$$

Then, by part (ii), it follows that

$$\mathrm{Ext}_{P_{\gamma,n}}(s_2, s_3) \to \infty \ \text{ as } \ |s_3| \to 0.$$

Hence, it follows by continuity that there is some intermediate $|\hat{s}_3|$ and corresponding $\hat{P}_{\gamma,n}$ such that

$$\mathrm{Ext}_{\Omega_{\gamma,n}}(s_2, s_3) = \mathrm{Ext}_{\hat{P}_{\gamma,n}}(s_2, s_3).$$

Using part (iii) of the Proposition and the conformal invariance of extremal length, we see that the fourth vertex $v \cap s_3$ must also be preserved. Thus, the normalized conformal biholomorphism

$$\zeta_{\gamma,n} : \Omega_{\gamma,n} \to \hat{P}_{\gamma,n}$$

is the desired edge-preserving map.

3. Verification of parametrizations

Let $X_{\gamma,n}$ on $\Omega_{\gamma,n}$ have the form (1), where

$$g(z)=z \quad \text{and} \quad dh=\frac{g(d\zeta_{\gamma,n})^2}{dz}.$$

Here we choose the base point of integration to be $0=s_1\cap s_2$ so that

$$X_{\gamma,n}(0)=0.$$

We seek to verify that the image of $X_{\gamma,n}$ gives a surface in $\mathbb{R}^3$ that can be extended by symmetry to the desired capillary graph over a regular $2n$-gon Q_n. By construction, we know $X_{\gamma,n}$ is a minimal immersion. What remains is to verify its image is also a graph over T_n that has the desired properties. To accomplish this, we investigate $X_{\gamma,n}$ along $\partial\Omega_{\gamma,n}$, and we separate this investigation into the three cases (C1)–(C3).

Case (C3). The first observation is that

(6) $$X_{\gamma,n} \textit{ is continuous on } \overline{\Omega}_{\gamma,n}.$$

To see this, let ϕ_j denote the angle between any two adjacent edges e_1 and e_2 on $\Omega_{\gamma,n}$, and let ψ_j denote the angle between the corresponding edges on $P_{\gamma,n}$. Then we have

$$\zeta_{\gamma,n}(z)=\zeta_{\gamma,n}(e_1\cap e_2)+(z-e_1\cap e_2)^{\psi_j/\phi_j}\zeta_0(z)$$

in an $\Omega_{\gamma,n}$-neighborhood of $e_1\cap e_2$, where ζ_0 is holomorphic and nonzero at $e_1\cap e_2$. Hence, it follows that

$$\zeta'_{\gamma,n}(z)^2=(z-e_1\cap e_2)^{2(\psi_j/\phi_j-1)}\tilde{\zeta}_0(z),$$

where $\tilde{\zeta}_0$ is holomorphic and nonzero at $e_1\cap e_2$. Clearly, from the geometry of $\Omega_{\gamma,n}$ and $\hat{P}_{\gamma,n}$ we have $\psi_j/\phi_j>\frac{1}{2}$, so

$$2\Big(\frac{\psi_j}{\phi_j}-1\Big)>-1.$$

Thus, it follows that

$$dh=\frac{g(d\zeta_{\gamma,n})^2}{dg}=z\zeta'(z)^2\,dz$$

is integrable on $\overline{\Omega}_{\gamma,n}$, proving (6).

Beginning our analysis on $\partial\Omega_{\gamma,n}$, we parametrize s_1 from 0 to $\sec\gamma-\tan\gamma$ by

$$z_1(t)=t, \qquad 0<t<\sec\gamma-\tan\gamma.$$

Then

$$dz(\dot{z}_1)=1,$$

and from the geometry of $\zeta_{\gamma,n}$ it follows that

$$d\zeta_{\gamma,n}(\dot{z}_1)^2 > 0.$$

Using this information, we compute

$$\begin{aligned} d\left(X_{\gamma,n}\right)_1(\dot{z}_1) &= \mathrm{Re}\left(\tfrac{1}{2}(1-t^2)\, d\zeta_{\gamma,n}(\dot{z}_1)^2\right) > 0, \\ d\left(X_{\gamma,n}\right)_2(\dot{z}_1) &= \mathrm{Re}\left(\tfrac{i}{2}(1+t^2)\, d\zeta_{\gamma,n}(\dot{z}_1)^2\right) = 0, \\ d\left(X_{\gamma,n}\right)_3(\dot{z}_1) &= \mathrm{Re}(t d\zeta_{\gamma,n}(\dot{z}_1)^2) > 0. \end{aligned} \tag{7}$$

Thus, the computations above show that

$$X_{\gamma,n}(s_1) \subset \{x_2 = 0\} \text{ is a curve of mirror symmetry}, \tag{8}$$

where the statement about mirror symmetry follows from the fact that $g(z) = z$. Moreover, the equations (7) yield

$$\left(X_{\gamma,n}\right)_1 \text{ and } \left(X_{\gamma,n}\right)_3 \text{ increase as } t \text{ increases},$$

so that

$$X_{\gamma,n}(s_1) \text{ is a graph over it projection into the } x_1x_2\text{-plane}. \tag{9}$$

Continuing, we parametrize s_2 from 0 to $e^{i\theta_n}$ by

$$z_2(t) = te^{i\theta_n}, \quad 0 < t < \rho, \qquad \text{where } \rho < 1.$$

Hence, it follows that

$$dz(\dot{z}_2) = e^{i\theta_n},$$

and since

$$\frac{1}{i} d\zeta_{\gamma,n}(\dot{z}_2)^2 > 0,$$

we have

$$\begin{aligned} d\left(X_{\gamma,n}\right)_1(\dot{z}_2) &= \mathrm{Re}\left(\frac{1}{2}(1-t^2e^{i2\theta_n})\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{e^{i\theta_n}}\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}\mathrm{Re}(i(e^{-i\theta_n}-t^2e^{i\theta_n})) = \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}(1+t^2)\sin\theta_n > 0, \\ d\left(X_{\gamma,n}\right)_2(\dot{z}_2) &= \mathrm{Re}\left(\frac{i}{2}(1+t^2e^{i2\theta_n})\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{e^{i\theta_n}}\right) \\ &= \frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}\mathrm{Re}(-(e^{-i\theta_n}+t^2e^{i\theta_n})) = -\frac{d\zeta_{\gamma,n}(\dot{z}_2)^2}{2i}(1+t^2)\cos\theta_n < 0, \\ d\left(X_{\gamma,n}\right)_3(\dot{z}_2) &= \mathrm{Re}(td\zeta_{\gamma,n}(\dot{z}_2)^2) = 0. \end{aligned}$$

It follows that

$$\frac{d(X_{\gamma,n})_2(\dot z_2)}{d(X_{\gamma,n})_1(\dot z_2)} = -\cot\theta_n,$$

and since $d\left(X_{\gamma,n}\right)_3(\dot z_2) = 0$,

(10) *$X_{\gamma,n}$ maps s_2 monotonically onto a straight segment contained in the ray* $R_{\theta_n} = \{(x_1, x_2, 0) \mid x_1 > 0 \text{ and } x_2 = -(\cot\theta_n)x_1\}$.

Finally, we parametrize the contact curve s_3 from $s_2 \cap s_3$ to $s_1 \cap s_3$ by

$$z_3(t) = \sec\gamma + \tan\gamma\, e^{it}, \quad T_{\gamma,n} < t < \pi.$$

Now, the value $T_{\gamma,n}$ is greatest in the borderline case $\gamma = \theta_n$. Here the circle C_γ intersects the line L_n tangentially at $z = e^{i\theta_n}$, and a simple calculation yields

$$T_{\theta_n,n} = \frac{\pi}{2} - \gamma.$$

Thus, we have

$$T_{\gamma,n} \le \frac{\pi}{2} - \gamma, \quad \theta_n \le \gamma < \frac{\pi}{2},$$

so that

$$\cos T_{\gamma,n} \le -\sin\gamma, \quad \theta_n \le \gamma < \frac{\pi}{2}. \tag{11}$$

Continuing, we have $dz(\dot z_3) = i\tan\gamma\, e^{it}$ and $d\zeta_{\gamma,n}(\dot z_3)^2 < 0$, so that

$$\begin{aligned}
d\left(X_{\gamma,n}\right)_1(\dot z_3) &= \operatorname{Re}\left(\frac{1}{2}(1-\sec^2\gamma - 2\sec\gamma\tan\gamma\, e^{it} - \tan^2\gamma\, e^{i2t})\frac{d\zeta_{\gamma,n}(\dot z_3)^2}{i\tan\gamma\, e^{it}}\right)\\
&= \tfrac{1}{2}d\zeta_{\gamma,n}(\dot z_3)^2 \operatorname{Re}(ie^{-it}(\tan\gamma + 2\sec\gamma\, e^{it} + \tan\gamma\, e^{i2t})) = 0,
\end{aligned}$$

$$\begin{aligned}
d\left(X_{\gamma,n}\right)_2(\dot z_3) &= \operatorname{Re}\left(\frac{i}{2}(1+\sec^2\gamma + 2\sec\gamma\tan\gamma\, e^{it} + \tan^2\gamma\, e^{i2t})\frac{d\zeta_{\gamma,n}(\dot z_3)^2}{i\tan\gamma\, e^{it}}\right)\\
&= \frac{d\zeta_{\gamma,n}(\dot z_3)^2}{2\tan\gamma}\operatorname{Re}\left(e^{-it}(1+\sec^2\gamma + 2\sec\gamma\tan\gamma\, e^{it} + \tan^2\gamma\, e^{i2t})\right)\\
&= \frac{d\zeta_{\gamma,n}(\dot z_3)^2}{\sin\gamma\cos\gamma}(\cos t + \sin\gamma) > 0 \qquad (12)
\end{aligned}$$

— the inequality being due to (11) — and

$$\begin{aligned}
d\left(X_{\gamma,n}\right)_3(\dot z_3) &= \operatorname{Re}\left((\sec\gamma + \tan\gamma\, e^{it})\frac{d\zeta_{\gamma,n}(\dot z_3)^2}{i\tan\gamma\, e^{it}}\right)\\
&= -\frac{d\zeta_{\gamma,n}(\dot z_3)^2}{\tan\gamma}\operatorname{Re}(ie^{-it}(\sec\gamma + \tan\gamma\, e^{it})) = -\frac{d\zeta_{\gamma,n}(\dot z_3)^2}{\sin\gamma}\sin t > 0.
\end{aligned}$$

Therefore, from our computations on s_3 and the fact that $g(z) = z$, we conclude that

(13) *$X_{\gamma,n}$ maps s_3 onto a contact curve of angle γ contained in a plane parallel to the x_2x_3-plane.*

Moreover, from (12) we get that

(14) *$X_{\gamma,n}(s_3)$ is a graph over its projection into the x_1x_2-plane.*

So, from (6), (8), (9), (10), (13) and (14) it follows that $X_{\gamma,n}(\overline{\Omega}_{\gamma,n})$ is compact and $X_{\gamma,n}(\partial\Omega_{\gamma,n})$ projects into the x_1x_2-plane in a one-to-one fashion onto the boundary of T_n. Using a theorem of Radó, it then follows that $X(\Omega_{\gamma,n})$ is a projection over T_n.

Case (C2). This case differs from case (C3) by the addition of the edge v into $\Omega_{\gamma,n}$ and $\hat{P}_{\gamma,n}$. Clearly, statements (6), (8), (9), (10) and (13) still hold. To show (14) is also true, we write

$$T_{\gamma,n} = \frac{\pi}{2} - \gamma, \quad 0 < \gamma < \theta_n.$$

Thus, inequality (11) holds, and this implies (14). So, it remains to check $X_{\gamma,n}$ along v.

Parameterizing v from $s_3 \cap v$ to $s_2 \cap v$ by

$$z_v = e^{it}, \quad \gamma < t < \theta_n,$$

we have

$$dz(\dot{z}_v) = ie^{it}$$

and

$$\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{i} < 0.$$

Computing, we obtain

$$d\left(X_{\gamma,n}\right)_1(\dot{z}_v) = \mathrm{Re}\left(\frac{1}{2}(1 - e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) = \frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{2i}\mathrm{Re}(e^{-it} - e^{it}) = 0,$$

$$d\left(X_{\gamma,n}\right)_2(\dot{z}_v) = \mathrm{Re}\left(\frac{i}{2}(1 + e^{i2t})\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) = \frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{2i}\mathrm{Re}(i(e^{-it} + e^{it})) = 0,$$

$$d\left(X_{\gamma,n}\right)_3(\dot{z}_v) = \mathrm{Re}\left(e^{it}\frac{d\zeta_{\gamma,n}(\dot{z}_v)^2}{ie^{it}}\right) < 0.$$

Thus, it follows that $X_{\gamma,n}$ *maps v monotonically onto a vertical line segment*. Fortunately, the theorem of Radó used in case (C3) can be generalized to allow for vertical line segments in the boundary. Hence, it follows that in case (C2) $X_{\gamma,n}(\Omega_{\gamma,n})$ is also a graph over T_n.

Case (C1). If $\gamma = 0$, we cannot use Radó's theorem to show $X_{0,n}(\Omega_{0,n})$ is a graph over some T_n because $X_{0,n}(\overline{\Omega}_{0,n})$ may no longer be compact. In particular, we can argue as above to show that $X_{0,n}$ *is continuous on* $\overline{\Omega}_{0,n} \backslash \{s_1 \cap v\}$, so that only neighborhoods of $s_1 \cap v$ may fail to be compact.

To show the surface $X_{0,n}(\Omega_{0,n})$ is a graph over some T_n, we consider it as a limit of graphs $X_{\gamma,n}(\Omega_{\gamma,n})$ corresponding to (C2). Indeed, in $\overline{\Omega}_{\gamma,n}$ it follows immediately that

$$s_1 \cap s_3,\ \ s_3 \cap v \to s_1 \cap v \text{ as } \gamma \to 0. \tag{15}$$

Furthermore, we have

$$\operatorname{Ext}_{\Omega_{\gamma,n}}(s_2, s_3) \to \infty \text{ as } \gamma \to 0,$$

which implies

$$|\hat{s}_3| \to 0 \text{ as } \gamma \to 0. \tag{16}$$

At this point, we consider the map $2X_{\gamma,n}$ obtained by extending the parametrization through reflection across s_1. From (15) and (16) it follows that the domains $2\Omega_{\gamma,n}$ and $2\hat{P}_{\gamma,n}$ converge to $2\Omega_{0,n}$ and $2P_{0,n}$, respectively, as γ approaches 0. Thus, we can use results from [Pommerenke 1992] to conclude that

$$(2\zeta_{\gamma,n}) \circ (2f_{\gamma,n}) \to 2\zeta_{0,n} \text{ as } \gamma \to 0,$$

where $f_{\gamma,n}$ maps $\Omega_{0,n}$ conformally onto $\Omega_{\gamma,n}$ in such a way that

$$f_{\gamma,n}(0) = 0, \quad f_{\gamma,n}(s_2 \cap v) = s_2 \cap v \ \text{ and } \ f_{\gamma,n}(s_1 \cap v) = s_1 \cap s_3.$$

Moreover, the convergence is uniform on compact subsets of

$$\overline{2\Omega}_{0,n} \backslash \{s_1 \cap v\},$$

and so we have that on this set $(2X_{\gamma,n}) \circ (2f_{\gamma,n})$ converges to $2X_{0,n}$.

Because of the convergence we can compute

$$\frac{X_{0,n}(s_2 \cap v) - X_{0,n}(\tilde{s}_2 \cap \tilde{v})}{|X_{0,n}(s_2 \cap v) - X_{0,n}(\tilde{s}_2 \cap \tilde{v})|} = \lim_{\gamma \to 0} \frac{X_{\gamma,n}(s_2 \cap v) - X_{\gamma,n}(\tilde{s}_2 \cap \tilde{v})}{|X_{\gamma,n}(s_2 \cap v) - X_{\gamma,n}(\tilde{s}_2 \cap \tilde{v})|} = (0, 1, 0),$$

where, for example, the notation $\tilde{s}_2$ refers to the image of s_2 under reflection across s_1. Therefore, we know the projection of $2X_{0,n}(2\Omega_{0,n})$ into the x_1x_2-plane is contained in some triangle $2T_n$. To show that this surface is actually a graph over $2T_n$, assume the contrary. That is, suppose there is a vertical line L_x over some point $x \in 2T_n$ such that L_x intersects $2X_{0,n}(2\Omega_{0,n})$ more than once or not at all. Then there must be some point $y \in 2T_n$ such that L_y is tangent to the surface. At such a point, the Gauss map must be horizontal, and this is a contradiction since no interior points of $2\Omega_{0,n}$ lie in the unit circle S^1.

4. Explicit parametrizations and regularity

The parametrizations of the Jenkins–Serrin graphs of case (C1) can be made explicit. To see this, we first conformally change coordinates to the upper half-plane $\mathbb{H}$ via the conformal map

$$\Phi = \Phi_n : \mathbb{H} \to \Omega_{\gamma,n},$$

normalized so that

$$\Phi(-1) = 0, \quad \Phi(0) = 1, \quad \Phi(\infty) = e^{i\theta_n}.$$

Then the Weierstrass data on $\mathbb{H}$ is given by

$$g = \Phi \quad dh = \frac{\Phi(d\Psi)^2}{d\Phi}, \tag{17}$$

where

$$\Psi : \mathbb{H} \to \hat{P}_{\gamma,n}$$

is the conformal map normalized so that

$$\Psi(-1) = 0, \quad \Psi(0) = 1, \quad \Psi(\infty) = \frac{1}{\sqrt{2}} e^{i\pi/4}.$$

To determine Φ explicitly, we map $\mathbb{H}$ to the first quadrant by the map $\sqrt{z}$, where we assume here and in what follows that any map z^q for $q \in \mathbb{R}$ is defined for $0 \le \theta < 2\pi$ by

$$re^{i\theta} \mapsto r^q e^{iq\theta}.$$

Then we compose with the Möbius transformation

$$z \to \frac{-z+i}{z+i},$$

taking the first quadrant onto the upper hemisphere of the unit disk. Finally, we map this upper hemisphere onto $\Omega_{\gamma,n}$ via the map $z^{\theta_n/\pi} = z^{(n-1)/(2n)}$. Thus

$$g(z) = \Phi(z) = \left(\frac{-\sqrt{z}+i}{\sqrt{z}+i} \right)^{(n-1)/(2n)}. \tag{18}$$

For Ψ, we can use the Schwarz–Christoffel formula to conclude that

$$\Psi(z) = \mathscr{C} \int_{-1}^{z} (w+1)^{-3/4} w^{-3/4} dz,$$

where $\mathscr{C}$ is determined by the fact that $\Psi(0) = 1$. In particular, we have

$$\mathscr{C} = \frac{1}{\int_{-1}^{0} (w+1)^{-3/4} w^{-3/4} dz},$$

and if we parametrize the interval $(-1,0)$ by $w=t-1$, this expression takes the form

$$\mathscr{C}=\frac{e^{i(3\pi/4)}}{\Lambda},$$

where

$$\Lambda=\int_0^1\frac{1}{(t-t^2)^{3/4}}dt.$$

Thus

$$(d\Psi)^2=\mathscr{C}^2(z+1)^{-3/2}z^{-3/2}dz=-\frac{i}{\Lambda^2}(z+1)^{-3/2}z^{-3/2},$$

and from (18) we can compute

$$\frac{\Phi}{d\Phi}=-\frac{i2n}{n-1}(z+1)\sqrt{z}.$$

Therefore, it follows from (17) that

$$dh=-\left(\frac{2n}{\Lambda^2(n-1)}\right)\frac{1}{z\sqrt{z+1}}.$$

Similarly, one can find explicit parametrizations for the graphs of case (C3). Here the map Ψ is a Schwarz–Christoffel map to the triangle $P_{\gamma,n}$ corresponding to (C3), and the Gauss map $\Phi=\Phi_{\gamma,n}$ is given in terms of hypergeometric functions. The procedure for finding the parametrization is similar for each n, and the interested reader is referred to [Huff and McCuan 2006] to see the result when $n=2$. Also, the reference [Carathéodory 1954] will prove useful in calculating the constants appearing in the hypergeometric functions. For the case (C2), the map $\Psi=\Psi_{\gamma,n}$ is again a Schwarz–Christoffel map onto the quadrilateral $\hat{P}_{\gamma,n}$, and the Gauss map $\Phi_{\gamma,n}$ is given in terms of hypergeometric functions. However, we can not determine Ψ explicitly as the exact locations of the vertices $s_3\cap v$ and $s_2\cap v$ are unknown. Additionally, the fact that $\Omega_{\gamma,n}$ is four-sided makes it difficult to determine the coefficients of Φ explicitly.

We can investigate the regularity of the graphs in the cases (C2) and (C3). The proofs are similar for each n; for the case $n=2$, see [Huff and McCuan 2006] (where the notation for our Q_2 is Ω). To begin with, we have a statement about the subcase of (C3) defined by $\theta_n<\gamma<\pi/2$. The dependency of the Hölder exponent on γ and n comes from the changing value of the angle between s_2 and s_3 of $\Omega_{\gamma,n}$ and the fact that $\zeta_{\gamma,n}$ always maps this angle to an angle of $\pi/4$ on $\hat{P}_{\gamma,n}$.

Theorem 2. *For $\theta_n<\gamma<\pi/2$, the graphing function $u_{\gamma,n}$ satisfies*

$$u_{\gamma,n}\in C^{1,\beta-\epsilon}(\overline{Q}_n)\backslash C^{1,\beta+\epsilon}(Q_n)$$

for small ϵ, where $0<\beta<1$ depends on γ and n.

In the boundary case $\gamma = \theta_n$, the unit normal is horizontal at v, and so $u_{\gamma,n}$ cannot be C^1. However, we can measure the continuity of $u_{\gamma,n}$ as recorded in the following theorem. In the conformal category, this case is distinguished by the fact that $\Omega_{\gamma,n}$ has an outward pointing cusp at $s_2 \cap s_3$. This means $\zeta_{\gamma,n}$ vanishes to all orders at this vertex, and it is this property that determines the range of the Hölder exponent.

Theorem 3. *If $\gamma = \theta_n$, then $u_{\gamma,n} \in C^{0,\beta}(\overline{Q}_n)$ for any $0 \le \beta < 1$.*

Functions $u_{\gamma,n}$ corresponding to (C2) are discontinuous at the vertices of Q_n, but we can investigate the regularity of the trace of $u_{\gamma,n}$ over an edge of Q_n. The crucial property from which the theorem below follows is that the function $\zeta_{\gamma,n}$ can be expressed in a neighborhood $\mathcal{U}$ of the vertex $v_2 = v \cap s_3$ of $\Omega_{\gamma,n}$ by the formula

$$\zeta_{\gamma,n}(z) = \zeta_{\gamma,n}(v_2) + (z - v_2)^{3/2}\zeta_0(z),$$

where ζ_0 is holomorphic and nonzero on $\mathcal{U}$.

Theorem 4. *If $0 < \gamma < \theta_n$, and $f_{\gamma,n}$ is the restriction of $u_{\gamma,n}$ to the interior of an edge S of Q_n, then*

$$f_{\gamma,n} \in C^{2/3}(\bar{S}) \setminus C^{2/3+\epsilon}(\mathring{S}).$$

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ON TOROIDAL ROTATING DROPS

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The existence of toroidal rotating drops was observed experimentally by Plateau in 1841. In 1983 Gulliver rigorously showed that toroidal solutions of the governing equilibrium equations do indeed exist. In this short note, we settle two questions posed by Gulliver concerning the existence of additional toroidal solutions. We use a general assertion concerning rotationally symmetric surfaces whose meridian curves have inclination angle given as a function of distance from the axis along with explicit estimates for rotating drops.

In 1843 Joseph Plateau challenged *geometricians* to find rotationally symmetric tori whose mean curvature is an even quadratic function of distance to the axis of rotation:

> I think it very probable that if calculation could approach the general solution of this great problem, and lead directly to the determination of all the possible figures of equilibrium, the annular figure would be included among them.

The figures of equilibrium to which Plateau refers are those of rotationally symmetric rotating liquid drops removed from the influence of gravity. Elementary considerations lead one to ordinary differential equations for the meridian curve of such an equilibrium, and solutions of these equations may be understood by considering only portions of meridian which are expressible as graphs $u = u(r)$ with respect to the radial variable r. For these portions, the prescribed mean curvature equation becomes

$$\frac{u''}{(1+u'^2)^{3/2}} + \frac{1}{r}\frac{u'}{\sqrt{1+u'^2}} = -4ar^2 + 2\lambda, \tag{1}$$

where $a = \rho\omega^2/(8\sigma)$ and λ are constants depending on the physical parameters density, angular velocity, surface tension, and enclosed volume. It follows that the solutions form a two-parameter family (up to scaling and rigid motion). Scaling so that $a = 1$, we will take λ and c as the two parameters where c is a constant

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of integration appearing below. For a more detailed discussion, see [Elms et al. 2003].

Except for certain well defined curves in the (λ, c) parameter space, solutions of equation (1) may be expressed, up to a constant of integration, as

$$u(r) = \int_{r_*}^{r} \frac{-at^4 + \lambda t^2 + c}{\sqrt{t^2 - (-at^4 + \lambda t^2 + c)^2}}\, dt$$

on a suitable interval $[r_*, R_*]$, and the question of existence of toroidal solutions is reduced to finding λ and c for which $u(R_*) = 0$. (If this condition holds, the meridian of the torus is described by joining the graphs of u and $-u$. We are assuming here, of course, that the constant of integration $u(r_*) = 0$; this normalization will be employed throughout this paper, though we note that, due to nonintegrability at r_*, the same normalization is not always possible in [Elms et al. 2003].)

R. Gulliver [1984] showed:

For each $c \geq \frac{3}{16}$, there is some $\lambda = \lambda(c)$ for which the corresponding solution is a torus with convex cross section.

There is some interval $c \in (0, \epsilon)$ and a smooth function $\lambda = \lambda(c)$ for which the corresponding solution is a (nonconvex) torus.

He went on to conjecture that there were toroidal parameter values (λ, c) for every $c > 0$. He also pointed out that only immersed toroidal solutions were possible for $c < 0$ but was unable to verify their existence. We prove the existence of toroidal solutions in both cases, that is, for all $c \neq 0$. In order to state our result precisely (and prove it), we must first discuss the limits of integration r_* and R_* and their dependence on the parameters λ and c.

Remark. Gulliver, in his paper, formulates equation (1) as

$$a + br^2 = 2H = \frac{dv}{dr} + \frac{v}{r},$$

where $v = \sin\psi = du/ds$, ψ is the inclination angle of the meridian, and s is an arclength parameter along the meridian. He does not explicitly specify the orientation of his arclength parameterization nor his choice of normal (into the drop or out of it) with respect to which he calculates the mean curvature, but by his specification $b > 0$, one can deduce that his formulation is consistent only if the mean curvature is calculated with respect to the inward normal and, hence, if the parameterization is "counterclockwise."

Since some formal solutions of the equations do not enclose a volume, and hence their meridians do not enclose an area, the notion of "counter-clockwise" does not always make sense. In order to avoid this ambiguity, we have formulated our equation for portions of the meridian on which the normal points upward and, hence, the rotating drop is formally below the meridian locally. It is easily verified that any portions of meridian for which the rotating drop is formally above the meridian are geometric reflections across the line $u = 0$ of those we consider.

For convenience of the reader, a loose translation between Gulliver's parameter notation and ours is as follows:

(1) Due to the reversal of the normal, Gulliver's H is our $-H$.

(2) Gulliver's rotation parameter $b > 0$ is for us $4a$.

(3) Gulliver's Lagrange parameter $a \in \mathbb{R}$ is our -2λ.

(4) Gulliver's constant of integration C is the same as $-c$.

(5) Having made these replacements and setting, without loss of generality, $v = u'/\sqrt{1+u'^2}$, Gulliver's equation (1) translates directly into our equation (1).

(6) Gulliver subsequently rescales his rotation parameter b to $4/3$ and gives C the new name $-\gamma^4$ (in harmony with his restriction $C < 0$). We rescale so that the rotation parameter a takes the value 1. Comparison leads to the translation formulae

$$\begin{cases} \lambda = -a\sqrt[3]{3}, \\ c = -C/\sqrt[3]{3} = -\gamma^4/\sqrt[3]{3}, \end{cases}$$

with our scaled parameters on the left and Gulliver's on the right. In particular, one sees that the $\gamma \geq \left(\frac{3}{8}\right)^{1/3}$ of Gulliver's Theorem 2 corresponds precisely to our $c > \frac{3}{16}$.

(7) The λ, c (or a, C) parameter space has been antipodally reflected through the origin and scaled according to the formulas in the previous item.

Inclination angle and the other toroidal solutions. We may rewrite (1) as

$$\left(\frac{ru'}{\sqrt{1+u'^2}}\right)' = -4r^3 + 2\lambda r$$

and integrate once to obtain

$$v = \sin\psi = \frac{u'}{\sqrt{1+u'^2}} = -r^3 + \lambda r + \frac{c}{r},$$

where ψ is the inclination angle of the graph of u with respect to the positive r-axis. For $c \neq 0$, the values r_* and R_* are solutions of the algebraic equations

$$|\sin\psi(r)| = \left|-r^3 + \lambda r + \frac{c}{r}\right| = 1 \tag{2}$$

($r_* = 0$ for $c = 0$).

In general, if we think of λ and c as fixed, we may consider the algebraic expression $v(r) = -r^3 + \lambda r + c/r$ also for values of r for which $|v(r)| > 1$. In this

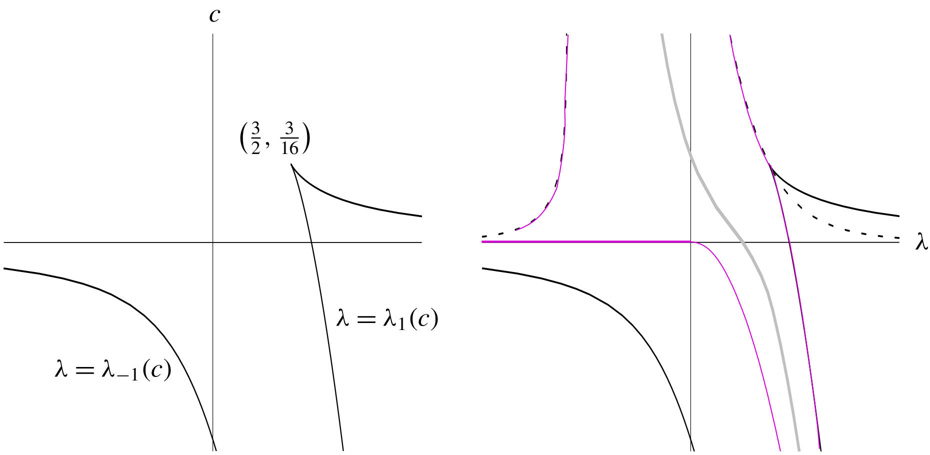

Figure 1. Left: Locus in parameter space where $|\sin\psi| = 1$ has a double root. Right: Numerically computed torus curve and region of parameter values possibly corresponding to tori (see page 288).

way, the equation $|v(r)| = 1$ appearing in (2) is evidently equivalent to a pair of (quartic) polynomial equations. This guarantees that there are only finitely many cases to consider. One must take into account however that the resulting intervals of definition (determined by roots $r_* = r_*(\lambda, c)$ and $R_* = R_*(\lambda, c)$) may depend discontinuously on λ and c. Let us begin, however, with the assumption that λ and c are fixed, and denote by $\mathcal{R} = \mathcal{R}_{\lambda,c}$ the collection of positive roots of $|v(r)| = 1$, counted with multiplicities.

Lemma 1. *Given $r_* \le R_*$ in $\mathcal{R}$ such that $|v(r)| < 1$ for $r_* < r < R_*$, there is a rotationally symmetric surface whose inclination angle $\psi(r)$ satisfies*

$$\sin\psi(r) = v(r) = -r^3 + \lambda r + \frac{c}{r} \tag{3}$$

for $r_* \le r \le R_*$. The surface is unique up to translation in the u direction.

Conversely, each complete rotationally symmetric surface that does not intersect $r = 0$ and whose inclination angle satisfies (3) projects onto an annulus $r_* \le r \le R_*$ with $|\sin\psi(r)| < 1$ for $r_* < r < R_*$.

Each double root $r_* = R_*$ corresponds to a cylinder. The parameter values for which this is possible lie along three curves in the (λ, c) parameter plane as depicted in Figure 1, left.

One can verify the following behavior in the neighborhood of the real roots of $|v(r)| = 1$:

Lemma 2. *Let r_* be a real number. If $v \in C^2[r_*, r_* + \epsilon]$ with $|v(r_*)| = 1$ and $|v(r)| < 1$ for $r_* < r \leq r_* + \epsilon$, the improper integral*

$$\left| \int_{r_*}^{r_*+\epsilon} \frac{v}{\sqrt{1 - v^2}} \, dr \right|$$

is finite if and only if $v'(r_*) \neq 0$. A similar statement holds on intervals of the form $[R_* - \epsilon, R_*]$.

Remark. In this result, v may be any function satisfying the hypotheses of the lemma, though we are only interested in applications in which $v = -r^3 + \lambda r + c/r$ as in (2).

The full significance of the curves in Figure 1, left, is explained in [Elms et al. 2003], but for our present purposes we need only verify some isolated facts about two of them. We start with a continuity assertion that was already observed by Gulliver as important for the case $c > \frac{3}{16}$.

Lemma 3. *For each $\lambda \in \mathbb{R}$ and $c > \frac{3}{16}$, the set $\mathcal{R} = \mathcal{R}_{\lambda,c}$ consists of exactly two positive roots $r_* = r_*(\lambda, c)$ and $R_* = R_*(\lambda, c)$ with $r_* < R_*$. In this region of the (λ, c)-plane r_* and R_* depend smoothly on λ and c. Consequently, we find that the quantity*

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1 - v^2}} \, dr$$

depends smoothly on λ and $c > \frac{3}{16}$. In particular, $u(R_*)$ is a continuous function of λ for fixed $c > \frac{3}{16}$.

Proof. We consider

$$v(r) = -r^3 + \lambda r + \frac{c}{r}$$

for $r > 0$ and fixed $c > \frac{3}{16}$. The assertion of the lemma follows from the fact that the equations

$$v(r_*) = -r_*^3 + \lambda r_* + \frac{c}{r_*} = 1 \tag{4}$$

and

$$v(R_*) = -R_*^3 + \lambda R_* + \frac{c}{R_*} = -1 \tag{5}$$

have unique solutions $r_* < R_*$ with $v' < 0$ on $[r_*, R_*]$.

First note that $\lim_{r \to 0} v(r) = +\infty$ and $\lim_{r \to +\infty} v(r) = -\infty$. Therefore, (4) and (5) have at least one solution each. It will be observed that v has a unique inflection point (and v' a unique maximum) at $t_{\max} = \sqrt[4]{c/3}$. If $\lambda \leq 2\sqrt{3c}$, then $v' \leq 0$ with equality only possible at $r = t_{\max}$ when $\lambda = 2\sqrt{3c}$. In this case, $v(t_{\max}) = 8(c/3)^{3/4} > 1$. Therefore, our assertions concerning (4) and (5) hold.

If $\lambda > 2\sqrt{3c}$, then v' has two zeros; the smaller, which corresponds to a local minimum of v, is given by

$$r_{\min} = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 12c}}{6}}.$$

Elementary computations show that

$$\frac{dr_{\min}}{d\lambda} = -\frac{r_{\min}}{2\sqrt{\lambda^2 - 12c}} < 0$$

and

$$\frac{d}{d\lambda}\, v(r_{\min}) = r_{\min} > 0. \tag{6}$$

In this case,

$$\lim_{\lambda \searrow 2\sqrt{3c}} v(r_{\min}) = 8(c/3)^{3/4} > 1.$$

In light of (6), we see that $v(r) > 1$ for $r \le r_{\max}$, where

$$r_{\max} = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 12c}}{6}} \tag{7}$$

is the larger zero of v'. In this case too, therefore, our assertions concerning (4) and (5) hold. $\square$

The situation for $c \le \frac{3}{16}$ is somewhat more complicated. Nevertheless, we find:

Lemma 4. *For each fixed $c \le \frac{3}{16}$, there is a unique value $\lambda = \lambda_1(c)$ determined by the equation*

$$v(r_{\max}) = 1,$$

where $r_{\max}$ is given by (7) (see Figure 1, left).

For $c \in \left(0, \frac{3}{16}\right]$ and $\lambda < \lambda_1$, the equation

$$|v(r)| = 1 \tag{8}$$

has exactly two positive solutions $r_* < R_*$. For $c = 0$, there is one solution R_*, and we may take $r_* = 0$.

For $c < 0$, there is a unique value $\lambda_{-1} = \lambda_{-1}(c) < \lambda_1$ determined by the equation

$$v(r_{\max}) = -1$$

(*considered as an equation for $\lambda = \lambda_{-1}$*). *For $c < 0$ and $\lambda_{-1} < \lambda < \lambda_1$ there are again exactly two positive solutions $r_* < R_*$ of* (8).

For $c \in \left[0, \frac{3}{16}\right]$, the expression

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1 - v^2}}\, dr$$

defines is a continuous function of λ *on* $(-\infty, \lambda_1)$; *the same function is continuous on* $\lambda_{-1} < \lambda < \lambda_1$ *for* $c < 0$. *In all cases,*

$$\lim_{\lambda \nearrow \lambda_1} u(R_*) = +\infty. \tag{9}$$

Proof. We observe that (7) is only real-valued for $0 \le c \le \frac{3}{16}$ if $\lambda \ge 2\sqrt{3c}$. For this range of parameters, a calculation similar to that leading to (6) yields

$$\frac{d}{d\lambda} v(r_{\max}) = r_{\max} > 0. \tag{10}$$

Furthermore,

$$\lim_{\lambda \searrow 2\sqrt{3c}} v(r_{\max}) = 8(c/3)^{3/4} \le 1$$

and

$$\lim_{\lambda \nearrow +\infty} v(r_{\max}) = +\infty. \tag{11}$$

Therefore λ_1 is well defined for $0 \le c \le \frac{3}{16}$. For $c < 0$, conditions (10) and (11) still hold. Furthermore,

$$\lim_{\lambda \searrow -\infty} v(r_{\max}) = -\infty.$$

Therefore, both λ_{-1} and λ_1 are well defined.

Considerations similar to those in the proof of Lemma 1 yield the uniqueness and continuity of r_* and R_* as functions of λ. The continuity of $u(R_*)$ follows as before. It remains to establish (9). To avoid certain technicalities, we restrict to the case $c < \frac{3}{16}$, but the case $c = \frac{3}{16}$ may be handled similarly. For $c < \frac{3}{16}$ and λ sufficiently close to λ_1, we know that $r_{\max}$ is well defined as described above with $r_{\max}$ and $v(r_{\max})$ increasing as functions of λ; see Figure 2. We set $R_1 = \lim_{\lambda \nearrow \lambda_1} r_{\max}$. Since

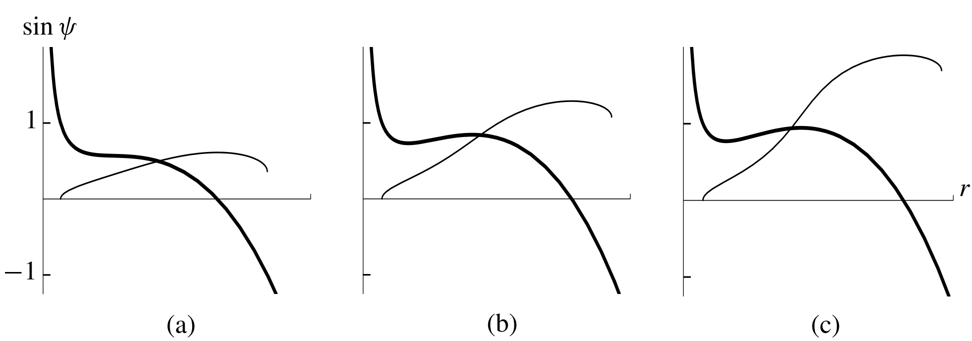

Figure 2. Profiles of $v = \sin\psi(r)$ for $c = \frac{3}{32}$ (thick curves). The values of λ are (a) 1, (b) 1.5, (c) 1.65. The corresponding antinodoid solutions $u(r)$ are superimposed (thin curves).

v' is nonvanishing at r_* and R_* (except in the case $c=0$ in which $r_*=0=\sin\psi(r_*)$ which causes no problem), we find from Lemma 2, that for all ϵ small enough and fixed, the integrals

$$\int_{r_*}^{r_{\max}-\epsilon} \frac{v}{\sqrt{1-v^2}}\,dr \quad \text{and} \quad \int_{r_{\max}+\epsilon}^{R_*} \frac{v}{\sqrt{1-v^2}}\,dr$$

are finite and may be bounded uniformly in λ as $\lambda \nearrow \lambda_1$. Thus,

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1-v^2}}\,dr \;\geq\; \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\,dr - M_\epsilon$$

for some constant M_ϵ.

Expanding $v(r)$ in a power series about $(\lambda, r) = (\lambda_1, R_1)$ on the other hand,

$$v = 1 + R_1(\lambda-\lambda_1) + (\lambda-\lambda_1)(r-R_1) + d(r-R_1)^2 + o(r-R_1)^2$$

where $d = v''(R_1)/2 < 0$. Therefore, we may fix ϵ small enough so that

$$v \geq 1 + \tfrac{1}{2}\big(R_1(\lambda-\lambda_1) + d(r-R_1)^2\big) > \tfrac{1}{2} \quad \text{for } r_{\max}-\epsilon \leq r \leq r_{\max}+\epsilon$$

uniformly in λ. Thus,

$$\begin{aligned} 1-v^2 &\leq -R_1(\lambda-\lambda_1) - d(r-R_1)^2 - R_1^2\frac{(\lambda-\lambda_1)^2}{4} - R_1 d\frac{(\lambda-\lambda_1)(r-R_1)^2}{2} \\ &\qquad - d^2\frac{(r-R_1)^4}{4} \\ &\leq -R_1(\lambda-\lambda_1) - d(r-R_1)^2, \end{aligned}$$

and

$$\begin{aligned} \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\,dr \;&\geq\; \frac{1}{2}\int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{1}{\sqrt{-R_1(\lambda-\lambda_1) - d(r-R_1)^2}}\,dr \\ &= \frac{1}{2\sqrt{-d}}\left(\sinh^{-1}\left(\sqrt{\frac{d}{R_1(\lambda-\lambda_1)}}\,(r_{\max}-R_1+\epsilon)\right)\right. \\ &\qquad \left. - \sinh^{-1}\left(\sqrt{\frac{d}{R_1(\lambda-\lambda_1)}}\,(r_{\max}-R_1-\epsilon)\right)\right). \end{aligned}$$

Since $r_{\max} \nearrow R_1$ as $\lambda \nearrow \lambda_1$ and $\epsilon > 0$, we see that

$$\lim_{\lambda \nearrow \lambda_1} \int_{r_{\max}-\epsilon}^{r_{\max}+\epsilon} \frac{v}{\sqrt{1-v^2}}\,dr = +\infty. \qquad \square$$

It remains to obtain a value $\lambda < \lambda_1$ (in the region of continuity) for which $u(R_*) < 0$. For this we use a general relation between the convexity of $v = \sin\psi$ and the height $u(R_*)$. A special case of this result was used implicitly by Gulliver in the case $c > \frac{3}{16}$.

Lemma 5 (Convexity and height for rotational surfaces). *Given* $0 < r_* < R_*$ *and any* v *decreasing from* 1 *to* -1 *on* $[r_*, R_*]$, *if* v *is convex*, *then*

$$u(R_*) = \int_{r_*}^{R_*} \frac{v}{\sqrt{1-v^2}} dr < 0.$$

Similarly, *if* v *is concave* ($v'' < 0$), *then* $u(R_*) > 0$.

This result is true for any real numbers $r_* < R_*$ and any function v satisfying the hypotheses stated in the lemma. We have used derivatives of v up to order two freely in the proof below, but the continuity of v resulting from convexity/concavity is adequate to obtain the result.

Remark. In the convex case, we say that the resulting surface is of *nodoid type*; in the concave case, of *antinodoid type*.

Proof of Lemma 5. Assume v is convex. The other case is handled similarly. There is a unique $r = r_{\text{crit}} \in (r_*, R_*)$ such that $v(r_{\text{crit}}) = 0$.

$$u(R_*) = \int_{r_*}^{r_{\text{crit}}} \frac{v}{\sqrt{1-v^2}} dr + \int_{r_{\text{crit}}}^{R_*} \frac{v}{\sqrt{1-v^2}} dr.$$

Again according to the monotonicity, the relation $v(r) = -v(t)$ defines a change of variables, and we obtain

$$u(R_*) = \int_{r_*}^{r_{\text{crit}}} \left(1 - \frac{v'(t)}{v'(r)}\right) \frac{v}{\sqrt{1-v^2}} dt < 0.$$

Notice that over the interval of integration, the second factor is positive; the first is negative by convexity. □

Referring back to the proofs of Lemma 3 and Lemma 4, one finds that for $c > 0$ and $\lambda << 0$, we have $v'(r) < 0$ on $[r_*, R_*]$ and $v(t_{\max}) < -1$ where $t_{\max} = \sqrt[4]{c/3}$ is the unique inflection point. It follows that $v = \sin\psi$ is convex on $[r_*, R_*]$ and $u(R_*) < 0$ by Lemma 5. Thus, by the intermediate value theorem, there is some λ for which $u(R_*) = 0$.

For $c = 0$ and $\lambda \le 0$, we have $v \le 0$ and $u(R_*) < 0$ so that the same conclusion holds. Technically, the resulting surface of rotation is not a torus in this case, since $r_* = 0$. However, one does obtain a pinched spheroid which encloses a volume.

For $c < 0$, the function $v = \sin\psi$ has a unique global maximum at the value $r_{\max}$ given in (7) and

$$\lim_{\lambda \searrow -\infty} v(r_{\max}) = -\infty.$$

As mentioned above, the monotonicity condition (10) holds, and it is clear that all values of λ in the interval for which $-1 < \sin\psi(r_{\max}) \le 0$ correspond to solutions with $u(R_*) < 0$.

Following through carefully the indicated calculations and applying Lemma 5 in situations similar to those above yields the following bounds for the parameters (λ, c) corresponding to toroidal solutions; see Figure 1, right.

Theorem. *For each fixed c there is at least one λ corresponding to a toroidal solution. If $c \geq \frac{3}{16}$, then*

$$-\sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}} < \lambda < \sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}}.$$

If $0 < c \leq \dfrac{3}{16}$, then

$$-\sqrt[4]{\frac{3}{c}} - 2\sqrt{\frac{c}{3}} < \lambda < \lambda_1(c).$$

If $c \leq 0$, then

$$2\sqrt{-c} < \lambda < \lambda_1(c).$$

No toroidal solutions can correspond to parameters outside the region defined by these inequalities.

A useful alternative characterization of λ_1 is given in [Elms et al. 2003]. We state it here for convenience.

$$\lambda_1(c) = 3r^2 + \frac{c}{r^2},$$

where $r = r(c)$ is the larger positive solution of $2r^4 - r + 2c = 0$.

As a final remark, we conjecture that there is exactly one value $\lambda_t = \lambda_t(c)$ corresponding to a toroidal solution. These values form a smooth curve in the

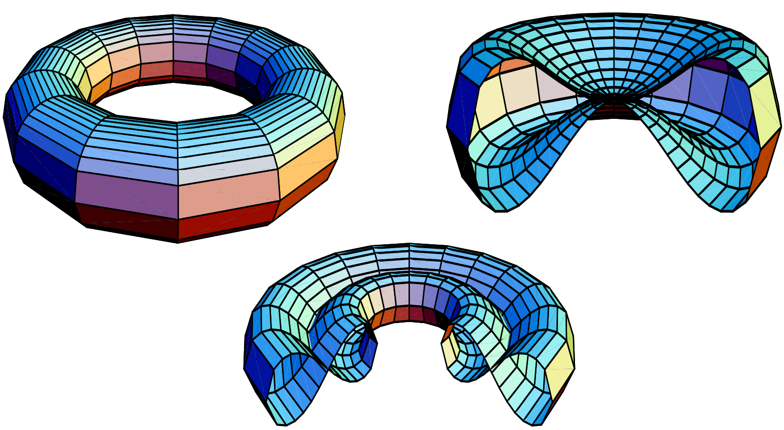

Figure 3. Toroidal surfaces: clockwise from top left, embedded torus ($c > 0$), pinched spheroid ($c = 0$), and immersed torus ($c < 0$).

interior of the region described in the Theorem; see Figure 1, right, where the thick gray curve is the numerically calculated curve of toroidal solutions.

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THE ASCENT OF A LIQUID ON A CIRCULAR NEEDLE

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Dedicated to the memory of Herbert Beckert

It is shown that there exists an asymptotic expansion of the ascent of a liquid on a circular needle if the radius of the cross section tends to zero. In particular, a formula derived formally by Derjaguin in 1945 is confirmed.

1. Introduction

We consider the following nonparametric capillary problem in the presence of gravity (see [Finn 1986, Chapter 1]). We seek a function $U = U(x)$, $x = (x_1, x_2)$, defined over the base domain $\Omega := \mathbb{R}^2 \setminus \overline{B_a(0)}$, where $B_a(0)$ is a disk with (small) radius a and center at $x = 0$, and satisfying the nonlinear elliptic boundary value problem

$$\operatorname{div} TU = \kappa U \quad \text{in } \Omega, \tag{1}$$

$$\nu \cdot TU = \cos\theta \quad \text{on } \partial\Omega, \tag{2}$$

where

$$TU = \frac{\nabla U}{\sqrt{1 + |\nabla U|^2}},$$

κ and θ are constants with $0 \le \theta \le \pi$, and ν is the exterior unit normal on $\partial\Omega$ (equivalently, the interior normal on $\partial B_a(0)$). The graph of U describes the capillarity-driven equilibrium interface in the exterior of a vertical cylinder (the needle) with cross section $B_a(0)$, in the presence of a constant gravity field directed downward; θ is the constant contact angle between the capillary surface and the tube and κ is the (positive) capillary constant, given by $\kappa = \rho g/\sigma$, where ρ is the density change across the interface, g is the acceleration of gravity, and σ is the surface tension.

No explicit solution of (1)–(2) is known. It was shown by Johnson and Perko [1968] that there exists a radially symmetric solution. From a maximum principle of Finn and Hwang [1989] for unbounded domains it follows that this symmetric solution is the only one.

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Set

$$u(r) = U(x), \quad r = \sqrt{x_1^2 + x_2^2}. \tag{3}$$

We will prove that there is an asymptotic expansion for the ascent $u(a)$ of the liquid in this problem. More precisely:

Theorem 1.1. *Set $B = \kappa a^2$ and let $\gamma = 0.5772\ldots$ be Euler's constant. Then the ascent $u(a)$ of a liquid on a circular needle with radius a satisfies*

$$\frac{u(a)}{a} = -\cos\theta\left(\tfrac{1}{2}\ln B + \gamma - 2\ln 2 + \ln(1+\sin\theta) + O(B^{1/5}\ln^2 B)\right)$$

as $B \to 0$, uniformly in $\theta \in [0, \pi]$.

Uniformly means that the remainder satisfies $|O(B^{1/5}\ln^2 B)| \le cB^{1/5}|\ln^2 B|$ for all $0 < B \le B_0$, if B_0 is sufficiently small, where the constant c depends only on B_0 and not on the contact angle θ.

It is noteworthy that the special nonlinearity of the problem implies that the expansion is uniform with respect to $\theta \in [0, \pi]$ although $|Du|$ tends to infinity as $\theta \to 0$ or $\theta \to \pi$ and therefore the differential equation (1) will be singular on $\partial\Omega$. Moreover, as a further consequence of the strong nonlinearity of the problem, we do not need any growth assumption at infinity.

In the case of complete wetting, that is, if $\theta = 0$, the formula

$$u(a) \sim -a\left(\tfrac{1}{2}\ln B - 0.809\ldots\right)$$

as $a \to 0$ was derived formally by Derjaguin [1946] by expansion matching. We recall that $B = \kappa a^2$. Higher-order approximations where obtained formally by James [1974] and Lo [1983], also by matching arguments.

(Matching means that some free constants which occur in two asymptotic expansions with an overlapping domain of their definition will be determined in an appropriate way; see [Van Dyke 1964; Fraenkel 1969], for example.)

Turkington [1980] proved that $u(a) \sim -\frac{1}{2}\cos\theta\, a\ln B$ as $a \to 0$ under an additional growth assumption at infinity. This assumption is superfluous because of the comparison principle of Finn and Hwang [1989].

The proof of the existence of the asymptotic expansion is based on a construction of an upper and a lower C^1-solution of (1)–(2) and on the maximum principle of Finn and Hwang for unbounded domains. We obtain the lower and the upper solution by gluing together a boundary layer expansion near the needle with a second expansion far from the needle such that the resulting function is in C^1. This method of composing of functions on different annular domains was used in [Miersemann 1996], where a numerical method for the circular tube was proposed.

Theorem 1.1 and the calculations of the appendix, together with those of [Lo 1983], suggest:

Conjecture. *For given $N \in \mathbb{N} \cup \{0\}$ the ascent $u(a)$ satisfies*

$$\frac{u(a)}{a} = -\cos\theta \left(\sum_{k=0}^{N} \sum_{l=0}^{M(k)} c_{kl}(\theta) B^k (\ln B)^l + o(B^N) \right)$$

as $B \to 0$, uniformly in $\theta \in [0, \pi]$.

2. Expansion near the needle

Since $U(x)$ is rotationally symmetric, the boundary value problem (1)–(2) reads, with the notation (3),

$$\frac{1}{r}\left(\frac{ru'(r)}{\sqrt{1+(u'(r))^2}}\right)' = \kappa u(r) \quad \text{in} \quad a < r < \infty,$$
$$\lim_{r \to a+0} \frac{u'(r)}{\sqrt{1+(u'(r))^2}} = -\cos\theta.$$

Set

$$r = as, \qquad v(s) = \frac{1}{a} u(as), \qquad B = \kappa a^2.$$

Then the problem becomes

$$\frac{1}{s}\left(\frac{sv'(s)}{\sqrt{1+(v'(s))^2}}\right)' = Bv(s) \quad \text{in} \quad 1 < s < \infty, \tag{4}$$

$$\lim_{s \to 1+0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = -\cos\theta. \tag{5}$$

For a fixed q, $1 < q < \infty$, $b_0 := -\cos\theta$, $\theta \in [0, \pi]$ and $b_1 \in [-1, 1]$ let

$$v_1(s) \equiv v_1(B, q, b_0, b_1; s)$$

be the solution of

$$\frac{1}{s}\left(\frac{sv'(s)}{\sqrt{1+(v'(s))^2}}\right)' = Bv(s) \quad \text{for } 1 < s < q, \tag{6}$$

$$\lim_{s \to 1+0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = b_0, \qquad \lim_{s \to q-0} \frac{v'(s)}{\sqrt{1+(v'(s))^2}} = b_1. \tag{7}$$

Set

$$\operatorname{div} Tv = \frac{1}{r}\left(\frac{rv'}{\sqrt{1+(v')^2}}\right)'.$$

It was shown in [Miersemann 1993; 1994] that for fixed q there exists a complete asymptotic expansion of v_1 as $B \to 0$, uniformly in $b_0,\ b_1 \in [-1, 1]$:

$$v_1 = \frac{C}{B} + \sum_{k=0}^{m} \varphi_k(s) B^k + O(B^{m+1}),$$

here $\varphi_k(s) \equiv \varphi_k(q, b_0, b_1; s)$ and

$$C \equiv C(q, b_0, b_1) = \frac{2(qb_1 - b_0)}{q^2 - 1}.$$

The function φ_0 is a solution of a boundary value problem for a nonlinear second order ordinary differential equation and the φ_k, for $k \geq 1$, are solutions of linear boundary value problems.

It turns out that we have to change q if $B \to 0$. More precisely, $q = B^{-\tau}$, for $\tau > 0$ small, will be an appropriate choice. Therefore, we need some information about how the functions, for example φ_k, depend on q.

Set

$$b_1 := \frac{b_0}{q}(1+\epsilon), \quad 0 \leq |\epsilon| < \epsilon_0 < 1,$$
$$\phi_k(s) \equiv \phi_k(q, b_0, \epsilon; s) := \varphi_k\Big(q, b_0, \frac{a_0}{q}(1+\epsilon); s\Big)$$

and for $m \geq 0$

$$v_{1,m}(s) \equiv v_{1,m}(B, q, b_0, \epsilon; s) := \frac{2\epsilon b_0}{B(q^2-1)} + \sum_{k=0}^{m} \phi_k(s) B^k. \tag{8}$$

Assume that

$$\lambda := Bq^2 \ln q \leq \lambda_0$$

for a sufficiently small positive λ_0, independent of B and q. We will choose $q = B^{-\tau}$ for $\tau \in \big(0, \frac{1}{2}\big)$.

Proposition 2.1. *Suppose $q \geq 3$. For a given $m \in \mathbb{N} \cup \{0\}$ there exist functions $\varphi_k(s) \equiv \varphi_k(q, b_0, b_1; s)$ for $k = 0, 1, \ldots, m$, analytic in $1 < s < q$ and continuous in $1 \leq s \leq q$, as well as functions $\phi_k(s) \equiv \phi_k(q, b_0, \epsilon; s)$, continuous in $|\epsilon| < \frac{1}{4}$, such that for $|\epsilon| \leq \frac{1}{4}$ and $s \in (1, q)$ we have*

$$\phi_k(s) = \sum_{l=0}^{N} \phi_{k,l}(q, b_0; s)\epsilon^l + R_{N+1}\epsilon^{N+1},$$

where

$$|\phi_{k,l}(q, b_0; s)| \leq c\,|b_0|\,(\ln q)^{k+1} q^{2k}, \quad |R_{N+1}| \leq c\,|b_0|\,(\ln q)^{k+1} q^{2k}$$

and

$$|\mathrm{div}\, T v_{1,m} - B v_{1,m}| \le c\,|b_0|\,(\ln q)^{m+1} q^{2m} B^{m+1}; \tag{9}$$

here $v_{1,m}$ *is the sum* (8). *The constants* c *depend only on* λ_0 *and on* k, N, m, *and not on* $b_0 \in [-1, 1]$.

In particular,

$$\phi_{0,0}(q, b_0; 1) = -b_0\Big(\ln q + \ln 2 - \tfrac{1}{2} - \ln\big(1 + \sqrt{1 - b_0^2}\big) + O(q^{-2}\ln q)\Big)$$

as $q \to \infty$.

The proof is given in Section A.1 of the Appendix.

3. Expansion far from the needle

Let $v_2(s) \equiv v_2(B, q, b_1; s)$ be the solution of

$$\frac{1}{s}\left(\frac{s v'(s)}{\sqrt{1 + (v'(s))^2}}\right)' = B v(s) \quad \text{in} \quad q < s < \infty, \tag{10}$$

$$\lim_{s \to q+0} \frac{v'(s)}{\sqrt{1 + v'(s)^2}} = b_1. \tag{11}$$

In contrast to the earlier expansion with respect to B near the needle, we expand v_2 with respect to b_1 for fixed Bond number $0 < B < 1$.

For small $|b_1|$ we have

$$v'(q) = \frac{b_1}{\sqrt{1 - b_1^2}} = b_1 \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(-b_1^2\right)^k.$$

We make the following ansatz for a solution of the differential equation (10), where $n \in \mathbb{N} \cup \{0\}$, $\rho \in \mathbb{R}$, $|\rho|$ small:

$$v_{2,n}(s) \equiv v_{2,n}(B, q, \rho; s) := \sum_{k=0}^{n} \psi_k(B, q; s)\, \rho^{2k+1} \tag{12}$$

with unknown functions $\psi_k(s) :\equiv \psi_k(B, q; s)$ such that

$$\psi_k'(q) = (-1)^k \binom{-\frac{1}{2}}{k}. \tag{13}$$

Since

$$v_{2,n}'(q) = \rho \sum_{k} \binom{-\frac{1}{2}}{k} \left(-\rho^2\right)^k,$$

it follows that $v_{2,n}$ satisfies the boundary condition (11) at $s=q$ if

$$\sum_{k=0}^{n}(-1)^k\binom{-\frac{1}{2}}{k}\rho^{2k+1}=\frac{b_1}{\sqrt{1-b_1^2}}. \tag{14}$$

Thus, since $b_1=b_0(1+\epsilon)/q$,

$$\rho=b_1+O(b_1^{2n+3})=\frac{b_0}{q}+\epsilon\frac{b_0}{q}+O\left(\left(\frac{b_0}{q}\right)^{2n+3}\right)$$

as $b_0/q\to 0$.

Definition 3.1. We write $w(\delta)=P(\delta,\ln\delta)$, where $0<\delta<\delta_0$, if for given $N\in\mathbb{N}$ we have

$$w(\delta)=\sum_{\alpha=1}^{N}\sum_{\beta=0}^{M(\alpha)}c_{\alpha\beta}\delta^{\alpha}(\ln\delta)^{\beta}+R_N(\delta),$$

where $c_{\alpha\beta}\in\mathbb{R}$, $R_N(\delta)$ is continuous in $0\le\delta<\delta_0$, $\lim_{N\to\infty}R_N(\delta)=0$ for fixed δ and $R_N(\delta)=o(\delta^N)$ as $\delta\to 0$.

Proposition 3.2. *Assume that* $0<B<1$, $q=B^{-\tau}$, $\tau\in(0,\tau_1]$, $0<\tau_1<\frac{1}{2}$ *and* $|\rho|<\rho_0$, *for* ρ_0 *sufficiently small. For a given* $n\in\mathbb{N}\cup\{0\}$ *there exist functions* $\psi_k(s)\equiv\psi_k(B,q;s)$, $k=0,\dots,n$, *analytic on* $q\le s<\infty$, *such that the sum* $v_{2,n}$ *of* (12) *satisfies*

$$|\operatorname{div}Tv_{2,n}-Bv_{2,n}|\le c|\rho|^{2n+3} \tag{15}$$

on $s\in[q,\infty)$, *where the constant* c *depends only on* τ_1, ρ_0 *and* n. *Further, for* $\delta:=\sqrt{B}q$ *there are functions* $w_k(\delta)=P(\delta,\ln\delta)$ *such that*

$$\psi_k(B,q;q)=\frac{1}{\sqrt{B}}w_k(\delta). \tag{16}$$

In particular,

$$\psi_0(B,q;q)=\frac{1}{\sqrt{B}}\frac{K_0(\delta)}{K_0'(\delta)},$$

where $K_0(\delta)$ *is a modified Bessel function of second kind and of order zero.*

The proof is given in Section A.2 of the Appendix.

Siegel [1980] observed that the function $\psi_0:=cK_0(\sqrt{B}s)$, where c is a positive constant, defines for a fixed $q>1$ a supersolution of the differential equation (10) on (q,∞). We will show that there is a positive constant A such that $v_{2,n}\pm A$ defines a supersolution and a subsolution, respectively, on (q,∞) if $q:=B^{-\tau}$ for appropriate τ satisfying $0<\tau\le\tau_1<\frac{1}{2}$ and if ρ is defined by (14). In particular,

$$v_{2,0}=\frac{K_0(\sqrt{B}s)}{\sqrt{B}K_0'(\sqrt{B}q)}\rho.$$

4. Composing of the inner and outer solutions

By the inner solution we mean the expansion $v_{1,m}$ near the needle and the outer solution is $v_{2,n}$, the expansion far from the needle.

We glue together these two expansions at $s = q$ in such a way that the composite function is in $C^1(1, \infty)$.

Set

$$v_{c,m,n}(s) := \begin{cases} v_{1,m}(B, q, b_0, \epsilon; s) & \text{for } 1 \le s \le q, \\ v_{2,n}(B, q, \rho; s) & \text{for } q < s < \infty. \end{cases}$$

This composite function is in $C^1(1, \infty)$ if and only if ρ satisfies (14) and $v_{1,m}$, $v_{2,n}$ coincide at $s = q$, that is, if

$$v_{1,m}(B, q, b_0, \epsilon; q) = v_{2,n}(B, q, \rho; q), \tag{17}$$

where $\rho = \rho(b_0, q, \epsilon)$ is defined by (14). Now set

$$\delta := \sqrt{B}q.$$

We choose $q = B^{-\tau}$ for a fixed $\tau \in \left(0, \frac{1}{2}\right)$; then $\delta \to 0$ if $B \to 0$.

Proposition 4.1. *Assume that $q = B^{-\tau}$ for a fixed $\tau \in \left(0, \frac{1}{2}\right)$. Then there is a solution ϵ of equation (17). In particular, we have*

$$\epsilon = \tfrac{1}{2}\delta^2 \ln \delta + \tfrac{1}{2}\left(\gamma - \ln 2 - \tfrac{1}{2}\right)\delta^2 + R(b_0, B, B^{-\tau})\delta^2$$

with

$$R(b_0, B, B^{-\tau}) = O(B^{2\tau}(\ln B)^{l+1}) + O(B^{1-2\tau} \ln^2 B)$$

uniformly in $b_0 \in [-1, 1]$ as $B \to 0$, where $l \in \mathbb{N} \cup \{0\}$ and

$$\gamma := \lim_{m \to \infty} \left(\sum_{k=1}^{m} \frac{1}{k} - \ln m \right) = 0.5772\ldots$$

is Euler's constant.

The proof is given in Section A.3 of the Appendix.

Assume that $q := B^{-\tau}$ for $0 < \tau \le \tau_1 < \frac{1}{2}$. Then, since

$$b_1 = \frac{b_0}{q}\left(1 + O(B^{1-2\tau} \ln B)\right),$$

it follows from the three propositions above that the $C^1(1, \infty)$ function $v_{c,m,n}$ satisfies, for $0 < B \le B_0 < 1$ with B_0 sufficiently small,

$$|\operatorname{div} T v_{c,m,n} - B v_{c,m,n}| \le \begin{cases} c\,|b_0|(-\ln B)^{m+1} B^{(1-2\tau)m+1} & \text{for } 1 \le s \le q, \\ c\,|b_0| B^{(2n+3)\tau} & \text{for } q < s < \infty. \end{cases}$$

The constant c depends only on m, n, B_0 and τ_1.

5. Asymptotic expansion

Let A be a positive constant. Set

$$v^+_{c,m,n} := v_{c,m,n} + A.$$

This function $v^+_{c,m,n}$ is in $C^1(1,\infty)$ and satisfies the boundary condition (5) at $s=1$. From the above estimate it follows

$$\begin{aligned}\operatorname{div} T v^+_{c,m,n} - B v^+_{c,m,n} &= \operatorname{div} T v_{c,m,n} - B v_{c,m,n} - AB \\ &\le B \begin{cases} c\,|b_0|\,(-\ln B)^{m+1} B^{(1-2\tau)m} - A & \text{for } 1 \le s \le q, \\ c\,|b_0|\,B^{(2n+3)\tau-1} - A & \text{for } q < s < \infty. \end{cases}\end{aligned}$$

The constant c depends only on m, n, B_0 and τ_1.

For $\tau \in \left(0, \frac{1}{2}\right)$ and $m, n \in \mathbb{N} \cup \{0\}$, set

$$p(m,n;\tau) := min\{(1-2\tau)m, (2n+3)\tau - 1\}$$

and let $\tau_0 \equiv \tau_0(m,n)$ be the solution of $(1-2\tau)m = (2n+3)\tau - 1$, that is,

$$\tau_0 = \frac{m+1}{2(m+1)+2n+1}.$$

Thus τ_0 is the solution of

$$\max_{0<\tau<1/2} p(m,n;\tau).$$

Set $p_0 \equiv p_0(m,n) := p(m,n;\tau_0)$; that is,

$$p_0 = \frac{2mn+m}{2m+2n+3}.$$

Choose

$$A := c\,|b_0|\,(-\ln B)^{m+1} B^{p_0}; \tag{18}$$

then the preceding inequality implies

$$\operatorname{div} T v^+_{c,m,n} - B v^+_{c,m,n} \le 0$$

for all B such that $0 < B \le B_0$ and for all s in $(1,q] \cup (q,\infty)$. The maximum principle of Finn and Hwang [1989] yields

$$v(s) \le v^+_{c,m,n}(s)$$

on $(1,\infty)$. By the same reasoning it follows that

$$v^-_{c,m,n} := v_{c,m,n} - A,$$

satisfies $v(s) \ge v^-_{c,m,n}(s)$ on $(1,\infty)$, where A is given by (18).

Summarizing, we have shown that $|v(s) - v_{c,m,n}(s)| \le c\,|b_0|(-\ln B)^{m+1}B^{p_0}$. We can choose p_0 arbitrarily large provided m and n are large enough; see the definition of p_0 above.

In particular, the height rise at $s = 1$ satisfies

$$|v(1) - v_{1,m}(1)| \le c\,|b_0|\,(-\ln B)^{m+1}B^{p_0}.$$

Thus

$$v(1) = \frac{C(q, b_0, b_1)}{B} + \sum_{k=0}^{m} \varphi_k(q, b_0, b_1; 1)B^k + O(b_0 B^{p_0} \ln^{m+1} B),$$

where $b_1 = b_0(1+\epsilon)/q$, $q = B^{-\tau_0}$ and ϵ is the solution of (17); see Proposition 4.1.

Thus, we consider

$$v_{1,m}(1) := \frac{C(q, b_0, b_1)}{B} + \sum_{k=0}^{m} \varphi_k(q, b_0, b_1; 1)B^k$$

as an *approximation of order* p_0 of the value $v(1)$.

Then, since $B = \kappa a^2$ and $u(a) = a v(1)$, we have

$$\frac{u(a)}{a} = v_{1,m}(1) + O(b_0 B^{p_0} \ln^{m+1} B) \tag{19}$$

as $B \equiv \kappa a^2 \to 0$.

Proof of Theorem 1.1. Set $m = 1$ and $n = 0$. Then $\tau_0 = \frac{2}{5}$, $p_0 = \frac{1}{5}$, $q \equiv B^{-\tau_0} = B^{-2/5}$ and $\delta \equiv \sqrt{B}q = B^{1/10}$. We obtain from Proposition 4.1

$$\epsilon = \tfrac{1}{2}\delta^2 \ln \delta + \tfrac{1}{2}(\gamma - \ln 2 - \tfrac{1}{2})\delta^2 + O(\delta^2 B^{1/5} \ln^2 B)$$

and Proposition 2.1 yields

$$\phi_0(1) = -b_0\Big(\ln q + \ln 2 - \tfrac{1}{2} - \ln\big(1 + \sqrt{1 - b_0^2}\big)\Big) + O(b_0 B^{1/5} \ln^2 B)$$

and $\phi_1(1)B = O(b_0 B^{1/5} \ln^2 B)$.

Thus

$$\begin{aligned}
v_{1,1}(1) &= \frac{2\epsilon b_0}{B(q^2-1)} + \phi_0(1) + \phi_1(1)B + O(b_0 B^{1/5} \ln^2 B) \\
&= b_0\left(\ln \delta - \ln 2 - \tfrac{1}{2} + \gamma + O(B^{1/5} \ln^2 B)\right)\left(1 - \frac{1}{q^2}\right)^{-1} \\
&\qquad - b_0\big(\ln q + \ln 2 - \tfrac{1}{2} - \ln(1 + \sqrt{1 - b_0^2})\big) + O(b_0 B^{1/5} \ln^2 B) \\
&= b_0\big(\tfrac{1}{2} \ln B - 2\ln 2 + \gamma + \ln(1 + \sqrt{1 - b_0^2})\big) + O(b_0 B^{1/5} \ln^2 B).
\end{aligned}$$

The theorem follows from formula (19) for $u(a)/a$. □

Appendix: Proof of the propositions

Here we prove the propositions of the previous sections. The argument concerns mainly expansions of nonlinear expressions with respect to appropriate parameters. In the expansion near the needle the special nonlinearity of the problem is exploited. The expansion far from the needle ensues by linearization of the problem with respect to the zero solution.

A.1. ***Expansion near the needle.*** Set for $0 < B < B_0$

$$v_m = \frac{C}{B} + \sum_{k=0}^{m} \varphi_k(s) B^k,$$

where C is a constant and φ_k are functions in $C^2(1, q)$, $1 < q < \infty$.

The sum v_m is said to be an *approximate solution* of (6)–(7) if v_m satisfies the boundary conditions (7) and if

$$|\mathrm{div}\, T v_m - B v_m| \le c B^{m+1}$$

on $(1, q)$, where $c = c(m, q)$ and c is independent on b_0, $b_1 \in [-1, 1]$.

In the following we will define C and φ_k so that v_m is an approximate solution. It turns out that C is given explicitely, φ_0 is the solution of a nonlinear boundary value problem for a second order differential equation and φ_k, for $k \ge 1$, are solutions of linear boundary value problems of second order, defined iteratively. The main idea here is to preserve the properties of the special nonlinearity also in the expansions.

In

$$\mathrm{div}\, T v_m \equiv \frac{1}{s} \left(\frac{s v_m'}{\sqrt{1 + {v_m'}^2}} \right)'$$

there appears the quotient $v_m' / \sqrt{1 + {v_m'}^2}$. We now derive some expansions in B related to this quotient.

Definition of C and φ_k. Since

$$\begin{aligned} 1+{v_m'}^2 &= 1 + \left(\sum_{l=0}^{m} \varphi_l' B^l \right)^2 \\ &= (1+{\varphi_0'}^2) \left(1 + 2 \frac{\varphi_0'}{\sqrt{1+{\varphi_0'}^2}} \sum_{l=1}^{m} \frac{\varphi_l'}{\sqrt{1+{\varphi_0'}^2}} B^l + \left(\sum_{l=1}^{m} \frac{\varphi_l'}{\sqrt{1+{\varphi_0'}^2}} B^l \right)^2 \right), \end{aligned}$$

it follows that

$$\frac{v_m'}{\sqrt{1+v_m'^2}} = \frac{v_m'}{\sqrt{1+\varphi_0'^2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-1/2}.$$

Set, for $l = 1, \dots, m$,

$$d_l := \frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}$$

and assume that

$$\text{(A–1)}\qquad \sup_{s\in(1,q)} |d_l| \le c_l^{(1)}(q) < \infty.$$

Then for $M \in \mathbb{N}$, provided $0 < B \le B_0(q)$ with B_0 sufficiently small, we have

$$\text{(A–2)}\qquad \frac{v_m'}{\sqrt{1+v_m'^2}} = \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} + \sum_{k=1}^{M} f_{m,k}(\varphi_0', \dots, \varphi_m')B^k + \tilde{f}_{m,M+1}B^{M+1},$$

where $f_{m,k}$ and $\tilde{f}_{m,M+1}$ are defined as follows. Set $g_m(B) := v_m'/\sqrt{1+v_m'^2}$, then

$$f_{m,k} = g_m^{(k)}(0)/k! \quad \text{and} \quad \tilde{f}_{m,k} = g_m^{(k)}(tB)/k! \quad \text{for } 0 < t < 1.$$

From assumption (A–1) on φ_k' we obtain

$$|f_{m,k}| \le c_{m,k}(q) < \infty \quad \text{and} \quad |\tilde{f}_{m,M+1}| \le \tilde{c}_{m,M+1}(q) < \infty.$$

We have, from (A–2), $f_{0,k} \equiv 0$ and $\tilde{f}_{0,k} \equiv 0$ for all $k \in \mathbb{N}$.

This argument exploits the special nonlinearity of the problem. More precisely, we have used that

$$\frac{|\varphi_0'|}{\sqrt{1+\varphi_0'^2}}$$

remains bounded even if $|\varphi_0'(s)| \to \infty$ if $s \to 1$ or $s \to q$.

We obtain from (A–2) the expansion

$$\text{(A–3)}\qquad \operatorname{div} Tv_m = \frac{1}{s}\left(\frac{s\varphi_0'}{\sqrt{1+\varphi_0'^2}}\right)' + \sum_{k=1}^{M}\frac{1}{s}\big(sf_{m,k}\big)'B^k + \frac{1}{s}\big(s\tilde{f}_{m,M+1}\big)'B^{M+1}.$$

We next need some information on how the derivatives $(f_{m,k})'$ and $(\tilde{f}_{m,l})'$ depend on b_0, b_1 and q.

Since $v_m' = \sum_{l=0}^{m} \varphi_l' B^l$ and

$$\text{(A–4)}\qquad \operatorname{div} Tv \equiv \frac{1}{s}v'(1+v'^2)^{-1/2} + v''(1+v'^2)^{-3/2}$$

it follows under assumption (A–1) that for $0 < B \le B_0 \equiv B_0(q)$, with B_0 sufficiently small,

$$\begin{aligned}\operatorname{div} T v_m &\\ &= \frac{1}{s}\frac{v_m'}{\sqrt{1+\varphi_0'^2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-1/2}\\ &\quad+\frac{v_m''}{(1+\varphi_0'^2)^{3/2}}\left(1+2\frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}}\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l+\left(\sum_{l=1}^{m}\frac{\varphi_l'}{\sqrt{1+\varphi_0'^2}}B^l\right)^2\right)^{-3/2}.\end{aligned}$$

Thus

$$\operatorname{div} T v_m = \frac{1}{s}\left(\frac{s\varphi_0'}{\sqrt{1+\varphi_0'^2}}\right)' + \sum_{k=1}^{M} F_{m,k}B^k + \tilde{F}_{m,M+1}B^{M+1}, \tag{A–5}$$

where $F_{m,k}$ and $\tilde{F}_{m,M+1}$ are defined as follows. Set

$$h_m(B) := \frac{1}{s}\frac{v_m'}{\sqrt{1+v_m'^2}} + v_m''\bigl(1+v_m'^2\bigr)^{-3/2}.$$

Then $F_{m,k} = h_m^{(k)}(0)/k!$ and $\tilde{F}_{m,k} = h_m^{(k)}(tB)/k!$ for $0 < t < 1$. We have $F_{0,k} \equiv 0$ and $\tilde{F}_{0,k} \equiv 0$ for all $k \in \mathbb{N}$.

Set for $l = 1, \dots, m$

$$e_l := \frac{\varphi_l''}{(1+\varphi_0'^2)^{3/2}}$$

and assume

$$\sup_{s\in(1,q)} |e_l| \le c^{(2)}(q) < \infty. \tag{A–6}$$

Then the functions $F_{m,k}$ and $\tilde{F}_{m,M+1}$ are bounded.

Since

$$\frac{1}{s}(sf_{m,k})' \equiv F_{m,k}, \quad \frac{1}{s}(s\tilde{f}_{m,k})' \equiv \tilde{F}_{m,k},$$

it follows, under assumptions (A–1) and (A–6), that the derivatives $(f_{m,k})'$, $(\tilde{f}_{m,k})'$ are bounded.

In the following considerations we derive boundary value problems which define the functions $\varphi_0, \varphi_1, \dots, \varphi_m$. Then we prove that these functions φ_l satisfy inequalities (A–1) and (A–6) *uniformly* in $q \ge 3$ and in $b_0 \in [-1, 1]$, where $b_1 = b_0(1+\epsilon)/q$, with $|\epsilon| \le \frac{1}{4}$.

The following lemma is useful in order to iteratively find the appropriate boundary value problem which defines φ_{m+1} for given $\varphi_0, \dots, \varphi_m$.

Lemma A.1.1. *Let assumption (A–1) on φ_l, for $l = 1, \dots, m+1$, be satisfied. Then*

$$\frac{v'_{m+1}}{\sqrt{1+{v'_{m+1}}^2}} = \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}} B^{m+1} + R,$$

where $|R| \le c(q)B^{m+2}$, $0 < B \le B_0(q)$, B_0 sufficiently small.

Proof.

$$\begin{aligned}
\frac{v'_{m+1}}{\sqrt{1+{v'_{m+1}}^2}} &= \left(v'_m + \varphi'_{m+1}B^{m+1}\right)\left(1+{v'_m}^2 + 2v'_m\varphi'_{m+1}B^{m+1} + {\varphi'_{m+1}}^2 B^{2m+2}\right)^{-1/2} \\
&= \left(v'_m + \varphi'_{m+1}B^{m+1}\right)\left(1+{v'_m}^2\right)^{-1/2} \\
&\qquad \cdot \left(1 + 2\frac{v'_m}{\sqrt{1+{v'_m}^2}}\frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1} + \frac{(\varphi'_{m+1})^2}{1+{v'_m}^2}B^{2m+2}\right)^{-1/2} \\
&= \left(\frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1}\right)\left(1 - \frac{v'_m\varphi'_{m+1}}{1+{v'_m}^2}B^{m+1} + R_1\right) \\
&= \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \left(-\frac{{v'_m}^2\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}} + \frac{\varphi'_{m+1}}{\sqrt{1+{v'_m}^2}}B^{m+1}\right) + R_2 \\
&= \frac{v'_m}{\sqrt{1+{v'_m}^2}} + \frac{\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}}B^{m+1} + R_2.
\end{aligned}$$

The remainders above satisfy $|R_1|,\ |R_2| \le c(q)B^{2m+2}$. Since

$$\begin{aligned}
&\frac{\varphi'_{m+1}}{(1+{v'_m}^2)^{3/2}} \\
&\quad = \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\left(1 + 2\frac{\varphi'_0}{\sqrt{1+{\varphi'_0}^2}}\sum_{l=1}^{m}\frac{\varphi'_l}{\sqrt{1+{\varphi'_0}^2}}B^l + \left(\sum_{l=1}^{m}\frac{\varphi'_l}{\sqrt{1+{\varphi'_0}^2}}B^l\right)^2\right)^{-3/2} \\
&\quad = \frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}} + R_3,
\end{aligned}$$

where $|R_3| \le c(q)B$, the expansion of the lemma is shown. $\square$

Lemma A.1.2. *Suppose assumptions (A–1) and (A–6) are satisfied. Then*

$$\operatorname{div} Tv_{m+1} = \operatorname{div} Tv_m + \frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)' B^{m+1} + O(B^{m+2})$$

as $B \to 0$, uniformly in $s \in (1, q)$.

Proof. We conclude from (A–4) and Lemma A.1.1 that

$$\begin{aligned}\operatorname{div} Tv_{m+1} &\equiv \frac{1}{s}\frac{v'_{m+1}}{\sqrt{1+{v'_m}^2}}+\frac{v''_{m+1}}{(1+{v'_m}^2)^{3/2}}\\ &= \frac{1}{s}\frac{v'_m}{\sqrt{1+{v'_m}^2}}+\frac{1}{s}\frac{\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}B^{m+1}+\frac{v''_{m+1}}{(1+{v'_{m+1}}^2)^{3/2}}+O(B^{m+2}).\end{aligned}$$

Since

$$\frac{v''_{m+1}}{(1+{v'_{m+1}}^2)^{3/2}}=\frac{v''_m}{(1+{v'_m}^2)^{3/2}}+\left(\frac{\varphi''_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}-\frac{3\varphi'_0\varphi''_0\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{5/2}}\right)B^{m+1}+O(B^{m+2}),$$

which follows by similar calculations as in the proof of Lemma A.1.1, we obtain

$$\operatorname{div} Tv_{m+1}=\frac{1}{s}\frac{v'_m}{\sqrt{1+{v'_m}^2}}+\frac{v''_m}{(1+{v'_m}^2)^{3/2}}+\frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)'B^{m+1}+O(B^{m+2}).$$

□

Lemma A.1.2 implies

$$\begin{aligned}&\operatorname{div} Tv_{m+1}-Bv_{m+1}\\ &\quad=\operatorname{div} Tv_m+\frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)'B^{m+1}-(C+B\varphi_0+\cdots+B^{m+1}\varphi_m)+O(B^{m+2}).\end{aligned}$$

Then from expansion (A–3) for $\operatorname{div} Tv_m$, with $M:=m+1$, and from the condition

$$\operatorname{div} Tv_{m+1}-Bv_{m+1}=O(B^{m+2})\quad \text{as } B\to 0,$$

there follows for $m\ge 0$ the differential equation

$$\text{(A–7)}\qquad \frac{1}{s}\left(\frac{s\varphi'_{m+1}}{(1+{\varphi'_0}^2)^{3/2}}\right)'+\frac{1}{s}(sf_{m,m+1})'=\varphi_m$$

on $1<s<q$. We recall that $f_{m,m+1}=g_m^{(m+1)}(0)/(m+1)!$, where $g_m(B)=v'_m/\sqrt{1+{v'_m}^2}$.

We conclude from $\operatorname{div} Tv_0-Bv_0=O(B)$ that

$$\text{(A–8)}\qquad \operatorname{div} T\varphi_0\equiv\frac{1}{s}\left(\frac{s\varphi'_0}{\sqrt{1+{\varphi'_0}^2}}\right)'=C$$

on $1<s<q$.

From the assumptions

$$\lim_{s\to 1+0}\frac{v'_m}{\sqrt{1+{v'_m}^2}}=b_0,\qquad \lim_{s\to q-0}\frac{v'_m}{\sqrt{1+{v'_m}^2}}=b_1$$

for fixed q and $0 < B \le B_0(q)$, and from the expansion (A–2), we get

(A–9) $$\lim_{s\to 1+0} \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} = b_0, \qquad \lim_{s\to q-0} \frac{\varphi_0'}{\sqrt{1+\varphi_0'^2}} = b_1.$$

Further, we obtain from Lemma A.1.1 that for $m \ge 1$

(A–10) $$\lim_{s\to 1+0} \frac{\varphi_{m+1}'}{(1+\varphi_0'^2)^{3/2}} = 0, \qquad \lim_{s\to q-0} \frac{\varphi_{m+1}'}{(1+\varphi_0'^2)^{3/2}} = 0,$$

and (A–2) implies the boundary conditions

(A–11) $$\lim_{s\to 1+0} f_{m,k}(\varphi_0', \dots, \varphi_m') = 0, \qquad \lim_{s\to q-0} f_{m,k}(\varphi_0', \dots, \varphi_m') = 0$$

for $k \ge 1$ and $m \ge 0$.

After integration of the differential equation from 1 to q it follows from the boundary conditions (A–11) and (A–12) that, for $m \ge 0$,

(A–12) $$\int_1^q s\varphi_m(s)\, ds = 0.$$

Applying the differential equation (A–8) for φ_0 and the boundary conditions (A–9), we find

(A–13) $$C = \frac{2(qb_1 - b_0)}{q^2-1}.$$

Set

(A–14) $$f(s) \equiv f(q, b_0, b_1; s) := b_0 f_0 + b_1 f_1,$$

where

$$f_0 := \frac{q^2-1-(s^2-1)}{s(q^2-1)}, \qquad f_1 := \frac{q(s^2-1)}{s(q^2-1)}.$$

Then it follows from (A–8) and the formula (A–13) for C that

(A–15) $$\frac{\varphi_0'(s)}{\sqrt{1+(\varphi_0'(s))^2}} = f(s)$$

or, equivalently,

(A–16) $$\varphi_0'(s) = \frac{f(s)}{\sqrt{1-f^2(s)}}.$$

Set for $1 \le s \le q$

(A–17) $$\tilde{\varphi}_0(s) := \int_1^s \frac{f(\tau)}{\sqrt{1-f^2(\tau)}}\, d\tau,$$

then $\varphi_0(s) = \tilde{\varphi}_0(s) + K$, where the constant K will be determined by the side condition (A–12). That is, $\varphi_0(s) \equiv \varphi_0(q, b_0, b_1; s)$ is given by

$$\varphi_0(s) = \tilde{\varphi}_0(s) - \frac{2}{q^2-1}\int_1^q \tau\tilde{\varphi}_0(\tau)\,d\tau. \tag{A–18}$$

Then we obtain $\varphi_l(s) \equiv \varphi_l(q, b_0, b_1; s)$ for $l \geq 1$, by the iterative application of (A–7), (A–9), (A–10) and (A–11). That is,

$$\varphi_{l+1}(s) = \tilde{\varphi}_{l+1}(s) - \frac{2}{q^2-1}\int_1^q \tau\tilde{\varphi}_{l+1}(\tau)\,d\tau, \tag{A–19}$$

where

$$\tilde{\varphi}_{l+1}(s) := \int_1^s \varphi'_{l+1}(\tau)\,d\tau \tag{A–20}$$

and

$$\varphi'_{l+1}(s) := (1+\varphi_0'^{\,2})^{3/2}\left(-f_{l,l+1} + \frac{1}{s}\int_1^s \tau\varphi_l(\tau)\,d\tau\right). \tag{A–21}$$

Set for the unknown b_1

$$b_1 := \frac{b_0}{q}(1+\epsilon), \tag{A–22}$$

where

$$|\epsilon| \leq \tfrac{1}{4} \quad \text{and} \quad q \geq 3. \tag{A–23}$$

We will determine ϵ in Section A.3 by gluing together two expansions at $s = q$, where $q = B^{-\tau}$ for $\tau > 0$ small.

Expansions with respect to ϵ. In this section we expand related functions with respect to ϵ.

Definition. Let $h \equiv h(q, b_0, \epsilon; s)$, where $1 \leq s \leq q$, $q \geq 3$, $|\epsilon| \leq \frac{1}{4}$ and $b_0 \in [-1, 1]$. We will write $h = \mathbb{O}(\epsilon; K)$ if for any fixed $M \in \mathbb{N} \cup \{0\}$

$$h = \sum_{l=0}^{M} h_l\epsilon^l + \tilde{h}_{M+1}\epsilon^{M+1},$$

where $h_l \equiv h_l(q, b_0; s)$, $\tilde{h}_{M+1} \equiv \tilde{h}_{M+1}(q, b_0, \epsilon; s)$, and $|h_l|$, $|\tilde{h}_{M+1}| \leq c_M|K|$. The constant c_M is independent on q, b_0, s, ϵ and K, it can depend on q, b_0 and s but not on ϵ.

From formula (A–14) for f and from (A–22) it follows that on $1 < s \leq q$

$$f = \frac{b_0}{s}\left(1 + \epsilon\frac{s^2-1}{q^2-1}\right). \tag{A–24}$$

Then

$$1-f^2=\Bigl(1-\Bigl(\frac{b_0}{s}\Bigr)^2\Bigr)(1+C_1\epsilon+C_2\epsilon^2), \tag{A–25}$$

where

$$C_1\equiv C_1(q,b_0;s)=-2b_0^2\frac{1}{q^2-1}\frac{s^2-1}{s^2-b_0^2},$$
$$C_2\equiv C_2(q,b_0;s)=-b_0^2\frac{1}{q^2-1}\frac{(s^2-1)^2}{s^2-b_0^2}.$$

Using (A–23), it follows that $|C_1\epsilon+C_2\epsilon^2|\le\frac{1}{2}$.

Set

$$\phi_k(s)\equiv\phi_k(q,b_0,\epsilon;s):=\varphi_k\Bigl(q,b_0,\frac{b_0}{q}(1+\epsilon);s\Bigr).$$

Then we obtain from formula (A–16) for φ_0'

$$\begin{aligned}\phi_0'&=\frac{b_0}{s}\Bigl(1+\epsilon\frac{s^2-1}{q^2-1}\Bigr)\Bigl(1-\Bigl(\frac{b_0}{s}\Bigr)^2\Bigr)^{-1/2}(1+C_1\epsilon+C_2\epsilon^2)^{-1/2}\\&=\frac{b_0}{\sqrt{s^2-b_0^2}}\bigl(1+\epsilon\mathbb{O}(\epsilon;1)\bigr).\end{aligned} \tag{A–26}$$

Formula (A–17) implies

$$\tilde\phi_0(s)=\tilde\phi_{0,0}(s)+\epsilon\mathbb{O}(\epsilon;b_0\ln s),$$

where

$$\tilde\phi_{0,0}(s)=b_0\Bigl(\ln\bigl(s+\sqrt{s^2-b_0^2}\,\bigr)-\ln\bigl(1+\sqrt{1-b_0^2}\,\bigr)\Bigr).$$

Finally, it follows from (A–18) that

$$\phi_0(s)=\phi_{0,0}(s)+\epsilon\mathbb{O}(\epsilon;b_0\ln q),$$

where

$$\begin{aligned}\phi_{0,0}(s)&=b_0\Bigl(\ln\bigl(s+\sqrt{s^2-b_0^2}\,\bigr)-\ln\bigl(1+\sqrt{1-b_0^2}\,\bigr)\Bigr)\\&+\frac{b_0}{q^2-1}\Bigl(\frac{q}{2}\sqrt{q^2-b_0^2}+\frac{b_0^2}{2}\ln\bigl(q+\sqrt{q^2-b_0^2}\,\bigr)-\frac{1}{2}\sqrt{1-b_0^2}-\frac{b_0^2}{2}\ln\bigl(1+\sqrt{1-b_0^2}\,\bigr)\Bigr).\end{aligned}$$

Using (A–24), (A–25) and (A–26), we immediately obtain

$$1+\phi_0'^2 \equiv (1-f^2)^{-1} = \frac{s^2}{s^2-b^2}\left(1+\epsilon\mathbb{O}(\epsilon;1)\right), \tag{A–27}$$

$$\frac{\phi_0'}{\sqrt{1+\phi_0'^2}} \equiv f = \frac{b_0}{s}\left(1+\epsilon\frac{s^2-1}{q^2-1}\right), \tag{A–28}$$

$$\frac{\phi_0''}{(1+\phi_0'^2)^{3/2}} \equiv f' = -\frac{b_0}{s^2}\left(1-\epsilon\frac{s^2+1}{q^2-1}\right). \tag{A–29}$$

Lemma A.1.3. *The functions ϕ_l, $l \geq 1$ are continuous in ϵ, $|\epsilon| \leq \frac{1}{4}$, and satisfy*

$$\phi_l(s) = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l}\right), \tag{A–30}$$

$$d_l \equiv \frac{\phi_l'}{\sqrt{1+\phi_0'^2}} = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l-1}\right), \tag{A–31}$$

$$e_l \equiv \frac{\phi_l''}{(1+\phi_0'^2)^{3/2}} = \mathbb{O}\left(\epsilon; b_0(\ln q)^l q^{2l-2}\right). \tag{A–32}$$

We will prove this lemma by induction based on formulas (A–15)–(A–17) and on the next lemma.

Lemma A.1.4. *Assume that equations (A–30)–(A–32) hold for $1 \leq l \leq m$. Then*

$$F_{m,m+1} = \mathbb{O}\left(\epsilon; b_0(\ln q)^{m+1}q^{2m}\right)$$

and, if $\lambda := Bq^2\ln q \leq \lambda_0$, for $\lambda_0 > 0$ sufficiently small, then

$$|\tilde{F}_{m,m+1}| \leq c_m|b_0|\,(\ln q)^{m+1}q^{2m},$$

where $c_m = c_m(\lambda_0)$ is independent on b_0 and q.

Proof. Set

$$h_m(B) = \frac{1}{s}(d_0+P)F(d_0,P) + (e_0+Q)G(c_0,P),$$

where $F = (1+2d_0P+P^2)^{-1/2}$, $G = (1+2d_0P+P^2)^{-3/2}$, $P = \sum_{l=1}^m d_lB^l$, $Q = \sum_{l=1}^m e_lB^l$.

From assumption (A–1) on d_l it follows $|2d_0P+P^2| \leq \frac{1}{2}$, provided λ_0 is sufficiently small. Since

$$F_{m,m+1} = \frac{h_m^{(m+1)}(0)}{(m+1)!} \quad\text{and}\quad \tilde{F}_{m,m+1} = \frac{h_m^{(m+1)}(tB)}{(m+1)!}, \quad \text{for } 0<t<1,$$

the lemma is a consequence of the Leibniz rule and the chain rule. We find from these rules for $\alpha = (\alpha_1,\dots,\alpha_m)$, $\alpha_l \in \mathbb{N}$ and $t = (t_1,\dots,t_m)$, $t_l \in \mathbb{N}\cup\{0\}$ and

$0 \le k \le m$ that

$$h_m^{(m+1)}(B) = \sum_{\sum_{l=1}^m \alpha_l t_l = m+1} \frac{1}{s} C_{m,\alpha,t} (P^{(\alpha_1)})^{t_1} \dots (P^{(\alpha_m)})^{t_m} + \sum_{k+\sum_{l=1}^m \alpha_l t_l = m+1} D_{m,k,\alpha,t} Q^{(k)} (P^{(\alpha_1)})^{t_1} \dots (P^{(\alpha_m)})^{t_m}, \tag{A–33}$$

where

$$C_{m,\alpha,t} = C_{m,\alpha,t}(d_0, e_0, P), \quad D_{m,k,\alpha,t} = D_{k,\alpha,t}(d_0, P)$$

and

$$\hat{C}_{m,\alpha,t} := C_{m,\alpha,t}(s, d_0, e_0, 0) = \mathbb{O}(\epsilon; 1), \quad \hat{D}_{m,\alpha,t} := D_{m,k,\alpha,t}(d_0, 0) = \mathbb{O}(\epsilon; 1).$$

We recall that $d_0 = \mathbb{O}(\epsilon; b_0/s)$ and $e_0 = \mathbb{O}(\epsilon; b_0/s^2)$. From (A–33) it follows that

$$h_m^{(m+1)}(0) = \sum_{\sum_{l=1}^m \alpha_l t_l = m+1} \frac{1}{s} \hat{C}_{m,\alpha,t} (d_{\alpha_1})^{t_1} \dots (d_{\alpha_m})^{t_m} + \sum_{k+\sum_{l=1}^m \alpha_l t_l = m+1} \hat{D}_{m,k,\alpha,t} e_k (d_{\alpha_1})^{t_1} \dots (d_{\alpha_m})^{t_m}.$$

Using the assumptions on d_l and e_l (Lemma A.1.3), we have

$$h_m^{(m+1)}(0) = \mathbb{O}\left(\epsilon; b_0 (\ln q)^{\sum_l^m \alpha_i t_i} q^{\sum_l^m (2\alpha_l t_l - 1)}\right) + \mathbb{O}\left(\epsilon; b_0 (\ln q)^{k+\sum_l^m \alpha_i t_i} q^{2k-2\sum_l^m (2\alpha_l t_l - 1)}\right),$$

where in the first term on the right we have $\sum_{l=1}^m \alpha_l t_l = m+1$, and $k + \sum_{l=1}^m \alpha_l t_l = m+1$ in the second term. Hence, since in the first term $\sum_{l=1}^m t_l \ge 2$ holds because of $\sum_{l=1}^m \alpha_l t_l \ge 2$, $\alpha_l \ge 1$ and $t_l \ge 0$, it follows that

$$h_m^{(m+1)}(0) = \mathbb{O}(\epsilon; b_0 (\ln q)^{m+1} q^{2m}).$$

The estimate of $h_m^{(m+1)}(tB)$, $0 < t < 1$, is a consequence of (A–33) since

$$|P^{(l)}| \le c_l \left(|d_l| + |d_{l+1}| B + \dots + |d_{m-l}| B^{m-l}\right). \qquad \square$$

We recall that $\lambda := Bq^2 \ln q \le \lambda_0$.

Corollary A.1.5. $f_{m,m+1} = \mathbb{O}(\epsilon; b_0 (\ln q)^{m+1} q^{(2m)} (s-1))$.

Proof. Since $F_{l,k} \equiv (1/s)(s f_{l,k})'$, it follows from the boundary condition $f_{l,k}(1) = 0$ (see (A–11)) that

$$f_{m,m+1} = \frac{1}{s} \int_1^s \tau F_{m,m+1}(\tau)\, d\tau. \tag{A–34}$$

$\square$

Proof. Proof of Lemma A.1.3 Assume that the lemma holds for $1 \le l \le m$. Then

$$\text{(A–35)} \qquad \frac{1}{s}\int_1^s \tau\phi_m(\tau)\,d\tau = \mathbb{O}\big(\epsilon;\,(\ln q)^m q^{2m}(s-1)\big).$$

Using formula (A–21) for φ'_{m+1}, Corollary A.1.5, (A–35) and the formula (A–27) for $1+\phi_0'^2$ we conclude that

$$\frac{\phi'_{m+1}}{\sqrt{1+\phi_0'^2}} = \mathbb{O}(\epsilon;\, b_0(\ln q)^{m+1}q^{2m+1})$$

and

$$\phi'_{m+1} = \mathbb{O}\left(\epsilon;\, b_0(\ln q)^{m+1}q^{2m+1}\frac{s^{3/2}}{(s-1)^{1/2}}\right).$$

Thus, it follows from (A–19) and (A–20) that

$$\phi_{m+1} = \mathbb{O}(\epsilon;\, b_0(\ln q)^{m+1}q^{2m+2}).$$

Formula (A–17) implies

$$\frac{\phi''_{m+1}}{(1+\phi_0'^2)^{3/2}} = 3\phi_0'\phi_0''\left(-f_{m,m+1}+\frac{1}{s}\int_1^s \tau\phi_m(\tau)\,d\tau\right) - (f_{m,m+1})' - \frac{1}{s^2}\int_1^s \tau\phi_m(\tau)\,d\tau + \phi_m.$$

Since, by (A–34),

$$f'_{m,m+1} = F_{m,m+1} - \frac{1}{s}f_{m,m+1},$$

it follows from formulas (A–27)–(A–29) for ϕ_0' and ϕ_0'', Lemma A.1.4, Corollary A.1.5, (A–35) and (A–30) that

$$\frac{\phi''_{m+1}}{(1+\phi_0'^2)^{3/2}} = \mathbb{O}(\epsilon;\, b_0(\ln q)^{m+1}q^{2(m+1)-2}).$$

It remains to show Lemma A.1.3 in the case $l=1$. Since $f_{0,1}\equiv 0$, we find from (A–21) that

$$\phi_1' = (1+\phi_0'^2)^{3/2}\,\frac{1}{s}\int_1^s \tau\phi_0(\tau)\,d\tau.$$

This equation implies Lemma A.1.3 in the case $l=1$ by using the properties of ϕ_0, see the formulas (A–27)–(A–29).

The continuity of ϕ_l in ϵ follows from formula (A–26) for ϕ_0' iteratively from (A–21), (A–20) and (A–19). □

Proof of Proposition 2.1. Because of Lemma A.1.3 it remains to show inequality (9) of Proposition 2.1, where $v_{1,m}\equiv v_m$. From Lemma A.1.4, (A–30) and the differential equations (A–8) for φ_0 and (A–7) for φ_l, where $m := l-1$ in (A–7), it follows that

$$\begin{aligned}\operatorname{div} Tv_m - Bv_m &= \frac{1}{s}\left(\frac{s\phi_0'}{\sqrt{1+\phi_0'^2}}\right)' + \sum_{k=1}^{m} F_{m,k}B^k + \tilde F_{m,m+1}B^{m+1} \\ &\qquad\qquad\qquad - B\left(\frac{C}{B}+\phi_0+\cdots+\phi_m B^m\right) \\ &= (\tilde F_{m,m+1}-\phi_m)B^{m+1} \\ &= \big(O(b_0(\ln q)^{m+1}q^{2m}) + O(b_0(\ln q)^m q^{2m})\big)B^{m+1} \\ &= O(b_0(\ln q)^{m+1}q^{2m})B^{m+1}. \qquad \square\end{aligned}$$

A.2. ***Expansion far from the needle.*** Set, for $0 < B < 1$, $q \geq 3$ and $|\rho| < \rho_0$,

$$v_n = \sum_{k=0}^{n} \psi_k(s)\rho^{2k+1},$$

where the $\psi_k(s) \equiv \psi_k(B, q; s)$ are twice continuously differentiable functions in $q \leq s < \infty$. Suppose that $\psi_k'(q)$ satisfies the condition (13) and that ρ is a solution of (14) for a given b_1. We will set $b_1 = b_0(1+\epsilon)/q$, where $|\epsilon|$ is small and q is large. Thus, ρ will be small. Then v_n satisfies the boundary condition (11).

The sum v_n is said to be an *approximate solution* of (10)–(11) if v_n satisfies the boundary condition (11) and if

$$|\operatorname{div} Tv_n - Bv_n| \leq c\,|\rho|^{2n+3}$$

on $[q, \infty)$, where the constant $c = c(n, \rho_0)$ is independent on B, ρ and s. We will see that ψ_k satisfies a linear second order boundary value problem, provided v_n is an approximate solution. In particular, ψ_0 is a solution of the linearized equation to (10) about the zero solution.

Definition of ψ_k. Assume for $k \in \mathbb{N} \cup \{0\}$ that

$$\sup_{s\in(q,\infty)} |\psi_k'(s)| < \infty, \tag{A–36}$$

uniformly in $0 < B < 1$ and $q \geq 3$.

Then, for given $N \in \mathbb{N}$ and $|\rho| < \rho_0$ with ρ_0 sufficiently small, we have

$$\begin{aligned}\frac{v_n'}{\sqrt{1+v_n'^2}} &\equiv \left(\sum_{k=0}^{n}\psi_k'\rho^{2k+1}\right)\left(1+\left(\sum_{k=0}^{n}\psi_k'\rho^{2k+1}\right)^2\right)^{-1/2} \\ &= \rho\psi_0' + \sum_{k=1}^{N} f_{n,k}(\psi_0', \ldots, \psi_n')\rho^{2k+1} + \tilde f_{n,N+1}\rho^{2N+3}.\end{aligned}$$

Set $g_n(\rho) := v_n'/\sqrt{1+v_n'^2}$. Then

$$f_{n,k} = g_n^{(2k+1)}(0)/(2k+1)! \quad \text{and} \quad \tilde{f}_{n,k} = g_n^{(2k+1)}(t\rho)/(2k+1)! \quad \text{for } 0<t<1.$$

From assumption (A–36) on ψ_k' it follows that

$$|f_{n,k}| \le c_{n,k}(q) < \infty \quad \text{and} \quad |\tilde{f}_{n,N+1}| \le \tilde{c}_{n,N+1}(q) < \infty.$$

Above we have used that $v_n(1+(v_n')^2)^{-1/2}$ is an odd function in ρ.

Thus

$$\text{(A–37)} \qquad \operatorname{div} Tv_n = \frac{1}{s}(s\psi_0')'\rho + \frac{1}{s}\sum_{k=1}^{N}(sf_{n,k})'\rho^{2k+1} + \frac{1}{s}(s\tilde{f}_{n,N+1})'\rho^{2N+3}.$$

As in the previous section we need estimates on the derivatives $(f_{n,k})'$ and $(\tilde{f}_{n,N+1})'$. Assume for $k \in \mathbb{N}\cup\{0\}$ that

$$\text{(A–38)} \qquad \sup_{s\in(q,\infty)} |\psi_k''(s)| < \infty,$$

uniformly in $0<B<1$ and $q \ge 3$.

Applying identity (A–4) and the assumptions (A–36) and (A–38) on ψ_k' and ψ_k'', we get

$$\operatorname{div} Tv_n = \frac{1}{s}(s\psi_0')'\rho + \sum_{k=1}^{N} F_{n,k}\rho^{2k+1} + \tilde{F}_{n,N+1}\rho^{2N+3}$$

and $F_{n,k}$, $\tilde{F}_{n,N+1}$ are bounded on $[q,\infty)$. Set

$$h_n(\rho) := \frac{1}{s}\frac{v_n'}{\sqrt{1+v_n'^2}} + v_n''\big(1+v_n'^2\big)^{-3/2}.$$

Then

$$F_{n,k} = \frac{h_n^{(2k+1)}(0)}{(2k+1)!} \quad \text{and} \quad \tilde{F}_{n,N+1} = \frac{h_n^{(2N+3)}(t\rho)}{(2N+3)k!} \quad \text{for } 0<t<1.$$

Lemma A.2.6. *Assume that ψ_l', $l=0,\dots,n+1$ satisfies (A–36). Then*

$$\frac{v_{n+1}'}{\sqrt{1+v_{n+1}'^2}} = \frac{v_n'}{\sqrt{1+v_n'^2}} + \psi_{n+1}'\rho^{2(n+1)+1} + R,$$

where $|R| \le c(q)\rho^{2(n+1)+3}$ and $0<\rho\le\rho_0(q)$ for ρ_0 sufficiently small.

Proof.

$$
\begin{aligned}
&\frac{v'_{n+1}}{\sqrt{1+{v'_{n+1}}^2}}\\
&\quad = (v'_n + \psi'_{n+1}\rho^{2(n+1)+1})\big(1+{v'_n}^2 + 2v'_n\psi'_{n+1}\rho^{2(n+1)+2} + (\psi'_{n+1})^2\rho^{4(n+1)+2}\big)^{-1/2}\\
&\quad = \big(v'_n + \psi'_{n+1}\rho^{2(n+1)+1}\big)\big(1+{v'_n}^2\big)^{-1/2}\\
&\qquad\qquad \cdot\left(1 + 2\,\frac{v'_n}{\sqrt{1+{v'_n}^2}}\,\frac{\psi'_{n+1}}{\sqrt{1+{v'_n}^2}}\rho^{2(n+1)+1} + \frac{(\psi'_{n+1})^2}{1+{v'_n}^2}\rho^{4(n+1)+2}\right)^{-1/2}\\
&\quad = \frac{v'_n}{\sqrt{1+{v'_n}^2}} + (1+{v'_n}^2)^{-3/2}\Big((1+{v'_n}^2)\psi'_{n+1} - {v'_n}^2\psi'_{n+1}\Big)\rho^{2(n+1)+1}\\
&\qquad\qquad + O\big(\rho^{4(n+1)+2}\big)\\
&\quad = \frac{v'_n}{\sqrt{1+{v'_n}^2}} + \frac{\psi'_{n+1}}{(1+{v'_n}^2)^{3/2}}\rho^{2(n+1)+1} + O\big(\rho^{4(n+1)+2}\big)\\
&\quad = \frac{v'_n}{\sqrt{1+{v'_n}^2}} + \psi'_{n+1}\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).
\end{aligned}
$$

The last line follows since $1+{v'_n}^2 = 1 + O(\rho)$. □

Lemma A.2.7. *Suppose the assumptions* (A–36) *and* (A–38) *on* ψ'_k *and* ψ''_k *are satisfied. Then*

$$\operatorname{div} Tv_{n+1} = \operatorname{div} Tv_n + \frac{1}{s}(s\psi'_{n+1})'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)$$

as $\rho \to 0$, *uniformly in* $s \in [q, \infty)$.

Proof. From (A–4) and Lemma A.2.6 it follows that

$$
\begin{aligned}
\operatorname{div} Tv_{n+1} &\equiv \frac{1}{s}\frac{v'_{n+1}}{\sqrt{1+{v'_n}^2}} + \frac{v''_{n+1}}{(1+{v'_n}^2)^{3/2}}\\
&= \frac{1}{s}\frac{v'_n}{\sqrt{1+{v'_n}^2}} + \frac{1}{s}\psi'_{n+1}\rho^{2(n+1)+1} + \frac{v''_{n+1}}{(1+{v'_{n+1}}^2)^{3/2}} + O\big(\rho^{2(n+1)+3}\big).
\end{aligned}
$$

Since

$$\frac{v''_{n+1}}{(1+{v'_{n+1}}^2)^{3/2}} = \frac{v''_n}{(1+{v'_n}^2)^{3/2}} + \psi''_{n+1}\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)$$

(see the proof of Lemma A.2.6), we find that

$$\begin{aligned}\operatorname{div} Tv_{n+1} &= \frac{1}{s}\frac{v_n'}{\sqrt{1+v_n'^2}} + \frac{v_n''}{(1+v_n'^2)^{3/2}} + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big)\\ &= \operatorname{div} Tv_n + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).\end{aligned}$$

Lemma A.2.7 implies

$$\begin{aligned}&\operatorname{div} Tv_{n+1} - Bv_{n+1}\\ &\quad= \operatorname{div} Tv_n + \frac{1}{s}(s\psi_{n+1}')'\rho^{2(n+1)+1} - B\big(v_n + \psi_{n+1}\rho^{2(n+1)+1}\big) + O\big(\rho^{2(n+1)+3}\big)\\ &\quad= \operatorname{div} Tv_n - Bv_n + \Big(\frac{1}{s}(s\psi_{n+1}')' - B\psi_{n+1}\Big)\rho^{2(n+1)+1} + O\big(\rho^{2(n+1)+3}\big).\end{aligned}$$

Then from the expansion (A–37) of $\operatorname{div} Tv_n$, with $N := n+1$, and the condition

$$\operatorname{div} Tv_{n+1} - Bv_{n+1} = O\left(\rho^{2(n+1)+3}\right)$$

as $\rho \to 0$, it follows on $q < s < \infty$ that

$$\frac{1}{s}(s\psi_0')' - B\psi_0 = 0 \tag{A–39}$$

and for $n \geq 0$

$$\frac{1}{s}(s\psi_{n+1}')' - B\psi_{n+1} = -\frac{1}{s}(sf_{n,n+1})'.$$

Thus (see Section 3) we define ψ_k, $k \in \mathbb{N}$, iteratively by the boundary value problem

$$\frac{1}{s}(s\psi_k')' - B\psi_k = -\frac{1}{s}\big(sf_{k-1,k}(\psi_0', \dots, \psi_{k-1}')\big)' \quad \text{on } (q, \infty), \tag{A–40}$$

$$\psi_k'(q) = (-1)^k \binom{-\frac{1}{2}}{k}, \qquad \limsup_{s\to\infty} |\psi_k(s)| < \infty. \tag{A–41}$$

□

Boundary value problem for ψ_k. The solution of the homogeneous equation (A–39) that satisfies the boundary conditions (A–41) is given by

$$\psi_0(s) = \frac{1}{\sqrt{B}}\frac{K_0(\sqrt{B}s)}{K_0'(\sqrt{B}q)}.$$

We obtain $\psi_1, \psi_2, \dots$ iteratively from the boundary value problem (A–40)–(A–41). The estimates (A–36), (A–38) on ψ_k', ψ_k'' and formula (16) of $\psi_k(B, q; q)$, see Proposition 3.2, follow iteratively from a formula for the solution ψ_k by using the properties of $f_{k-1,k}(\psi_0', \dots, \psi_{k-1}')$. Once we have shown (A–36) and (A–38), we arrive at the estimate (15) of Proposition 3.2, since

$$\begin{aligned}\operatorname{div} Tv_{2,n} - Bv_{2,n} &= \frac{1}{s}(s\psi_0')'\rho + \frac{1}{s}\sum_{k=1}^{n}(sf_{n,k})'\rho^{2k+1} + \tilde{F}_{n,n+1}\rho^{2n+3} \\ &\qquad\qquad\qquad\qquad\qquad\qquad - B(\psi_0\rho + \cdots + \psi_n\rho^{2n+1}) \\ &= \tilde{F}_{n,n+1}\rho^{2n+3}.\end{aligned}$$

The proof of Theorem 1.1 requires Proposition 3.2 in the case $n = 0$ only. That is, we have to confirm the estimates (A–36), (A–38) for ψ_0', ψ_0'' and the property (16) of Proposition 3.2. Since

$$\psi_0(B, q; s) = \frac{1}{\sqrt{B}}\frac{K_0(\sqrt{B}s)}{K_0'(\delta)}, \quad \delta = \sqrt{B}q,$$

the expansion of $w_0(\delta)$ (see Proposition 3.2) follows from the expansions of $K_0(\delta)$ and $K_0'(\delta)$ as $\delta \to 0$. Since $\lim_{s\to\infty}\psi_0'(s) = 0$, where $B > 0$ is fixed, and since $K_0''(z) > 0$ for, $z > 0$, it follows that $|\psi_0'(s)| \le 1$ on $[q, \infty)$. From the differential equation (A–39) we conclude that

$$|\psi_0''(s)| \le \frac{1}{q} + \sqrt{B}\sup_{s\in(q,\infty)}\frac{K_0(\sqrt{B})}{|K_0'(\delta)|} \le \frac{1}{q} + \sqrt{B}\frac{K_0(\delta)}{|K_0'(\delta)|},$$

where we have used that $K_0'(z) < 0$, where $z > 0$. Thus

$$\sup_{s\in(q,\infty)}|\psi_0''(s)| \le \frac{1}{q} + \sqrt{B}\; O(\delta\ln\delta) \quad \text{as } \delta \to 0.$$

We will now prove iteratively the existence of ψ_k, the estimates (A–36) and (A–38), and the formula (16) for ψ_k if $k \ge 1$.

Let $K_0(z)$ and $I_0(z)$ be the modified Bessel functions of second kind of order zero. Concerning properties of the Bessel functions $K_0(z)$ and $I_0(z)$, see [Abramowitz and Stegun 1964] and the considerations in [Siegel 1980].

For $k \in \mathbb{N}$, set

$$f := f_{k-1,k}(\psi_0', \dots, \psi_{k-1}'), \quad F := -\frac{1}{s}(sf)', \quad \eta := (-1)^k\binom{-\frac{1}{2}}{k}.$$

Any solution of the differential equation (A–40) can be written as

$$\text{(A–42)}\quad \begin{aligned}\psi(s) = \left(c_1 - \int_q^s tI_0(\sqrt{B}t)F(t)\,dt\right)K_0(\sqrt{B}s) \\ + \left(c_2 + \int_q^s tK_0(\sqrt{B}t)F(t)\,dt\right)I_0(\sqrt{B}s),\end{aligned}$$

where $c_1,\ c_2 \in \mathbb{R}$. From the boundary conditions (A–41) it follows that

$$c_2 = -\int_q^\infty tK_0(\sqrt{B}t)F(t)\,dt, \tag{A–43}$$

$$c_1 = \frac{1}{\sqrt{B}K_0'(\delta)}\left(\eta + \sqrt{B}I_0'(\delta)\int_q^\infty tK_0(\sqrt{B}t)F(t)\,dt\right). \tag{A–44}$$

Since

$$f_{0,1}(\psi_0') = \tfrac{1}{2}\big(\psi_0'(t)\big)^3 = \frac{1}{2}\big(K_0'(\delta)\big)^{-3}\big(K_0'(\sqrt{B}t)\big)^3,$$

we expect that $f_{k-1,k}$ is a sum of such products too.

Definition. A function $f(t)$ is said to be of type (SP) if

(i) there exists an $M \in \mathbb{N}$ such that f can be written as

$$f(t) = \sum_{l=1}^{M} A_l(\delta)B_l(\sqrt{B}t),$$

where $A_l,\ B_l \in C^\infty(0,\infty)$,

(ii) there is a $k_l \in \mathbb{N}\cup\{0\}$ such that $A_l(\delta) = \delta^{k_l}P(\delta,\ln\delta),\ \ B_l(\delta) = \delta^{-k_l}P(\delta,\ln\delta)$ as $\delta \to 0$, where the expression $P(\delta,\ln\delta)$ is explained in Definition 3.1, and

(iii) $B_l(u) = O(\mathrm{e}^{-2u})$ as $u \to \infty$.

Suppose f is of type (SP). Applying (A–42)–(A–44), we find

$$\psi(s) = \frac{1}{\sqrt{B}}\big(F_1(\delta,\sqrt{B}s)K_0(\sqrt{B}s) + F_2(\delta,\sqrt{B}s)I_0(\sqrt{B}s)\big), \tag{A–45}$$

where

$$\begin{aligned} F_1 := \frac{\eta}{K_0'(\delta)} + \frac{I_0'(\delta)}{K_0'(\delta)}\Big(\sum_l A_l(\delta)\int_\delta^\infty uK_0'(u)B_l(u)\,du + \delta K_0(\delta)\sum_l A_l(\delta)B_l(\delta)\Big)\\ -\sum_l A_l(\delta)\int_\delta^{\sqrt{B}s} uI_0'(u)B_l(u)\,du + \sqrt{B}s\ I_0(\sqrt{B}s)\sum_l A_l(\delta)B_l(\sqrt{B}s)\\ -\delta I_0(\delta)\sum_l A_l(\delta)B_l(\delta) \end{aligned}$$

and

$$F_2 := -\sum_l A_l(\delta)\int_{\sqrt{B}s}^\infty uK_0'(u)B_l(u)\,du - \sqrt{B}s\ K_0(\sqrt{B}s)\sum_l A_l(\delta)B_l(\sqrt{B}s).$$

The derivative ψ' is given by

$$\psi'(s) = F_1(\delta,\sqrt{B}s)K_0'(\sqrt{B}s) + F_2(\delta,\sqrt{B}s)I_0'(\sqrt{B}s). \tag{A–46}$$

We conclude from (A–46) that ψ'_k is of type (SP), provided the function $f := f_{k-1,k}(\psi'_0, \dots, \psi'_{k-1})$ is of type (SP). Property (i) of the definition follows immediately from formula (A–46). We omit here the considerations that (ii) and (iii) are also satisfied. Then $f_{k,k+1}(\psi'_0, \dots \psi'_k)$ is of type (SP) since

$$\begin{aligned} f_{k,k+1}(\psi'_0, \dots, \psi'_k) &= \frac{1}{(2k+3)!} \frac{d^{2k+3} g_k}{d\rho^{2k+3}}(0) \\ &= \sum_{\sum_{l=0}^{k}(2\alpha_l+1)t_l = 2k+3} r_{k,\alpha,t} (\psi'_{\alpha_0})^{t_0} \dots (\psi'_{\alpha_k})^{t_k}, \end{aligned}$$

where $\alpha = (\alpha_0, \dots, \alpha_k)$, $t = (t_0, \dots, t_k)$, α_l, $t_l \in \mathbb{N} \cup \{0\}$ and $r_{k,\alpha,t} \in \mathbb{R}$. We recall that $g_k(\rho) = v'_k / \sqrt{1 + (v'_k)^2}$ and $v'_k = \sum_{l=0}^{k} \psi'_l \rho^{2l+1}$.

Finally, we find iteratively from (A–45), (A–46) and the differential equation (A–40) that the estimates (A–36), (A–38) for ψ'_k, ψ''_k hold and that

$$\sqrt{B}\psi_k(B, q; q) = P(\delta, \ln \delta)$$

(see Proposition 3.2).

A.3. ***Composing of the inner and outer solutions.*** Set $q := B^{-\tau}$ for some $\tau \in (0, \frac{1}{2})$. Then we will show that there is a solution $\epsilon \in (-\frac{1}{4}, \frac{1}{4})$ of equation (17), that is of $G(\epsilon) = 0$, where

$$G(\epsilon) := \frac{2\epsilon b_0}{B(q^2-1)} + \sum_{k=0}^{m} \phi_k(q, b, \epsilon; q) B^k - \frac{1}{\sqrt{B}} \sum_{k=0}^{n} w_k(\delta) \rho^{2k+1}.$$

Here is $\delta = \sqrt{B} q$, $b_1 = b_0(1+\epsilon)/q$ and $\rho = \rho(b_0, q, \epsilon)$ is given by (14). In particular,

$$\rho = b_1 + O(b_1^{2n+3}) = \frac{b_0(1+\epsilon)}{q} + O\left(\frac{b_0}{q^{2n+3}}\right)$$

as $q \to \infty$. The existence of a zero of the continuous function $G(\epsilon)$ follows from the intermediate value theorem. Propositions 2.1 and 3.2 imply

$$\begin{aligned} G(\epsilon) = \frac{2\epsilon b_0}{B(q^2-1)} + \phi_0(q, b_0, \epsilon; q) + O(b_0 q^2 (\ln q)^2 B) \\ - \frac{1}{\sqrt{B}} \left(w_0(\delta)\rho + O\left(\frac{b_0}{q^3} \delta (\ln \delta)^l \right) \right) \end{aligned}$$

for some $l \in \mathbb{N} \cup \{0\}$. Since, by Proposition 2.1 and the formula for $\phi_{0,0}$ (page 307), we have

$$\begin{aligned} \phi_0(q, b_0, \epsilon; q) &= \phi_{0,0}(q, b_0; q) + O(b_0 \epsilon \ln q) \\ &= \tfrac{1}{2} b_0 + O\left(b_0 \frac{\ln q}{q^2}\right) + O(b_0 \epsilon \ln q) \end{aligned}$$

and

$$w_0(\delta) = \frac{K_0(\delta)}{K_0'(\delta)} = \delta\big(\ln\delta + \gamma - \ln 2 + O(\delta^2(\ln\delta)^2)\big)$$

as $\delta \to 0$, it follows that

$$G(\epsilon) = \frac{2\epsilon b_0}{\delta^2} + \frac{b_0}{2} - b_0(\ln\delta + \gamma - \ln 2) + O\Big(b_0 \frac{\epsilon}{\delta}\frac{1}{q^2}\Big) + O\Big(b_0 \frac{\ln q}{q^2}\Big) + O(b_0\epsilon\ln q) \\ + O(b_0 q^2(\ln q)^2 B) + O(b_0\delta^2(\ln\delta)^2) + O(b_0\epsilon\ln\delta) + O\Big(b_0(\ln\delta)^l \frac{1}{q^2}\Big).$$

For R real, $|R| \le 1$, set

$$\epsilon(R) := \tfrac{1}{2}\delta^2\ln\delta + \tfrac{1}{2}(\gamma - \ln 2 - \tfrac{1}{2})\delta^2 + R\delta^2.$$

then $|\epsilon| < \frac{1}{4}$ if $\delta < \delta_0$, for δ_0 sufficiently small. We have $G(\epsilon(1)) > 0$ and $G(\epsilon(-1)) < 0$ if $0 < \delta < \delta_0$, for δ_0 sufficiently small.

Finally, we obtain from $G(\epsilon(R)) = 0$ an estimate of R. Since

$$R = O\left(\frac{1}{q^2}\ln\delta\right) + O\left(\frac{\ln q}{q^2}\right) + O(\delta^2\ln\delta\ln q) + O(\delta^2(\ln\delta)^2) + O\left(\frac{1}{q^2}(\ln\delta)^l\right),$$

we find

$$R \equiv R(b_0, B, B^{-\tau}) = O\left((\ln B)^{k+1}B^{2\tau}\right) + O\left((\ln B)^2 B^{1-2\tau}\right)$$

uniformly in $b_0 \in [-1, 1]$. Thus, Proposition 4.1 is shown.

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CAPILLARY SURFACES AT A REENTRANT CORNER

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A capillary surface expressible as a graph over a domain containing a protruding corner can exhibit strikingly varying behavior, with discontinuous transitions, depending on local boundary conditions. Korevaar in 1980 showed that very different kinds of behavior must be expected when the corner opening exceeds π, and later Lancaster and Siegel extended that result to indicate a remarkable range in the kinds of behavior that can occur. This work characterizes all possible modes of behavior for this case, subject to a conjecture of Concus and Finn for the protruding angle case.

1. Introduction

A *capillary surface* S is the interface separating two immiscible fluids adjacent to each other. In this work we discuss interfaces that are ideally thin and can be represented as graphs $f(x, y)$ over a base domain Ω. We only consider equilibrium configurations. As shown initially by Laplace, for incompressible fluids in a vertical cylinder (a *capillary tube*), the shape of the surface is governed by the equation

$$\nabla \cdot Tf = \kappa f + \lambda \quad \text{in } \Omega, \tag{1}$$

$$Tf \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega, \tag{2}$$

where Tf is defined as $\nabla f/(\sqrt{1+|\nabla f|^2})$ and $\kappa = \rho g/\sigma$ is the *capillarity constant*, with ρ the density change across the surface, g the gravitational attraction, and σ the surface tension of the interface. The constant λ is a Lagrange parameter arising from a possible volume constraint, ν is the exterior normal vector on $\partial\Omega$, and $\gamma = \gamma(s)$ is a function of position on $\partial\Omega$, satisfying $0 \le \gamma(s) \le \pi$. The surface $z = f(x, y)$ describes the shape of the static liquid-gas interface in a vertical cylindrical tube of bounded cross-section Ω. In this paper, we assume $\kappa > 0$, corresponding to the case of a vertically downward gravity field, with the denser fluid below the surface.

MSC2000: primary 76B45; secondary 35J65, 53A10.
Keywords: capillary surfaces, reentrant corner, corner behavior, Concus–Finn conjecture.

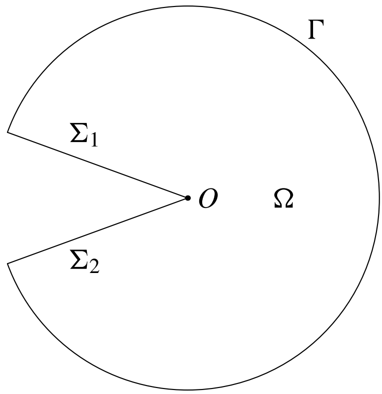


Figure 1. Reentrant wedge domain.

I will address the particular case in which Ω is a wedge domain with a reentrant corner. Specifically, I will assume that the corner of the wedge is the origin O of coordinates, and $\partial\Omega$ consists of three smooth portions:

$$\Gamma = \{(\cos\varphi, \sin\varphi) : -\alpha \le \varphi \le \alpha\},$$
$$\Sigma_1 = \{(\rho\cos\alpha, \rho\sin\alpha) : 0 < \rho < 1\},$$
$$\Sigma_2 = \{(\rho\cos\alpha, -\rho\sin\alpha) : 0 < \rho < 1\}.$$

See Figure 1. Σ_1 and Σ_2 can be relaxed to only asymptotic straight lines when approaching the origin. Let γ equal $\gamma_1(s)$, $\gamma_2(s)$ along the sides Σ_1, and Σ_2, respectively, where $s = 0$ corresponds to the point O.

Capillary surfaces can exhibit strikingly idiosyncratic behavior at corner points of the domain Ω of definition, as a consequence of the characteristic nonlinearities in Equation (1). This was initially observed by Concus and Finn [1974b; 1974a; 1974]. Later this behavior was further delineated in [Simon 1980; Tam 1986a; 1986b; Lieberman 1988; Miersemann 1985; Concus and Finn 1994; 1996; Finn 1986; Lancaster and Siegel 1996; Shi and Finn 2004].

We take [Lancaster and Siegel 1996] as the starting point for this work. There the authors considered the limiting values Rf of f along radial approaches within Ω to the vertex O. They showed that under very general hypotheses this limit always exists, and they delineated various possibilities for its behavior in terms of direction of approach to O. For protruding corners (opening angle $2\alpha < \pi$), their results can be considered close to definitive, subject to a conjecture of Concus and Finn. The present work addresses the complementary case of reentrant corners (opening angle $2\alpha > \pi$), for which the behavior can be very different.

We will proceed by indirect reasoning using methods of geometric measure theory. In Section 3 we will introduce the concept of a generalized solution of the

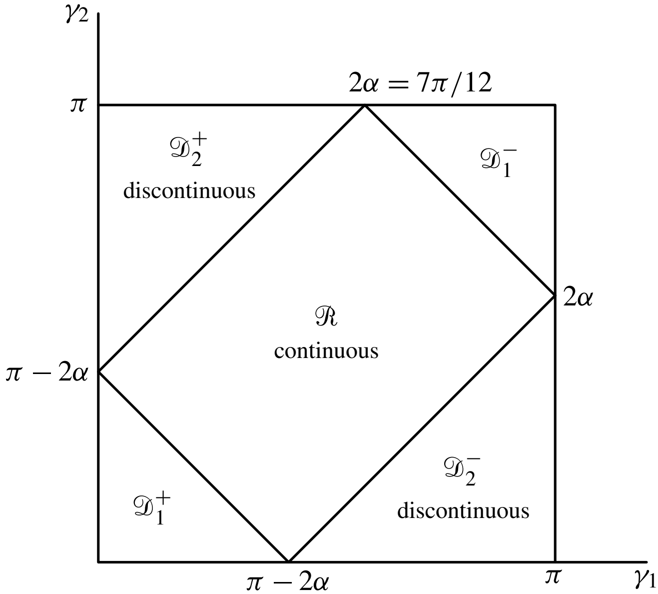


Figure 2. The Concus–Finn rectangle: regions in parameter space corresponding to different behaviors at a convex corner of angle 2α. In regions $\mathcal{D}_1^{\pm}$ there is no bounded graph.

minimal surface equations in the sense of Miranda [1964]. In Section 4 we adapt Giusti's minimal cone theory [1977]. These two sections, together with some other earlier results, will play an essential role later on in the proof of the main result.

At the end of this work, we distinguish various cases, according to the contact angles γ_1, γ_2 on the two sides of the wedge domain formed near O. These cases can be characterized geometrically, using a diagram analogous to one introduced by Concus and Finn [1996] for the protruding angle case. We show the Concus–Finn diagram in Figure 2. The central rectangle $\mathcal{R}$ is uniquely determined by the angle 2α. Its vertices along the coordinate axes have coordinate $\pi - 2\alpha$ (in the reentrant case the corresponding value will be $2\alpha - \pi$; see Figure 3). The four triangular regions are denoted $\mathcal{D}_1^+, \mathcal{D}_1^-, \mathcal{D}_2^+, \mathcal{D}_2^-$.

A central new result of this paper is that, assuming the truth of the Concus–Finn conjecture for the protruding angle case, *any solution arising from data outside the rectangle $\mathcal{R}$ is necessarily discontinuous at O for the reentrant angle case as well, in the sense that different limit values must occur for different radial directions of approach.* We will completely characterize the ways in which the discontinuous behavior can be manifested. This can change according to the particular domain of the diagram where (γ_1, γ_2) lies.

For a reentrant corner, data within $\mathcal{R}$ can lead to a discontinuous solution; this contrasts with the protruding angle case, for which all solutions arising from such data are known to be not only continuous but Hölder differentiable up to O.

2. The Concus–Finn diagram

We start by reviewing results and notation from [Lancaster and Siegel 1996], and categorizing them using the Concus–Finn diagram.

Protruding wedge domains. Solutions can exist at corners for any data (γ_1, γ_2); but for any transition from $\mathcal{R}$ to $\mathcal{D}_1^{\pm}$ across a common boundary point (γ_1^*, γ_2^*), there is a discontinuous change in behavior, from uniform boundedness at O of the solution for all data up to and including (γ_1^*, γ_2^*), to unboundedness for all data in $\mathcal{D}_1^{\pm}$, with asymptotic behavior depending only on the local geometry.

The radial limits of f at the vertex of the corner will be denoted by $Rf(\theta) = \lim_{r\to 0+} f(r\cos\theta, r\sin\theta)$, $-\alpha < \theta < \alpha$ and $Rf(\pm\alpha) = \lim_{x,y\to 0} f(x, y)$ where $(x, y) \in \Sigma_{1,2}$, which are the limits of the boundary values of f on the two sides of the corner.

Theorem 2.1 [Lancaster and Siegel 1996]. *Let f be a bounded solution to (1) satisfying the boundary condition (2) on $\partial^{\pm}\Omega \setminus O$, discontinuous at O, with $0 < \gamma_0 \le \gamma^{\pm}(s) \le \gamma_1 < \pi$. If $\alpha \ge \pi/2$, then $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. If $\alpha < \pi/2$ and there exist constants $\underline{\gamma}^{\pm}$ and $\bar{\gamma}^{\pm}$ satisfying*

$$0 < \underline{\gamma}^{\pm} \le \pi/2, \quad \pi/2 \le \bar{\gamma}^{\pm} < \pi, \quad \underline{\gamma}^{+} + \underline{\gamma}^{-} > \pi - 2\alpha, \quad \bar{\gamma}^{\pm} + \bar{\gamma}^{-} < 2\alpha + \pi,$$

so that $\underline{\gamma}^{\pm} \le \gamma^{\pm}(s) \le \bar{\gamma}^{\pm}$ for all $s \in (0, s_0)$ for some s_0, then again $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. Furthermore, in either case $Rf(\theta)$ is a continuous function on $[-\alpha, \alpha]$ behaving in one of the following ways:

(i) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these cases* (I) and (D), respectively.

(ii) There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R \le \alpha, \alpha_R = \alpha_L + \pi$, and Rf is constant on $[-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and either increasing on $[\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2]$ or decreasing on $[\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2]$. Label these cases (ID) and (DI), respectively.

Corollary 2.2 [Lancaster and Siegel 1996]. *Let f be a bounded solution to (1) satisfying*

$$\begin{aligned} Tf\cdot\nu &= \cos\gamma_1 \quad on \quad \Sigma_1, \\ Tf\cdot\nu &= \cos\gamma_2 \quad on \quad \Sigma_2, \end{aligned} \tag{3}$$

with constant $\gamma_1, \gamma_2 \in (0, \pi)$. *For* $\alpha < \pi/2$, *assume in addition*

$$\pi - 2\alpha < \gamma_1 + \gamma_2 < \pi + 2\alpha.$$

Then

Case (I) *cannot hold if* $\gamma_1 - \gamma_2 \leq \pi - 2\alpha$.

Case (D) *cannot hold if* $\gamma_2 - \gamma_1 \leq \pi - 2\alpha$.

For $\alpha > \pi/2$, *case* (ID) *cannot hold if* $\gamma_1 + \gamma_2 \leq 3\pi - 2\alpha$.

For $\alpha > \pi/2$, *case* (DI) *cannot hold if* $\gamma_1 + \gamma_2 \geq 2\alpha - \pi$.

Corollary 2.3 [Lancaster and Siegel 1996]. *Let* f *be a bounded solution to* (1) *satisfying* (3), *with* $\gamma_1, \gamma_2 \in (0, \pi)$, $\alpha < \pi/2$, *and* $(\gamma_1, \gamma_2) \in \overline{\mathcal{R}}$, *then* f *must be continuous at* O.

Concus and Finn have conjectured that for data from $\mathcal{D}_2^{\pm}$, there could only be discontinuous solutions.

Conjecture 2.4 (Concus–Finn). *Suppose* $\alpha < \pi/2$, *and* $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{\pm}(2\alpha)$, *then the solution* f *has a discontinuity at the vertex* O.

In this paper, everything is based on the assumption that Conjecture 2.4 is true. First, as indicated in Figure 2, we can summarize the situation for a protruding wedge domain in terms of the regions $\mathcal{R}$, $\mathcal{D}_1^{\pm}$, $\mathcal{D}_2^{\pm}$:

- The pair (γ_1, γ_2) lies in $\overline{\mathcal{R}}(2\alpha)$ if and only if both f and its outward unit normal are continuous up to O.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{+}(2\alpha)$, f must be a (D) case.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_2^{-}(2\alpha)$, f must be an (I) case.
- For $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{\pm}(2\alpha)$, f is no longer a bounded graph near O.

Hence with this conjecture, we have completed categorizing the continuity of the solutions at the corner of the domain in terms of the Concus–Finn diagram.

Reentrant wedge domains. We now develop parallel results for reentrant wedge domains. When $2\alpha = \pi$, the boundary is smooth, which is not a case considered in this paper.

Theorem 2.5. *Assume* $\alpha > \pi/2$. *Let* f *be a solution to* (1) *satisfying* (3), *with* $\gamma_1, \gamma_2 \in (0, \pi)$.

(i) *For* $(\gamma_1, \gamma_2) \in \overline{\mathcal{R}}(2\alpha)$, f *can be continuous at* O *or in one of the cases* (I) *or* (D).

(ii) *For* $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{+}(2\alpha)$, f *can be in one of the cases* (DI), (I), *or* (D).

(iii) *For* $(\gamma_1, \gamma_2) \in \mathcal{D}_1^{-}(2\alpha)$, f *can be in one of the cases* (ID), (I), *or* (D).

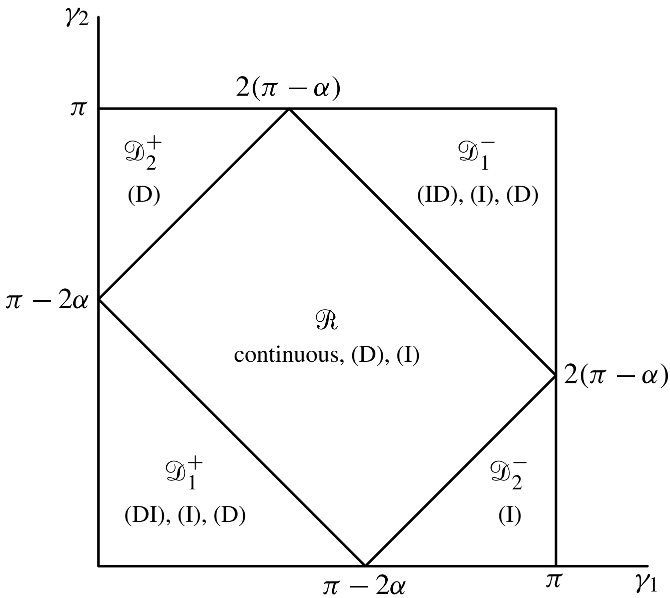


Figure 3. Analogue of the Concus–Finn rectangle for reentrant wedges.

(iv) *For* $(\gamma_1, \gamma_2) \in \mathcal{D}_2^+(2\alpha)$, f *must be in case* (D).

(v) *For* $(\gamma_1, \gamma_2) \in \mathcal{D}_2^-(2\alpha)$, f *must be in case* (I).

In Section 6 we construct examples showing that each of these cases actually occurs.

By Corollaries 2.2 and 2.3 we deduce:

(i′) For $\gamma_1, \gamma_2 \in \overline{\mathcal{R}}(2\alpha)$, f can only be continuous up to O, or be in case (I) or (D).

(ii′) For $\gamma_1, \gamma_2 \in \mathcal{D}_1^+(2\alpha)$, f cannot be in case (ID).

(iii′) For $\gamma_1, \gamma_2 \in \mathcal{D}_1^-(2\alpha)$, f cannot be in case (DI).

(iv′) For $\gamma_1, \gamma_2 \in \mathcal{D}_2^+(2\alpha)$, f can be continuous up to O, or be in case (D).

(v′) For $\gamma_1, \gamma_2 \in \mathcal{D}_2^-(2\alpha)$, f can be continuous up to O, or be in case (I).

In other words, all that is left to prove is that continuity is excluded from the $\mathcal{D}_1^\pm, \mathcal{D}_2^\pm$ regions, and that each of the cases can occur.

3. Generalized solutions

To discuss discontinuous capillary surfaces further, we introduce the definition of generalized (or weak) solution of the minimal surface equations in the sense of Miranda, and prove some results for capillary surfaces over a reentrant wedge domain.

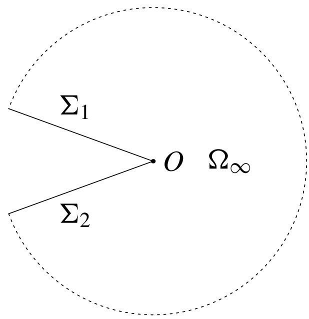


Figure 4. The infinite domain.

Set $\Omega_\infty = \{(\rho\cos\varphi, \rho\sin\varphi) : -\alpha < \varphi < \alpha,\ \rho > 0\}$. We redefine the symbols for the boundary pieces (see Figure 4):

$$\Sigma_1 = \{\rho > 0,\ \varphi = \alpha\},$$
$$\Sigma_2 = \{\rho > 0,\ \varphi = -\alpha\}$$

Definition 3.1. A function $u : \Omega_\infty \to [-\infty, \infty]$ is called a *generalized solution* of the equations

$$\begin{aligned} \nabla \cdot Tu &= 0 && \text{in } \Omega_\infty, \\ Tu \cdot \nu &= \cos\gamma_1 && \text{on } \Sigma_1, \\ Tu \cdot \nu &= \cos\gamma_2 && \text{on } \Sigma_2 \end{aligned} \tag{4}$$

if the subgraph of u defined by

$$U = \{(x, y, z) : (x, y) \in \Omega_\infty, z < u(x, y)\}$$

minimizes the functional

$$\int_{\Omega_\infty \times \mathbb{R}} |D\varphi_U| - \cos\gamma_1 \int_{\Sigma_1 \times \mathbb{R}} \varphi_U \, dH_2 - \cos\gamma_2 \int_{\Sigma_2 \times \mathbb{R}} \varphi_U \, dH_2,$$

where φ_U is the characteristic function of U and H_2 is 2-dimensional Hausdorff measure in $\mathbb{R}^3$. That means that for every Caccioppoli set (set of locally finite perimeter) $E \subset \Omega_\infty \times \mathbb{R}$ coinciding with U outside some compact set $K \subset \mathbb{R}^3$ we have

$$W(K, U) \le W(K, E), \tag{5}$$

where

$$W(K, U) = \int_{(\Omega_\infty \times \mathbb{R}) \cap K} |D\varphi_U| - \cos\gamma_1 \int_{(\Sigma_1 \times \mathbb{R}) \cap K} \varphi_U \, dH_2 - \cos\gamma_2 \int_{(\Sigma_2 \times \mathbb{R}) \cap K} \varphi_U \, dH_2$$

and likewise for $W(K, E)$.

A sequence of functions v_n is said to *converge locally* to a function v in a domain Ω if the characteristic functions of the subgraphs of v_n converge almost everywhere to the characteristic function of the subgraph of v in $\Omega \times \mathbb{R}$.

The function u is allowed to take the values $\pm\infty$. It follows from [Miranda 1964] that every classical solution of equations (4) is a generalized solution; conversely, every locally bounded generalized solution is a classical solution of equations (4).

We introduce the sets

$$\begin{aligned} P(u) &= \{(x, y) \in \Omega_\infty : u(x, y) = +\infty\}, \\ N(u) &= \{(x, y) \in \Omega_\infty : u(x, y) = -\infty\}, \\ G(u) &= \Omega_\infty - (\bar{P}(u) \cup \bar{N}(u)). \end{aligned}$$

It follows that P minimizes the functional

$$\begin{aligned} \Phi(A) &= \iint_{\Omega_\infty} |D\chi_A| - \cos\gamma_1 \int_{\Sigma_1} \chi_A \, dH_1 - \cos\gamma_2 \int_{\Sigma_2} \chi_A \, dH_1 \\ &= H_1(\Omega_\infty \cap \partial A) - \cos\gamma_1 H_1(\Sigma_1 \cap \partial A) - \cos\gamma_2 H_1(\Sigma_2 \cap \partial A) \end{aligned}$$

in Ω_∞, where H_1 is 1-dimensional Hausdorff measure in $\mathbb{R}_2$.

Similarly, N minimizes the functional

$$\begin{aligned} \Psi(A) &= \iint_{\Omega_\infty} |D\chi_A| + \cos\gamma_1 \int_{\Sigma_1} \chi_A \, dH_1 + \cos\gamma_2 \int_{\Sigma_2} \chi_A \, dH_1 \\ &= H_1(\Omega_\infty \cap \partial A) + \cos\gamma_1 H_1(\Sigma_1 \cap \partial A) + \cos\gamma_2 H_1(\Sigma_2 \cap \partial A) \end{aligned}$$

After modification by a set of measure zero, the two sets

$$\partial P \cap \Omega_\infty \quad \text{and} \quad \partial N \cap \Omega_\infty$$

consist of straight lines that do not intersect inside of Ω_∞. Moreover:

Lemma 3.2. (i) *Let L be the portion of ∂P which lies inside Ω_∞. Suppose L is not empty, then L is a straight line which either meets Σ_i in an angle γ_i, or passes through the point O meeting the sides in angles β_i with $\beta_i \geq \gamma_i$, $i = 1, 2$. Here the angles β_i are measured inside P.*

(ii) *Let L' be the portion of ∂N which lies inside Ω_∞. Suppose L' is not empty, then L' is a straight line which either meets Σ_i in an angle $\pi - \gamma_i$ provided that $\Sigma_i \cap \partial N \neq \varnothing$, or passes point O in an angle ϑ_i with $\vartheta_i \geq \pi - \gamma_i$, $i = 1, 2$. Here the angles ϑ_i are measured inside N.*

From now on consider specifically the reentrant corner domain. For this case we can also say:

Theorem 3.3. *Suppose v is a generalized solution of (4) on Ω_∞ with $\alpha > \pi/2$. The P, N, G regions of v are as defined at the beginning of this section. Assume P and N are nonempty.*

(i) *Each component of $\partial P \cap \Omega_\infty$ and $\partial N \cap \Omega_\infty$ is infinite.*

(ii) *Suppose a component of P or N has two boundary lines inside Ω_∞. The lines either meet on $\partial\Omega_\infty$, or their extensions meet outside Ω_∞, with an angle η, measured from the side containing P, or N respectively. Then $\eta \geq \pi$.*

(iii) *There are at most two components of $\partial P \cap \Omega_\infty$, or $\partial N \cap \Omega_\infty$.*

Proof. (i) No line segment inside Ω_∞ can meet both sides of a reentrant corner domain.

(ii) We work by contradiction. Suppose there is a component of P which has two boundary lines inside Ω_∞. Their extension lines meet with an angle $\eta < \pi$. See Figure 5. Comparing P with $P - OACDB$, we get $AC - OA\cos\gamma_1 + BD - OB\cos\gamma_2 \leq CD$. Hence, successively,

$$AC + AE + BD + BE \leq CD + OA\cos\gamma_1 + OB\cos\gamma_2 + AE + BE,$$
$$CE + DE - CD \leq AE + BE + OA\cos\gamma_1 + OB\cos\gamma_2.$$

Move CD to infinity, parallel to itself. Then $CE + DE - CD \to \infty$, while $AE + BE + OA\cos\gamma_1 + OB\cos\gamma_2$ remains fixed. Contradiction.

(iii) If $\partial P \cap \Omega_\infty$ has a whole line as a component, there must be no other component. Because of (ii), it is easy to see $\partial P \cap \Omega_\infty$ cannot contain three or more half-lines. $\square$

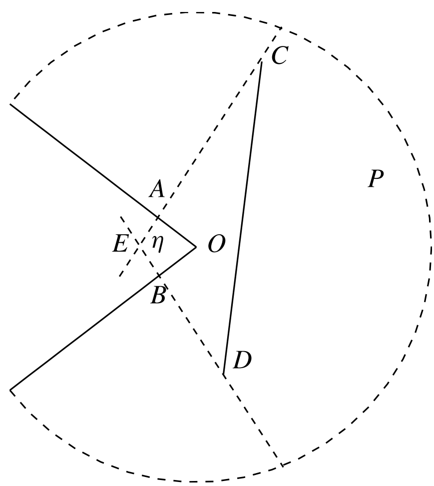


Figure 5. Proof of Theorem 3.3. Impossible case for P.

The following theorem describes the structure of the infinite sets P and N, the proof of the theorem is similar to [Tam 1986a, Theorem 2.4].

Corollary 3.4. *Under the assumptions of Theorem 3.3, the only possibilities for P other than $\varnothing$, Ω_∞ are the ones shown in Figure 6, namely*:

(i) *P consists of a single component, which is bounded between Σ_1 and a line L_1 with an opening angle β_1. L_1 meets Σ_1 at point A. Either $A \in \Sigma_1$ and $\beta_1 = \gamma_1$, or $A = O$ and $\beta_1 \geq \gamma_1$.*

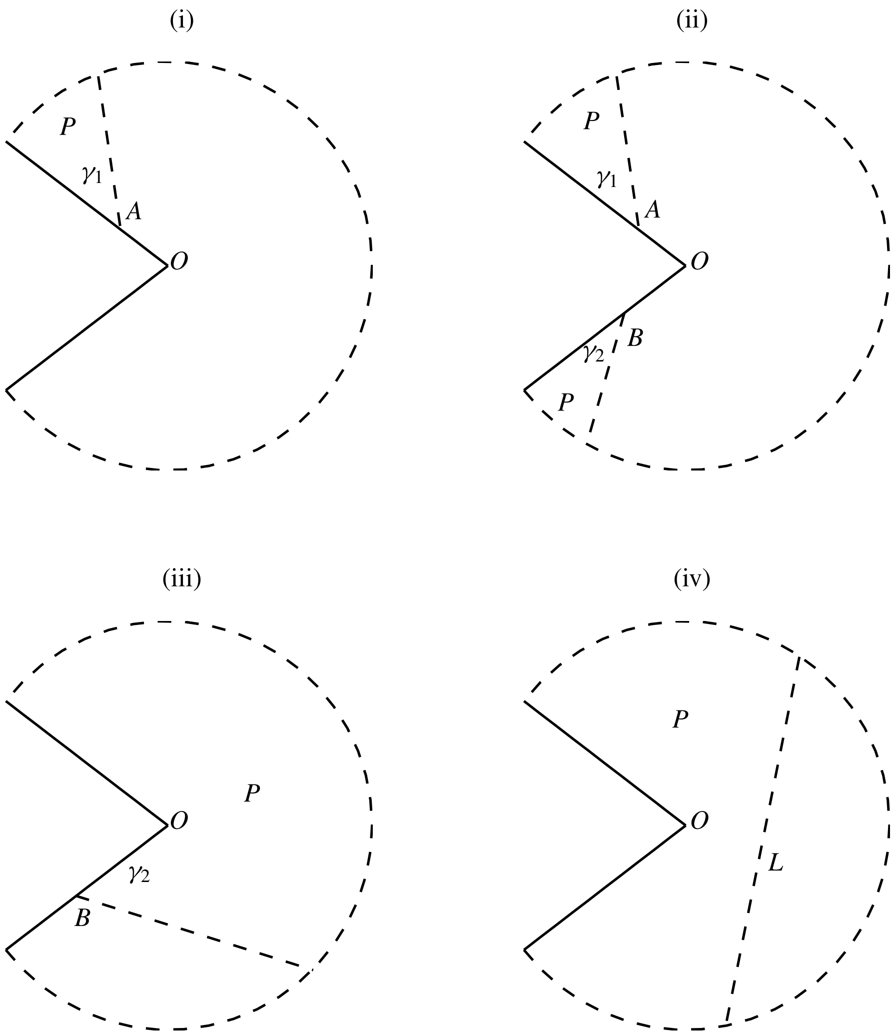


Figure 6. All possibilities for the region P.

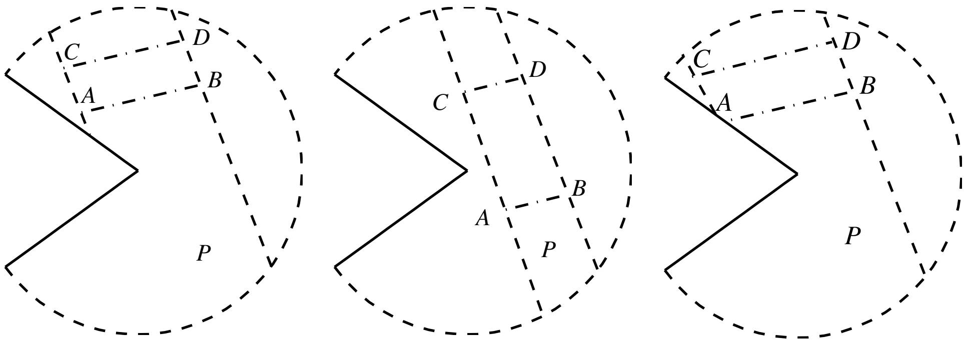


Figure 7. Impossible cases for the region P.

(ii) *P consists of two components. On is bounded between Σ_1 and a line L_1 with an opening angle β_1 and the other is bounded between Σ_2 and a line L_2 with an opening angle β_2. For $i = 1, 2$, suppose L_i meets Σ_i at point A_i. Either $A_i \in \Sigma_i$ and $\beta_i = \gamma_i$, or $A_i = O$ and $\beta_i \geq \gamma_i$.*

(iii) *P consists of a single component, which is bounded by Σ_1, $OB \subset \Sigma_2$, and line L_2 which meets Σ_2 at point B with an opening angle γ_2.*

(iv) *There is a whole line L which lies inside Ω_∞ and P is either the region bounded by $\partial\Omega_\infty$ and L, or the half plane bounded by L.*

Proof. Again we work by contradiction.

From Theorem 3.3 we obtain that P has at most two components. All the possibilities for the structure of P are indicated in Figures 6 and 7.

Next we prove that those cases in Figure 7 are impossible. Compare P with $P - ABDC$ in that figure. By definition,

$$\Phi(P) \leq \Phi(P - ABDC) \quad \text{and} \quad 2|AC| \leq 2|AB|.$$

Moving $\overline{CD}$ to infinity, parallel to itself, we get $\infty \leq |AB|$. Contradiction. □

4. Radial linearity

In this section, we extend Giusti's work on minimal graphs [1977] to general H-graphs. We use the "blow-up" procedure to expand the capillary surface about 0 and show that the limit set C exists and that C is a minimal cone. Furthermore, this limit cone is unique.

The results in this section apply for any n-dimensional space. However, for our specific problem (1), we only need to consider $n = 3$.

Definition 4.1. Suppose Ω_∞ is the wedge domain in $\mathbb{R}^2$ defined as before. Define

$$Q = \Omega_\infty \times \mathbb{R}, \quad \delta Q = \partial\Omega_\infty \times \mathbb{R}.$$

Now suppose K is any open set in $\mathbb{R}^3$. Define

$$\delta K = \delta Q \cap K, \quad \delta^* K = Q \cap \partial K.$$

Denote by $\mathrm{BV}(K)$ the space of functions of bounded variation on K.

Definition 4.2. If $f \in \mathrm{BV}(K)$, set

$$\begin{aligned} F_K(f) &= \int_{Q\cap K} |Df| - \int_{\delta K} \cos\gamma \cdot f\, dH_2 \\ \zeta(f, K) &= \inf\{F_K(g) : g \in \mathrm{BV}(K), \operatorname{supp}(g - f) \subset K\}, \\ \psi(f, K) &= F_K(f) - \zeta(f, K). \end{aligned}$$

If f is the characteristic function of some set E with finite perimeter, we shall write $F_K(E)$, $\nu(E, K)$ and $\psi(E, K)$ instead of $F_K(\varphi_E)$, $\nu(\varphi_E, K)$ and $\psi(\varphi_E, K)$.

Definition 4.3. A set E is a *minimal* in K if $\psi(E, K) = 0$.

Definition 4.4. We call C a *cone* in $\mathbb{R}^3$ if

$$C = \{t(x, y, z) : t \geq 0, (x, y, z) \in A\},$$

for some set $A \in \mathbb{R}^3$.

This is the main theorem of this section.

Theorem 4.5. *Suppose that E minimizes the functional*

$$F_{B_1}(W) + \int_{Q\cap B_1} H\varphi_W,$$

for some H uniformly bounded on B_1, such that $0 \in \partial E$. For $t > 0$, let

$$E_t = \{x \in \mathbb{R}^3 : tx \in E\}.$$

Then as $t \to 0$, E_t converges locally in $\mathbb{R}^3$ to a set C. Moreover, C is a minimal cone.

In the rest of this section, we adopt the notation of trace in the sense of [Giusti 1977, Chapter 2]: f^- is the trace of f from above $z = 0$, and f^+ is the trace from below.

To prove Theorem 4.5 we need some lemmas. The first is an adaptation of [Giusti 1977, Lemma 5.3].

Lemma 4.6. *Let $f \in \mathrm{BV}(B_R)$ and let $0 < \rho < r < R$. Then*

$$\int_{\delta^* B_1} |f^-(rx) - f^-(\rho x)|\, dH_2 \le \int_{Q\cap(B_r - B_\rho)} \Big|\Big\langle \frac{x}{|x|^3}, Df\Big\rangle\Big|.$$

Proof. If $g \in C^1(A; \mathbb{R}^3)$, then $\int_A |\langle g, Df\rangle|$ is the total variation in A of the measure $\langle g, Df\rangle$; that is ,

$$\textstyle\int_A |\langle g, Df\rangle| = \sup\big\{\int f\nabla\cdot(\mu g)\, dx : \mu \in C_0^1(A), |\mu| \le 1\big\}.$$

Now set $g(x) = x/|x|^3$ and let h be any C^1 function. Define α by

$$\alpha(x) = h(x/|x|).$$

Then $(\nabla\cdot\alpha g) = 0$ in $\mathbb{R}^3 - \{0\}$, so from [Giusti 1977, 2.14] we have

$$\begin{aligned}\int_{Q\cap(B_r-\bar{B}_\rho)} \alpha\langle g, Df\rangle &= \int_{\delta^* B_r} \alpha f^-\Big\langle g, \frac{x}{|x|}\Big\rangle dH_2 - \int_{\delta^* B_\rho} \alpha f^+\Big\langle g, \frac{x}{|x|}\Big\rangle dH_2 \\ &\qquad + \int_{\delta(B_r-\bar{B}_\rho)} \alpha f^-\langle g, \vec{n}\rangle\, dH_2 \\ &= r^{-2}\int_{\delta^* B_r} \alpha f^-\, dH_2 - \rho^{-2}\int_{\delta^* B_\rho} \alpha f^+\, dH_2 \\ &= \int_{\delta^* B_1} h(x)\big(f(rx) - f(\rho x)\big)\, dH_2,\end{aligned}$$

where $\vec{n}$ is the outward unit normal of $\delta(B_r - \bar{B}_\rho)$. (Recall that $\langle x, \vec{n}\rangle = 0$ anywhere on δK.)

Next we restrict h so that $|h(x)| \le 1$ and hence $|\alpha(x)| \le 1$. By the definition of $\int_A |\langle g, Df\rangle|$, we have

$$\int_{\delta^* B_1} h(x)[f^-(rx) - f^+(\rho x)]\, dH_2 \le \int_{Q\cap(B_r-\bar{B}_\rho)} |\langle g, Df\rangle|$$

for any function h such that h is C^1 and $|h| \le 1$.

Now for almost all $\rho < r$ we have $\int_{\delta^* B_\rho} |Df| = 0$ and $f^+ = f^- = f$, by [Giusti 1977, Remark 2.13], so that

$$\text{(6)}\qquad \int_{\delta^* B_1} h(x)[f^-(rx) - f^-(\rho x)]\, dH_2 \le \int_{Q\cap(B_r-B_\rho)} |\langle g, Df\rangle|$$

for almost all $\rho < r$. Thus if we take any $\rho < r$, we can choose a sequence $\{\rho_j\}$ such that $\rho_j \to \rho$, (6) holds for each ρ_j and $f^-(\rho_j x) \to f^-(\rho x)$ in $L^1(\partial B_1)$. Taking the limit as $j \to \infty$ we obtain (6) for every $\rho < r$. Now on taking the supremum over all h with $|h| \le 1$ we arrive at the desired inequality. □

The next three results correspond to [Giusti 1977, Lemmas 5.5, 5.6, 5.8].

Lemma 4.7. *Suppose $f \in \mathrm{BV}(B_R)$ and $\rho < R$. If $\{\rho_j\}$ is a sequence such that $\rho_j \le \rho$ and $\rho_j \to \rho$, then*

$$\lim_{j\to\infty} \zeta(f, B_{\rho_j}) = \zeta(f, B_\rho) \quad \textit{and} \quad \lim_{j\to\infty} \psi(f, B_{\rho_j}) = \psi(f, B_\rho).$$

Proof. Given $\varepsilon > 0$, by the definition of $\zeta(f, B_\rho)$ we can choose a function $g \in \mathrm{BV}(B_\rho)$ such that $\mathrm{supp}(g - f) \subset B_\rho$ and

$$F_{B_\rho}(g) \le \zeta(f, B_\rho) + \varepsilon.$$

For j large enough we have $\mathrm{supp}(g - f) \subset B_{\rho_j}$ and hence

$$F_{B_\rho}(g) \ge F_{B_{\rho_j}}(g) - \int_{\delta(B_\rho - B_{\rho_j})} \cos\gamma \cdot g\, dH_2 \ge \zeta(f, B_{\rho_j}) - \int_{\delta(B_\rho - B_{\rho_j})} \cos\gamma \cdot g\, dH_2.$$

Since $\varepsilon > 0$ is arbitrary,

$$\zeta(f, B_\rho) \ge \limsup_{j\to\infty} \zeta(f, B_{\rho_j}).$$

On the other hand, for $j \in N$, we can choose $g_j \in \mathrm{BV}(B_\rho)$ such that $g_j - f$ is supported in B_{ρ_j} and

$$\zeta(f, B_{\rho_j}) + \frac{1}{j} \ge F_{B_{\rho_j}}(g_j).$$

Also notice that

$$F_{B_{\rho_j}}(g_j) = F_{B_\rho}(g) - F_{B_\rho - B_{\rho_j}}(f),$$

so we have

$$\zeta(f, B_{\rho_j}) + \frac{1}{j} \ge \zeta(f, B_\rho) - F_{B_\rho - B_{\rho_j}}(f)$$

and therefore

$$\limsup_{j\to\infty} \zeta(f, B_{\rho_j}) \ge \zeta(f, B_\rho).$$

Thus we have proved the first equation. The second follows immediately from

$$\lim_{j\to\infty} F_{B_{\rho_j}}(f) = F_{B_\rho}(f). \qquad \square$$

Lemma 4.8. *Suppose $f, g \in \mathrm{BV}(B_R)$ and $\rho < R$. Then*

$$|\zeta(f, B_\rho) - \zeta(g, B_\rho)| \le \int_{\delta^* B_\rho} |f^- - g^-|\, dH_2.$$

Proof. Since the equality to be proved is symmetric in f and g, it is sufficient to show that

$$\zeta(f, B_\rho) - \zeta(g, B_\rho) \le \int_{\delta^* B_\rho} |f^- - g^-|\, dH_2.$$

Given $\varepsilon > 0$, we can choose $\varphi \in \mathrm{BV}(B_R)$ such that $\mathrm{supp}(\varphi - f) \subset B_\rho$ and

$$F_{B_\rho}(\varphi) \le \zeta(f, B_\rho) + \varepsilon.$$

Let $\{\rho_j\}$ be a sequence such that $\rho_j \le \rho$, $\rho_j \to \rho$,

$$\int_{\partial B_{\rho_j}} |Df| = \int_{\partial B_{\rho_j}} |Dg| = 0$$

and $\operatorname{supp}(f - \varphi) \subset B_{\rho_j}$. For every j, define

$$g_j = \begin{cases} \varphi & \text{in } B_{\rho_j}, \\ g & \text{in } B_R - \bar{B}_{\rho_j}. \end{cases}$$

Then, by [Giusti 1977, Proposition 2.8], we have $g_j \in \mathrm{BV}(B_R)$ and

$$\begin{aligned}
&\zeta(g, B_\rho) \\
&\quad \le F_{B_\rho}(g_j) \\
&\quad = \int_{Q\cap B_\rho} |Dg_j| - \int_{\delta B_\rho} \cos\gamma \cdot g_j \, dH_2 \\
&\quad = \int_{Q\cap B_{\rho_j}} |D\varphi| + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |\varphi - g| \, dH_2 - \int_{\delta B_\rho} \cos\gamma \cdot g_j \, dH_2 \\
&\quad \le \left(\int_{Q\cap B_\rho} |D\varphi| - \int_{\delta B_\rho} \cos\gamma \cdot \varphi dN_2 \right) + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |\varphi - g| \, dH_2 \\
&\qquad\qquad + \int_{\delta B_\rho} \cos\gamma (\varphi - g_j) \, dH_2 \\
&\quad = F_{B_\rho}(\varphi) + \int_{Q\cap(B_\rho - \bar{B}_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |f - g| \, dH_2 + \int_{\delta B_\rho} \cos\gamma (\varphi - g_j) \, dH_2 \\
&\quad \le \zeta(f, B_\rho) + \varepsilon + \int_{Q\cap(B_\rho - B_{\rho_j})} |Dg| + \int_{\delta^* B_{\rho_j}} |f - g| \, dH_2 + \int_{\delta B_\rho} \cos\gamma (\varphi - g_j) \, dH_2.
\end{aligned}$$

As $\varepsilon > 0$ is arbitrary and the terms $\int_{Q\cap(B_\rho - B_{\rho_j})} |Dg|$ and $\int_{\delta B_\rho} \cos\gamma (\varphi - g_j) \, dH_2$ vanish as $j \to \infty$, the lemma follows by taking the limit. □

Lemma 4.9. *Suppose* $f \in \mathrm{BV}(B_R)$ *and* $0 < \rho < r < R$. *Then*

$$\left(\int_{Q_\rho^r} \left| \left\langle \frac{x}{|x|^3}, Df \right\rangle \right| \right)^2 \le \left(r^{-2} F_{B_r}(f) - \rho^{-2} F_{B_\rho}(f) + 2 \int_\rho^r t^{-3} \psi(f, B_t) \, dt \right) \times 2 \int_{Q_\rho^r} |x|^{-2} |Df|, \tag{7}$$

where $Q_\rho^r = Q \cap (B_r - B_\rho)$.

Proof. Suppose first that $f \in C^1(B_R)$ and then, for $0 < t < R$, define

$$f_t(x) = \begin{cases} f(x) & \text{for } t < |x| < R, \\ f(tx/|x|) & \text{for } |x| < t. \end{cases}$$

Then we have

$$\int_{Q\cap B_t} |Df_t|\,dx = \frac{t}{2}\int_{\delta^* B_t} |Df|\left(1-\frac{\langle x, Df\rangle^2}{|x|^2|Df|^2}\right)^{1/2} dH_2,$$

$$\int_{\delta B_t} \cos\gamma\cdot f_t\,dH_2 = \frac{t}{2}\int_{\delta(\partial B_t)} \cos\gamma\cdot f\,dH_1$$

which is to say

$$F_{B_t}(f_t) = \frac{t}{2}\left(\int_{\delta^* B_t} |Df|\left(1-\frac{\langle x, Df\rangle^2}{|x|^2|Df|^2}\right)^{1/2} dH_2 - \int_{\delta(\partial B_t)} \cos\gamma\cdot f dH_1\right),$$

and hence

$$\begin{aligned}\zeta(f, B_t) &= F_{B_t}(f) - \psi(f, B_t)\\ &\le F_{B_t}(f_t)\\ &\le \frac{t}{2}\left(\int_{\delta^* B_t} |Df|\,dH_2 - \frac{1}{2}\int_{\delta^* B_t} \frac{\langle x, Df\rangle^2}{|x|^2|Df|}\,dH_2 - \int_{\delta(\partial B_t)} \cos\gamma\cdot f dH_1\right)\end{aligned}$$

and

$$\tfrac{1}{2}t^{-2}\int_{\delta^* B_t} \frac{\langle x, Df\rangle^2}{|x|^2|Df|}\,dH_2 \le \frac{d}{dt}\big(t^{-2}F_{B_t}(f)\big) + 2t^{-3}\psi(f, B_t).$$

Now integrating with respect to t between ρ and r, we have

$$\frac{1}{2}\int_{Q\cap(B_r-B_\rho)} \frac{\langle x, Df\rangle^2}{|x|^4|Df|}\,dx \le r^{-2}F_{B_r}(f) - \rho^{-2}F_{B_\rho}(f) + 2\int_\rho^r t^{-3}\psi(f, B_t)\,dt.$$

On the other hand, from the Schwarz inequality we have

$$\left(\int_{Q\cap(B_r-B_\rho)} \left|\left\langle \frac{x}{|x|^3}, Df\right\rangle\right| dx\right)^2 \le \int_{Q\cap(B_r-B_\rho)} \frac{|Df|}{|x|^2}\,dx \int_{Q\cap(B_r-B_\rho)} \frac{\langle x, Df\rangle^2}{|x|^4|Df|} dx,$$

so Equation (7) holds for $f \in C^1(B_R)$.

Now suppose that $f \in \mathrm{BV}(B_R)$. By [Giusti 1977, Remarks 2.12 and 2.13] we can approximate f by C^1 functions f_k such that

$$\int_{Q\cap B_t} |Df_k| \to \int_{Q\cap B_t} |Df| \quad\text{and}\quad \int_{\partial(Q\cap B_t)} |f - f_k|\,dH_2 \to 0$$

for almost all t. If we write (7) for f_k and observe that, by Lemma 4.8, $\psi(f_k, B_t) \to \psi(f, B_t)$, we see that (7) holds for $f \in \mathrm{BV}(B_R)$ and almost all ρ, r. Finally we obtain (7) for every ρ and r by approximating with increasing sequences $\{\rho_j\} \to \rho$ and $\{r_j\} \to r$ for which (7) holds. □

Remark 4.10. By approximating at the final step with sequences decreasing to r and ρ we obtain (7) with $\bar{B}_r, \bar{B}_\rho$ instead of B_r and B_ρ.

Remark 4.11. From (7) it follows that, for every $\rho < r$,

$$\rho^{-2} F_{B_\rho}(f) \le r^{-2} F_{B_r}(f) + 2 \int_\rho^r t^{-3} \psi(f, B_t)\, dt. \tag{8}$$

In particular, $\psi(f, B_r) = 0$ implies $\rho^{-2} F_{B_\rho}(f) \le r^{-2} F_{B_r}(f)$. Hence $\rho^{-2} F_{B_\rho}$ is an increasing function of ρ.

The next result is adapted from [Giusti 1977, Lemma 9.1].

Lemma 4.12. *Let K be an open set in $\mathbb{R}^3$, and let $\{E_j\}$ be a sequence of Caccioppoli sets such that*

$$\lim_{j\to\infty} \psi(E_j, A) = 0 \quad \textit{for all } A \Subset K.$$

Suppose that there exists a set E such that

$$\varphi_{E_j} \to \varphi_E \textit{ in } L^1_{\text{loc}}(K).$$

Then E is a minimal set in K, that is,

$$\psi(E, A) = 0 \quad \textit{for all } A \Subset K.$$

Moreover, if $L \Subset K$ is such that $\int_{\delta^* L} |D\varphi_E| = 0$, we have

$$\lim_{j\to\infty} F_L(E_j) = F_L(E).$$

Proof. Let $A \Subset K$. We may suppose that ∂A is smooth, so that for every j,

$$\begin{aligned} F_A(E_j) &= \zeta(E_j, A) + \psi(E_j, A) \le F_A(E_j - A^\circ) + \psi(E_j, A) \\ &\le H_2((\delta^* A) \cap E_j) + \psi(E_j, A) \le H_2(\delta^* A) + \psi(E_j, A). \end{aligned}$$

By [Giusti 1977, Theorem 9.1],

$$F_A(E) \le \liminf_{j\to\infty} F_A(E_j) \le H_2(\delta^* A).$$

For $t > 0$, set

$$A_t = \{x \in K : \operatorname{dist}(x, A) < t\}.$$

We have

$$\lim_{j\to\infty} \int_{Q\cap A_t} |\varphi_{E_j} - \varphi_E|\, dx = 0$$

so there exists a subsequence $\{E_{k_j}\}$ such that, for almost every t close to 0,

$$\lim_{j\to\infty} \int_{\delta^* A_t} |\varphi_{E_{k_j}} - \varphi_E|\, dH_2 = 0.$$

From Lemma 4.8 we have $\lim_{j\to\infty} \zeta(E_{k_j}, A_t) = \zeta(E, A_t)$ for these values of t, and therefore from [Giusti 1977, Theorem 1.9] we get

$$\psi(E, A_t) = 0.$$

Now let $L \Subset K$ be such that

$$\int_{\delta^* L} |D\varphi_E| = 0,$$

and let A be a smooth open set with $L \Subset A \Subset K$. Let $\{F_j\}$ be any subsequence of $\{E_j\}$. Reasoning as above we can find a set A_t and a subsequence $\{F_{k_j}\}$ such that

$$\lim_{j\to\infty} \zeta(F_{k_j}, A_t) = \zeta(E, A_t).$$

Since $\lim_{j\to\infty} \psi(F_{k_j}, A_t) = \psi(E, A_t) = 0$ we have

$$\lim_{j\to\infty} F_{A_t}(F_{k_j}) = F_{A_t}(E),$$

and hence from [Giusti 1977, Proposition 1.13]:

$$\lim_{j\to\infty} F_L(F_{k_j}) = F_L(E). \qquad \square$$

Proof of Theorem 4.5. We first prove the conclusion for every sequence $\{t_j\}$ tending to zero; that is, for every sequence $\{t_j\}$ tending to zero there exists a subsequence $\{s_j\}$ such that E_{s_j} converges locally in $\mathbb{R}^3$ to a set C. Moreover C is a minimal cone.

Then we will prove that this limit cone C does not depend on the specific sequence $\{s_j\}$, and hence is the limit for E_t.

Suppose $t_j \to 0$. We show that for every $R > 0$ there exists a subsequence $\{\sigma_j\}$ such that E_{σ_j} converges in B_R. We have

$$F_{B_R}(E_t) = \frac{1}{t^2} F_{B_{tR}}(E),$$

so choosing t sufficiently small (so that $tR < 1$ and $t < 1$), we guarantee that E_t minimizes

$$F_{B_R}(E_t) + t \int_{Q\cap B_R} H\varphi_{E_t} \le H_2(\delta^* B_R) + tM \cdot H_3(Q \cap B_R) = 4\alpha R^2 + tM \cdot \tfrac{4}{3}\alpha R^3,$$

where $|H| < M$.

Hence, by [Giusti 1977, Theorem 1.19] on compactness, a subsequence E_{σ_j} will converge to a set C_R in B_R. Taking a sequence $R_i \to \infty$ we obtain, by means of a diagonal process, a set $C \subset \mathbb{R}^3$ and a sequence $\{s_j\}$ such that $E_{s_j} \to C$ locally. Now applying Lemma 4.12, we see that C is minimal and it remains only to show that C is a cone.

Also by Lemma 4.12 we have

$$F_{B_R}(E_{s_j}) \to F_{B_R}(C) \quad \text{for almost all } R > 0.$$

Hence if we define

$$p(t) = \frac{1}{t^2} F_{B_t}(E) + \frac{16}{3}\alpha t M = F_{B_1}(E_t) + \frac{16}{3}\alpha t M,$$

we have, for almost all $R > 0$,

$$\lim_{j\to\infty} p(s_j R) = \frac{1}{R^2} F_{B_R}(C).$$

Moreover,

$$\psi(E, B_t) = F_{B_t}(E) - \zeta(E, B_t) = F_{B_t}(E) + \int_{Q\cap B_t} H\varphi_E - \int_{Q\cap B_t} H\varphi_E - \zeta(E, B_t).$$

By the definition of ζ, we can say for all $\varepsilon > 0$, there is a Caccioppoli set E_ε satisfying $\operatorname{supp}(\varphi_{E_\varepsilon} - \varphi_E) \subset B_t$ and $0 < F_{B_t}(E_\varepsilon) - \zeta(E, B_t) < \varepsilon$. Hence,

$$\begin{aligned}\psi(E, B_t) &\le F_{B_t}(E_\varepsilon) + \int_{Q\cap B_t} H\varphi_{E_\varepsilon} - \int_{Q\cap B_t} H\varphi_E - \zeta(E, B_t)\\ &\le \int_{Q\cap B_t} H(\varphi_{E_\varepsilon} - \varphi_E) + \varepsilon \le \frac{8}{3}\alpha M t^3 + \varepsilon.\end{aligned}$$

Then, by (8) and letting $\varepsilon \to 0$, we get

$$\rho^{-2} F_{B_\rho}(E) + \tfrac{16}{3}\alpha M\rho \le r^{-2} F_{B_r}(E) + \tfrac{16}{3}\alpha M r,$$

so that $p(t)$ is increasing in t.

If $\rho < R$, there exists for every j an $m_j > 0$ such that $s_j\rho > s_{j+m_j}R$. Thus $p(s_{j+m_j}R) \le p(s_j\rho) \le p(s_j R)$, so that

$$\lim_{j\to\infty} p(s_j\rho) = \lim_{j\to\infty} p(s_j R) = \frac{1}{R^2} F_{B_R}(C).$$

Thus we have proved that

$$\frac{1}{\rho^2} F_{B_\rho}(C)$$

is independent of ρ and so, from Lemmas 4.6, 4.8 and 4.9, we conclude that

$$\int_{\delta^* B_1} \left|\varphi_C(\rho x) - \varphi_C(r x)\right| dH_2 = 0$$

for almost all $\rho, r > 0$. Hence the set C differs only on a set of measure zero from a cone with vertex at the origin.

Now suppose we have two sequences $\{s_j\}$ and $\{s'_j\}$ that give us two minimal cones in the limit, C and C'. Recall that $p(t)$ is increasing in t. Therefore $R^{-2}F_{B_R}(C)$ is independent of both R and C; that is, for almost all R,

$$\lim_{t\to 0} p(t) = \frac{1}{R^2} F_{B_R}(C) = \frac{1}{R^2} F_{B_R}(C').$$

As on the previous page, we apply Lemmas 4.6, 4.8 and 4.9 to the set E and get, for $s_j > s'_k$,

$$\begin{aligned}
&\left(\int_{\delta^* B_1} |\varphi_E(s_j x) - \varphi_E(s'_k x)|\, dH_2\right)^2 \\
&\quad \le 2\int_{Q\cap(B_{s_j} - B_{s'_k})} \frac{|D\varphi_E|}{|x|^2}\left(s_j^{-2}F_{B_{s_j}}(E) - s_k'^{-2}F_{B_{s'_k}}(E) + 2\int_{s'_k}^{s_j} t^{-3}\psi(\varphi_E, B_t)\, dt\right) \\
&\quad \le 2\int_{Q\cap(B_{s_j} - B_{s'_k})} \frac{|D\varphi_E|}{|x|^2}\left(F_{B_1}(E_{s_j}) - F_{B_1}(E_{s'_k}) + \frac{16}{3}\alpha M(s_j - s'_k)\right).
\end{aligned}$$

Suppose $j, k \to \infty$. We have

$$\int_{\delta^* B_1} |\varphi_E(s_j x) - \varphi_E(s'_k x)|\, dH_2 \to 0;$$

that is,

$$\int_{\delta^* B_1} |\varphi_{E_{s_j}}(x) - \varphi_{E_{s'_k}}(x)|\, dH_2 \to 0,$$

which implies

$$\int_{\delta^* B_1} |\varphi_C(x) - \varphi_{C'}(x)|\, dH_2 = 0.$$

Hence C and C' are almost equal, completing the proof of the theorem. □

5. Continuity at a reentrant wedge

We now prove the well-definedness of a new boundary condition and we subdivide the reentrant wedge domain along the new boundary to get two protruding wedge domains, which enables us to apply Concus–Finn conjecture to prove the main theorem.

First we introduce a uniformity lemma for P, N:

Lemma 5.1 [Finn 1986, Lemma 7.1]. *Suppose we have a wedge domain Ω_∞ with $2\alpha \ge \pi$. A sequence of functions $\{f_j\}$ converges locally to a generalized solution of the corresponding minimal surface problem, f. Denote their subgraphs as V_j and V, respectively. Then for some point $(x_0, y_0) \in \Omega$, there exists $r_0 > 0$ and $C > 0$ not depending on j such that for all $t \in \mathbb{R}$, the following is true:*

If $\left|V'_{j,r}(x_0, y_0, t)\right| > 0$ *and* $\left|V_{j,r}(x_0, y_0, t)\right| > 0$ *for all* $r > 0$, *then*

$$|V_{j,r}(x_0, y_0, t)| \geq Cr^3 \quad \text{and} \quad |V'_{j,r}(x_0, y_0, t)| \geq Cr^3$$

for all $0 < r \leq r_0$, *where*

$$C_r(x_0, y_0, t_0) = \left\{(x, y, t) \in \mathbb{R}^3 : \operatorname{dist}((x, y),\ (x_0, y_0)) < r,\ |t - t_0| < r\right\},$$
$$V'_{j,r}(x_0, y_0, t) = C_r(x_0, y_0, t) - V_j.$$

Lemma 5.2 [Chen et al. 1998, Lemma 6.1]. *If f is a classical minimal surface over $\mathbb{R}^2$ and is linear in every radial direction* (*that is, its restriction to each radial direction is a linear function*), *then f is either a plane or a helicoid.*

Lemma 5.3 [Chen et al. 1998, Lemma 6.2]. *If G is a nonempty domain, the only possibility that there is a classical minimal surface f defined on G which is linear in every radial direction is that $G = \Omega_\infty$ and $(\gamma_1, \gamma_2) \in \mathfrak{R}$. Moreover, f is a plane.*

Proof. It follows from Lemma 5.2 that f is a plane or a helicoid defined on G, say, $f = a \tan^{-1}(y/x)$ for some constant a.

If P or $N \neq \varnothing$, then either $\partial P \cap \partial G \neq \varnothing$ or $\partial N \cap \partial G \neq \varnothing$. Without loss of generality, we may assume that $L = \partial P \cap \partial G \neq \varnothing$. It follows from Corollary 3.4 that L is either a line or a half-line in Ω_∞.

Let f, defined over G, be a helicoid or a plane and take $x_0 \in L$ distinct from O. Then $f \in C^1$ in $\overline{G} \cap B_\rho(x_0)$, where $B_\rho(x_0)$ is a small open disk with radius ρ and center x_0 such that the disk belongs to the sector Ω_∞. The subgraph F of f can not be a minimal surface in a small neighborhood of $(x_0, f(x_0))$ since F will violate the inequality (5) if K is small enough so that $K \cap (\partial\Omega_\infty \times \mathbb{R}) = \varnothing$; hence $P = N = \varnothing$. To see that F violates (5) we will construct a "better" comparison set as follows.

Let T_0 be the tangential plane to the surface $S : z = f(x)$, $x \in G$, at the point $(x_0, f(x_0))$. Take a plane E_0 parallel to the edge $T_0 \cap (L \times \mathbb{R})$ which intersects the vertical plane $L \times \mathbb{R}$ and the plane T_0 in a distance h from the edge $T_0 \cap (L \times \mathbb{R})$.

Choose the plane that lies above of $T_0 \cap (L \times \mathbb{R})$. Take two further planes perpendicular to the edge $T_0 \cap (L \times \mathbb{R})$ and with distances $\pm a$ from $(x_0, f(x_0))$.

This construction defines a prismatic set W. Set $V = F \cup W$ and $h = a^2$, and let ω be the opening angle of the edge $T_0 \cap (L \times \mathbb{R})$ as indicated in the figure. Then

$$\int_{(\Omega_\infty \times \mathbb{R}) \cap K} |D\varphi_F| - \int_{(\Omega_\infty \times \mathbb{R}) \cap K} |D\varphi_V| = 4a^3(1 - \sin \tfrac{1}{2}\omega) + o(a^3) \quad \text{as } a \to 0.$$

Thus $G = \Omega_\infty$ and f satisfies equations (4). Since a helicoid $f = a \tan^{-1}(y/x)$ cannot make constant contact angles on Σ_1 and Σ_2, the only possibility is that f is a plane. The conclusion that $\pi + (\gamma_2 - \gamma_1) < \theta < \pi + (\gamma_2 + \gamma_1)$ then follows. □

Lemma 5.4. *Suppose $f \in C^2(\Omega_\infty)$ solves the problem*

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \quad && \textit{in } \Omega_\infty, \\ Tf \cdot \nu_1 &= \cos\gamma_1 && \textit{on } \Sigma_1, \\ Tf \cdot \nu_2 &= \cos\gamma_2 && \textit{on } \Sigma_2, \end{aligned}$$

where $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$, $\kappa \geq 0$ and f is bounded on Ω_∞. Then there is a radial line L and some curve $\mathcal{C}$, such that $\mathcal{C}$ is tangent to L when approaching the vertex and

$$\lim_{\substack{(x,y)\in\mathcal{C} \\ (x,y)\to(0,0)}} Tf(x,y) = \nu_L,$$

where ν_L is the unit normal vector of L pointing toward P.

Proof. Denote by E the subgraph of f, and set

$$E_t = \{x \in \mathbb{R}^3 : tx \in E\}.$$

By Theorem 4.5 we see that as $t \to 0$, E_t converges locally in $\mathbb{R}^3$ to a minimal cone C.

It is well known that the only minimal cone in $\mathbb{R}^3$ is a plane. Thus C is a plane.

By the assumption that $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$ we see that C has to be a vertical plane passing through z-axis. Let the line L be the projection of C onto the xy-plane.

To complete the proof of the lemma we just need to show the following:

Claim. There is a level curve $\mathcal{C}$ of $f(0,0)$ that reaches $(0,0)$ and is tangent to L.

Proof. The level set of $f(0,0)$ must lie within a cusp region with $(0,0)$ as its tip. For assume to the contrary that there exists a sequence $\{(x_k, y_k)\}$ in the level set with the property that

$$(x^*, y^*) := \lim_{k\to\infty}\left(\frac{x_k}{\sqrt{x_k^2+y_k^2}}, \frac{y_k}{\sqrt{x_k^2+y_k^2}}\right) \notin L. \tag{9}$$

Take the blow-up sequence

$$f_k(x,y) = \frac{1}{\varepsilon_k}\big(f(\varepsilon_k x, \varepsilon_k y) - f(0,0)\big),$$

where $\varepsilon_k = \sqrt{x_k^2+y_k^2}$. Then

$$f_k\left(\frac{x_k}{\varepsilon_k}, \frac{y_k}{\varepsilon_k}\right) = 0 \quad \text{for all } k.$$

By Lemma 5.1, then, the limit minimal cone must go through $(x^*, y^*, 0)$, which is impossible.

Next we check that, in some neighborhood of the origin, there is no point of the level set of $(0, 0)$ where the gradient of f vanishes. For suppose to the contrary that there exists a sequence $\{(x_k, y_k)\} \to (0, 0)$ of points in the level set satisfying

$$\nabla f(x_k, y_k) = 0 \quad \text{for all } k.$$

(Recall that f is C^2 in Ω_∞ and satisfies the problem in Lemma 5.4.) Construct another blow-up sequence as above and let

$$f_k(x, y) = \frac{1}{\varepsilon_k}\big(f(\varepsilon_k x, \varepsilon_k y) - f(0, 0)\big),$$

where $\varepsilon_k = \sqrt{x_k^2 + y_k^2}$. Then $\nabla f_k(x_k/\varepsilon_k, y_k/\varepsilon_k) = \nabla f(x_k, y_k) = 0$ for all k. In the notation of (9), we have

$$(x^*, y^*) = \lim_{k\to\infty}\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}\right) \in L;$$

therefore [Massari and Pepe 1975, Theorem 3] yields

$$\vec{\nu}_k\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, 0\right) \to \vec{\nu}_C(x^*, y^*, 0) = \vec{\nu}_L,$$

contradicting the equality

$$\vec{\nu}_k\left(\frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, 0\right) = \vec{\nu}_f(x_k, y_k, 0) = (0, 0, 1).$$

Thus we see that the level set of $f(0, 0)$ is locally a union of *unbranched* level curves, which do not stop at any interior point of Ω_∞; moreover $(0, 0)$ is an accumulation point of this level set. To conclude the proof of the claim, assume to the contrary that no one level curve approaches $(0, 0)$; in other words, there is a sequence of distinct level curves approaching the vertex. On each of them, choose a point (x_k, y_k) nearest to $(0, 0)$. Then $\vec{\nu}_f(x_k, y_k) \cdot \vec{\nu}_L \to 0$, which again contradicts Massari's theorem, which says that $\vec{\nu}_f(x_k, y_k) \cdot \vec{\nu}_L \to 1$. □

Now we exclude continuity from $\mathcal{D}$-regions for reentrant corner domains, assuming Conjecture 2.4.

Theorem 5.5. *Let f be a bounded solution to (1) satisfying (3), with $(\gamma_1, \gamma_2) \notin \overline{\mathcal{R}}$ and $\alpha > \pi/2$. Under the assumption that Conjecture 2.4 is true, f must have discontinuous radial limits at O.*

Proof. Assume to the contrary that $\gamma_1, \gamma_2 \notin \overline{\mathcal{R}}$ and f has continuous radial limits at $(0, 0)$.

Denote by Σ_1', Σ_2' the lines extending Σ_1', Σ_2'. Consider, for $t \to 0$, the blow-up functions

$$f_t(x, y) = \frac{1}{t}\big(f(tx, ty) - f(0, 0)\big)$$

which converge locally to a generalized solution $v(x, y)$ to the corresponding minimal surface problem defined in Ω_∞.

By Lemma 5.3, the G region of $v(x, y)$ must be empty. Hence v defines a vertical plane, whose projection L onto the xy-plane equals $\partial P \cap \partial N$. By the conclusion of the previous section, L must be either a half-line or a whole line passing through the origin.

Assume L is a half-line; we claim that it must lie between Σ_1' and Σ_2', inclusive (Figure 8, left). Otherwise either the P or the N region will cover a subdomain which is again a reentrant wedge. This leads to a contradiction with Lemma 5.1 on uniformity.

By Lemma 5.4 we know that along any radial half-line approaching the origin, $\lim Tf \cdot \nu$ is well defined. Therefore we can split the domain Ω_∞ from L, and get two subproblems:

$$\begin{aligned} &\nabla \cdot Tf = \kappa f + \lambda && \text{between } \Sigma_1 \text{ and } L, \\ &Tf \cdot \nu = \cos \gamma_1 && \text{on } \Sigma_1, \\ &Tf \cdot \nu = \pm 1 && \text{on } L \end{aligned}$$

and

$$\begin{aligned} &\nabla \cdot Tf = \kappa f + \lambda && \text{between } L \text{ and } \Sigma_2, \\ &Tf \cdot \nu = \mp 1 && \text{on } L, \\ &Tf \cdot \nu = \cos \gamma_2 && \text{on } \Sigma_2. \end{aligned}$$

Both of them admit a continuous solution.

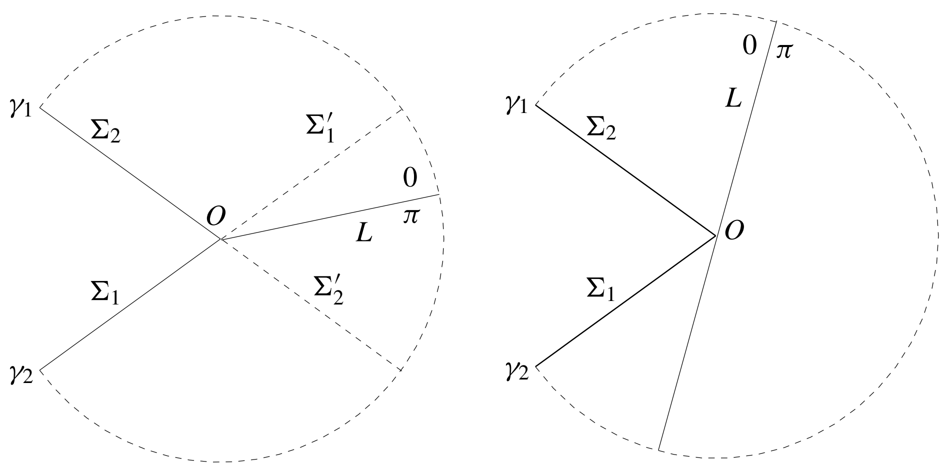


Figure 8. Proof of Theorem 5.5. Left: case where L is a half-line. Right: case where L is a whole line.

By the Concus–Finn conjecture, the angle between Σ_1 and L is either $\pi - \gamma_1$ or γ_1, while the angle between L and Σ_2 is correspondingly either γ_2 or $\pi - \gamma_2$. This leads to $|\gamma_1 - \gamma_2| = 2\alpha - \pi$, which is impossible.

Now assume instead that L is a whole line passing through the origin. Then the region between Σ_1' and Σ_2' must lie on one side of L (see Figure 8, right). We again split into two subproblems:

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \text{between } \Sigma_1 \text{ and } L_1, \\ Tf \cdot \nu &= \cos\gamma_1 \quad \text{on } \Sigma_1, \\ Tf \cdot \nu &= \pm 1 \qquad \text{on } L_1 \end{aligned}$$

and

$$\begin{aligned} \nabla \cdot Tf &= \kappa f + \lambda \text{between } L_2 \text{ and } \Sigma_2, \\ Tf \cdot \nu &= \pm 1 \qquad\qquad\qquad \text{on } L_2, \\ Tf \cdot \nu &= \cos\gamma_2 \qquad\qquad \text{on } \Sigma_2. \end{aligned}$$

Using a similar reasoning as in the previous case, we see that

$$\gamma_1 + \gamma_2 = 2\alpha - \pi, \text{ or } \gamma_1 + \gamma_2 = 3\pi - 2\alpha,$$

which is again impossible. □

6. Examples

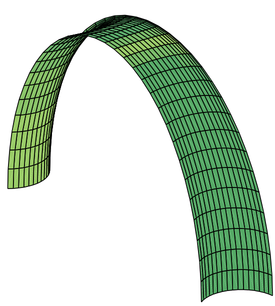
We now construct explicit examples for some of the discontinuous cases given in Theorem 2.5. The notation (D), (I), (DI) and (ID) is defined in Section 3.

Example 6.1. For any $(\gamma_1, \gamma_2) \in \mathscr{R}(2\alpha) \cup \mathscr{D}_2^+ \cup \mathscr{D}_1^-$, we have case (D).

Referring to Figures 9–11, consider the region $\Omega_1 \in \Omega$ bounded by two (close) parallel sides Σ_1, $\tilde{\Sigma}_1$ and two circular arcs C, $\tilde{C}$ which are symmetric about a line orthogonal to Σ_1 $\tilde{\Sigma}_1$.

The arcs are constrained to meet Σ_1 in the angle γ_1, and $\tilde{\Sigma}_1$ in the (fixed) angle $\tilde{\gamma}_1$, which is chosen in the interval $(0, \pi - \gamma_1)$. Thus each arc is part of a circle, with radius ε/a_1, and with $a_1 = \cos\gamma_1 + \cos\tilde{\gamma}_1$.

We distinguish two cases, according as $\gamma_1 < 2\alpha - \pi$ or $\gamma_1 > 2\alpha - \pi$; they are indicated in Figures 9 and 10, respectively. In the former case, we can position a disk of radius δ (independent of ε) in Ω and tangent to Σ_2 at O. Following a construction initiated by Korevaar [1980], we next construct an upper half of the inner side of a torus containing the (horizontal) arcs C, $\tilde{C}$ (see figure on the right). This has the appearance of a Japanese footbridge. It can be represented as a function $g(x, y)$ over Ω_1, with



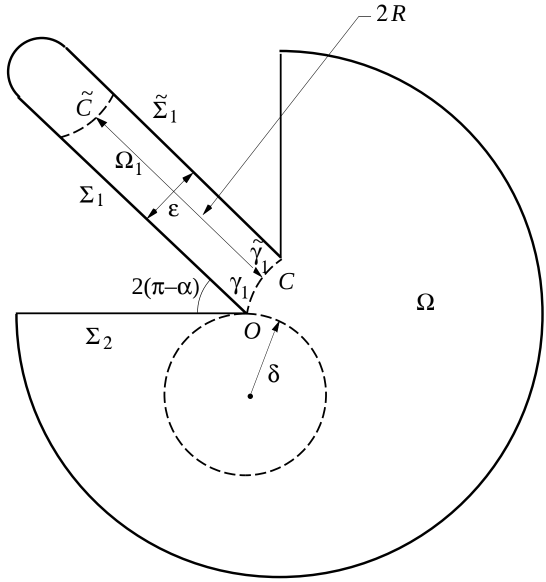


Figure 9. Construction of example of case (D), with $(\gamma_1, \gamma_2) \in \mathcal{R} \cup \mathcal{D}_2^+ \cup \mathcal{D}_1^-$ and $\gamma_1 < 2\alpha - \pi$.

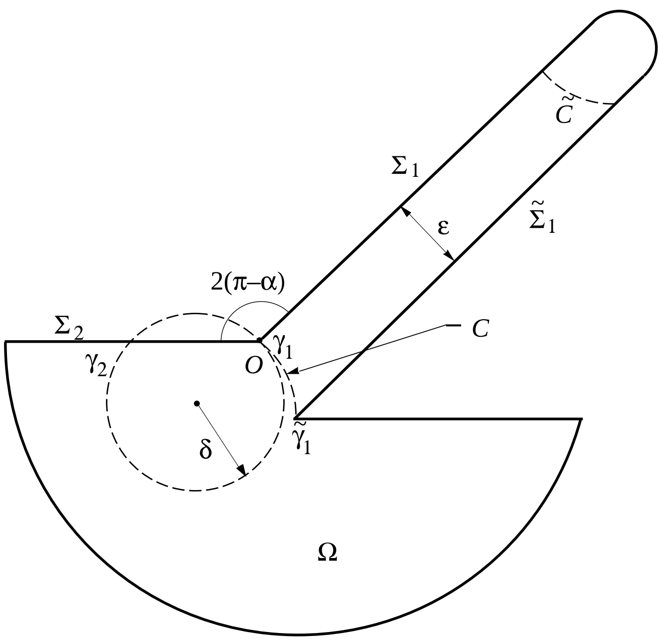


Figure 10. Construction of example of case (D), with $(\gamma_1, \gamma_2) \in \mathcal{R} \cup \mathcal{D}_2^+ \cup \mathcal{D}_1^-$ and $\gamma_1 > 2\alpha - \pi$.

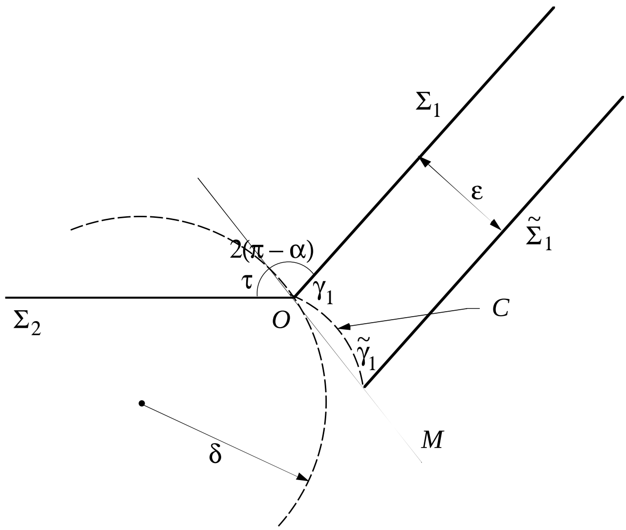


Figure 11. Detail of construction.

$\nu \cdot Tg = -1$ on C and $\tilde{C}$, ν being unit exterior normal.On Σ_1 and $\tilde{\Sigma}_1$, the torus meets vertical walls over the sides in the constant angles γ_1, $\tilde{\gamma}_1$, so $\nu \cdot Tg = \cos\gamma_1$ on Σ_1, $\nu \cdot Tg = \cos\tilde{\gamma}_1$ on $\tilde{\Sigma}_1$.

We extend γ_1, $\tilde{\gamma}_1$ smoothly to the remaining boundary of Ω. By theorems of Emmer [1973] and of Finn and Gerhardt [1977], a (unique) solution $f(x, y)$ of (1)+(3) exists in Ω, with $\nu \cdot Tf = \cos\gamma_1$ on Σ_1, $\nu \cdot Tf = \cos\tilde{\gamma}_1$ on $\tilde{\Sigma}_1$. On C, $\tilde{C}$ there holds $\nu \cdot Tf > -1$, since $|Tf| < 1$ for any function f.

We adjust f by an additive constant so that $\max g = (1/K)(a_1/\varepsilon - 1/R - \lambda)$, where R (the inner radius of the torus) is half the distance between C and $\tilde{C}$. The mean curvature of the torus is given by $\frac{1}{2}\nabla \cdot (Tg)$, and is minimized at the upper symmetry point, at which $\nabla \cdot (Tg) = (a_1/\varepsilon - 1/R)$. We thus find $\nabla \cdot (Tg) \geq \kappa g + \lambda$. By the comparison principle of Concus and Finn [1996, Theorem 5.1], we then obtain $f \geq g > \max g - (R + \varepsilon/a_1)$ in Ω_1, and thus $\lim_{\varepsilon\to 0} f = \infty$, uniformly over Ω_1, and in particular for any radial approach to O from within Ω_1. On the other hand, we find using [Concus and Finn 1996, Theorem 5.2] that $f < 2/(\kappa\delta) + \delta + \lambda/\kappa$ throughout the interior of the disk of radius δ, and hence also for any radial approach to O within that disk. We conclude that for small enough ε, the behavior at O must be either (D) or else (DI). However, (DI) is excluded by [Lancaster and Siegel 1996, Corollary 3].

If $\gamma_1 > 2\alpha - \pi$ the construction above does not work, because with decreasing ε the segment $\tilde{\Sigma}_1$ would enter the δ-disk. We start instead by constructing that disk to be tangent to C at O (Figure 10). We then have $\tau = \pi - 2\alpha + \gamma_1$. But

$\gamma_1 - \gamma_2 < 2\alpha - \pi$ by assumption, and thus $\tau < \gamma_2$. We now rotate the disk slightly about O, so that its circumference meets C at a positive angle, but still maintaining the condition $\tau < \gamma_2$ (see Figure 11). From [Concus and Finn 1996, Theorem 5.3] we now derive that again $f < 2/(\kappa\delta) + \delta + \lambda/\kappa$ in the intersection of the disk with Ω. It remains to narrow the Ω_1 region in a way ensuring that $\tilde{\Sigma}_1$ does not enter the δ-disk; but that again elevates the solution height unboundedly within that Ω_1.

We can do that simply by introducing a tangent line M to the δ-disk at O. For each choice of ε (tending to zero) we choose C to be the unique circular arc meeting Σ_1 at O in the angle γ_1 and meeting $\tilde{\Sigma}_1$ at the intersection of $\tilde{\Sigma}_1$ with M. The angle $\tilde{\gamma}_1$ will not remain constant in this construction, but the radius of C tends to zero with ε, and thus the comparison principle can again be applied to show that the solutions then uniformly to infinity throughout the domains $\Omega_1(\varepsilon)$. We are done.

Example 6.2. For any $(\gamma_1, \gamma_2) \in \mathcal{R}(2\alpha) \cup \mathcal{D}_1^+ \cup \mathcal{D}_2^-$, we are in case (I).

Use the same construction of Example 6.1, interchanging the sides Σ_1 and Σ_2.

Example 6.3. For any $(\gamma_1, \gamma_2) \in \mathcal{R}(2\alpha)$, we have a capillary surface continuous up to the vertex.

First we introduce a lemma on the general existence of a continuous solution of the capillary equation

$$\nabla \cdot Tf = \kappa f + \lambda \quad \text{in } \Omega, \tag{10}$$

which, for rotationally symmetric solutions, becomes

$$(r \sin\psi)_r = \kappa r u, \; u_r = \tan\psi, \tag{11}$$

where ψ is the inclination angle of the vertical surface section $u(r)$ with the r-axis.

Lemma 6.4 [Johnson and Perko 1968]. *Given $u_0 > 0$, there exists a unique $R_0(u_0) > 0$ and solution $u(r; u_0)$ of (11) in $0 < r < R_0$ such that*

(1) *u and ψ extend continuously to the closed interval $0 \le r \le R_0$,*

(2) *$u(0; u_0) = u_0$, $\psi(0; u_0) = 0$, $\psi(R_0; \psi) = \pi$, and*

(3) *the functions u and ψ both are monotone increasing on $0 \le r \le R_0$.*

This guarantees the existence of a convex rotation surface S with a single “bottom” point, that become vertical on a horizontal circular ring, and for which the height $u(x, y)$ is a solution of (10).

Lemma 6.5 [Finn 1986, pp. 67–69]. *Let $u_0 > 0$ and let Π be any nonvertical plane. There is a* (*unique*) *plane parallel to Π and tangent to the solution surface of Lemma 6.4.*

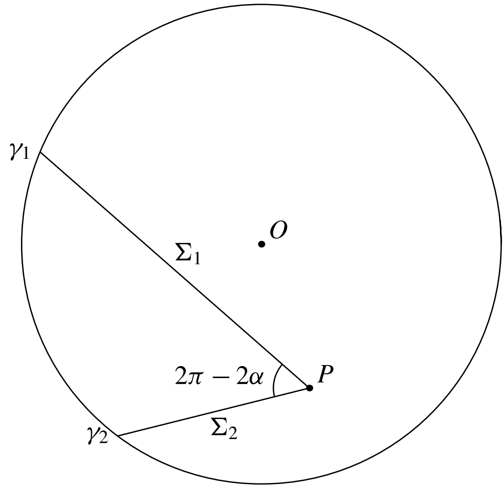


Figure 12. Projection of the continuous capillary surface on the xy-plane.

Proof. First move Π parallel to itself in a direction orthogonal to itself until it doesn't meet S. Then move Π parallel to itself toward S. If the first point of contact with S is an interior point of the surface, that gives us the plane we want. If the first contact point is with the ring on which S is vertical, continue moving Π until there is a last point of contact with S. That is then the plane we want. $\diamond$

So any nonvertical plane can be realized up to rigid motion preserving the normal, as a tangent plane of S. But [Concus and Finn 1996, Theorem 1] characterizes the interior of $\mathcal{R}$ as exactly the set of intersection angles (γ_1, γ_2) with the wedge planes, that arise from all possible nonvertical planes. Since we can get any nonvertical plane as a tangent to S, we can get any point of $\mathcal{R}$.

Now, by [Concus and Finn 1996, Theorem 1], starting from any point (γ_1, γ_2) in $\mathcal{R}$, a unique nonvertical plane is determined. There is a unique point p of the plane that lies above the vertex of the wedge. Position S to be tangent to the plane at p.

Then consider a neighborhood of p on S. Recall that $(\gamma_1, \gamma_2) \in \mathcal{R}$ guarantees that $2\alpha - \pi < \gamma_1 + \gamma_2 < 3\pi - 2\alpha$ and $|\gamma_1 - \gamma_2| < 2\alpha - \pi$. Therefore we can always find suitable positions for sides Σ_1, Σ_2; see Figure 12 for the projection of the surface onto the xy-plane. Now we have constructed a continuous solution to the equation that also satisfies the boundary condition for the given $(\gamma_1, \gamma_2) \in \mathcal{R}$. $\square$

Example 6.6. For any given $(\gamma_1, \gamma_2) \in \mathcal{D}_1^+(2\alpha)$, we are in case (DI).

On each of the regions Ω_i, $i = 1, 2$, consider two circles which is symmetric about the dashed line. Each circle has radius ε_i/a_i, $a_i = \cos\gamma_i + \cos\tilde{\gamma}_i$, where

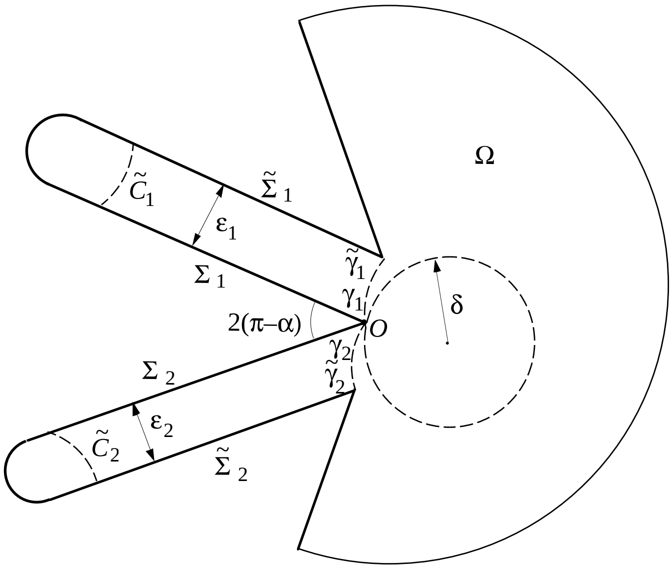


Figure 13. Construction for case (DI) (Example 6.6): $\mathcal{D}_1^+$ and O can be joined within Ω by a circle of fixed radius δ, independent of ε_1 and ε_2.

$0 < \tilde{\gamma}_i < \pi - \gamma_i$, and meets side Σ_i with an angle γ_i and side $\tilde{\Sigma}_i$ with an angle $\tilde{\gamma}_i$. Let g_1, g_2 both be the portion of a torus obtained by rotating one of the circular arcs above the xy-plane, like g_1 in Example 6.1.

Again it follows from the comparison principle that

$$f \geq g_i \geq \frac{1}{\kappa}\Big(\frac{a_1}{\varepsilon_i} - \frac{1}{R-\varepsilon_i}\Big) - R \quad \text{in } \Omega_i,\, i = 1, 2.$$

Since now $(\gamma_1, \gamma_2) \in \mathcal{D}_1^+$, which says that $\gamma_1 + \gamma_2 < 2\alpha - \pi$, we can construct a ball Ω_0 of radius δ which is also contained in Ω not overlapping with Ω_1 or Ω_2. Thus

$$f \leq \frac{2}{\kappa\delta} + \delta \quad \text{in } \Omega_0.$$

By making ε_i sufficiently small we can make

$$\frac{2}{\kappa\delta} + \delta \leq \frac{1}{\kappa}\Big(\frac{a_i}{\varepsilon_i} - \frac{1}{R-\varepsilon_i}\Big) - R$$

for $i = 1, 2$, forcing f to be in case (DI) at O.

Example 6.7. For any $(\gamma_1, \gamma_2) \in \mathcal{D}_1^-(2\alpha)$, we are in case (ID).

Similar to Example 6.6. Consider $-f$.

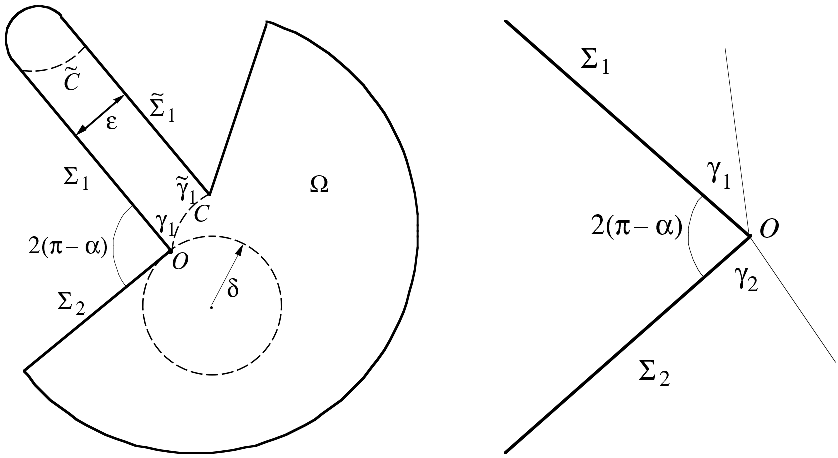


Figure 14. Left: Construction for case (D) in region $\mathfrak{D}_1^+$. Right: Exclusion of central fan.

Example 6.8. For any $(\gamma_1, \gamma_2) \in \mathfrak{D}_1^+(2\alpha)$, we are in case (D).

When (γ_1, γ_2) is in this region there are more possibilities of discontinuous solutions. So when constructing a (D) case there is an additional complexity: differ case (D) from case (DI).

We use a construction very similar to that given in Example 6.1.

Referring to Figure 14, left, consider the region Ω_1 bounded by two parallel straight sides, Σ_1, $\tilde{\Sigma}_1$, and two circular arcs which are symmetric about the dashed line.

First choose an angle θ_1 such that $\gamma_1 \le \theta_1 < 2\alpha - \pi$ and $\theta_1 + \gamma_2 > 2\alpha - \pi$. Then let each arc meet side Σ_1 with an angle θ_1 and side $\tilde{\Sigma}_1$ with an angle $\tilde{\theta}_1$. Here $\tilde{\theta}_1$is any value in $(0, \pi - \theta_1)$. Hence each arc is a part of a circle with radius ε_1/a_1, where $a_1 = \cos\theta_1 + \cos\tilde{\theta}_1$.

Region Ω_2 contains a disk B with some radius δ. Σ_2 is a part of the boundary $\partial\Omega_2$. We fix δ, and make Ω_1 and Ω_2 not overlap each other for any $\varepsilon_1 > 0$. Notice that when $(\gamma_1, \gamma_2) \in \mathfrak{D}_1^+$ this is always possible.

Construct the torus in region Ω_1 and the lower hemisphere in region Ω_2 the same way as in Example 6.1. By making ε_1, and ε_2 sufficiently small we can force f to have a jump discontinuity at O.

Now the only thing to do before we can say for sure this is a (D) case is to eliminate the possibility of a (DI) case.

Lancaster and Siegel [1996, Theorem 2] proved that in a (DI) case, there exist fans of constant radial limits adjacent to Σ_1 and Σ_2. And the size of the fan on side Σ_i is no less than γ_i for $i = 1, 2$. This indicates a jump discontinuity of the radial limits happens in radial directions away from Σ_2 by an angle at least γ_2. On the other hand, by the above construction we know another jump discontinuity happens in radial directions away from Σ_1 by an angle at least θ_1. Since $\theta_1 + \gamma_2 > 2\alpha - \pi$, between the two discontinuities there is no enough space for a half plane constant radial limits to happen. Therefore a (DI) case is impossible.

Finally we proved that this construction gives us a (D) case.

Example 6.9. For any given $(\gamma_1, \gamma_2) \in \mathcal{D}_1^-(2\alpha)$, this is an example of case (I).

Similar configuration as in Example 6.8. Reverse the sides Σ_1 and Σ_2.

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APPROXIMATING SYMMETRIC CAPILLARY SURFACES

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An iterative method is introduced for approximating symmetric capillary surfaces which makes use of the known exact volume. For the interior and annular problems this leads to upper and lower bounds at the center or inner boundary and at the outer boundary, and to an asymptotic expansion in powers of the Bond number. For the exterior problem we determine the leading order asymptotics of the boundary height as the Bond number tends to zero, obtaining a result first proved by B. Turkington.

1. Introduction

The study of capillary surfaces goes back to Laplace [1805–1806]. The canonical modern reference is [Finn 1986]. We will consider symmetric capillary surfaces with gravity in one of three cases: interior, annular and exterior. A vertical circular cylindrical tube immersed in an infinite reservoir of fluid will create an interior and an exterior capillary surface. Two concentric circular tubes will create an annular capillary surface between them.

Let r be the radial variable and let ψ be the inclination angle of the surface $z = u(r)$. Then $\sin\psi = u_r/\sqrt{1+u_r^2}$ and $Nu = (1/r)(r\sin\psi)_r$ is twice the mean curvature of the surface. A capillary surface is determined by the capillary equation $Nu = Bu$, where B is a positive constant, the Bond number, and by specifying the contact angle $\gamma \in [0, \pi]$ on the boundary. The contact angle is the angle between the interface cross–section and vertical, measured inside the fluid. Thus, the inclination angle will be prescribed on the boundary. In order for the annular problem to be similar to the interior problem we take the contact angle to be $\pi/2$ on the inner boundary and γ on the outer boundary.

The interior and annular problems can be written

$$Nu = Bu, \quad a < r < 1, \quad \sin\psi(a) = 0, \quad \sin\psi(1) = \cos\gamma, \tag{1}$$

where $a = 0$ for the interior problem and $0 < a < 1$ for the annular problem.

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The exterior problem is

$$Nu = Bu, \quad r > 1, \quad \sin\psi(1) = -\cos\gamma, \quad u \to 0 \text{ as } r \to \infty. \tag{2}$$

For all three problems we take

$$0 \le \gamma < \pi/2. \tag{3}$$

If $\gamma = \pi/2$ then $u = 0$. If $\pi/2 < \gamma \le \pi$ then $\bar{u} = -u$ satisfies $N\bar{u} = B\bar{u}$ with contact angle $\bar{\gamma} = \pi - \gamma$, so $0 \le \bar{\gamma} < \pi/2$. Scaling allows us to take one boundary at $r = 1$. It is known [Siegel 1980] that for a solution to (2), u and u_r decay exponentially fast as $r \to \infty$. Also, under (3), the solution u is positive in every case by the Comparison Principle [Finn 1986, Theorem 5.1; Siegel 1980, Theorem 1].

The volume lifted can be determined for all three problems:

$$B \int_I r u(r)\, dr = \cos\gamma,$$

where $I = [a, 1]$ for (1) and $I = [1, \infty)$ for (2).

We wish to employ approximate solutions that have the correct volume. The key observation is that if v_1 is a nonnegative function with the correct volume then we may define v_2 by $Nv_2 = Bv_1$ and v_2 will satisfy the correct boundary conditions.

Theorem 1.1. *Let v_1 be a nonnegative continuous function on I which satisfies $B\int_I r v_1(r)\, dr = \cos\gamma$ where I is $[a, 1]$ or $[1, \infty)$. Assume that v_1 is nondecreasing when I is $[a, 1]$ and $v_1(r) = O(\frac{1}{r^3})$ as $r \to \infty$ when $I = [1, \infty)$. Here $B > 0$, $0 \le \gamma < \frac{\pi}{2}$ and $0 \le a < 1$. Then there is a function v_2 defined and continuous on I, satisfying $Nv_2 = Bv_1$, given as a quadrature of v_1, which satisfies the boundary conditions of problem (1) or (2). Let ψ_2 be the inclination angle of v_2 and let*

$$h_2 = \frac{\sin\psi_2}{\sqrt{1 - \sin^2\psi_2}}.$$

For $I = [a, 1]$, let $\sin\psi_2(r) = (B/r)\int_a^r s v_1(s)\, ds$ and $v_2(r) = v_2(a) + \int_a^r h_2(s)\, ds$. Then v_2 is nondecreasing, $\sin\psi(a) = 0$ and $\sin\psi(1) = \cos\gamma$.

For $I = [1, \infty)$, let $\sin\psi_2(r) = -(B/r)\int_r^\infty s v_1(s)\, ds$ and $v_2(r) = -\int_r^\infty h_2(s)\, ds$. Then v_2 is nonincreasing, $\sin\psi_2(1) = -\cos\gamma$ and $v_2(r) = O(r^{-1})$ and $v_{2r}(r) = O(r^{-2})$ as $r \to \infty$.

For $I = [a, 1]$, by choosing

$$v_2(a) = \frac{1}{1 - a^2}\left(\frac{2\cos\gamma}{B} - \int_a^1 (1 - r^2)\frac{\sin\psi_2(r)}{\sqrt{1 - \sin^2\psi_2(r)}}\, dr\right), \tag{4}$$

v_2 will satisfy the volume condition $B\int_I r v_2\, dr = \cos\gamma$. With this choice v_2 will be nonnegative when $B \le 6$.

Proof. First consider $I = [a, 1]$. Since v_1 is nonnegative we have $\sin\psi_2 \geq 0$, which implies that v_2 is nondecreasing. Since v_1 is nondecreasing we have

$$\sin\psi_2 \leq \frac{Bv_1(r^2-a^2)}{2r} \leq \frac{Brv_1}{2}.$$

It follows that

$$\left(\frac{\sin\psi_2}{r}\right)_r = \frac{2}{r^2}\left(\frac{Brv_1}{2} - \sin\psi_2\right) \geq 0.$$

Thus, $(\sin\psi_2)/r \leq \cos\gamma$ or $\sin\psi_2 \leq r\cos\gamma \leq r$. Since $p/\sqrt{1-p^2}$ is increasing on $[0, 1)$, we have

$$\frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}} \leq \frac{r}{\sqrt{1-r^2}},$$

so $v_2(r) \leq v_2(a) + \sqrt{1-a^2} - \sqrt{1-r^2}$. Thus v_2 is defined and continuous on I. Requiring $B\int_I rv_2\,dr = \cos\gamma$, after changing the order of integration, results in (4). Now for $B \leq 6$, use

$$\frac{\sin\psi_2}{\sqrt{1-\sin^2\psi_2}} \leq \frac{r\cos\gamma}{\sqrt{1-r^2\cos^2\gamma}} \leq \frac{r\cos\gamma}{\sqrt{1-r^2}}$$

in (4) to see that

$$v_2(a) \geq \frac{\cos\gamma}{1-a^2}\left(\frac{2}{B} - \int_0^1 r\sqrt{1-r^2}\,dr\right) = \frac{\cos\gamma}{1-a^2}\left(\frac{2}{B} - \frac{1}{3}\right) \geq 0.$$

Thus v_2 is nonnegative.

Next consider $I = [1, \infty)$. Since v_1 is nonnegative, $\sin\psi_2 \leq 0$, which implies that v_2 is nonincreasing. From the volume condition on v_1, $\sin\psi_2(1) = -\cos\gamma$. From $(\sin\psi_2)_r = Bv_1 - (\sin\psi_2)/r \geq 0$, we get $\sin\psi_2 \geq -\cos\gamma$. Since $v_1 = O(r^{-3})$, $\sin\psi_2 = O(r^{-2})$, giving $v_{2r} = O(r^{-2})$ as $r \to \infty$. Since v_2 is nonincreasing and tends to zero, v_2 is nonnegative. From the formula for v_2, we see that $v_2 = O(r^{-1})$ as $r \to \infty$. As $(\sin\psi_2)_r(1) = Bv_1(1) + \cos\gamma > 0$, the integral for $v_2(1)$ is finite. Thus v_2 is continuous on I.

Finally, by the defining formulas, in all cases, $Nv_2 = Bv_1$ in the interior of I. □

For interior or annular capillary surfaces and $B \leq 6$, Theorem 1.1 provides a sequence of iterates $\{v_n\}$, where $Nv_{n+1} = Bv_n$ for $n \geq 0$. The simplest initial function is the constant function satisfying the volume condition

$$v_0 = \frac{2\cos\gamma}{B(1-a^2)}. \tag{5}$$

The properties of this sequence are explored in Section 2. An asymptotic expansion in powers of B is obtained. The theory is then applied to the interior problem and a formula of Rayleigh for measuring surface tension is proved.

The exterior problem is considered in Section 3. Two approximations are used to prove a result of Bruce Turkington [1980] on the asymptotic boundary height as B tends to zero.

An attractive feature of the method employed in this paper is its applicability to capillary problems with $\gamma = 0$. The general asymptotic series result in [Miersemann 1993] excludes the case $\gamma = 0$. However, for the interior problem, Miersemann [1994] has established an asymptotic expansion with $0 \le \gamma < \pi/2$.

The annular problem certainly merits further work. A start on this has been made by Alan Elcrat, Tae-Eun Kim and Ray Treinen [Elcrat et al. 2004].

2. Interior and annular capillary surfaces

The sequence of iterates $\{v_n\}$ for the interior and annular capillary problem (1) introduced after Theorem 1.1 has the properties listed in Theorem 2.3 below. The proof will make use of two lemmas whose proof is straightforward. Denote the inclination angles of two functions v and w defined on $[a, 1]$ by ψ_v and ψ_w, respectively.

Lemma 2.1. *Let $a < b < 1$. If $Nv < Nw$ for $a < r < b$ and $\psi_v(a) = \psi_w(a)$ then $\psi_v < \psi_w$, for $a < r \le b$. If $Nv < Nu$ for $b < r < 1$ and $\psi_v(1) = \psi_w(1)$, then $\psi_w < \psi_v$, for $b \le r < 1$.*

Lemma 2.2. *If $\psi_v < \psi_w$ on $(a, 1)$ and $\int_a^1 rv\,dr = \int_a^1 rw\,dr$ then there exists $b \in (a, 1)$ such that $v(b) = w(b)$ and $w(r) < v(r)$ for $r < b$ and $v(r) < w(r)$ for $r > b$.*

Theorem 2.3. *Let u be the solution to (1) and ψ its inclination angle. For $B \le 6$, the iterates provided by Theorem 1.1 with v_0 given by (5) have the following properties*:

$$\psi_0 < \psi_2 < \cdots < \psi < \cdots \psi_3 < \psi_1;$$

$$v_1(a) < v_3(a) < \cdots < u(a) < \cdots < v_2(a) < v_0 \quad \text{for } a < r < 1;$$

$$v_0 < v_2(1) < \cdots < u(1) < \cdots < v_3(1) < v_1(1),$$

$$|u - v_n| < C(\gamma, a)\left(B\frac{\sqrt{1-a^2}}{1+a^2}\right)^n, \quad \text{where } C(\gamma, a) = \frac{\sqrt{1-a^2\cos^2\gamma} - \sin\gamma}{\cos\gamma}.$$

Proof. From the defining equations, $\psi_1 > 0$ and $v_1 > 0$ on $(a, 1]$ and so $\sin\psi_2 > 0$ on $(a, 1]$.

Since u is positive, it follows that $\sin\psi = \frac{B}{r}\int_a^r su(s)\,ds > 0$ for $r > a$. Since v_0 is constant, $\psi_0 = 0$. Thus, $\psi_0 < \psi$.

We proceed to prove a number of statements of a recursive nature, using Lemmas 2.1 and 2.2. First we show that $\psi_{2k} < \psi$ implies that $\psi < \psi_{2k+1}$ for $k \ge 0$. By Lemma 2.2 there exists $b_{2k} \in (a, 1)$ with $v_{2k}(b_{2k}) = u(b_{2k})$, $u < v_{2k}$ for $r < b_{2k}$

and $u > v_{2k}$ for $r > b_{2k}$. Since $Nv_{2k+1} = Bv_{2k}$ and $Nu = Bu$, we conclude that $\psi < \psi_{2k+1}$ by Lemma 2.1 by arguing on the intervals $[a, b_{2k}]$ and $[b_{2k}, 1]$.

In a similar fashion, one proves that $\psi < \psi_{2k+1}$ implies that $\psi_{2k+2} < \psi$ for $k \geq 0$. Combining statements, we have $\psi_{2k} < \psi < \psi_{2k+1}$ for $k \geq 0$.

We know that $\psi_0 < \psi_2$ for $r > a$. Next we show that $\psi_{2k} < \psi_{2k+2}$ implies that $\psi_{2k+3} < \psi_{2k+1}$ for $k \geq 0$. By Lemma 2.2 there exists $c_k \in (a, 1)$ with $v_{2k}(c_k) = v_{2k+2}(c_k)$, $v_{2k+2} > v_{2k}$ for $r < c_k$ and $v_{2k+2} < v_{2k}$ for $r > c_k$. Using $Nv_{2k+3} = Bv_{2k}$ and $Nv_{2k+1} = Bv_{2k}$, we get $\psi_{2k+3} < \psi_{2k+1}$ by Lemma 2.1.

Likewise, one proves that $\psi_{2k+3} < \psi_{2k+1}$ implies that $\psi_{2k+4} > \psi_{2k+2}$ for $k \geq 0$. Combing statements gives that $\{\psi_{2k}\}$ is increasing and $\{\psi_{2k+1}\}$ is decreasing.

From $\psi_{2k} < \psi$ it follows that $u(a) < v_{2k}(a)$ and $v_{2k}(1) < u(1)$ by Lemma 2.2. From $\psi < \psi_{2k+1}$ it follows that $v_{2k+1}(a) < u(a)$ and $u(1) < v_{2k+1}(1)$ again by Lemma 2.2.

Similarly, $\psi_{2k} < \psi_{2k+2}$ implies that $v_{2k+2}(a) < v_{2k}(a)$ and $v_{2k}(1) < v_{2k+2}(1)$ for $k \geq 0$; and $\psi_{2k+3} < \psi_{2k+1}$ implies that $v_{2k+1}(a) < v_{2k+3}(a)$ and $v_{2k+3}(1) < v_{2k+1}(1)$ for $k \geq 0$. Thus $\{v_{2k+1}(a)\}$ is increasing, $\{v_{2k+1}(1)\}$ is decreasing, $\{v_{2k}(a)\}$ is decreasing and $\{v_{2k}(1)\}$ is increasing. The proof of the interleaving properties is complete.

Finally, we establish the error bound. Since $u(a) < v_0$ and $v_0 < u(1)$, and u is increasing, we have $|u - v_0| < u(1) - u(a) < v_1(1) - v_1(a)$. The latter expression can be estimated. By the defining equations we have

$$\sin \psi_1 = \frac{\cos \gamma}{1 - a^2} \frac{(r^2 - a^2)}{r} \quad \text{and} \quad v_1(1) - v_1(a) = \int_a^1 \frac{\sin \psi_1}{\sqrt{1 - \sin^2 \psi_1}}\, dr.$$

Using the inequality $\sin \psi_1 \leq r \cos \gamma$ to estimate the integral, we get

$$v_1(1) - v_1(a) \leq \int_a^1 \frac{r \cos \gamma}{\sqrt{1 - r^2 \cos^2 \gamma}}\, dr = C(\gamma, a).$$

Thus, $|u - v_0| < C(\gamma, a)$. This is the case $n = 0$ of the bound to be established and we proceed by induction. Assume

$$|u - v_n| < \mathcal{B}_n := C(\gamma, a)\left(B \frac{\sqrt{1 - a^2}}{1 + a^2}\right)^n.$$

From the defining equations for $\{v_n\}$ and the equation for u we have

$$\sin \psi - \sin \psi_{n+1} = \frac{B}{r} \int_a^r s(u(s) - v_n(s))\, ds \quad \text{or} \quad -\frac{B}{r} \int_r^1 s(u(s) - v_n(s))\, ds\,.$$

This gives $|\sin \psi - \sin \psi_{n+1}| < (\mathcal{B}_n B)/(2r) \min\{r^2 - a^2,\ 1 - r^2\}$. Using the fact that

$$\min\{r^2 - a^2, 1 - r^2\} \leq \frac{2(r^2 - a^2)(1 - r^2)}{1 + a^2},$$

we have

$$|\sin\psi - \sin\psi_{n+1}| < \frac{\mathcal{B}_n B}{1+a^2} r(1-r^2). \tag{6}$$

If $m := n+1$ is even, then since $\psi_m < \psi$ and $v_m(a) > u(a),\ v_m(1) < u(1)$,

$$|u - v_m| \le \max\{v_m(a) - u(a),\ u(1) - v_m(1)\} < (u(1) - v_m(1)) - (u(a) - v_m(a)).$$

Similarly, if m is odd, then $|u - v_m| < (v_m(1) - u(1)) - (v_m(a) - u(a))$. Thus $|u - v_m| < \left|\int_a^1 (u_r - v_{m\,r})\,dr\right| \le \int_a^1 |u_r - v_{m\,r}|\,dr$. We use the Mean Value Theorem to estimate the integrand, noting that

$$u_r = \frac{\sin\psi}{\sqrt{1-\sin^2\psi}} \quad \text{and} \quad v_{m\,r} = \frac{\sin\psi_m}{\sqrt{1-\sin^2\psi_m}}, \quad u_r - v_{m\,r} = \frac{\sin\psi - \sin\psi_m}{(1-\xi^2)^{3/2}},$$

where ξ is between $\sin\psi$ and $\sin\psi_m$. Using $\xi < \sin\psi_1 \le r$, we have $|u_r - v_{m\,r}| < |\sin\psi - \sin\psi_m|/(1-r^2)^{3/2}$. Combining this with previous bound (6), we have

$$|u - v_{n+1}| < \frac{\mathcal{B}_n B}{a^2+1} \int_a^1 \frac{r}{\sqrt{1-r^2}}\,dr = \mathcal{B}_n B \frac{\sqrt{1-a^2}}{1+a^2} = \mathcal{B}_{n+1}.$$

This completes the induction argument. □

The upper bound $\mathcal{B}_n = C(\gamma, a)\bigl(B\sqrt{1-a^2}/(1+a^2)\bigr)^n$ is at most B^n, so we have an upper bound independent of γ and a. For the interior problem, the result $v_1(0) < u(0)$ and $u(1) < v_1(1)$ was first proved in [Finn 1981] and the result $\psi < \psi_1$ was first proved in [Siegel 1989]. For the interior problem with $\gamma = 0$, Theorem 2.3 gives $|u - v_1| < B$, whereas [Siegel 1989] has the better estimate $|u - v_1| < B/3$.

The iterates $\{v_n\}$ can be used to establish an asymptotic expansion for u in powers of B. Denote differentiation with respect to B by D_B.

Theorem 2.4. *Let $0 \le \gamma < \pi/2$ and $0 < B \le 6$. The solution $u(r, B)$ to (1) has an asymptotic expansion in powers of B,*

$$u(r, B) = v_0 + u_0(r) + u_1(r)B + u_2(r)B^2 + \cdots,$$

where $u_n(r) = D_B^n w_k(r, 0)/n!$ with $w_k = v_k - v_0$ for $k > n \ge 0$. There are constants C_n such that $\bigl|u - \bigl(v_0 + u_0(r) + \cdots + u_n(r)B^n\bigr)\bigr| \le C_n B^{n+1}$ for $n \ge 0$.

Proof. The idea is to show that the w_n's have Taylor expansions in powers of B and combine that with Theorem 2.3. To do this we need to show that $D_B^{\ell} w_k$ exists and is continuous for $0 \le B \le 6,\ 0 \le \gamma \le \frac{\pi}{2}$ and $\ell \ge 0,\ k \ge 0$. The inclination angle for w_k is ψ_k since w_k differs by a constant from v_k. The w_k's are generated recursively by

$$
\begin{cases}
\sin\psi_{k+1} = \sin\psi_1 + \dfrac{B}{r}\displaystyle\int_a^r s w_k(s)\,ds,\\[2ex]
w_{k+1}(r) = w_{k+1}(a) + \displaystyle\int_a^r \frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds,\\[2ex]
w_{k+1}(a) = -\displaystyle\int_a^1 (1-s^2)\frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds,
\end{cases} \tag{7}
$$

for $k \ge 0$. We have $w_0 = 0$ and

$$\sin\psi_1 = \frac{\cos\gamma}{1-a^2}\frac{r^2-a^2}{r}.$$

From the volume condition for v_k it follows that

$$\int_a^1 r w_k\,dr = 0\,. \tag{8}$$

We will show by induction on k that $D_B^\ell w_k$ and $D_B^\ell \sin\psi_k$ are continuous for $\ell \ge 0$ and $D_B^\ell \sin\psi_k = O(1-r)$ for $\ell \ge 1$.

We will differentiate the recursion relation (7) repeatedly with respect to B, so we need the equality

$$D_B^\ell \frac{\sin\psi_k}{\sqrt{1-\sin^2\psi_k}} = \sum_{j=0}^{\ell} \frac{h_{\ell,j}}{(1-\sin^2\psi_k)^{(2j+1)/2}}, \tag{9}$$

where each $h_{\ell,j}$, for $\ell \ge 0$, is a homogeneous polynomial of degree $2j+1$ in $\sin\psi_k,\ D_B\sin\psi_k,\ \ldots,\ D_B^\ell\sin\psi_k$ which is of degree at least j in $D_B\sin\psi_k,\ \ldots,\ D_B^\ell\sin\psi_k$. This is seen by induction on ℓ. Statement (9) is true for $\ell = 0$. Assume it is true for ℓ; differentiation gives

$$D_B^{\ell+1}\frac{\sin\psi_k}{\sqrt{1-\sin^2\psi_k}} = \sum_{j=0}^{\ell} \frac{D_B h_{\ell,j}}{(1-\sin^2\psi_k)^{(2j+1)/2}} - \frac{(2j+1)h_{\ell,j}\sin\psi_k D_B\sin\psi_k}{(1-\sin^2\psi_k)^{(2j+3)/2}},$$

so $h_{\ell+1,\,j} = D_B h_{\ell,\,0}$,

$$h_{\ell+1,\,j} = D_B h_{\ell,\,j} + (2j-1)h_{\ell,\,j-1}\sin\psi_k D_B\sin\psi_k \quad \text{for } 1 \le j \le \ell,$$

and $h_{\ell+1,\,\ell+1} = (2\ell+1)h_{\ell,\,\ell}\sin\psi_k D_B\sin\psi_k$. Since $D_B h_{\ell,j}$ is homogeneous of degree $2j+1$ in $\sin\psi_k,\ D_B\sin\psi_k,\ \ldots,\ D_B^\ell\sin\psi_k$ and of degree at least j in $D_B\sin\psi_k,\ \ldots,\ D_B^\ell\sin\psi_k$, statement (9) holds with ℓ replaced by $\ell+1$.

Now, back to the induction argument on k. The case for $k = 0$ is true since $w_0 = 0,\ \sin\psi_0 = 0$. Assume the statement is true for k. Taking ℓ derivatives of (7) with respect to B we obtain

$$D_B^\ell \sin\psi_{k+1} = \frac{B}{r}\int_a^r s\,D_B^\ell w_k(s)\,ds + \frac{\ell}{r}\int_a^r s\,D_B^{\ell-1} w_k(s)\,ds,$$

$$D_B^\ell w_{k+1}(r) = D_B^\ell w_{k+1}(a) + \int_a^r D_B^\ell \frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds,$$

$$D_B^\ell w_{k+1}(a) = -\int_a^1 (1-s^2) D_B^\ell \frac{\sin\psi_{k+1}}{\sqrt{1-\sin^2\psi_{k+1}}}\,ds.$$

Differentiating the volume condition (8), we have $\int_a^1 r D_B^\ell w_k\,dr = 0$ for all $\ell \geq 0$. Thus we see that $D_B^\ell \sin\psi_{k+1}$ is continuous and $D_B^\ell \sin\psi_{k+1} = O(1-r)$ for $\ell \geq 1$. It follows that the integrals defining $D_B^\ell w_{k+1}$ are convergent, so $D_B^\ell w_{k+1}$ is continuous. The induction argument is complete.

Now, for a given positive n, take $k > n$. By Taylor's Theorem, $w_k(r, B) = w_k(r, 0) + D_B w_k(r, 0)B + \cdots + D_B^n w_k(r, 0)B^n + O(B^{n+1})$, and by Theorem 2.3, $u(r, B) = v_0 + w_k(r, B) + O(B^{k+1})$. Thus $u = v_0 + w_k(r, 0) + D_B w_k(r, 0)B + \cdots + D_B^n w_k(r, B) + O(B^{n+1})$. By the uniqueness of asymptotic expansions, this may be written $u(r, B) = v_0 + u_0(r) + u_1(r)B + u_2(r)B^2 + \cdots + u_n(r)B^n + O(B^{n+1})$. $\square$

Example 2.5. As an example of Theorem 2.4, consider the interior capillary problem (1) with $a = 0$ and $\gamma = 0$. Then $v_0 = 2/B$, $\sin\psi_1 = r$, $w_1 = \frac{2}{3} - \sqrt{1-r^2}$, so $u = 2/B + \frac{2}{3} - \sqrt{1-r^2} + O(B)$. Similarly, $\sin\psi_2 = r + \frac{1}{3}(B/r)\big((1-r^2)^{3/2} + r^2 - 1\big)$ so that $w_2(r, 0) = \frac{2}{3} - \sqrt{1-r^2}$ and $D_B w_2(r, 0) = -\frac{1}{6} + \frac{1}{3}\ln\big(1 + \sqrt{1-r^2}\,\big)$, giving

$$u(r, B) = \frac{2}{B} + \frac{2}{3} - \sqrt{1-r^2} + \Big(-\frac{1}{6} + \frac{1}{3}\ln\big(1+\sqrt{1-r^2}\,\big)\Big)B + O(B^2).$$

Setting $r = 0$, we have $u(0, B) = 2/B - \frac{1}{3} + \frac{1}{3}\big(\ln 2 - \frac{1}{2}\big)B + O(B^2)$. Inverting this relationship and setting $u_0 = u(0, B)$, we obtain

$$B = \frac{2}{u_0} - \frac{2}{3u_0^2} + \frac{\frac{4}{3}\big(\ln 2 - \frac{1}{2}\big) + \frac{2}{9}}{u_0^3} + O\Big(\frac{1}{u_0^4}\Big) \quad \text{as } u_0 \to \infty.$$

This is a formula due to Rayleigh [1915]. It is the basis for the technique of measuring surface tension by means of the rise of liquid in a narrow tube.

3. Exterior capillary surface

In the exterior case, since the domain is unbounded, we must proceed differently in finding an initial approximation v_1.

Set $v_1 = AK(r)$, where $K(r) = (1/\sqrt{B})K_0(\sqrt{B}r)$ (K_0 being a modified Bessel function of the second kind) and A is a positive constant. We will make use of the fact [Siegel 1980] that v_1, which satisfies $v_{1\,rr} + v_{1\,r}/r = Bv_1$ for $r > 0$, is a supersolution: $Nv_1 < Bv_1$ for $r > 0$. The Bessel function $K_0(r)$ has the following properties [Lebedev 1965]:

$$K_0(r) > 0, \quad K_0'(r) < 0, \quad K_0(r) \sim \frac{e^{-r}}{\sqrt{2\pi r}} \text{ as } r \to \infty, \quad K_0(r) \sim -\ln r \text{ as } r \to 0.$$

We also need that $(rK_0')' = rK_0$ for $r > 0$ and $K_0'(r) \sim -r^{-1}$ as $r \to 0$. Now choose A so that $B\int_1^\infty rv_1\,dr = \cos\gamma$: namely, $A = -(\cos\gamma)/K_0'(\sqrt{B})$.

Theorem 3.1. *Let $v_1(r) = AK(r)$ be as chosen above and let v_2 be determined according to Theorem 1.1, so that $Nv_2 = Bv_1$, $v_2(r)$, $v_{2r}(r) \to 0$ as $r \to \infty$. Then $\psi_2(r) < \psi(r)$ for $r > 1$, $\psi_2(1) = \psi(1) = \gamma - \pi/2$ and $v_1(1) < u(1) < v_2(1)$. It follows that $u(1) = -\cos\gamma \ln\sqrt{B} + O(1)$ as $B \to 0$.*

Proof. By Theorem 1.1, $\psi_2(1) = \psi(1) = \gamma - \pi/2$ and $v_2(r)$, $v_{2r}(r) \to 0$ as $r \to \infty$.

If $v_1(1) \geq u(1)$, then $v_1(r) > u_1(r)$ for $r > 1$ by the comparison principle. This contradicts the volume condition. Thus $v_1(1) < u(1)$. Note that

$$v_1(1) = -\frac{K_0(\sqrt{B})\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})} = -\cos\gamma \ln\sqrt{B} + O(1) \text{ as } B \to 0. \tag{10}$$

Also, because of the volume condition, there exists a $b > 1$ so that $v_1(b) = u(b)$. Since v_1 is a supersolution, $v_1(r) > u(r)$ for $r > b$ and $v_1(r) < u(r)$ for $r < b$. This implies that $Nv_2 < Nu$ for $r < b$ and $Nv_2 > Nu$ for $r > b$. Using that $\psi_2(1) = \psi(1)$, $r\sin\psi_2(r)$, $r\sin\psi(r) \to 0$ as $r \to \infty$ and integrating on $[1, b]$ and $[b, \infty]$ gives that $\sin\psi_2(r) < \sin\psi(r)$ for $r > 1$. Thus $\psi_2(r) < \psi(r)$ for $r > 1$ or, equivalently, $v_{2r} < u_r$ for $r > 1$. Using that $u(r)$, $v_2(r) \to 0$ as $r \to \infty$, and integrating on $[1, \infty)$, gives that $u(1) < v_2(1)$.

Finally, we have

$$r\sin\psi_2(r) = -B\int_r^\infty sv_1(s)\,ds = -rv_{1r}(r),$$

so $\sin\psi_2 = AK_0'(\sqrt{B}r)$. Using that $v_{2r} = \sin\psi_2/\sqrt{1-\sin^2\psi_2}$ and integrating on $[1, \infty)$ gives

$$v_2(1) = \frac{\cos\gamma}{K_0'(\sqrt{B})}\int_1^\infty \frac{K_0'(\sqrt{B}r)}{\sqrt{1-\left(\dfrac{\cos\gamma}{K_0'(\sqrt{B})}K_0'(\sqrt{B}r)\right)^2}}\,dr. \tag{11}$$

We will show that there is an upper bound on $v_2(1)$ which is asymptotically the same as (10). Change variables in the integral with the substitution $s = \sqrt{B}r$ and write the integral as the sum of two terms, where δ is an arbitrary fixed positive number: $v_2(1) = I_1 + I_2$, $I_1 = \int_{\sqrt{B}}^{\delta} F\,ds$, $I_2 = \int_{\delta}^{\infty} F\,ds$, where

$$F = \frac{\cos\gamma}{\sqrt{\left(\dfrac{\sqrt{B}K_0'(\sqrt{B})}{K_0'(s)}\right)^2 - B\cos^2\gamma}} < F_1 = \frac{\cos\gamma}{\sqrt{s^2 - B\cos^2\gamma}}$$

and

$$F = \frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0'(s)}{\sqrt{1-\left(\dfrac{\cos\gamma K_0'(s)}{K_0'(\sqrt{B})}\right)^2}} < F_2 = \frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0'(s)}{\sqrt{1-\left(\dfrac{\cos\gamma K_0'(\delta)}{K_0'(\sqrt{B})}\right)^2}}.$$

The upper bound F_1 was obtained by using that $(rK_0')' = rK_0 > 0$, so that

$$\left|\sqrt{B}K_0'(\sqrt{B})\right| > |sK_0'(s)|$$

for $s > \sqrt{B}$. Using the upper bounds F_1 and F_2 for the integrals I_1 and I_2, we obtain

$$\begin{aligned} I_1 &< \cos\gamma\Big(\ln\big(\delta + \sqrt{\delta^2 - B\cos^2\gamma}\big) - \ln\big(\sqrt{B}(1+\sin\gamma)\big)\Big) \\ &= -\cos\gamma\ln\sqrt{B} + O(1), \\ I_2 &= -\frac{\cos\gamma}{\sqrt{B}K_0'(\sqrt{B})}\frac{K_0(\delta)}{\sqrt{1-\left(\dfrac{\cos\gamma K_0'(\delta)}{K_0'(\sqrt{B})}\right)^2}} = O(1). \end{aligned}$$

Thus $u(1) < v_2(1) = I_1 + I_2 < -\cos\gamma\ln\sqrt{B} + O(1)$. Combining this with the lower bound (10), we have that $u(1) = -\cos\gamma\ln\sqrt{B} + O(1)$ as $B \to 0$. □

Translating [Turkington 1980, Theorem 3.3] to the notation of this paper gives $u(1) \sim -\cos\gamma\ln\sqrt{B}$ as $B \to 0$. Theorem 3.1 gives a better estimate of the error.

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CONVEX, ROTATIONALLY SYMMETRIC LIQUID BRIDGES BETWEEN SPHERES

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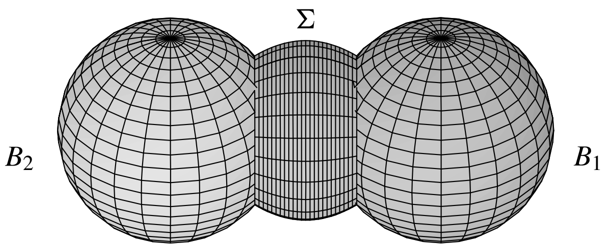
A liquid bridge between two balls will have a free surface which has constant mean curvature, and the angles of contact between the free surface and the fixed surfaces of the balls will be constant (although there might be two different contact angles: one for each ball). If we consider rotationally symmetric bridges, the free surface must be a Delaunay surface, which may be classified as a unduloid, a nodoid, or a catenoid, with spheres and cylinders as special cases. In this paper, it is shown that a convex unduloidal bridge between two balls is a constrained local energy minimum for the capillary problem, and a convex nodoidal bridge between two balls is unstable.

1. Introduction

The stability and energy minimality of a liquid bridge between parallel planes has been well studied [Finn and Vogel 1992; Vogel 1987; 1989; 2002; Zhou 1997]. That of the related problem of a liquid bridge between fixed balls, as in the figure,



has been studied less (but see [Basa et al. 1994; Vogel 2005; Vogel 1999]). We give a simple way of determining if a convex, rotationally symmetric bridge between fixed balls is an energy minimum. Namely, if a convex bridge between spheres is a section of an unduloid, it is a constrained local energy minimum, and if it is a section of a nodoid, it is unstable, and in particular not an energy minimum. (For

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rotationally symmetric bridges, we will use "convex" to mean that the profile curve of the free surface is a convex function.)

Someone familiar with [Vogel 1989] might be suspicious of this claim, because it is shown there that convex bridges between planes are always stable. How could reducing the radius of the spheres from infinity to a finite amount change the behavior so drastically? The resolution of this apparent paradox is that in looking at bridges between parallel planes, one deals with stability or energy minimality modulo translations parallel to the planes: there are perturbations which are automatically energy neutral. Changing the fixed surfaces from planes to spheres will change the boundary contribution of the relevant quadratic form $\mathcal{M}$, defined in (1–2), and in particular the value of the quadratic form as applied to the perturbations which were energy neutral for the bridge between planes. This is in fact the key point of the paper. If the bridge is a section of a nodoid, then in changing the fixed surfaces from planes to spheres, the energy neutral perturbations change to energy reducing perturbations, causing instability. On the other hand, if the bridge is a section of an unduloid, then in changing the fixed surfaces from planes to spheres, the energy neutral perturbations change to energy increasing ones, which we will show implies that the bridge is a constrained local energy minimum.

Definitions. In considering the stability and energy minimality of a liquid bridge between solid balls, some concepts from the general theory of capillary surfaces must be recalled [Finn 1986; Vogel 2000; Vogel 2002]. Suppose that Γ is the boundary of a fixed solid region in space, and that we put a drop of liquid in contact with Γ. Let Ω be the region in space occupied by the liquid, and Σ the free boundary of Ω (the part of $\partial\Omega$ not contained in Γ). In the absence of gravity or other external potentials, the shape of the drop results from minimizing the functional

$$\mathscr{E}(\Omega) = |\Sigma| - c|\Sigma_1|, \tag{1–1}$$

where $|\Sigma|$ is the area of the free surface of the drop, $|\Sigma_1|$ is the area of the region on Γ wetted by the drop, and $c \in [-1, 1]$ is a material constant. The minimization is under the constraint that the volume of the drop is fixed. The first-order necessary conditions for a drop to minimize (1–1) are that the mean curvature of Σ be a constant H (this is a Lagrange multiplier arising from the volume constraint) and that the angle between the normals to Σ and to Γ along the curve of contact be constantly $\gamma = \arccos c$ (see [Finn 1986]).

A capillary surface Σ is a *constrained local energy minimum* if it is the free boundary of a drop Ω such that $\mathscr{E}(\Omega) < \mathscr{E}(\Omega')$ for any comparison drop Ω' near (but not equal to) Ω in an appropriate sense, and containing the same volume of liquid. The question of what sense of "nearness" is appropriate is a complex one, but one approach is based on curvilinear coordinates [Vogel 2000].

In the common special case that there is a group of symmetries taking Γ to itself, we say that Σ is a *constrained local energy minimum modulo symmetries* if $\mathscr{E}(\Omega) \le \mathscr{E}(\Omega')$ for comparison drops Ω' that are near Ω, and if $\mathscr{E}(\Omega) = \mathscr{E}(\Omega')$ implies that Ω' is obtained by applying an element of the symmetry group to Ω. The specific example that we will deal with in this paper is that of a liquid bridge between parallel planes. No bridge could be a constrained local energy minimum, since translations parallel to the planes leave energy unchanged. However, in certain circumstances one can show that a given bridge is a minimum modulo these translations: any nearby bridge with the same energy (and volume) will be a translation of the original one [Vogel 2002].

Suppose that $\Sigma = \Sigma(0)$ is embedded in a smoothly parameterized family of drops $\Sigma(\varepsilon)$, all of which contain the same volume. If $(d^2/d\varepsilon^2)\mathscr{E}(\Sigma(\varepsilon))$ is negative at $\varepsilon = 0$ for that family, Σ is said to be *unstable*. Otherwise, Σ is *stable*.

The quadratic form related to stability and energy minimality is

$$\mathcal{M}(\phi,\phi) = \iint_\Sigma |\nabla\phi|^2 - |S|^2\phi^2\, d\Sigma + \oint_\sigma \rho\phi^2\, d\sigma. \tag{1–2}$$

Here $|S|^2$ is the square of the norm of the second fundamental form of Σ. (In terms of mean curvature H and Gaussian curvature K we have $|S|^2 = 2(2H^2 - K)$, and in terms of the principal curvatures, $|S|^2 = k_1^2 + k_2^2$.) We write σ for $\partial\Sigma$. The coefficient ρ is given by

$$\rho = \kappa_\Sigma \cot\gamma - \kappa_\Gamma \csc\gamma, \tag{1–3}$$

where κ_Σ is the curvature of the curve $\Sigma \cap \Pi$ and κ_Γ is the curvature of $\Gamma \cap \Pi$, if Π is a plane normal to the contact curve $\partial\Sigma$. These planar curvatures are signed: in Figure 2, left, both κ_Σ and κ_{Γ_n} are negative.

We will denote the subspace of $H^1(\Sigma)$ of all ϕ for which $\iint_\Sigma \phi d\Sigma = 0$ by $1^\perp$, since this subspace is the collection of functions which are perpendicular to the constant function 1 in the H^1 inner product. The relationship between $\mathcal{M}$ and stability is that Σ is stable if and only if $\mathcal{M}(\phi,\phi) \ge 0$ for all $\phi \in 1^\perp$. If Σ is a local energy minimum or a local energy minimum mod symmetries, then Σ is stable. However, stability does not imply that Σ is any sort of local energy minimum. It is not known whether the stronger condition $\mathcal{M}(\phi,\phi) > 0$ for all nontrivial $\phi \in 1^\perp$ is enough to imply that a capillary surface is some sort of energy minimum. (See [Zhou 1997, Editorial comment] and [Vogel 2000] for a discussion of this point. If the contact curves are "pinned" rather than free to move on Γ, the strengthened condition will imply energy minimality; see [Grosse-Brauckmann 1996].) In [Vogel

2000], it was shown that if for some $\varepsilon > 0$, we have $\mathcal{M}(\phi, \phi) \geq \varepsilon \, \|\phi\|^2$ holding on $1^\perp$, where $\|\cdot\|$ is the $H^1(\Sigma)$ norm, then Σ is a volume constrained local minimum for energy. If $\mathcal{M}(\phi, \phi) \geq \varepsilon \, \|\phi\|^2$ on a subspace for an $\varepsilon > 0$, $\mathcal{M}$ is said to be *strongly positive* on that subspace.

The quadratic form $\mathcal{M}$ is analyzed in [Vogel 2000; 2002] by considering an eigenvalue problem arising from integration by parts. Define the differential operator $\mathcal{L}$ by

$$\mathcal{L}(\psi) = -\Delta\psi - |S|^2\psi,$$

where Δ is the Laplace–Beltrami operator on Σ. The eigenvalue problem we study is given by

$$\begin{aligned} \mathcal{L}(\psi) &= \lambda\psi && \text{on } \Sigma, \\ \boldsymbol{b}(\psi) \equiv \psi_1 + \rho\psi &= 0 && \text{on } \partial\Sigma, \end{aligned} \tag{1–4}$$

where ψ_1 is the outward normal derivative of ψ. If the eigenvalue problem has no nonpositive eigenvalues, the bridge is stable, and in fact a constrained local energy minimum. If there are two or more negative eigenvalues, then the bridge is unstable. If there is one negative eigenvalue, and the rest are positive, then there is a further condition which must be checked to see if the bridge is stable (see [Vogel 2005; Vogel 1987]). In [Vogel 2002] it is shown that a bridge between parallel planes must always have zero as a double eigenvalue, corresponding to energy neutral translations. The relationship between the bilinear form $\mathcal{M}$ and the operator $\mathcal{L}$ is that

$$\mathcal{M}(\phi, \psi) = \iint_\Sigma \phi\mathcal{L}(\psi)\, d\Sigma + \oint_\sigma \phi\boldsymbol{b}(\psi)\, d\sigma, \tag{1–5}$$

after an integration by parts.

This general theory must be modified when we consider bridges between fixed balls, at least when we want to allow for different contact angles on the different balls B_1 and B_2. In that case, there will be two material constants c_1 and c_2, and the energy functional will be

$$\mathscr{E}(\Omega) = |\Sigma| - c_1|\Sigma_1| - c_2|\Sigma_2|,$$

where Σ_1 and Σ_2 are the wetted regions on B_1 and B_2 respectively. The contact angles with the B_i will be $\gamma_i = \arccos c_i$. The bilinear form $\mathcal{M}$ must also be modified. If we write σ_i for the curve of contact of Σ with B_i, we have

$$\mathcal{M}(\phi, \phi) = \iint_\Sigma |\nabla\phi|^2 - |S|^2\phi^2\, d\Sigma + \oint_{\sigma_1} \rho_1\phi^2\, d\sigma + \oint_{\sigma_2} \rho_2\phi^2\, d\sigma. \tag{1–6}$$

The boundary conditions for the eigenvalue problem for the operator $\mathcal{L}$ must similarly be adjusted.

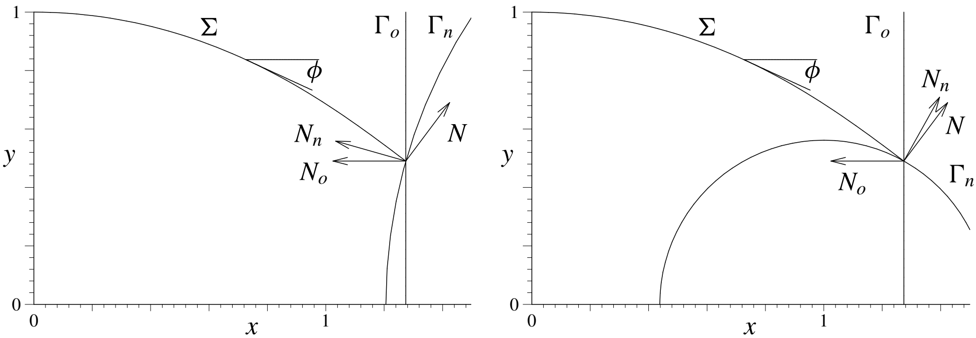
2. Comparing bridges between planes and bridges between spheres

In the absence of gravity, a capillary surface is a surface Σ of constant mean curvature which makes a constant contact angle γ with a fixed surface Γ. Suppose that we have such a surface, and, while keeping Σ and its boundary fixed, we replace Γ by a new surface Γ', which still contains $\partial\Sigma$. Suppose this new surface Γ' makes a new constant contact angle γ' with Σ. The general question is how this will effect stability or energy minimality of Σ. At first glance, this question may seem artificial. However, rotationally symmetric liquid bridges between solid spheres are the same surfaces as those between parallel planes. Since much is known about stability of bridges between planes, our hope is that from this knowledge we can infer some information about stability of bridges between spheres.

From (1–2), we can conclude that changing the fixed surface Γ may change the value of ρ, but that the surface integral in $\mathcal{M}$ remains unchanged. It therefore makes sense to compare ρ values for bridges between planes and bridges between spheres. It is known (see [Vogel 1987]) that a bridge between parallel planes must be a surface of revolution. (However, there are bridges between spheres which are not surfaces of revolution. See Note 1.) Surfaces of revolution having constant mean curvature are called Delaunay surfaces. Their profile curves may be obtained by rolling a conic section along an axis and tracing the path of a focus. Rolling an ellipse results in a curve called an undulary, and the resulting surface is an unduloid. Rolling a hyperbola yields a nodary as a profile curve and a nodoid as the surface. Parabolas give catenaries and catenoids, cylinders come from rolling circles, and spheres come from "rolling" line segments. See [Kenmotsu 2003] for more information about Delaunay surfaces.

To make things specific, consider the following situation. Suppose that we have a Delaunay surface generated by a profile passing through the point (x_0, y_0), and that the axis of rotation of the Delaunay surface is the x axis. Suppose that κ_Σ is the curvature of the profile at the point (x_0, y_0) (this agrees with the terminology in (1–3)). The bridge is only part of the Delaunay surface, so let's assume that the bridge lies to the left of the plane $x = x_0$. The profile curves of one case is illustrated in Figure 2, left, where the center of the sphere is to the right of Γ_o. The other case, where the center is to the left of Γ_o, but the sphere still does not cross the free surface Σ, is in Figure 2, right. The point of the following calculation is to determine how the value of ρ along the curve of contact will change in going from the Delaunay surface forming a bridge between planes to the Delaunay surface forming a bridge between spheres.

Lemma 2.1. *Suppose that the fixed surface that the bridge Σ contacts is the plane $x = x_0$, whose profile is labeled Γ_o in Figure 2, and let γ_o be the contact angle between the normals N and N_o to Σ and Γ_o, respectively. Let ρ_o be the value of ρ*



Case 1 Case 2

Figure 2. Changing the fixed surface.

for this configuration. Now consider replacing the plane by a sphere going through (x_0, y_0), *whose profile is labeled* Γ_n *in Figure 2.* (The subscripts o and n stand for "old" and "new".) *Assume that this sphere has radius a and center on the x-axis. The contact angle has changed to* γ_n, *and the value of* ρ *has changed to* ρ_n. *Set* $\eta = \gamma_o - \gamma_n$. *Then*

$$\rho_n - \rho_o = \frac{1}{(\cot\eta - \cot\gamma_o)\sin^2\gamma_o}\left(\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}\right), \tag{2–1}$$

Proof. Case 1: We have $\rho_o = \kappa_\Sigma \cot\gamma_o$, since the curvature of the fixed surface is zero. Now replace the plane by Γ_n. The contact angle is now the angle between N and N_n, and has changed to $\gamma_n = \gamma_o - \eta$, where $\eta = \arcsin(y_0/a)$. Therefore the new value of ρ is

$$\rho_n = \kappa_\Sigma \cot(\gamma_o - \eta) + \frac{1}{a}\csc(\gamma_o - \eta),$$

since the sectional curvature of the fixed surface has decreased from 0 to $-1/a$. Trigonometric identities for $\cot(A - B)$ and $\csc(A - B)$ give, as desired,

$$\begin{aligned}
&\rho_n - \rho_o \\
&\quad = \kappa_\Sigma\left(\frac{\cot\gamma_o\cot\eta + 1}{\cot\eta - \cot\gamma_o} - \cot\gamma_o\right) + \frac{1}{a\sin\gamma_o\sin\eta}\left(\frac{1}{\cot\eta - \cot\gamma_o}\right)\\
&\quad = \kappa_\Sigma\left(\frac{\cot\gamma_o\cot\eta + 1 - \cot\gamma_o\cot\eta + \cot^2\gamma_o}{\cot\eta - \cot\gamma_o}\right) + \frac{1}{\sin\gamma_o\sin\eta}\left(\frac{1}{\cot\eta - \cot\gamma_o}\right)\\
&\quad = \frac{1}{\cot\eta - \cot\gamma_o}\left(\kappa_\Sigma\csc^2\gamma_o + \frac{1}{a\sin\gamma_o\sin\eta}\right)\\
&\quad = \frac{1}{(\cot\eta - \cot\gamma_o)\sin^2\gamma_o}\left(\kappa_\Sigma + \frac{\sin\gamma_o}{y_0}\right).
\end{aligned}$$

The calculation for case 2 is similar, except that now $\eta = \pi/2 - \arcsin(y_0/a)$, and is omitted. $\square$

Now, suppose that we have a bridge with a convex profile. In both case 1 and case 2, one can show that $0 < \eta < \gamma_o < \pi$, so that $\cot\eta - \cot\gamma_o > 0$. Therefore, the sign of $\kappa_\Sigma + (\sin\gamma_o)/y_0$ will determine whether the value of ρ has increased or decreased. From this we will be able to determine stability of convex bridges between spheres. We first need to recall some facts about the profiles of Delaunay surfaces.

If $(x(s), y(s))$ is an arclength parametrization of the profile of a Delaunay surface, with inclination angle $\phi(s)$ (see Figure 2 for ϕ) and mean curvature H, we have the following system of ordinary differential equations (see [Vogel 1989]):

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi, \qquad \frac{d\phi}{ds} = \frac{\cos\phi}{y} + 2H. \tag{2–2}$$

From this system, it's easy to see that

$$\frac{d}{ds}\left(y\cos\phi + Hy^2\right) = 0,$$

so that $y\cos\phi + Hy^2$ is constant along Delaunay profiles. The value of this constant has a geometric meaning.

Lemma 2.2. *Let the constant value of $y\cos\phi + Hy^2$ on the profile of a Delaunay surface be called c. If $Hc > 0$, the profile is a nodary, and if $Hc < 0$ the profile is an undulary.*

Proof. This is already known (see [Oprea 2000], for example), but I was not able to locate a proof in the literature, and it is not hard to present one. It is easy to check that $c = 0$ for a sphere, so this case will not occur. Substitute the definition of c into the last equation in (2–2) to see that

$$\frac{d\phi}{ds} = H + \frac{c}{y^2}.$$

If H and c have the same signs, $\phi(s)$ is monotone on the profile. This rules out undularies, and a catenary is not possible for $H \neq 0$, hence we must have a nodary. On the other hand, suppose that H and c have different signs. From the definition of c it is clear that $\phi = \pi/2$ cannot be on the profile. The only possibility in this case is an undulary (of which a circular cylinder is a special case). $\square$

Lemma 2.3. *Suppose that we have a rotationally symmetric bridge Σ with a convex profile contacting a plane as in Figure 2. Suppose that we replace the plane Γ_o with a sphere Γ_n as in the figure. If Σ is a portion of an unduloid, then $\rho_n > \rho_o$, and if Σ is a portion of a nodoid, then $\rho_n < \rho_o$. In particular, if we take a convex bridge between parallel planes and replace the planes by spheres, both values of*

ρ in (1–6) will increase if Σ is a portion of an unduloid, and decrease if Σ is a portion of a nodoid.

Proof. As noted before, the sign of the change of ρ is the same as the sign of $\kappa_\Sigma + (\sin\gamma_o)/y_0$. But this last quantity will be equal to

$$\frac{d\phi}{ds} + \frac{\cos\phi_o}{y_0} = 2\left(\frac{\cos\phi_o}{y_0} + H\right) = \frac{2}{y_0^2}\left(y_0\cos\phi_o + Hy_0^2\right),$$

where ϕ_o is the inclination angle of the profile at the right endpoint, so $\phi_o = \pi/2 - \gamma_o$. Thus

$$\kappa_\Sigma + \frac{\sin\gamma_o}{y_0} = 2\frac{c}{y_0^2},$$

where c has the same meaning as in Lemma 2.2. From the last equation in (2–2), it is clear that for a convex profile we have $H < 0$, so $c > 0$ for an undulary and $c < 0$ for a nodary. □

Theorem. *Suppose that Σ is a rotationally symmetric bridge between spheres, whose profile is given as a solution to (2–2), and that $d\phi/ds < 0$ and $dx/ds > 0$ on the bridge profile including the endpoints. If Σ is a section of a nodoid, it is unstable. If Σ is a section of an unduloid or a sphere, it is stable, and is in fact a local constrained energy minimum.* (We do not assume that the spheres have equal radius or that the contact angles are equal.)

Proof. It is known that for bridges between parallel planes, a convex bridge is a constrained local energy minimum modulo translations in directions parallel to the planes [Vogel 2002; [1989]]. In the proof in [Vogel 2002], we considered the quadratic form

$$\mathcal{M}_o(\phi,\phi) = \iint_\Sigma |\nabla\phi|^2 - |S|^2\phi^2\,d\Sigma + \oint_{\sigma_1}\rho_{o,1}\phi^2\,d\sigma + \oint_{\sigma_2}\rho_{o,2}\phi^2\,d\sigma.$$

We write $\rho_{o,i}$ for the old value of ρ_i as in Lemma 2.1. It was shown that this is strongly positive (i.e., that there is an $\varepsilon > 0$ so that $\mathcal{M}_o(\phi,\phi) \ge \varepsilon\,\|\phi\|^2$, where $\|\cdot\|$ is the $H^1(\Sigma)$ norm) on the subspace of $1^\perp$ of ϕ's which are also orthogonal in $H^1(\Sigma)$ to infinitesimal translations parallel to the fixed planes. This strong positivity leads directly to the statement about energy minimality. However, if μ corresponds to a translation parallel to the fixed planes, we must have $\mathcal{M}_o(\mu,\mu) = 0$, since $\mathcal{M}$ is the second Fréchet derivative of energy, and energy is unchanged by translations. In fact, the eigenvalue problem (1–4) will have a single negative eigenvalue, 0 as an eigenvalue of multiplicity two, and all other eigenvalues positive. Using the same notation as in [Vogel 2002], we let μ_1 and μ_2 span the subspace of infinitesimal translations parallel to the fixed planes. With the parametrization of Σ given in

Section 5 of that paper, we have

$$\mu_1(u,v) = \frac{\cos v}{\sqrt{1+(f')^2}} \quad \text{and} \quad \mu_2(u,v) = \frac{\sin v}{\sqrt{1+(f')^2}}$$

(the profile being given as the graph of $r = f(u)$). These functions also span the kernel of the eigenvalue problem (1–4).

If the corresponding new values $\rho_{n,i}$ satisfy $\rho_{n,1} < \rho_{o,1}$ and $\rho_{n,2} < \rho_{o,2}$, the bridge is unstable for the new configuration of fixed surfaces, and hence not a constrained local energy minimum. The reason is simple: we must have $\mathcal{M}_n(\phi,\phi) < \mathcal{M}_o(\phi,\phi)$ for any ϕ which is nonzero on a set of positive measure on the boundary of Σ. In particular, $\mathcal{M}_n(\mu_1,\mu_1) < 0$. But translations in the original configuration also conserve volume, so $\iint_\Sigma \mu_1 \, d\Sigma = 0$, i.e., $\mu_1 \in 1^\perp$. The second variation of energy is negative for this infinitesimally volume-conserving perturbation, so we have instability in the case that $\rho_{n,i} < \rho_{o,i}$. From Lemma 2.3, we therefore have instability when the bridge is a portion of a nodoid.

If $\rho_{n,i} > \rho_{o,i}$, so the bridge is a portion of an unduloid, we expect the new configuration to be more stable in some sense than the old one. In fact, we will see that in this case $\mathcal{M}_n$ is strongly positive on all of $1^\perp$. For suppose that this is not the case. We certainly know that $\mathcal{M}_n$ is nonnegative on this space, since $\mathcal{M}_o$ is nonnegative on this space and $\mathcal{M}_n(\phi,\phi) \geq \mathcal{M}_o(\phi,\phi)$. So, if $\mathcal{M}_n$ is not strongly positive on $1^\perp$, there must exist a sequence $\{\phi_k\}$ in $1^\perp$ for which $\|\phi_k\| = 1$ and $\lim_{k\to\infty} \mathcal{M}_n(\phi_k,\phi_k) = 0$.

Projecting this sequence onto the span of μ_1 and μ_2, we write

$$\phi_k = a_k\mu_1 + b_k\mu_2 + \phi_k^*.$$

Note that since $\iint_\Sigma \mu_i \, d\Sigma = 0$, we have $\phi_k^* \in 1^\perp$. By going to a subsequence, we may assume that $\{a_k\}$ and $\{b_k\}$ converge to a and b, respectively. Now,

$$\begin{aligned}
&\mathcal{M}_n(\phi_k,\phi_k) \\
&\quad \geq \mathcal{M}_o(\phi_k,\phi_k) \\
&\quad = \mathcal{M}_o\left(a_k\mu_1 + b_k\mu_2 + \phi_k^*,\ a_k\mu_1 + b_k\mu_2 + \phi_k^*\right) \\
&\quad = \mathcal{M}_o(a_k\mu_1 + b_k\mu_2,\ a_k\mu_1 + b_k\mu_2) + 2\mathcal{M}_o\left(a_k\mu_1 + b_k\mu_2,\ \phi_k^*\right) + \mathcal{M}_o\left(\phi_k^*,\phi_k^*\right) \\
&\quad = \mathcal{M}_o\left(\phi_k^*,\phi_k^*\right) \geq \varepsilon \left\|\phi_k^*\right\|^2,
\end{aligned}$$

where the terms $\mathcal{M}_o(a_k\mu_1 + b_k\mu_2, a_k\mu_1 + b_k\mu_2)$ and $\mathcal{M}_o\left(a_k\mu_1 + b_k\mu_2, \phi_k^*\right)$ vanish by (1–5) and the fact that $\mathcal{L}(\mu_i) = 0$ on Σ, $\boldsymbol{b}(\mu_i) = 0$ on σ.

From the inequality above and because $\mathcal{M}_n(\phi_k,\phi_k)$ converges to 0, we conclude that $\lim_{k\to\infty}\{\phi_k^*\} = 0$ in $H^1(\Sigma)$; thus

$$\lim_{k\to\infty} \phi_k = a\mu_1 + b\mu_2$$

in $H^1(\Sigma)$. An immediate consequence is that a and b cannot both be zero, since all of the ϕ_k's have length 1 in $H^1(\Sigma)$. This leads to a contradiction. Since $\mathcal{M}_n(\phi, \phi)$ is continuous on $H^1(\Sigma)$,

$$\mathcal{M}_n(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) = \lim_{k\to\infty} \mathcal{M}_n(\phi_k, \phi_k) = 0.$$

However, $a\mu_1 + b\mu_2$ is not identically zero on $\partial\Sigma$. The reason is that it represents the component normal to Σ of a nontrivial translation parallel to the original fixed planes. Therefore

$$\mathcal{M}_n(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) > \mathcal{M}_o(a\mu_1 + b\mu_2, a\mu_1 + b\mu_2) = 0,$$

a contradiction. Thus $\mathcal{M}_n$ is strongly positive on all of $1^\perp$, proving that a bridge between spheres which is convex and part of an unduloid must be a local energy minimum. □

Note 1. No claim about energy minimality was made in the case that the bridge is a section of a sphere. In this case, the spectrum of the eigenvalue problem (1–4) remains the same as in the problem of a bridge between parallel planes, so that 0 is an eigenvalue of multiplicity two. What is happening at the symmetrically placed spherical bridge is that there is a "wine cup" bifurcation. By shooting arguments, one can show that this spherical bridge is embedded in a family of Delaunay surfaces which form bridges between the balls. But by simple trigonometric arguments, one can also construct a family of asymmetrically placed spherical bridges, as in Figure 3. For every volume larger than the volume V_0 of the symmetrically placed spherical bridge, there is a one-parameter family of asymmetric spherical bridges, all of which rotate into each other. As the volume decreases to V_0, these all collapse to the symmetrically placed spherical bridge, so that the symmetrically placed spherical bridge is a limiting member of this family as well.

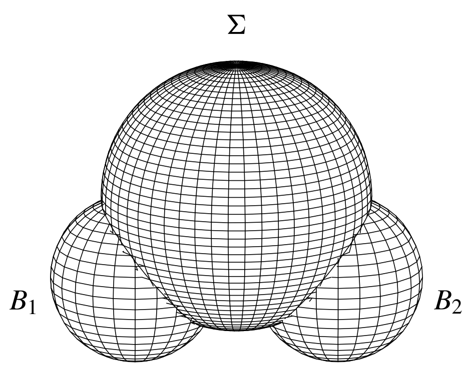


Figure 3. Asymmetrically placed spherical bridge.

Note 2. A cylindrical bridge between spheres is a limiting case of unduloids. Conditions under which the cylinder is a local energy minimum are derived in [Vogel 1999].

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NEW EXOTIC CONTAINERS

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We describe the construction of new exotic containers in a gravitational field. These are containers which, for certain volumes of fluid, possess a continuum of noncongruent equilibrium configurations. Unlike the constructions of Gulliver and Hildebrandt (1986) and Concus and Finn (1991), our containers need not be rotationally symmetric. One of their features is that the equilibria seem likely to be local minimizers of energy, in contrast to earlier constructions where the equilibria were always unstable.

1. Introduction

Consider a container Σ, partially filled with a fluid of density ρ, occupying a region T, sitting in a gravitational field of intensity g. We denote the liquid-air interface by Λ, enabling us to write down the potential energy of the configuration,

$$E = \sigma|\Lambda| + \rho g \int_T z\, dv - \sigma\tau|\Sigma'|, \quad |\tau| < 1. \tag{1–1}$$

Here $|\Lambda|$ is the area of the free surface, $|\Sigma'|$ is the wetted area of container wall, and τ is the wetting energy of the fluid in contact with the wall. In equilibrium the configuration will be such that the potential energy is an extremum with respect to the volume constraint, $|T| = V_0$. The case of positive g corresponds to the gravitational force acting downward. However, the cases $g = 0$ and $g < 0$ are also of interest to us.

The Euler–Lagrange equations determine the following conditions for equilibrium. First, the mean curvature of the free surface satisfies $2H = \kappa z + \lambda$, where $\kappa = \rho g/\sigma$ and λ is a constant arising as a Lagrange multiplier. The sign of the mean curvature is determined relative to the unit normal on Λ directed away from the fluid T. Secondly, the boundary conditions stipulate that the free surface meets the container wall at a contact angle γ where $\cos\gamma = \tau$. Here the angle γ is measured interior to the fluid.

An exotic container is a vessel with a smooth wall such that for some particular volume of fluid there will exist a continuum of geometrically distinct equilibrium

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configurations. In the published examples the exotic containers are always taken to be rotationally symmetric about a vertical axis. The corresponding interfaces are also rotationally symmetric and usually contain the flat surface $u \equiv 0$ as one member of the family. The first example appeared in the paper by R. Gulliver and S. Hildebrandt [1986], who considered the case of zero gravity, $g = 0$, and with wetting energy, $\tau = 0$, so that the contact angle was $\gamma = \pi/2$. In this case the free surfaces are spherical caps and the construction of the container wall is very geometric. The remaining cases where $g \geq 0$ and arbitrary contact angle γ, $0 < \gamma < \pi$ were studied in a series of papers by P. Concus and R. Finn.

It turns out that in all of these cases the rotationally symmetric equilibria are not stable. This feature was first discussed in [Concus and Finn 1989]. A complete proof was given in [Wente 1999]. This behavior was verified in a drop tower experiment, discussed in a paper by Concus, Finn, and Weislogel [Concus et al. 1992]. The experiment was also reproduced later in gravity-free conditions in a space lab mission.

This paper shows the construction of many new exotic containers. Consider, first, a round spherical container containing some fluid, with the free surface being a circular planar disk meeting the container wall with contact angle γ. This is an equilibrium configuration in gravity-free conditions with wetting energy $\tau = \cos\gamma$. It is part of a two-parameter family of congruent configurations (not geometrically distinct). They are minimizers of the appropriate energy. What happens if we add gravity? Can we construct exotic containers whose extrema are local minimizers of energy? This is our goal.

Our method of construction is as follows. Start with a one-parameter family of extremal surfaces satisfying the Euler–Lagrange equation. (For the Gulliver–Hildebrandt construction the family consisted of spherical caps symmetric about the z-axis. The Concus–Finn examples use the rotationally symmetric solutions to the sessile drop equation when $g > 0$.) The class of admissible surfaces we shall use are those extrema of cylindrical type. Namely, they are ruled surfaces with a generating curve lying in the xz-plane. The rulings are straight lines parallel to the y-axis. Since at every point one of the two principle curvatures is zero, the surfaces are metrically flat. For $g = 0$, the surfaces we shall use are tilted planes. For $g \neq 0$ the generating curves are determined by the condition that the signed curvature of the generating curve is a linear function of height. Such curves have an interesting history, having first been studied by Euler, and are called elastic curves. We shall refer to the corresponding extremal surfaces as elastic surfaces.

Start with our family, $\{\Lambda_t\}$, parametrized by t. The surface Λ_0 will be the horizontal plane. Λ_t will be that elastic surface whose generating curve passes through the origin, whose curvature k equals κz, and whose inclination angle at $(0, 0)$ is t. These curves will have an inflection point at the origin, bending one way when z is positive and the opposite way when z is negative. We must also allow the surfaces Λ_t to be transported vertically by an amount $h(t)$ to be determined. Call this new family $\tilde{\Lambda}_t$. Pick a region $X_0 \subset \Lambda_0$ with bounding curve Γ_0. One now constructs the rigid bounding surface Σ by analogy with the Monge cone construction. Σ will consist of the union of curves Γ_t lying in $\tilde{\Lambda}_t$. The requirement is that each elastic surface $\tilde{\Lambda}_t$ meet the rigid surface Σ at a prescribed contact angle γ. It is still necessary that the volume enclosed by Σ and $\tilde{\Lambda}_t$ remain fixed. This is achieved by an appropriate choice of $h(t)$. For $\gamma = 90°$ the construction is easier. The volume condition is satisfied by setting $h(t) \equiv 0$ and the surface Σ is generated by taking orthogonal trajectories emanating from the base curve, Γ_0. Generally, we shall assume that Γ_0 is symmetric about the y-axis, but this is not a necessary condition for our construction.

Consider the case $g = 0$, where Λ_t is the family of tilted planes. If $\Gamma_0 \subset \Lambda_0$ is a circle with center at the origin, our construction recovers the round sphere. However, if Γ_0 is some other closed curve, say an ellipse, we obtain some new bounding surfaces which are not so easy to describe.

The construction of exotic containers is also of interest in one lower dimension. This may be imagined as some fluid sitting between two vertical planes. In this case the free surface is a narrow ribbon. We shall analyze this case as well. We refer to this situation as the planar case.

In Section 2 we discuss elastic curves, those for which the signed curvature is a linear function of height. Such curves will generate our elastic surfaces, Λ_t. They were studied by Euler and have other potential applications to capillary theory besides the construction of exotic containers. They determine potential sessile or pendant drops in one lower dimension. In Section 3 we carry out the construction of the exotic containers. We shall analyze the planar case first and then the more physical situation in dimension three.

In Section 4 we discuss the stability question. As mentioned above, for the gravity-free case and where our initial curve Γ_0 is a circle, the corresponding container wall is a sphere. Here the configurations are all energy minimizers (all congruent as well). We would like this to be true more generally. We do show that in the planar case our construction does produce a container for which the initial flat surface Λ_0 is a strong local minimizer. It seems likely that this should be true in the three-dimensional case as well assuming that Γ_0 is chosen properly.

2. Elastic curves

In this section we construct a particular set of solutions $z = u(x, y)$ for the free-surface interface, namely those that depend only on one variable x, so $z = u(x)$. The solutions (regarded as curves in the xz-plane) are called elastic curves; the corresponding ruled surfaces in $\mathbb{R}^3$ will be called elastic surfaces. By a vertical translation we may set the Lagrange multiplier λ to 0. By a rescaling we may suppose that $\kappa = \rho g/\sigma = 1$. This gives us the differential equation

$$u''/(1+(u')^2)^{3/2} = u, \quad z = u(x). \tag{2–1}$$

The signed curvature is precisely u. It is a linear function of position. As noted in the introduction, this is the governing equation for the equilibrium configuration of a planar elastic rod. See [Giaquinta and Hildebrandt 1996] for a nice discussion. Let the curve $\mathscr{C}$ be parametrized by arc length, $\langle x(s), u(s)\rangle$. If we let $\theta(s)$ be the angle of inclination, the differential equation (2–1) can be rewritten as

$$\theta'(s) = k(s) = u(s), \quad k(s) = \text{curvature}. \tag{2–2}$$

Here are some elementary observations concerning solutions to (2–2).

- If $\langle x(s), u(s)\rangle$ solves (2–2) so does $\langle x(s)+x_0, u(s)\rangle$.
- If $\langle x(s), u(s)\rangle$ solves (2–2) so does $\langle x(s), -u(s)\rangle$.
- If we reverse the orientation of the curve by setting $\sigma = -s$, the new curve satisfies the equation $\theta'(\sigma) = -u(\sigma)$.
- Suppose $\theta'(s) = u(s)$. If $u(s) > 0$ the curve is bending counterclockwise, if $u(s) < 0$ it is bending in a clockwise manner, there is an inflection point as the curve crosses the x-axis.
- For $\theta'(s) = u(s)$, $\rho g/\sigma = 1$ the fluid is taken to be on the right in the direction of increasing s.
- The strong touching principle: Let $\mathscr{C}_1$, $\mathscr{C}_2$ be oriented curves both satisfying (2–2), with $\theta_1'(s) = u_1(s)$ and $\theta_2'(s) = u_2(s)$. If $\theta_1 = \theta_2$ at some level $u = \bar{u}$, then the two curves are congruent. If they touch at some point they are identical.

Upon differentiating (2–2) we have $\theta''(s) = u'(s) = \sin\theta$, which can be integrated giving

$$\tfrac{1}{2}\theta'(s)^2 + \cos\theta = E, \qquad u^2(s) = 2(E - \cos\theta). \tag{2–3}$$

We analyze solutions to (2–3) via phase-plane analysis. Clearly the energy E is at least -1. In fact:

(a) If $E = -1$, then $\theta = \pi$ and $u(s) = 0$. The solution is the horizontal line, $u = 0$, being traversed right to left so that the fluid lies above the z-axis.

(b) If $E = 1$, there is the constant solution $\theta(s) = 0$. Again $u(s) = 0$ and the curve is traversed left to right with the fluid below the z-axis.

Now consider the case $-1 < E < 1$. The phase portrait of (2–3) is shown in Figure 1. The solution $\theta(s)$ will be oscillatory with a minimum value $\theta(0) = \theta_0$,

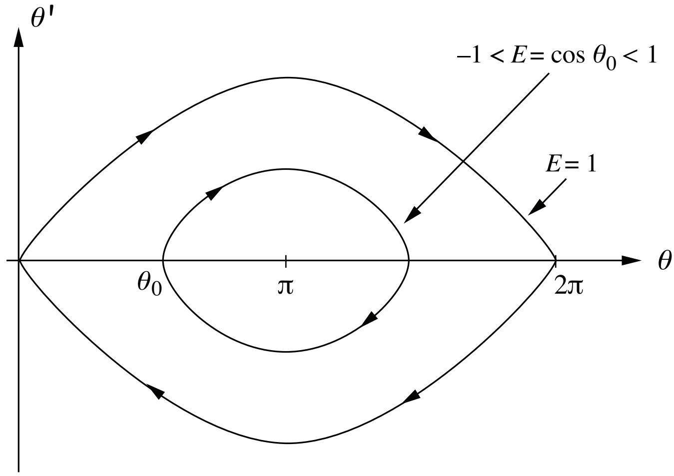

Figure 1. Phase portrait of the solution (2–3).

where $0 < \theta_0 < \pi$ and $E = \cos\theta_0$. We have $\theta_{\min} = \theta_0$ and $\theta_{\max} = 2\pi - \theta_0$. Since $E = \cos\theta_0$, the second equation in (2–3) becomes

$$u^2(s) = 2(\cos\theta_0 - \cos\theta).$$

This expresses u as a function of θ. We can use θ itself as a parameter, $\theta_0 \le \theta \le 2\pi - \theta_0$. Since $dx/d\theta = \cos\theta$ one finds

$$x(\theta) = \int_{\theta_0}^{\theta} \frac{\cos\varphi\, d\varphi}{\sqrt{\cos\theta_0 - \cos\varphi}}.$$

We have a parametrization $\langle x(\theta), y(\theta)\rangle$ valid for $\theta_0 \le \theta \le 2\pi - \theta_0$ with initial condition $\langle x(\theta_0), u(\theta_0)\rangle = \langle 0, 0\rangle$. Let this curve be expressed in terms of arc length, $0 \le s \le \ell$. It may be extended using (2–3) to the interval $-\ell \le s \le \ell$ with $\langle x(-s), u(-s)\rangle = -\langle x(s), u(s)\rangle$. Finally the solution extends for all s by setting $x(s+2\ell) = x(s) + x(2\ell)$, and $u(s+2\ell) = u(s)$. The complete curve bends to the left when u is positive and to the right when u is negative. A complete picture is obtained by observing the curve $\langle x(\theta), u(\theta)\rangle$, $\theta_0 \le \theta \le \pi$.

Theorem 2.1. *Consider solutions to $\theta'(s) = u(s)$ satisfying (2–3) with $-1 < E < 1$. By setting $E = \cos\theta_0$, $0 < \theta_0 < \pi$ we have*

$$u^2(\theta) = 2(\cos\theta_0 - \cos\theta).$$

Consider that portion of the complete curve where $\theta_0 \le \theta \le \pi$; for $\theta = \theta_0$ the curve passes through the origin. One has three types of graphs, depending on whether $0 < \theta_0 < \pi/2$, $\theta_0 = \pi/2$, or $\pi/2 < \theta_0 < \pi$ (Figure 2).

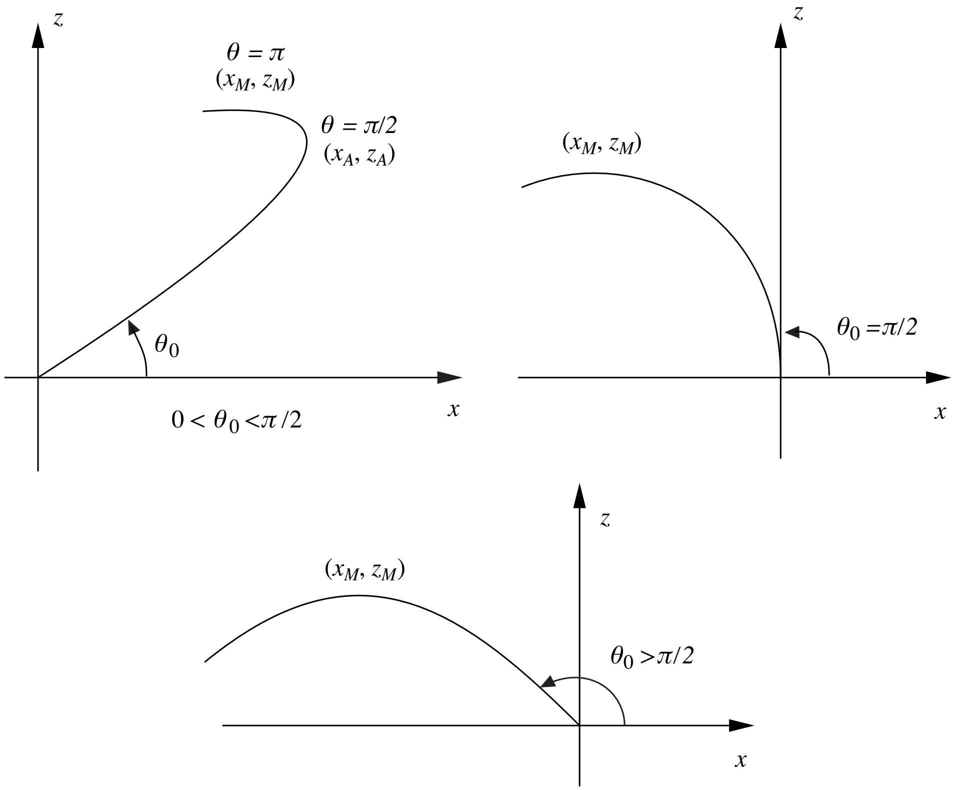

Figure 2. Qualitative appearance of elastic curves.

We set $(x_M, z_M) = \langle x(\pi), u(\pi)\rangle$, *and also* $(x_A, z_A) = \langle x(\pi/2), u(\pi/2)\rangle$ *in the case* $0 < \theta_0 < \pi/2$. *We identify* z_M *with* u_M *and* z_A *with* u_A.

(1) *We have* $u_A^2 = u^2(\pi/2) = 2E$; *equivalently*, $u_A = \sqrt{2E} = \sqrt{2\cos\theta_0}$. *The function* u_A *is strictly increasing in* E, *for* $0 < E < 1$, *with range* $0 < u_A < \sqrt{2}$.

(2) x_A *is strictly increasing in* E, *for* $0 < E < 1$, *with range* $0 < x_A < \infty$, *and* x_A *becomes infinite as* E *approaches* 1.

(3) *We have* $u_M^2 = 2(E+1)$, *or equivalently*, $u_M = \sqrt{2(E+1)}$, *for* $-1 < E < 1$. *Therefore* u_M *is strictly increasing in* E, *for* $-1 < E < 1$, *with range* $0 < u_M < 2$.

(4) x_M *is strictly increasing in* E, *for* $-1 < E < 1$, *with range* $-\pi < x_M < \infty$.

Proof. We need only verify (2) and (4). Suppose $\mathscr{C}_1 : \langle x_1(\theta), u_1(\theta)\rangle$ and $\mathscr{C}_2 : \langle x_2(\theta), u_2(\theta)\rangle$ are two solutions with initial inclination angles $0 < \alpha_2 < \alpha_1 \le \pi/2$ so that $0 \le \cos\alpha_1 = E_1 < E_2 = \cos\alpha_2 < 1$. It follows that $u_{A_1} < u_{A_2}$ by (1). Initially the curve $\mathscr{C}_2$ lies to the right of $\mathscr{C}_1$. I claim that it does so for $0 \le u \le u_{A_1}$.

If this were not the case, the two curves would intersect at some smallest value $\bar{u}$, with $0 < \bar{u} < u_A$. At this level the inclination angles would satisfy $\theta_1 \le \theta_2$. With equality we are done, since by the touching principle we would have $\mathscr{C}_1 = \mathscr{C}_2$. But $\theta_1 < \theta_2$ is not possible either, for then at some smaller value $\tilde{u}$, $0 < \tilde{u} < \bar{u}$, we would have $\theta_1 = \theta_2$. This would mean that $\mathscr{C}_2$ is a horizontal translate of $\mathscr{C}_1$, which is impossible. It follows that x_A is a strictly increasing function of E, $0 < E < 1$. The integral formula for x_A is

$$x_A = \int_{\theta_0}^{\pi/2} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E - \cos\varphi)}} = \int_{\theta_0}^{\pi/2} \frac{\cos\varphi\, d\varphi}{\sqrt{2(\cos\theta_0 - \cos\varphi)}}.$$

It follows that the range of values for x_A, for $0 < \theta_0 < \pi/2$, is $0 < x_A < +\infty$. Statement (2) follows.

We now consider statement (4).

First take the case $0 < \theta_0 < \pi/2$ or $0 < E < 1$. Here $x_M = x(\pi)$. We find

$$x_M = x_A + \int_{\pi/2}^{\pi} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E - \cos\varphi)}} = x_A - \int_0^{\pi/2} \frac{\cos\psi\, d\psi}{\sqrt{2(E + \cos\psi)}}.$$

Thus, since x_A is strictly motonic in E, so is x_M. Moreover, $\theta_0 \to 0$ and $x_M \to +\infty$ as $E \to 1$.

We remark that $x_M < 0$ for $\theta_0 = \pi/2$. There is exactly one value $\theta_c \in (0, \pi/2)$ with $x_M = 0$. The corresponding complete curve is a closed curve in a figure eight shape.

Now consider the case $\pi/2 \le \theta_0 < \pi$. Parametrizing by the arc length s, we have a curve traced right to left starting at the origin with $E = \cos\theta_0$, and $-1 < E < 0$. Consider the curve obtained by reflection about the z-axis. This will resemble a sine curve traversed left to right satisfying the curvature equation $\theta'(s) = -u(s)$, with $\theta(0) = \psi_0 = \pi - \theta_0$. The curve bends clockwise and the inclination angle decreases from ψ_0 to 0. For $\theta = 0$ the reflected curve has coördinates $\langle x_R, u_R\rangle$, where $u_R = u_M = 2(1 + E)$ and $x_R = -x_M$ is positive. One finds

$$x_R = \int_0^{\psi_0} \frac{\cos\varphi\, d\varphi}{\sqrt{2(E + \cos\varphi)}} = \int_0^{\psi_0} \frac{\cos\varphi\, d\varphi}{\sqrt{2(\cos\varphi - \cos\psi_0)}}, \quad 0 < \psi_0 < \pi/2.$$

We claim that x_R is a strictly decreasing function of E for $-1 < E < 0$; that is, it is strictly increasing in ψ_0, $0 < \psi_0 < \pi/2$, and $E = -\cos\psi_0$.

We parametrize the curve as a function of u, getting

$$\frac{d}{du}\left(\frac{x'(u)}{\sqrt{1 + x'(u)^2}}\right) = -u.$$

This gives $x'^2/(1+x'^2) = (a^2+u^2)^2/4$ for $0 \le u \le u_M$, where $u_M^2 = 2-a^2$, for a positive constant $a < \sqrt{2}$. We obtain the integral formula

$$x_R = \int_0^{\sqrt{2-a^2}} \frac{(a^2+u^2)\,du}{\sqrt{4-(a^2+u^2)^2}}.$$

Let $u = \sqrt{2}v$ and $a = \sqrt{2}\alpha$. Rewrite the integral and then change variables, setting $\alpha^2 + v^2 = t$, to find

$$x_R = \frac{1}{\sqrt{2}} \int_\beta^1 \frac{t\,dt}{\sqrt{1-t^2}\sqrt{t-\beta}}, \quad 0 < \beta = \alpha^2 < 1.$$

Finally, set $t = (1-\beta)x + \beta$, obtaining

$$x_R = \frac{\sqrt{1-\beta}}{\sqrt{2}} \int_0^1 \frac{\big((1-\beta)x+\beta\big)\,dx}{\sqrt{x}\sqrt{1-\big((1-\beta)x+\beta\big)^2}}.$$

This integral is an increasing function of β, $0 < \beta < 1$. As one can see by differentiation,

$$\frac{dx_R}{d\beta} = \frac{(1-\beta)^{3/2}}{2\sqrt{2}} \int_0^1 \frac{(1-x)^2\big(2+((1-\beta)x+\beta)\big)}{\sqrt{x}\big(1-((1-\beta)x+\beta)\big)^{3/2}}\,dx.$$

This is positive for $0 < \beta < 1$.

For $\beta = 0$, one has $\theta_0 = \pi/2$, $u_M = \sqrt{2}$ and

$$x_R = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sqrt{\cos\varphi}\,d\varphi = \frac{1}{\sqrt{2}} \int_0^1 \frac{\sqrt{x}\,dx}{\sqrt{1-x^2}}.$$

For $\beta \approx 1$, $\theta_0 \approx \pi$ and $\psi_0 \approx 0$ and we see that $u_M \to 0$ and $x_R \to \pi/2$; hence $x_M \to -\pi/2$. □

Figure 3 illustrates the family of curves just discussed: solutions $\langle x(s), u(s)\rangle$ of $\theta'(s) = u(s)$ going through $\langle x(0), u(0)\rangle = \langle 0, 0\rangle$ and satisfying $\theta(0) = \theta_0$, $0 < \theta_0 < 1$, where s ranges from 0 to $s_M(\theta_0)$ with $u(s_M) = u_M = \sqrt{2(E+1)}$, for $-1 < E < 1$. These are the generating curves of the family of elastic surfaces we shall use to construct our exotic containers. We see from our discussion that any two curves in this family only intersect at the origin.

The family of curves where $E \ge 1$ are also interesting but are not relevant to our discussion here. For $E = 1$ one also obtains the soliton solution, while for $E > 1$ the curves remain away from the u-axis and always turn in one direction.

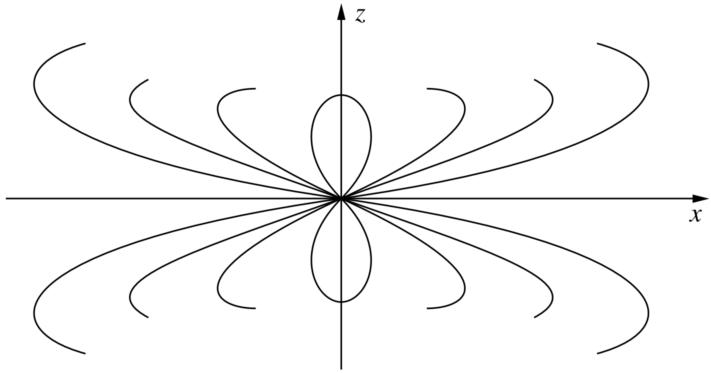

Figure 3. The family of elastic curves, $g > 0$.

3. Construction of exotic containers

There are two theorems in this section. The first carries out the construction of the exotic container in the planar case. Here the free surface is a curve, as is the container wall. Physically, this amounts to the situation of a liquid held between two parallel vertical plates. Though somewhat easier to handle, this construction is worth a separate discussion. The second theorem treats the three-dimensional case.

We start with a one-parameter family Λ_t of immersed surfaces in $\mathbb{R}^3$, each of which has mean curvature $2H = \kappa z$ with $\kappa = \rho g/\sigma$. Specifically we shall choose the family of elastic surfaces described in Section 2, although other choices would work equally well. In the planar case we use the corresponding elastic curves.

We have a map, $F_0 : \Omega \times I$ into $\mathbb{R}^3$ where $\Omega = \mathbb{R}^2$ and I is an open interval centered about $t = 0$. Points in Ω are labeled (u, v), points in I by t, and the target space is (x, y, z). We have

$$F_0(u, v, t) = \langle f(u, t), v, g(u, t)\rangle,$$

where the pair $\langle f(u, t), g(u, t)\rangle$, for a given value of t, describes the generating curve of the elastic family parametrized by arc length with $f(0, 0) = g(0, 0) = 0$ and such that the inclination angle at the origin is t. For each t, the map $F_0(u, v, t)$ is a flat isometric immersion of an elastic surface with

$$|(F_0)_u| = |(F_0)_v| = 1 \quad \text{and} \quad \langle (F_0)_u \cdot (F_0)_v\rangle = 0.$$

We set $\Omega_{t_0} = \Omega \times \{t_0\}$ and call $\Lambda_{t_0} = F_0(\Omega_{t_0})$, our immersed surface with prescribed mean curvature $2H = \kappa z$. For $\kappa = 0$, Λ_t is a tilted plane containing the y-axis with inclination angle t, while for $\kappa > 0$ we have the elastic surfaces described in Section 2, where the fluid lies below the surface (sessile drop case).

For any κ, Λ_0 is the horizontal plane and Λ_{-t} is the reflection of Λ_t in the xy-plane. In particular, $F_0(u, v, 0) = (u, v, 0)$.

We denote by $\xi(u, v, t) = (F_0)_u \wedge (F_0)_v$ the unit normal vector to Λ_t pointing out of the fluid.

Theorem 3.1 (two-dimensional case). *Suppose given a one-parameter family of oriented curves* $F_0(u, t) = \langle f(u, t), g(u, t)\rangle$ *satisfying*

(1) $2H = \kappa z$, *where* $\kappa = \rho g/\sigma$ *and* $2H$ *is the signed curvature*;

(2) *the curves are parametrized by arc length so that* $|(F_0)_u| = 1$ *with* $F_0(u, 0) = \langle u, 0\rangle$; *and*

(3) $F_0(0, t) = \langle 0, 0\rangle$ *and* $(F_0)_u(0, t) = \langle \cos t, \sin t\rangle$. (For $\kappa > 0$ our curves represent the free surface of sessile drops, for $\kappa = 0$ we have $F_0(u, t) = \langle u \cos t, u \sin t\rangle$, while for $\kappa < 0$ we have the pendant drop situation.)

Let initial values for $u_1(t)$, $u_2(t)$ *be given. We want* $u_1(t) < u_2(t)$. *For this reason we assume that* $u_1(0) = u_1$, $u_2(0) = u_2$ *with* $u_2 = -u_1 > 0$. *There exist two smooth functions* $u_1(t)$, $u_2(t)$ *and a function* $h(t)$ *with* $h(0) = 0$, *all defined in a neighborhood of* $t = 0$ *with the following property. Let*

$$F(u, t) = \langle f(u, t), g(u, t)+h(t)\rangle$$

be our family of extremals, characterized by the following properties:

(a) $F(\mathbb{R} \times \{t\}) = \Lambda_t$, *our immersed free curve.*

(b) Σ_1 *is the left bounding wall*, $F(u_1(t), t)$, *while* Σ_2 *is the right bounding wall*, $F(u_2(t), t)$.

We have $F(u_1(t), t) \in \Lambda_t \cap \Sigma_1$ *and* $F(u_2(t), t) \in \Lambda_t \cap \Sigma_2$. *The curve* $F(u, t)$, *with* $u_1(t) \le u \le u_2(t)$, *is an extremal curve for the variational problem and each such curve*, Λ_t, *will meet the container walls* Σ_1, Σ_2 *at an interior contact angle* γ, *with* $0 < \gamma < \pi$.

Let $V(t)$ *be the "volume" enclosed by* Λ_t *and the container walls* Σ_1, Σ_2. *We may suppose that the curves* Σ_1, Σ_2 *are connected from below to form a closed container.*

The volume enclosed by the container and any free surface, Λ_t, *is a constant.*

Proof. Given $h(t)$, a smooth function, we shall use the contact angle condition to determine a first-order differential equation for $u_1(t)$, $u_2(t)$ satisfying $u_1(0) = u_1$, $u_2(0) = u_2$, where we shall assume that $u_2 = -u_1 > 0$. Then we use the fixed volume condition to obtain an equation for $h'(t)$ in terms of $u_1(t)$, $u_2(t)$. We end up with a first-order system for the pair $\{u_1(t), u_2(t)\}$ with initial conditions $\langle u_1(0), u_2(0)\rangle = \langle u_1, u_2\rangle$. The existence theorem for ordinary differential equations gives us our solution $\langle u_1(t), u_2(t)\rangle$. We are then able to find $h(t)$ where we set (without loss of generality) $h(0) = 0$.

We have the functions

$$\begin{aligned} F_0(u,t) &= \langle f(u,t), g(u,t)\rangle, \\ F(u,t) &= \langle f(u,t), g(u,t)+h(t)\rangle, \\ F_u &= \langle f_u, g_u\rangle \quad \text{with } f_u^2 + g_u^2 = 1, \\ F_t &= \langle f_t, g_t + h'(t)\rangle, \\ \xi(u,t) &= \langle -g_u, f_u\rangle, \end{aligned}$$

the latter being the unit normal vector. We set $e_1 = F_u$, the unit tangent vector to Λ_t, while $e_2 = \xi(u,t)$ is the unit normal vector. We are to find $u_2(t)$ so that $F(u_2(t), t)$ will describe the right container wall, Σ_2. The tangent vector to this curve is to be parallel to $w = (\cos\gamma)e_1 + (\sin\gamma)e_2 = (\cos\gamma)F_u + (\sin\gamma)\xi$. That is: $\dot{u}_2 F_u + F_t = \alpha w$ for some scalar α. We write this as

$$F_t = -\dot{u} F_u + \alpha w.$$

Now F_u and w are linearly independent, $0 < \gamma < \pi$. By taking the inner product of the preceding equation with ξ and F_u successively we find

$$\begin{aligned} \alpha &= \csc\gamma\,(F_t \cdot \xi), \\ \dot{u} &= \cot\gamma\,(F_t \cdot \xi) - (F_t \cdot F_u). \end{aligned}$$

The second of these equations is our differential equation for $u_2(t)$, which we can rewrite more explicitly as

$$\dot{u}_2(t) = F_t \cdot (\cot\gamma\,\xi - F_u) = \cot\gamma\,(f_u g_t - f_t g_u) + h'(t)(\cot\gamma\, f_u - g_u). \tag{3–1}$$

The right-hand side is an "explicit" function of (u, t).

The differential equation for $u_1(t)$ is almost identical. The only change is that the outward unit tangent vector to Λ_t is $e_1 = -F_u$. We are led to the following differential equation for $u_1(t)$:

$$\dot{u}_1(t) = -\cot\gamma\,(F_t \cdot \xi) - (F_t \cdot F_u). \tag{3–2}$$

Given $h(t)$ and initial data for $u_1(t)$ and $u_2(t)$, there exists a unique solution $\langle u_1(t), u_2(t)\rangle$ such that the curves $F(u_1(t), t)$, $F(u_2(t), t)$ describing Σ_1, Σ_2 meet the free surfaces $\Lambda_t : F(u,t)$, $u_1(t) \le u \le u_2(t)$, with the desired interior contact angle γ.

We now use the conservation of volume condition to determine the function $h(t)$. For each t, the free surface Λ_t is given by the map $F(u,t)$, $u_1(t) \le u \le u_2(t)$. As we vary our family, the normal component of the variation is $\varphi = (F_t \cdot \xi)$. The first order variation of "volume" should be zero:

$$\dot{V}(t) = \int_{u_1(t)}^{u_2(t)} (F_t \cdot \xi)\, du = 0.$$

Substituting our formulae for F and ξ we get

$$h'(t) \int_{u_1(t)}^{u_2(t)} f_u du = -\int_{u_1(t)}^{u_2(t)} (f_u g_t - f_t g_u)\, du. \tag{3–3}$$

(For $t = 0$ we have $f_u \equiv 1$ so that the expression for $h'(t)$ is well defined.) Equation (3–3) determines $h'(t)$ as a function of (u_1, u_2). Substitute this expression for $h'(t)$ back into equations (3–1) and (3–2) to obtain a first order system of differential equations for $\{u_1(t), u_2(t)\}$. Given initial conditions $u_1(0) = u_1$, $u_2(0) = u_2$ with $u_2 = -u_1 > 0$ we have solutions $\{u_1(t), u_2(t)\}$. Setting $h(0) = 0$ we obtain $h(t)$ using (3–3). □

Remark. Given symmetric initial data, the generated solutions will have the following properties.

(i) $h(t)$ is an even function of t.

(ii) The boundary curves Σ_1, Σ_2 are symmetric about the z-axis.

(iii) The free surface Λ_t meet the container walls Σ_1, Σ_2 at an interior contact angle γ, $0 < \gamma < \pi$.

(iv) The "volume" enclosed by Λ_t and Σ_1, Σ_2 is a constant independent of t.

(v) For $t = 0$, Λ_t is the horizontal line segment $u_1 \le u \le u_2$.

(vi) For $\gamma = \pi/2$ with $\cot\gamma = 0$ we have $h(t) \equiv 0$ and the curves Σ_1, Σ_2 are simply orthogonal trajectories of the family $F_0(u, t)$, assuming $u_1(0) = -u_2(0)$.

We now consider the three-dimensional case. Our elastic surfaces are described by functions $F_0(u, v, t) = \langle f(u, t), v, g(u, t)\rangle$ with $F_0(u, v, 0) = \langle u, v, 0\rangle$ and $(F_0)_u(0, v, t) = \langle \cos t, 0, \sin t\rangle$. For each t the equilibrium surface $F_0(\Omega \times \{t\})$ has mean curvature $2H = \kappa z$. Now let $h(t)$ be a smooth function of t with $h(0) = 0$ and set

$$F(u, v, t) = \langle f(u, t), v, g(u, t) + h(t)\rangle. \tag{3–4}$$

For each t, $F(\Omega \times \{t\}) = \Lambda_t$ is a potential equilibrium surface, where $h(t)$ is a Lagrange multiplier. Our parameter space has coördinates (u, v, t), while the coördinates in the target space are labeled (x, y, z).

Our construction proceeds as follows. Start with a base curve $C_0 : \langle a(s), b(s), 0\rangle$ with $u = a(s)$, $v = b(s)$. We assume this is a smooth curve parametrized by arc length. Let its length be L, so $a(s)$, $b(s)$ are periodic functions of period L defined on $\mathbb{R}$. Assume also that C_0 is a convex curve in the u-v plane, symmetric about the v-axis. Since $F(u, v, 0) = \langle u, v, 0\rangle$ we see that $\Gamma_0 = F(C_0)$ is identical to C_0. With C_0 as our base curve, we form the cylinder $C_0 \times \mathbb{R}$.

We will consider surfaces S in the parameter space which are normal graphs over this cylinder. The unit normal vector to $C_0 \times \mathbb{R}$ is $n(s) = \langle b'(s), -a'(s), 0\rangle$. Let $\varphi(s, t)$ be a smooth function that is periodic in s of period L and satisfies $\varphi(s, 0) = 0$. Set

$$\begin{aligned} u(s,t) &= a(s) + \varphi(s,t)b'(s) \\ v(s,t) &= b(s) - \varphi(s,t)a'(s). \end{aligned} \tag{3–5}$$

The surface, S, is then described by the map

$$S : \langle u(s,t), v(s,t), t\rangle. \tag{3–6}$$

We set $\Sigma \equiv F(S)$. This is a parametric surface in the target space. We need to determine $h(t)$, $\varphi(s, t)$ so that Σ satisfies the contact angle condition with the surfaces Λ_t. We shall determine $h(t)$ so that the volume enclosed by the container wall Σ and any given free surface Λ_t remains constant. Specifically:

Theorem 3.2 (the three-dimensional case). *Let the base curve $C_0 : \langle a(s), b(s), 0\rangle$ be as described above. It is convex and symmetric about the v-axis, and periodic with period L. There exist*

(a) *a function $\varphi(s, t)$ defined for t in an interval about* 0 *and for all s, periodic in s of period L, satisfying $\varphi(s, 0) = 0$;* and

(b) *a smooth function $h(t)$ that is even in t and satisfies $h(0) = 0$;*

the whole satisfying the following property.

Let S be the surface in the parameter space given as a normal graph over the cylinder $C_0 \times \mathbb{R}$ by (3–5), (3–6), and let $\Sigma = F(S)$ be the image surface in the target space under the map F. For each t, let Λ_t be the equilibrium surface $F(\Omega \times \{t\})$. The wall Σ and the equilibrium surface Λ_t intersect along a curve $\Gamma_t = F\big(u(s,t), v(s,t), t\big)$. The two surfaces intersect transversally with a contact angle γ, where $0 < \gamma < \pi$.

Finally suppose that the container Σ is closed off from below so that Σ and the free surface Λ_t enclose a volume $V(t)$. We can choose $h(t)$ so that the volume remains constant.

Proof. Given the convex base curve C_0 as described, we have the surface S as a normal graph over the cylinder $C_0 \times \mathbb{R}$ given by (3–5) and (3–6) for any function $\varphi(s, t)$. We want $\varphi(s, t)$ to be periodic in s of period L, defined in some interval about $t = 0$, and with $\varphi(s, 0) = 0$. We designate

$$\Sigma = F(S), \quad \Gamma_0 = F(C_0) \text{ (the base curve)}, \quad \Gamma_t = F(C_t).$$

Here Γ_t is a curve lying on Λ_t and on Σ. Given a smooth function $h(t)$ even in t with $h(0)=0$, we shall derive a first-order partial differential equation for $\varphi(s,t)$ that can be solved by the method of characteristics. This will produce a surface Σ satisfying the correct contact angle condition with Λ_t for any given contact angle γ in the interval $0<\gamma<\pi$. We then determine $h(t)$ so that the volume condition is satisfied.

Let

$$\mathscr{F}(s,t) \equiv F\big(u(s,t), v(s,t), t\big)$$

describe Σ, where $u(s,t)$ and $v(s,t)$ are given in (3–5). The vectors F_u and F_v are unit tangent vectors to Λ_t with $(F_u \cdot F_v)=0$. Let $\xi(u,v,t)=F_u \wedge F_v$ be the unit normal vector on Λ_t. We observe that

$$\mathscr{F}_s = u_s F_u + v_s F_v \quad \text{and} \quad \mathscr{F}_t = u_t F_u + v_t F_v$$

are tangent vectors to Σ, and $\mathscr{F}_s$ is tangent to the curve Γ_t as well. We set

$$e_2 = \frac{1}{\sqrt{u_s^2+v_s^2}}\,\mathscr{F}_s = \frac{1}{\sqrt{u_s^2+v_s^2}}\,(u_s F_u + v_s F_v),$$

the unit tangent vector to $\Gamma_t = \Sigma \cap \Lambda_t$. With ξ as unit normal vector to Λ_t we complete the orthonormal frame along Γ_t by setting

$$e_1 = \frac{1}{\sqrt{u_s^2+v_s^2}}\,(v_s F_u - u_s F_v).$$

This is a tangent vector on the surface Λ_t which is a conormal along the bounding curve Γ_t. As in Theorem 3.1 we set

$$w = \cos\gamma\, e_1 + \sin\gamma\, \xi.$$

This vector is to be tangent to Σ. Since $\mathscr{F}_s$ and $\mathscr{F}_t$ span the tangent space we may write

$$\alpha w = \lambda \mathscr{F}_s + \mathscr{F}_t \tag{3–7}$$

for suitable scalers α, λ. Using our expressions above for $\mathscr{F}_s$ and $\mathscr{F}_t$ we rewrite (3–7) to obtain

$$F_t = -(\lambda u_s + u_t)F_u - (\lambda v_s + v_t)F_v + \alpha w.$$

We obtain three equations by taking the inner product of F_t with ξ, F_u, and F_v respectively. First,

$$(F_t \cdot \xi) = \alpha(w \cdot \xi) = \sin\gamma\, \alpha;$$

hence $\alpha = \csc\gamma\ (F_t \cdot \xi)$. For the other two equations one uses the fact that $(F_t \cdot F_v) = 0$ and $(F_u \cdot F_v) = 0$. Taking the inner product of F_t with F_u leads to

$$(\lambda u_s + u_t) = -(F_t \cdot F_u) + \cot\gamma \left(\frac{v_s}{\sqrt{u_s^2 + v_s^2}} (F_t \cdot \xi) \right).$$

Taking the inner product of F_t with F_v yields

$$(\lambda v_s + v_t) = \cot\gamma \left(\frac{-u_s}{\sqrt{u_s^2 + v_s^2}} (F_t \cdot \xi) \right). \tag{3–8}$$

We use these two equations to eliminate λ and find

$$u_s v_t - u_t v_s = (F_t \cdot F_u) v_s - (F_t \cdot \xi) \cot\gamma \sqrt{u_s^2 + v_s^2}. \tag{3–9}$$

Using the expressions for $u(s,t)$, $v(s,t)$ in (3–5) gives us a first order PDE for $\varphi(s,t)$:

$$\begin{aligned} \big(1 + (a'b'' - a''b)\varphi\big)\varphi_t + (F_t \cdot F_u)(b' - a'\varphi_s - a''\varphi) \\ - (F_t \cdot \xi)(\cot\gamma)\sqrt{u_s^2 + v_s^2} = 0, \end{aligned} \tag{3–10}$$

where $u_s^2 + v_s^2 = 1 + \varphi_s^2 + (a''^2 + b''^2)\varphi^2 + 2(a'b'' - a''b')\varphi + 2(a'a'' + b'b'')\varphi\varphi_s$.

This is a first order PDE which we can solve by the method of characteristics subject to the initial condition $\varphi(s,0) = 0$.

As in the two-dimensional case (Theorem 3.1), we use the conservation of volume condition to determine $h'(t)$. Let $V(t)$ be the volume enclosed by the surface Λ_t and the container wall Σ. We may suppose that the bottom of the container is closed off so that the computed volume is finite. The rate of change of volume is obtained by integrating the normal variation over the part of Λ_t that lies inside the container.

$$\dot V(t) = \iint_{F(A_t)} (F_t \cdot \xi)\, dS = 0. \tag{3–11}$$

Here A_t is the domain in the parameter space whose image under F is the desired region. Now $F(u,v,t)$ is given by (3–4). We have $F_t = \langle f_t, 0, g_t + h'(t)\rangle$ and $\xi = F_u \wedge F_v = \langle -g_u, 0, f_u\rangle$. Substitute these expressions into (3–11) and we find

$$\begin{aligned} (F_t \cdot \xi) &= (-f_t g_u + f_u g_t) + h' f_u, \\ (F_t \cdot F_u) &= (f_u f_t + g_u g_t) + h' g_u. \end{aligned} \tag{3–12}$$

We use these expressions to rewrite (3–11) as

$$h'(t) \iint_{A_t} f_u\, du\, dv + \iint_{A_t} (-f_t g_u + f_u g_t)\, du\, dv = 0. \tag{3–13}$$

Here $F(A_t)$ lies on the surface Λ_t, with boundary Γ_t. The identity (3–13) determines $h'(t)$ as the ratio of two integrals over the region A_t, with known integrands. The region A_t in the parameter space is bounded by the curve $\langle u(s,t), v(s,t)\rangle$ as given by (3–5), which is the normal graph over the base curve $\mathscr{C}_0$ and depends solely on the function $\varphi(s,t)$. We use (3–13) to substitute this integral expression for $h'(t)$ into the differential equations (3–9) or (3–10). This occurs in the expressions for $(F_t \cdot F_u)$ and $(F_t \cdot \xi)$ as in (3–12). Our differential equation (3–10) takes the form

$$\varphi_t = \mathscr{G}(s, t, \varphi, \varphi_s, h'), \tag{3–14}$$

where h' is the ratio of two integrals depending only on $\varphi(s,t)$. We can still use the method of characteristics to obtain a solution $\varphi(s,t)$, defined in a neighborhood of $t=0$, periodic in s and satisfying $\varphi(s,0)=0$.

We can show existence of a solution using the Picard iteration process. Insert into (3–14) the expression for $h'(t)$ determined by the fixed volume condition (3–13). The method of characteristics gives a system of differential equations for $s=s(\sigma,t)$, $\varphi=\varphi(\sigma,t)$, $p(\sigma,t)=\varphi_s(\sigma,t)$ and $\varphi_t(\sigma,t)=q(\sigma,t)$, with initial data when $t=0$ determined by the base curve Γ_0. One writes the differential equation, in t, as an integral equation with the initial data built in. This allows us to set up an iteration process. Start with an initial function $\langle s_0(\sigma,t), \varphi_0(\sigma,t), p_0(\sigma,t), q_0(\sigma,t)\rangle$. Use this input to calculate $h_0'(t)$ using (3–14). The Picard process allows us to compute $\langle s_1(\sigma,t), \varphi_1(\sigma,t), p_1(\sigma,t), q_1(\sigma,t)\rangle$. The iteration continues, and convergence to a unique solution follows. Having obtained the solution to (3–14) we use (3–13) to find $h'(t)$ and $h(t)$, setting $h(0)=0$. Finally, we use the map $F(u,v,t)$ to obtain the exotic container Σ.

The solution might be implemented as follows. Let P be a partition of an interval, $[0,T]$ into $0=t_1<t_2<\cdots<t_N=T$. Construct a piecewise linear function $h_N(t)$ by using (3–13) to compute $h_N'(t_k)$ and extending $h_N(t)$ linearly over the interval $[t_k, t_{k+1}]$. Now use (3–10) to evolve $\varphi(s,t)$ through this interval as well.

Our base curve Γ_0 was symmetric about the y-axis. This implies that $h'(0)=0$, so $h(t)\equiv 0$ on the interval $[0,t_1]$. At $t=t_1$ we recompute $h'(t_1)$ using (3–13) and extend linearly onto the next subinterval, $[t_1,t_2]$. The process continues. □

Remarks. (1) If the base curve Γ_0 is symmetric about the y-axis and if the contact angle is $\gamma=\pi/2$, then the volume condition is satisfied by setting $h(t)\equiv 0$. The bounding surface Σ is generated by the set of orthogonal trajectories to the elastic surfaces, $F_0(u,v,t)$, which cut through the base curve Γ_0.

(2) One could set up the surface S in the parameter space as a normal graph over the round cylinder $\mathscr{C}_0\times\mathbb{R}$, where $\mathscr{C}_0=\langle\cos s, \sin s, 0\rangle$. This somewhat simplifies the differential equation (3–10), but the initial data $\varphi(s,0)$ will no longer be zero.

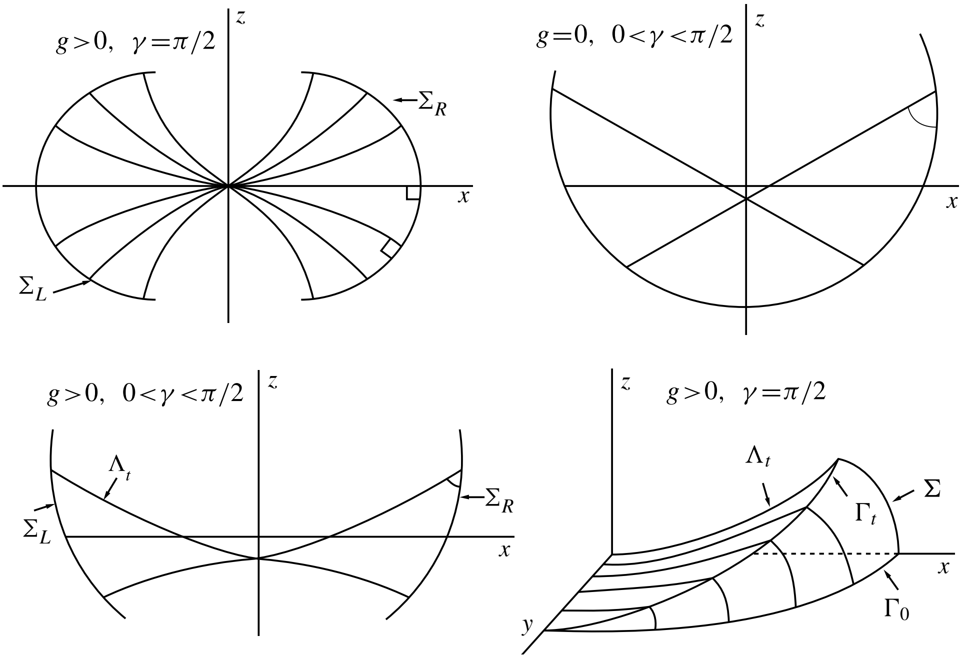

Figure 4. Qualitative appearance of a few exotic containers, showing the relationship of liquid surface and wall. Top row and bottom left: planar case; bottom right: three-dimensional case.

(3) The surface Σ constructed in Theorem 3.2 needs to be filled out. This surface is the union of boundary curves Γ_t. This set of curves Γ_t has an envelope that creates an edge for Σ. The curve Γ_t and the corresponding equilibrium surface Λ_t touch the envelope at points where the normal component of the variation $F_t \cdot \xi$ vanishes. The fixed volume condition (3–11) shows that Λ_t is divided into two regions determined by the sign of $F_t \cdot \xi$. The nodal curve on Λ_t will touch the envelope in two points. Consider the case $g = 0$, so that the extremals are tilted planes. Let Γ_0 be a circle centered at $(0, 0)$. For $\gamma = \pi/2$ the generated surface is a pair of sections of a sphere resembling orange peels (as the referee astutely remarked). In this case the envelope degenerates, becoming two points. To complete the surface one must extend Σ smoothly so that each curve Γ_t lies inside. By continuity the contact angle condition will prevail at the envelope. For $\gamma \neq \pi/2$ the envelope is a curve with each extremal touching the envelope in two points. Again we can extend Σ smoothly so that each extremal surface satisfies the contact angle condition everywhere.

The same discussion applies to the case when $g \neq 0$, with Γ_0 other than a circle. How one fills out the exotic container surface could affect the stability question. Letting the new pieces of surface bulge out increases the chance for stability.

4. Minimization

Given our family of elastic extrema for fixed $\kappa_0 = \rho g/\sigma$ and contact angle γ_0, we have shown how to construct an exotic container with the property that there exists a one-parameter family of equilibria (including the horizontal plane, $u = 0$), each of which meets the container wall at contact angle γ_0 and encloses the same volume V_0. It follows that each member of this family has the same potential energy. Are these equilibria local minimizers of energy subject to the volume constraint? I now outline an argument which indicates that this is the case in the planar case.

4.1. If $\kappa_0 = 0$ the equilibria are tilted lines and the exotic container will be a section of a circle. These are all minimizers.

4.2. Suppose $\kappa = \kappa_0 = \rho g/\sigma$ and $\tau = \tau_0 = \cos\gamma_0$. The corresponding exotic container consists of two curves: Σ_L on the left and Σ_R on the right. Both are symmetric about the z-axis. Let t be a vertical coördinate. Denote by $p(t)$ that point on Σ_L at level t and by $q(t)$ the corresponding point on Σ_R. For $t = 0$ we set $p(0) = p_0$, $q(0) = q_0$ with connecting extremal, $u \equiv 0$. By our exotic container construction this is part of a one-parameter family of extremal curves connecting Σ_L to Σ_R, each enclosing volume V_0 and making a contact angle γ_0 at each end. They are all extremals of the energy functional

$$E = |C| + \kappa_0 \int_T z\,dv - \tau_0|\Sigma'| \equiv E_0 - \tau_0|\Sigma'|. \tag{4–1}$$

Each of these extremals connect some point $p \in \Sigma_L$ to some point q on Σ_R. As p descends the corresponding q will rise.

4.3. There is a continuous map $(p, q) \mapsto C(p, q)$ defined for $p \in \Sigma_L$, $q \in \Sigma_R$ in a neighborhood of (p_0, q_0), where $C(p, q)$ is an extremal for E_0 connecting p to q, enclosing volume V_0, and with $C(p_0, q_0)$ being the extremal $u \equiv 0$. Each $C(p, q)$ will be a strong local minimizer of the energy E_0 for the fixed endpoint problem and subject to the volume constraint.

This is because the solution $u \equiv 0$, with contact angle γ_0, has the property that for the free boundary problem, the second variation is nonnegative for all volume-preserving perturbations with a one-dimensional kernel. The kernel of the corresponding variational problem is our given one-parameter family. The boundary values are not fixed here. For fixed boundary values, the extremal $u = 0$ is a strong local minimizer of energy, subject to the volume constraint. It follows that for any pair of points $p \in \Sigma_L$, $q \in \Sigma_R$ close to p_0, q_0 there will exist exactly one strong local minimizer of energy for the fixed boundary problem and enclosed volume V_0.

4.4. Let $p \in \Sigma_L$ be fixed and let $C(t) = C(p, q(t))$ be the extremal curve that joins p to $q(t)$. Let $\gamma(t)$ be the contact angle of $C(t)$ with Σ_R. We claim that $\gamma(t_1) > \gamma(t_2)$ when $t_1 < t_2$.

Recall that for the extremal $u \equiv 0$, the second variation of the energy was nonnegative with a one-dimensional kernel for the free endpoint problem and subject to the volume constraint. Now fix the point $p_0 \in \Sigma_L$ but let $q \in \Sigma_R$ vary. This is a semifree variational problem. The energy is $E = E_0 - \cos\gamma_0\,|\Sigma_R'|$, where Σ_R' is the wetted part of Σ_R. Because of the fixed point restriction the second variation of this functional is positive definite for volume preserving perturbations.

By continuity the same remains true for nearby extremals $C(p, q)$. Let the contact angle of $C(p, q)$ with Σ_R be γ_q. The relevant energy functional now is

$$E = E_0 - \cos\gamma_q\,|\Sigma_R'|.$$

Let $C(t)$ be the extremal connecting p to $q(t)$. Each $C(t)$ is an extremal for its energy

$$\frac{dE}{dt} = \frac{dE_0}{dt} - \cos\gamma(t)\,\frac{d|\Sigma_R'|}{dt} = 0.$$

Now fix $q_1 = q(t_1)$ with extremal $C(t_1)$. We compute the energy

$$E = E_0 - \cos\gamma_1\,|\Sigma_R'|$$

along $C(t)$. The energy functional will have a minimum for $t = t_1$. Let

$$e(t) = E_0 - \cos\gamma_1\,|\Sigma_R'|$$

denote this energy. We have $e'(t_1) = 0$, whereas $e'(t) > 0$ for $t > t_1$ and $e'(t) < 0$ for $t < t_2$. However,

$$e'(t) = \bigl(\cos\gamma(t) - \cos\gamma_1\bigr)\frac{d|\Sigma_R'|}{dt}.$$

Since $d|\Sigma_R'|/dt$ is positive, our assertion follows.

4.5. Let Γ be any curve connecting $p \in \Sigma_L$ to $q \in \Sigma_R$ that encloses volume V_0 and is C^0-close to the extremal $u \equiv 0$. We claim that the energy E of (4–1) applied to Γ is not less than the same energy for the curve, $u \equiv 0$.

First, we may replace Γ by $C(p, q)$. This decreases the energy E_0, and thus the energy E, since the wetted energy is unchanged. Now $C(p, q)$ connects p to $q = q(t)$ for some t. Let t^* be that value such that the extremal $C(p, q(t^*))$ meets Σ_R at contact angle γ_0. Then the contact angle at p for $C(t^*)$ is also γ_0. We apply the discussion in the preceding section to conclude that $E(C(t^*)) \le E(C(t))$, with E given by (4–1).

But $C(t^*)$ is part of our one-parameter family all having the same contact angle, γ_0. It follows that $E(\Gamma) \ge E(C(t^*)) = E(C_0)$. This concludes the argument.

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