Pacific Journal of Mathematics

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Volume 251 No. 2 June 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 251, No. 2, 2011

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We prove two Kazdan–Warner-type identities involving the renormalized volume coefficients $v^{(2k)}$ of a Riemannian manifold (M^n, g), the Gauss–Bonnet curvature G_{2r}, and a conformal Killing vector field on (M^n, g). In the case when the Riemannian manifold is locally conformally flat, we find

$$v^{(2k)} = (-2)^{-k}\sigma_k \quad \text{and} \quad G_{2r}(g) = \frac{4^r(n-r)!r!}{(n-2r)!}\sigma_r$$

and our results reduce to earlier ones established by Viaclovsky in 2000 and the second author in 2006.

1. Introduction

Theorem A [Viaclovsky 2000b; Han 2006a]. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, let $\sigma_k(g^{-1} \circ A_g)$ be the σ_k curvature of g, and let X be a conformal Killing vector field on (M, g). When $k \geq 3$, assume also that (M, g) is locally conformally flat. Then*

$$\int_M \langle X, \nabla\sigma_k(g^{-1} \circ A_g)\rangle dv_g = 0. \tag{1-1}$$

Recall that on an n-dimensional Riemannian manifold (M, g) with $n \geq 3$, the full Riemannian curvature tensor Rm decomposes as

$$Rm = W_g \oplus (A_g \odot g),$$

where W_g denotes the Weyl tensor of g,

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\Big)$$

denotes the Schouten tensor, and $\odot$ is the Kulkarni–Nomizu wedge product. Under a conformal change of metrics $g_w = e^{2w}g$, where w is a smooth function over the

Haizhong Li is supported by NSFC grant number 10971110.
MSC2000: primary 53C20; secondary 53A30.
Keywords: renormalized volume coefficients, $v^{(2k)}$ curvature, conformal transformation, locally conformally flat, σ_k curvature, Gauss–Bonnet curvatures, Kazdan–Warner.

manifold, the Weyl curvature changes pointwise as $W_{g_w} = e^{2w} W_g$. Thus, essential information about the Riemannian curvature tensor under a conformal change of metrics is reflected by the change in the Schouten tensor. One often tries to study the Schouten tensor through the elementary symmetric functions $\sigma_k(g^{-1} \circ A_g)$ (which we later denote as $\sigma_k(g)$) of the eigenvalues of the Schouten tensor, called the σ_k curvatures of g, by studying how they deform under conformal change of metrics.

Question. For all $k \geq 1$, can we generalize Theorem A without the condition that (M, g) is locally conformally flat?

In this note, we show the answer is yes. The renormalized volume coefficients $v^{(2k)}(g)$ of a Riemannian metric g, were introduced in the physics literature in the late 1990s in the context of AdS/CFT correspondence — see [Graham 2009] for a mathematical discussion — and were shown in [Graham and Juhl 2007] to be equal to $\sigma_k(g^{-1} A_g)$, up to a scaling constant, when (M, g) is locally conformally flat. In fact, in the normalization we are going to adopt,

$$v^{(2)}(g) = -\tfrac{1}{2}\sigma_1(g) \quad \text{and} \quad v^{(4)}(g) = \tfrac{1}{4}\sigma_2(g). \tag{1-2}$$

For $k = 3$, Graham and Juhl [2007, page 5] have also listed the formula

$$v^{(6)}(g) = -\frac{1}{8}\Big(\sigma_3(g) + \frac{1}{3(n-4)}(A_g)^{ij}(B_g)_{ij}\Big), \tag{1-3}$$

where

$$(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj}$$

is the *Bach* tensor of the metric. Just as $\int_M \sigma_k(g^{-1} \circ A_g)\, dv_g$ is conformally invariant when $2k = n$ and (M, g) is locally conformally flat, Graham [2009] showed that $\int_M v^{(2k)}(g)\, dv_g$ is also conformally invariant on a general manifold when $2k = n$. Chang and Fang [2008] showed that, for $n \neq 2k$, the Euler–Lagrange equations for the functional $\int_M v^{(2k)}(g)\, dv_g$ under conformal variations subject to the constraint $\mathrm{Vol}_g(M) = 1$ satisfies $v^{(2k)}(g) = \text{const}$, which is a generalized characterization for the curvatures $\sigma_k(g^{-1} \circ A_g)$ when (M, g) is locally conformally flat, as given by Viaclovsky [2000a].

In this note, we will first show that the curvatures $v^{(2k)}(g)$ will play the role of $\sigma_k(g^{-1} \circ A_g)$ in (1-1) for a general manifold. Graham [2009] also gives an explicit expression of $v^{(8)}(g)$, but the explicit expression of $v^{(2k)}(g)$ for general k is not known because they are algebraically complicated; see [Graham 2009, page 1958]. Thus the study of the $v^{(2k)}(g)$ curvatures involves significant challenges not shared by that of $\sigma_k(g)$: First, $v^{(2k)}(g)$ for $k \geq 3$ depends on derivatives of curvature of g; in fact, these depend on derivatives of curvatures of order up to $2k-4$. Second, the $v^{(2k)}(g)$ are defined in [Graham 2009] via an indirect, highly nonlinear inductive algorithm. Despite these difficulties, we can use some properties of these $v^{(2k)}(g)$ curvatures to prove the following.

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$, and let X be a conformal Killing vector field on (M^n, g). For $k \geq 1$, we have*

$$\int_M \langle X, \nabla v^{(2k)}(g)\rangle dv_g = 0. \tag{1-4}$$

Remark 1.2. From (1-2), we know that Theorem 1.1 is equivalent to Theorem A when $k = 1, 2$, or when (M^n, g) is locally conformally flat for $k \geq 3$.

One main reason for interest in identities such as (1-1) and (1-4) is that they play crucial roles in analyzing potentially blowing up conformal metrics with a prescribed curvature function, with $v^{(2k)}(g)$ prescribed in this case. Although little is known about this problem at this stage, Theorem 1.1 establishes one ingredient for attacking this problem.

Our second result involves the Gauss–Bonnet curvatures G_{2r} for $2r \leq n$, introduced by H. Weyl in 1939 and defined by

$$G_{2r}(g) = \delta^{j_1 j_2 \cdots j_{2r-1} j_{2r}}_{i_1 i_2 \cdots i_{2r-1} i_{2r}} R^{i_1 i_2}{}_{j_1 j_2} \cdots R^{i_{2r-1} i_{2r}}{}_{j_{2r-1} j_{2r}},$$

where $\delta^{j_1 j_2 \cdots j_{2r-1} j_{2r}}_{i_1 i_2 \cdots i_{2r-1} i_{2r}}$ is the generalized Kronecker symbol; see also [Labbi 2008]. Note that $G_2 = 2R$, with R the scalar curvature.

Theorem 1.3. *Let (M^n, g) be a compact Riemannian manifold, and let X be a conformal Killing vector field. Then for the Gauss–Bonnet curvatures defined above, we have*

$$\int_M \langle X, G_{2r}(g)\rangle dv_g = 0.$$

Remark 1.4. When (M, g) is locally conformally flat, we see that the Gauss curvature satisfies

$$G_{2r}(g) = \frac{4^r (n-r)! r!}{(n-2r)!} \sigma_r,$$

so Theorem 1.3 reduces to Theorem A.

Remark 1.5. M. Labbi [2008] proved that the first variation of the functional $\int_M G_{2r} dv_g$ within metrics with constant volume gave the so-called generalized Einstein metric, and this functional has the variational property for $2r < n$ and is a topological invariant for $2r = n$. In fact, if $n = 2r$, this functional is the Gauss–Bonnet integrand up to a constant [Chern 1944].

In the next section, we first provide a general proof for Theorem 1.1 by adapting an ingredient in a preprint version [Han 2006b] of [Han 2006a], and using of a variation formula for $v^{(2k)}(g)$ established in [Graham 2009] and [Chang and Fang 2008]. Because of the explicit expression for $v^{(6)}(g)$ and potential applications to other related problems in low dimensions, we provide in Section 3 a self-contained proof for Theorem 1.1 in the case $k = 3$. We prove Theorem 1.3 in Section 4.

2. Proof of Theorem 1.1

We will need the following variation formula for $v^{(2k)}(g)$; see [Graham 2009].

Proposition 2.1. *Under the conformal transformation $g_t = e^{2t\eta} g$, the variation of $v^{(2k)}(g_t)$ is given by*

$$\frac{\partial}{\partial t}\Big|_{t=0} v^{(2k)}(g_t) = -2k\eta v^{(2k)} + \nabla_i(L^{ij}_{(k)}\eta_j), \tag{2-1}$$

where $L^{ij}_{(k)}$ is defined as in [Graham 2009] *by*

$$L^{ij}_{(k)} = -\sum_{l=1}^{k} \frac{1}{l!} v^{(2k-2l)}(g)\partial_\rho^{l-1} g^{ij}(\rho)\big|_{\rho=0},$$

with $g_{ij}(\rho)$ denoting the extension of g such that

$$g_+ = \frac{(d\rho)^2 - 2\rho g(\rho)}{4\rho^2}$$

is an asymptotic solution to $\mathrm{Ric}(g_+) = -ng_+$ *near $\rho = 0$.*

An integral version of (2-1) first appeared in [Chang and Fang 2008]:

$$\int_M \Big(\frac{\partial}{\partial t}\Big|_{t=0}(v^{(2k)}(g_t)) + 2k\eta v^{(2k)}(g)\Big) dv_g = 0.$$

Proof of Theorem 1.1 in the case $n \neq 2k$. Let X be a conformal vector field on M. Let ϕ_t denote the local one-parameter family of conformal diffeomorphisms of (M, g) generated by X. Thus for some smooth function ω_t on M, we have

$$\phi_t^*(g) = e^{2\omega_t} g =: g_t.$$

We have the properties

$$\phi_t^* v^{(2k)}(g) = v^{(2k)}(\phi_t^* g) = v^{(2k)}(e^{2\omega_t} g), \tag{2-2}$$

$$\dot\omega := \frac{d}{dt}\Big|_{t=0}\omega_t = \frac{\mathrm{div} X}{n}, \tag{2-3}$$

$$\frac{\partial}{\partial t}\Big|_{t=0}(g_t^{-1}\circ A(g_t)) = -\nabla^2\dot\omega - 2\dot\omega g^{-1}\circ A(g), \tag{2-4}$$

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathrm{div}_{g_t} X = nX\eta = n\langle X, \nabla\eta\rangle. \tag{2-5}$$

Using (2-2), (2-3), and (2-1), we have

$$\begin{aligned}\langle X, \nabla v^{(2k)}(g)\rangle &= \frac{\partial}{\partial t}\Big|_{t=0}(v^{(2k)}(g_t))\\ &= -2k\dot{\omega}v^{(2k)} + \nabla_i(L^{ij}_{(k)}\dot{\omega}_j)\\ &= -\frac{2k}{n}(\operatorname{div} X)v^{(2k)} + \nabla_i(L^{ij}_{(k)}\dot{\omega}_j)\\ &= -\frac{2k}{n}\operatorname{div}(v^{(2k)}X) + \frac{2k}{n}\langle X, \nabla v^{(2k)}(g)\rangle + \frac{1}{n}\nabla_i(L^{ij}_{(k)}(\operatorname{div} X)_j),\end{aligned}$$

from which it follows that

$$\Big(1 - \frac{2k}{n}\Big)\langle X, \nabla v^{(2k)}(g)\rangle = -\frac{2k}{n}\operatorname{div}(v^{(2k)}X) + \frac{1}{n}\nabla_i(L^{ij}_{(k)}(\operatorname{div} X)_j). \tag{2-6}$$

Theorem 1.1 in the case $2k \neq n$ follows directly by integrating (2-6) over M. □

Proof of Theorem 1.1 in the case $2k = n$. As in [Han 2006b], we will prove that for any conformal metric $g_1 = e^{2\eta}g$ of g,

$$\int_M \langle X, v^{(2k)}(g_1)\rangle dv_{g_1} = \int_M \langle X, v^{(2k)}(g)\rangle dv_g = -\int_M \operatorname{div}_g X v^{(2k)}(g) dv_g,$$

that is, $\int_M \langle X, v^{(2k)}(g)\rangle dv_g$ is independent of the particular choice of metric in the conformal class. We only have to prove that

$$\frac{\partial}{\partial t}\Big|_{t=0}\int_M \operatorname{div}_{g_t} X v^{(2k)}(g_t) dv_{g_t} = 0 \quad \text{for } g_t = e^{2t\eta}g. \tag{2-7}$$

We prove (2-7) by direct computations using Proposition 2.1. Indeed,

$$\begin{aligned}&\frac{\partial}{\partial t}\Big|_{t=0}\int_M \operatorname{div}_{g_t} X v^{(2k)}(g_t) dv_{g_t}\\ &\quad= \int_M \big(n\langle X, \nabla\eta\rangle v^{(2k)} + \operatorname{div} X(-2k\eta v^{(2k)} + \nabla_i(L^{ij}_{(k)}\eta_j)) + n\eta \operatorname{div} X v^{(2k)}\big)dv_g\\ &\quad= \int_M \big(n\langle X, \nabla\eta\rangle v^{(2k)} + \operatorname{div} X\nabla_i(L^{ij}_{(k)}\eta_j)\big)dv_g\\ &\quad= \int_M \big(\langle nv^{(2k)}X, \nabla\eta\rangle - L^{ij}_{(k)}(\operatorname{div} X)_i\eta_j\big)dv_g\\ &\quad= \int_M \big(-\operatorname{div}(nv^{(2k)}X) + \nabla_j(L^{ij}_{(k)}(\operatorname{div} X)_i)\big)\eta dv_g = 0\end{aligned}$$

in the case $n = 2k$ by (2-6).

The remaining argument is an adaptation of an argument of Bourguignon and Ezin [1987]: either the connected component of the identity of the conformal group $C_0(M, g)$ is compact, and then there is a metric $\hat{g}$ conformal to g admitting $C_0(M, g)$ as a group of isometries, from which it follows that $\operatorname{div}_{\hat{g}} X \equiv 0$ and therefore (1-4) holds; or, $C_0(M, g)$ is noncompact, and then by a theorem of

Obata and Ferrand, (M, g) is conformal to the standard sphere, in which case we can pick the canonical metric to compute the integral on the left hand side of (1-4) and conclude that it is zero. $\square$

3. A self-contained proof of Theorem 1.1 in the case $k = 3$

We aim to give a direct, self-contained derivation for a more explicit version of (2-1); namely, under conformal change of metric $g_t = e^{2t\eta} g$,

$$\frac{\partial}{\partial t}\Big|_{t=0} v^{(6)}(g_t) = -6v^{(6)}(g)\eta + \nabla^j\left(\left(\frac{T_{ij}^{(2)}(g)}{8} + \frac{B_{ij}(g)}{24(n-4)}\right)\nabla^i\eta\right), \tag{3-1}$$

where $T_{ij}^{(2)}(g)$ is the Newton tensor associated with A_g, as defined in Reilly [1977]:

Definition. For an integer $k \geq 0$, the k-th Newton tensor is

$$T_{ij}^{(k)} = \frac{1}{k!}\sum \delta_{i_1\cdots i_k i}^{j_1\cdots j_k j} A_{i_1 j_1}\cdots A_{i_k j_k},$$

where $\delta_{i_1\cdots i_k i}^{j_1\cdots j_k j}$ is the generalized Kronecker symbol.

With (3-1) we can repeat the proof in the last section to prove Theorem 1.1 in the case $k = 3$.

First we recall the transformation laws for the tensors B_{ij} and A_{ij} under conformal change of metric $g_t = e^{2t\eta} g$ — see [Chang and Fang 2008]:

$$A_{ij}(g_t) = A_{ij} - t\nabla_{ij}^2\eta + t^2\nabla_i\eta\nabla_j\eta - \tfrac{1}{2}t^2|\nabla\eta|_g^2 g_{ij},$$
$$B_{ij}(g_t) = e^{-2t\eta}\big(B_{ij} + (n-4)t(C_{ijk} + C_{jik})\nabla^k\eta + (n-4)t^2 W_{ikjl}\nabla^k\eta\nabla^l\eta\big),$$

where $C_{ijk} := A_{ij,k} - A_{ik,j}$ are the components of the *Cotton* tensor, with $A_{ij,k}$ the components of the covariant derivative of the Schouten tensor A_{ij}.

Thus

$$\frac{\partial}{\partial t}\Big|_{t=0} A^{ij}(g_t) = -\nabla^{ij}\eta - 4A^{ij}(g)\eta,$$
$$\frac{\partial}{\partial t}\Big|_{t=0} B_{ij}(g_t) = (n-4)(C_{ijk} + C_{jik})\nabla^k\eta - 2\eta B_{ij}.$$

Proposition 3.1 [Viaclovsky 2000a; Han 2006b; Hu and Li 2004]. *We have*

(i) $k\sigma_k(g) = \sum_{i,j} T_{ij}^{(k-1)} A_{ij}$

(ii) $\sum_i T_{ii}^{(k)} = (n-k)\sigma_k(g).$

(iii) $\sum_l \nabla^l W_{lijk} = -(n-3)C_{ijk}.$

Using the relation between $v^{(6)}$ and $\sigma_3(g)$, and with $A^{ij}B_{ij}$ as in (1-3), we find

$$\begin{aligned}
&-8\frac{\partial}{\partial t}\Big|_{t=0} v^{(6)}(g_t)\\
&= T^{(2)}_{ij}(g)\big(-\nabla^{ij}\eta - 2\eta A^{ij}(g)\big)\\
&\quad + \frac{1}{3(n-4)}\big(-B_{ij}(g)\nabla^{ij}\eta + (n-4)A^{ij}(g)(C_{ijk}+C_{jik})\nabla^k\eta - 6\eta A^{ij}B_{ij}\big)\\
&= -6\Big(\sigma_3(g) + \frac{1}{3(n-4)}A^{ij}B_{ij}\Big)\eta - \Big(T^{(2)}_{ij}(g) + \frac{B_{ij}(g)}{3(n-4)}\Big)\nabla^{ij}\eta + \tfrac{2}{3}A^{ij}(g)C_{ijk}\nabla^k\eta\\
&= 48v^{(6)}(g)\eta - \nabla^j\Big(\Big(T^{(2)}_{ij}(g) + \frac{B_{ij}(g)}{3(n-4)}\Big)\nabla^i\eta\Big)\\
&\quad + \Big(\sum_j\Big(T^{(2)}_{ij,j}(g) + \frac{B_{ij,j}(g)}{3(n-4)}\Big) + \tfrac{2}{3}A^{kl}C_{kli}\Big)\nabla^i\eta,
\end{aligned}$$

where we used (1-3) and Proposition 3.1(i). The following lemma implies that

$$\sum_j\Big(T^{(2)}_{ij,j}(g) + \frac{B_{ij,j}(g)}{3(n-4)}\Big) + \tfrac{2}{3}A^{kl}C_{kli} = 0,$$

thus establishing (3-1).

Lemma 3.2. (i) $\displaystyle\sum_j T^{(2)}_{ij,j} = -A^{pq}C_{pqi}$.

(ii) $\displaystyle\sum_j B_{ij,j} = (n-4)A^{kl}C_{kli}$.

Proof of (i). In normal coordinates, we have

$$\sum_j T^{(2)}_{ij,j} = \sum\Big(\frac{1}{2!}\sum \delta^{j_1j_2j}_{i_1i_2i}A_{i_1j_1}A_{i_2j_2}\Big)_j = \sum \delta^{j_1j_2j}_{i_1i_2i}A_{i_1j_1}A_{i_2j_2,j} = -A^{pq}C_{pqi},$$

where we used

$$\delta^{j_1j_2j}_{i_1i_2i} = \begin{vmatrix} \delta_{i_1j_1} & \delta_{i_1j_2} & \delta_{i_1j} \\ \delta_{i_2j_1} & \delta_{i_2j_2} & \delta_{i_2j} \\ \delta_{ij_1} & \delta_{ij_2} & \delta_{ij} \end{vmatrix}$$

and $\sum_i A_{ii,j} = \sum_i A_{ij,i}$, itself a consequence of the second Bianchi identity. □

Proof of (ii). First, using Proposition 3.1(iii) and substituting R_{ij} in terms of A_{ij} in the definition of the Bach tensor B_{ij}, we obtain

$$\begin{aligned}
B_{ij} &= -\sum_k C_{ikj,k} + \sum_{k,l} A_{kl}W_{likj}\\
&= -\sum_k (A_{ik,jk} - A_{ij,kk}) + \sum_{k,l} A_{kl}W_{likj}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\sum_j B_{ij,j}\\
&= -\sum_{j,k}(A_{ik,jkj}-A_{ij,kkj})+\sum_{k,l,j}(A_{kl,j}W_{likj}+A_{kl}W_{likj,j})\\
&= -\sum_{j,k}(A_{ik,jkj}-A_{ik,jjk})+\sum_{k,l,j}A_{kl,j}W_{likj}-(n-3)\sum_{k,l}A_{kl}C_{kil}\\
&= -\sum_{j,k,m}(A_{ik,m}R_{mjkj}+A_{im,j}R_{mkkj}+A_{mk,j}R_{mikj})\\
&\quad +\sum_{k,l,j}A_{kl,j}W_{likj}+(n-3)\sum_{k,l}A_{kl}C_{kli}\\
&= \sum_{j,k,m}(-A_{mk,j}R_{mikj}+A_{km,j}W_{mikj})+(n-3)\sum_{k,l}A_{kl}C_{kil}\\
&= \sum_{j,k,m}A_{mk,j}(-A_{mk}g_{ij}+A_{mj}g_{ik}-g_{mk}A_{ij}+g_{mj}A_{ik})+(n-3)\sum_{k,l}A_{kl}C_{kli}\\
&= \sum_{m,k}(-A_{mk,i}A_{mk}+A_{mi,k}A_{mk}-A_{mk,j}g_{mk}A_{ij}+A_{mj,k}g_{mk}A_{ij})\\
&\quad +(n-3)\sum_{k,l}A_{kl}C_{kli}\\
&= \sum_{m,k}A_{mk}(A_{mi,k}-A_{mk,i})+(n-3)\sum_{k,l}A_{kl}C_{kli}\\
&= \sum_{m,k}A_{mk}C_{mik}+(n-3)\sum_{k,l}A_{kl}C_{kli}\\
&= (n-4)\sum_{k,l}A_{kl}C_{kli},
\end{aligned}
$$

where we have used

$$R_{mikj}=W_{mikj}+A_{mk}g_{ij}-A_{mj}g_{ik}+g_{mk}A_{ij}-g_{mj}A_{ik}. \qquad \square$$

Proof of Theorem 1.1 in the special case $k=3$. We use the notation of Section 2. Let ϕ_t be the local one-parameter family of conformal diffeomorphisms of (M,g) generated by X. For $g_t=\phi_t^*(g)=e^{2\omega_t}g$, similarly to (3-1), we have

$$
\begin{aligned}
\langle X, v^{(6)}\rangle &= \frac{\partial}{\partial t}\Big|_{t=0} v^{(6)}(g_t)\\
&= -6v^{(6)}(g)\dot\omega+\sum_{i,j}\nabla^j\left(\left(\frac{T^{(2)}_{ij}(g)}{8}+\frac{B_{ij}(g)}{24(n-4)}\right)\nabla^i\dot\omega\right),
\end{aligned}
\tag{3-2}
$$

if $n \neq 2k$. Then integrating (3-2) we can get Theorem 1.1.

If $n = 2k$, then by use of (3-1) and (3-2), we can prove that $\int_M \langle X, v^{(6)}(g)\rangle dv_g$ is independent of the particular choice of the metric within the conformal class. The remainder of the proof repeats verbatim that of Section 2. □

4. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 using a method similar to the one used in Section 2. Let (M^n, g) be a compact Riemannian manifold, and denote by R_{ijkl} the Riemann curvature tensor in local coordinates. Define a tensor P_r by

$$P_r{}_i^j = \delta^{jj_1j_2\cdots j_{2r-1}j_{2r}}_{ii_1i_2\cdots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_1j_2} \cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \quad \text{for } 2r \le n,$$

where $\delta^{jj_1j_2\cdots j_{2r-1}j_{2r}}_{ii_1i_2\cdots i_{2r-1}i_{2r}}$ is the generalized Kronecker symbol.

Lemma 4.1. *The tensor P_r is divergence free, that is,*

$$P_r{}_{i,j}^j = 0 \quad \textit{for any } i.$$

This property was present in [Labbi 2008] and [Lovelock 1971], although with different notation and formalism. Since we define the tensor P_r explicitly as above, and the property of P_r in Lemma 4.1 is a direct consequence of the Bianchi identity, we include a proof here.

Proof. We have

$$\begin{aligned} P_r{}_{i,j}^j &= r\delta^{jj_1j_2\ldots j_{2r-1}j_{2r}}_{ii_1i_2\ldots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_1j_2,j} \cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\ &= -r\delta^{jj_1j_2\cdots j_{2r-1}j_{2r}}_{ii_1i_2\cdots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_2j,j_1} \cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\ &\quad - r\delta^{jj_1j_2\cdots j_{2r-1}j_{2r}}_{ii_1i_2\cdots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{jj_1,j_2} \cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\ &= -2r\delta^{jj_1j_2\cdots j_{2r-1}j_{2r}}_{ii_1i_2\cdots i_{2r-1}i_{2r}} R^{i_1i_2}{}_{j_1j_2,j} \cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}} \\ &= -2P_r{}_{i,j}^j, \end{aligned}$$

where we have used the second Bianchi identity. It then follows that $P_r{}_{i,j}^j = 0$. □

Lemma 4.2. *The generalized Kronecker symbol satisfies*

$$\sum_{i,j=1}^n \delta^i_j \delta^{jj_1\ldots j_r}_{ii_1\ldots i_r} = (n-r)\delta^{j_1\ldots j_r}_{i_1\ldots i_r} \quad \textit{for any } 1 \le i_1, \ldots, j_r \le n \textit{ and } r \le n.$$

The proof follows by a direct calculation from the definition.

Let X be a conformal vector field, and denote by ϕ_t the one-parameter subgroup of diffeomorphisms generated by X. Then there exists a family of functions ω_t such that $g_t=\phi_t^*g=e^{2\omega_t}g$. We have (2-3), $\omega_0=0$, and

$$G_{2r}(g_t)=\phi_t^*G_{2r}(g). \tag{4-1}$$

Under the conformal change of metric $g_t=e^{2\omega_t}g$, we have the formula (see for example [Chow et al. 2006])

$$R^{ij}{}_{kl}(g_t)=e^{-2\omega_t}(R^{ij}{}_{kl}-(\alpha\odot g)^{ij}{}_{kl}), \tag{4-2}$$

where we denote $\alpha_{ij}=(\omega_t)_{ij}-(\omega_t)_i(\omega_t)_j+\frac{1}{2}|\nabla\omega_t|^2g_{ij}$ for convenience (note that $(\omega_t)_{ij}$ is the covariant derivative with respect to the fixed metric g) and $\odot$ is the Kulkarni–Nomizu product, defined by

$$(\alpha\odot g)_{ijkl}=\alpha_{ik}g_{jl}+\alpha_{jl}g_{ik}-\alpha_{il}g_{jk}-\alpha_{jk}g_{il}.$$

From (4-2) we see that

$$\begin{aligned} G_{2r}(g_t)=e^{-2r\omega_t}&\delta^{j_1j_2\dots j_{2r-1}j_{2r}}_{i_1i_2\dots i_{2r-1}i_{2r}}\\ &\cdot\big(R^{i_1i_2}{}_{j_1j_2}-(\alpha\odot g)^{i_1i_2}{}_{j_1j_2}\big)\cdots\big(R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}}-(\alpha\odot g)^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}}\big). \end{aligned} \tag{4-3}$$

Taking derivative with respect to t on both sides of (4-1) and using (4-3), we see by using (2-3) that

$$\begin{aligned} &\langle X, G_{2r}(g)\rangle\\ &=\frac{\partial}{\partial t}\Big|_{t=0}G_{2r}(g_t)\\ &=-2r\dot\omega G_{2r}(g)-r\delta^{j_1j_2\cdots j_{2r-1}j_{2r}}_{i_1i_2\cdots i_{2r-1}i_{2r}}\Big(\frac{\partial\alpha}{\partial t}\Big|_{t=0}\odot g\Big)^{i_1i_2}{}_{j_1j_2}R^{i_3i_4}{}_{j_3j_4}\cdots R^{i_{2r-1}i_{2r}}{}_{j_{2r-1}j_{2r}}\\ &=-2r\dot\omega G_{2r}(g)-4r(n-2r+1)P_{r-1}{}_i^j\,\dot\omega^i_j\\ &=-2r\frac{\operatorname{div}X}{n}G_{2r}(g)-\frac{4r(n-2r+1)}{n}P_{r-1}{}_i^j(\operatorname{div}X)^i{}_j\\ &=-2r\frac{\operatorname{div}X}{n}G_{2r}(g)-\frac{4r(n-2r+1)}{n}\nabla_j\big(P_{r-1}{}_i^j(\operatorname{div}X)^i\big). \end{aligned} \tag{4-4}$$

where we have used Lemma 4.2 in the third equality and Lemma 4.1 in the last. Integrating (4-4) over M and using the divergence theorem, we see that

$$\int_M\langle X, G_{2r}(g)\rangle dv=-2r\int_M\frac{\operatorname{div}X}{n}G_{2r}(g)dv=\frac{2r}{n}\int_M\langle X, G_{2r}(g)\rangle dv, \tag{4-5}$$

Hence, if $n>2r$, it follows from (4-5) that $\int_M\langle X, G_{2r}(g)\rangle dv=0$. If $n=2r$, we follow ideas in Section 2, that is, we need to prove that the integral

$$\int_M G_{2r}(g)\operatorname{div}_g Xdv_g,$$

is independent of a particular choice of metric within a conformal class. Let $g_1 = e^{2\eta} g (\eta \in C^\infty(M))$ be any metric in the conformal class $[g]$. Considering a family of metrics $g_t = e^{2t\eta} g$ connecting g and g_1, we need to prove that

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_M G_{2r}(g_t) \operatorname{div}_{g_t} X \, dv_{g_t} = 0.$$

By a direct computation, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0} \int_M G_{2r}(g_t) \operatorname{div}_{g_t} X \, dv_{g_t} \\
&= \int_M \Big(\frac{\partial}{\partial t}\Big|_{t=0} G_{2r}(g_t) \operatorname{div} X + G_{2r}(g) \frac{\partial}{\partial t}\Big|_{t=0} \operatorname{div}_{g_t} X + n\eta G_{2r}(g) \operatorname{div} X \Big) dv_g \\
&= \int_M \big(-2r\eta G_{2r}(g) \operatorname{div} X - 4r(n-2r+1) {P_{r-1}}_i^{\,j} \eta^i{}_j \operatorname{div} X \\
&\qquad\qquad + nG_{2r}(g)\langle \nabla\eta, X\rangle + nG_{2r}(g) \operatorname{div} X\eta \big) dv_g \\
&= \int_M \big(-2r\eta G_{2r}(g) \operatorname{div} X - 4\eta r(n-2r+1) {P_{r-1}}_i^{\,j} (\operatorname{div} X)^i_j \\
&\qquad\qquad - n\eta \langle \nabla G_{2r}(g), X\rangle \big) dv_g \\
&= 0,
\end{aligned}$$

where we have used (2-5) in the second equality, the divergence theorem in the third and (4-4) in the last. The remainder of the proof follows the idea of [Bourguignon and Ezin 1987] as in Section 2. Hence we complete the proof of Theorem 1.3.

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Received May 18, 2010.

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PACIFIC JOURNAL OF MATHEMATICS

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Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The subscription price for 2011 is US $420/year for the electronic version, and $485/year for print and electronic.
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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 251 No. 2 June 2011

0030-8730(201106)251:2;1-C