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Let C be an ACM (projectively normal) nonsingular curve in $\mathbb{P}^3_{\mathbb{C}}$ not contained in a plane, and suppose C is general in its Hilbert scheme — this is irreducible once the postulation is fixed. Answering a question posed by Peskine, we show the gonality of C is $d - l$, where d is the degree of the curve and l is the maximum order of a multisecant line of C. Furthermore $l = 4$ except for two series of cases, in which the postulation of C forces every surface of minimum degree containing C to contain a line as well. We compute the value of l in terms of the postulation of C in these exceptional cases. We also show the Clifford index of C is equal to $\operatorname{gon}(C) - 2$.

1. Introduction

Let C be a nonsingular projective curve over an algebraically closed field $\mathbb{K}$. The *gonality* of C, written $\operatorname{gon}(C)$, is the minimum degree of a surjective morphism $C \to \mathbb{P}^1$, or equivalently the minimum positive integer k such that there exists a g^1_k on C.

For curves of genus $g \geq 1$ the gonality varies between 2, the value it takes on hyperelliptic curves, and $\left[\frac{1}{2}(g+3)\right]$, which by Brill–Noether theory is the gonality of a general curve of genus g. It may be regarded as the most fundamental invariant of the algebraic structure of C after the genus, providing a stratification of the moduli space of curves of genus g.

When a curve is embedded in some projective space, it is natural to wonder whether the gonality may be related to extrinsic properties of the curve. Here is a classical result in this direction, already known to Noether [Ciliberto 1984; Hartshorne 1986]:

Theorem 1.1. *A smooth curve $C \subset \mathbb{P}^2$ of degree $d \geq 3$ has gonality $\operatorname{gon}(C) = d-1$, and any morphism $C \to \mathbb{P}^1$ of degree $d-1$ is obtained by projecting C from one of its points.*

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See [Hartshorne 2002] for a proof and references. It is a simple exercise to prove the statement using the method of [Lazarsfeld 1986], which associates a vector bundle on $\mathbb{P}^2$ to a basepoint-free pencil on C. It is this method that we will exploit in the proof of our result.

One may ask a similar question for a curve $C \subset \mathbb{P}^3$. If L is a line in $\mathbb{P}^3$, projection from L induces a morphism $\pi_L : C \to \mathbb{P}^1$, whose degree is the degree of C minus the number of points of intersection of C and L. Thus the morphisms π_L of minimal degree are those corresponding to maximal order multisecant lines. We define

$$l = l(C) = \max\{\deg(C \cap L) : L \text{ a line in } \mathbb{P}^3\}$$

By analogy with the plane curves case one might wonder whether

$$\operatorname{gon}(C) = \deg(C) - l(C) \tag{1-1}$$

for a curve in $\mathbb{P}^3$, in which case following the terminology of [Hartshorne 2002] we say the gonality of C is computed by multisecants. Of course, this is usually not the case. For example, a general curve of genus g has gonality $\left[\frac{1}{2}(g+3)\right]$ and can be embedded in $\mathbb{P}^3$ as a nonspecial linearly normal curve of degree $g+3$. Since the Grassmannian of lines in $\mathbb{P}^3$ has dimension 4, and the set of lines meeting C is a codimension one subvariety, one expects $l(C)$ to be 4, and so

$$\deg(C) - l(C) = g - 1 > \left[\tfrac{1}{2}(g+3)\right] = \operatorname{gon}(C).$$

See [Hartshorne 2002, Examples 2.8 and 2.9] for specific counterexamples.

On the other hand, if the embedding of C in $\mathbb{P}^3$ is very special, one may hope the gonality of C is computed by multisecants. In this vein Peskine raised the question:

Question 1.2. *If C is a smooth ACM curve in $\mathbb{P}^3$, is its gonality computed by multisecants?*

Here ACM means arithmetically Cohen–Macaulay: a curve in $\mathbb{P}^3$ is ACM if the natural maps $H^0(\mathbb{P}^3, \mathcal{O}(n)) \to H^0(C, \mathcal{O}_C(n))$ are surjective for every $n \geq 0$.

Some special cases have been treated in the literature. Early results about uniqueness of the linear series $|\mathcal{O}_C(1)|$ for complete intersections and other ACM curves are in [Ciliberto and Lazarsfeld 1984]. Basili [1996] has proven that the gonality of a smooth complete intersection is indeed computed by multisecants. Ellia and Franco [2001] showed that the maximum order l of a multisecant to a general complete intersection of type (a, b) is 4 if $a \geq b \geq 4$ as one expects. Lazarsfeld [1997, 4.12] finds lower bounds for the gonality of a complete intersection curve in $\mathbb{P}^n$.

Results from [Martens 1996] and [Ballico 1997] show that the gonality of a smooth curve $C \subset \mathbb{P}^3$ on a smooth quadric surface is computed by multisecants.

In [Hartshorne 2002] it is shown that if a smooth curve $C \subset \mathbb{P}^3$ is ACM, lies on a smooth cubic surface X, and is general in its linear system on X, then its gonality is computed by multisecants. Farkas [2001] has shown that smooth ACM curves $C \subset \mathbb{P}^3$ lying on certain smooth quartic surfaces that do not contain rational or elliptic curves have gonality computed by multisecants.

In this paper, we show that, with the exception of very few cases we cannot decide, the gonality of a *general* ACM curve is indeed computed by multisecants. We have to make sense of the expression general ACM curve. To obtain an irreducible parameter space for ACM curves one needs to fix the *Hilbert function*, that is, the sequence of integers $h^0(\mathbb{O}_C(n))$. This is more conveniently expressed by its second difference or h-vector:

$$h_C(n) = h^0(\mathbb{O}_C(n)) - 2h^0(\mathbb{O}_C(n-1)) + h^0(\mathbb{O}_C(n-2)).$$

which has the advantage of being finitely supported while still nonnegative. We will denote by $A(h)$ the Hilbert scheme parametrizing ACM curves in $\mathbb{P}^3$ with h-vector h. By a theorem of Ellingsrud (see Remark 6.4), the Hilbert scheme $A(h)$ is *smooth and irreducible*. Thus by a general ACM curve we will mean a curve in a Zariski open nonempty subset of $A(h)$. We believe it is reasonable to assume that C is general in the statement of our theorem, because it might happen that a special ACM curve had a low degree pencil unrelated to the line bundle $\mathbb{O}_C(1)$.

Theorem 1.3. *Assume $\mathbb{K} = \mathbb{C}$ is the field of complex numbers. Let $C \subset \mathbb{P}^3$ be a nonplanar smooth ACM curve. If C is general in the Hilbert scheme $A(h_C)$, then*

$$\operatorname{gon}(C) = d - l,$$

where $d = \deg(C)$ and $l = l(C)$ is the maximum order of a multisecant line to C, except perhaps when the degree d, the genus g and the least degree s of a surface containing C form one of the following triples: $(15, 26, 5)$, $(16, 30, 5)$, $(21, 50, 6)$, $(22, 55, 6)$, $(23, 60, 6)$, $(28, 85, 7)$, $(29, 91, 7)$, $(36, 133, 8)$.

For curves C contained in a quadric or a cubic surface, the statement follows from the references cited above. So our contribution is for curves not lying on a cubic surface.

We can also determine the integer $l(C)$ in terms of the h-vector of C. Most of the time $l(C) = 4$, with two families of exceptions. These exceptional cases arise because the h-vector forces surfaces of minimal degree containing C to contain a line as well; this line is then a multisecant of order higher than expected. If s as above denotes the *least degree* of a surface containing C, we let

$$t = \min\{n : h^0(\mathcal{I}_C(n)) - h^0(\mathbb{O}_{\mathbb{P}^3}(n-s)) > 0\},$$

so that (s,t) is the smallest type of a complete intersection containing C. We denote by e the *index of speciality* of C: $e = \max\{n : h^1\mathcal{O}_C(n) > 0\}$.

The value of $l(C)$ is given as follows:

Theorem 1.4. *Let $C \subset \mathbb{P}^3_{\mathbb{C}}$ be a general smooth ACM curve with $s \geq 4$. Let $l = l(C)$ denote the maximum order of a multisecant line to C. Then $l = 4$, unless*

- *the h-vector of C satisfies $h(e+1) = 3$ and $h(e+2) = 2$, in which case $l = e+3$ and C has a unique $(e+3)$-secant line, or*
- *$t > s+3$ and the h-vector of C satisfies $h(t) = s-2$ and $h(t+1) = s-3$, but not $h(e+1) = 3$, $h(e+2) = 2$, in which case $l = t-s+1$ and C has a unique $(t-s+1)$-secant line.*

Nollet [1998] has found a sharp bound for the maximal order $l = l(C)$ of a multisecant line in terms of the h-vector of C, valid for *any* irreducible ACM curve. If C is not a complete intersection, the bound is the largest integer n for which $h_C(n-1) - h_C(n) > 1$. Since this number is at least s, we see that $l(C)$ and the gonality of C *vary* in the family $A(h)$, provided $s \geq 5$, and the gonality of the general curve is $d-4$ (in fact the argument of Theorem 4.1 shows that $l(C)$ varies in the linear system $|C|$ on a smooth surface X of degree $s \geq 5$). On the other hand, in the special case $h(e+1) = 3$ and $h(e+2) = 2$, then Nollet's bound is precisely $e+3$, so that $l(C)$ is constant in $A(h)$.

Finally, in most cases we can prove that *every* pencil computing the gonality of C arises from a maximum order multisecant: the finite list of exceptions is given in Theorem 9.1. In particular, C has a finite number of pencils of minimal degree, and therefore its Clifford index is $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2 = d - l(C) - 2$.

It would be interesting to investigate linear series g^r_k on general ACM curves also for $r \geq 2$. For results in this direction we refer to [Lopez and Pirola 1995].

Outline of proof and structure of the paper. Since the conclusions of our result are semicontinuous on the Hilbert scheme $A(h)$, it suffices to show the existence of a *single* curve C for which the result holds. Let C be a smooth ACM curve in $\mathbb{P}^3$ with given h-vector h, not lying on any surface of degree at most 3. In Section 3 we review the classical result that for every smooth space curve D of degree at least 10 there exists a line L that is at least a 4-secant line of D. Thus $\mathrm{gon}(C) \leq d-4$. Next, if C is general in $A(h)$, it is contained in a *smooth* surface X of degree s. We prove in Corollary 4.2 that, if C is general in its linear system on X and L is an l-secant line of C with $l \geq 5$, then L is contained in X. In fact, we prove a slightly more general result, which gives explicit conditions for a space curve not to have 5-secant lines:

Theorem 4.1. *Let $C \subset \mathbb{P}^3_{\mathbb{K}}$ be a curve contained in an irreducible surface X of degree s. Suppose C is a Cartier divisor on X and*

$$H^0(\mathbb{P}^3, \mathcal{I}_C(s-2)) = 0, \quad H^1(\mathbb{P}^3, \mathcal{I}_C(m)) = 0 \text{ for } m = s-2, s-3, s-4.$$

If C is general in its linear system on X, then $\deg(C \cap L) \leq 4$ *for every line L not contained in X, and C has only finitely many* 4*-secant lines not contained in X.*

In particular, if X does not contain a line, then C does not have an l-secant line for any $l \geq 5$.

At this point to prove our main theorem we need to show that every pencil of minimal degree arises from a multisecant line. The proof uses the technique from [Lazarsfeld 1986], which associates to a basepoint-free pencil on C a vector bundle $\mathscr{E}$ on the surface X, as explained in Section 5. In Section 6 we review enough liaison theory for ACM curves to be able to show that the bundle $\mathscr{E}$ is Bogomolov unstable. Thus it has a destabilizing divisor $A \in \mathrm{Pic}(X)$, whose degree $x = A.H$ satisfies stringent numerical restrictions in terms of the intersection numbers A^2, $A.C$ and C^2.

To use these constraints effectively we need to control the Picard group of X. The hypothesis that the ground field is $\mathbb{C}$ allows us to apply the Noether–Lefschetz type theorem of [Lopez 1991, II.3.1] or the more recent [Brevik and Nollet 2008] to conclude that, if C is general in $A(h)$ and X is very general among surfaces of minimal degree containing C, then $\mathrm{Pic}(X)$ is freely generated by H and the irreducible components of a curve Γ that is general among curves minimally linked to C. Such a Γ is a general ACM curve, but it may not be irreducible. Thus we are led to establish a structure theorem for general ACM curves. Section 7 is devoted to the proof of this result. It generalizes Gruson–Peskine's theorem [1978], according to which the general ACM curve in $A(h)$ is smooth and irreducible if h is of decreasing type ("has no gaps"):

Theorem 7.21. *Let $A(h)$ denote the Hilbert scheme parametrizing ACM curves in $\mathbb{P}^3_{\mathbb{K}}$ with h-vector h. If Γ is general in $A(h)$, then*

$$\Gamma = D_1 \cup D_2 \cup \cdots \cup D_r,$$

where $r - 1$ is the number of Gruson–Peskine gaps of h, and the D_i are distinct smooth irreducible ACM curves whose h-vectors are determined by the gap decomposition of h as explained in Section 7. Furthermore, for every $1 \leq i_1 < i_2 < \cdots < i_h \leq r$, the curve

$$D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_h}$$

is still ACM.

Thus we can write the destabilizing divisor as $A = aH + \sum a_i D_i$. In the proof of the main Theorem 9.1, using the fact that the curves D_i and their unions are ACM, together with the numerical constraints on $x = A.H$ we show $-s-1 \leq x < 0$. We then play this inequality against the bounds of Corollary 8.9, which are essentially upper bounds for the genus of an ACM curve lying on X in terms of the degree of the curve and of degree of X. In fact, these bounds are a refinement of the bounds for the genus of an ACM curve proven in [Gruson and Peskine 1978] (see Remark 8.8). The end result is that there are only two possibilities for A: either $-A = H$ (the plane section) or $-A = H - L$ for some line L on X.

Corollary 5.7 shows that in case $A = -H$ the pencil arises from a multisecant line not contained in X, while in case $A = L - H$ the pencil arises from L. This shows pencils of minimal degree on C all arise from multisecant lines, thus completing the proof of the theorem.

2. Notation and terminology

A linear system of degree k and projective dimension r on C is denoted with the symbol g^r_k, and a g^1_k is called a pencil. The *gonality* of C, written $\mathrm{gon}(C)$, is the least positive integer k such that there exists a g^1_k on C. Since a pencil of least degree is automatically basepoint-free, the gonality of C is the least degree of a surjective morphism $C \to \mathbb{P}^1$. One can further notice that a g^1_k with $k = \mathrm{gon}(C)$ is complete, so that $h^0(C, \mathcal{O}_C(Z)) = 2$ for every divisor Z in the pencil.

Definition 2.1. Assume $C \subset \mathbb{P}^3$ is a nonplanar curve. Given a line L, let $\pi_L : C \to \mathbb{P}^1$ be obtained projecting C from L, and let $\mathfrak{X}(L)$ denote the g^1_k corresponding to π_L. Note that $\mathfrak{X}(L)$ is obtained from the pencil cut out on C by planes through L removing its base locus, which coincides with the scheme theoretic intersection $C \cap L$. In particular,

$$\deg(\pi_L) = \deg \mathfrak{X}(L) = \deg(C) - \deg(C.L)$$

and $\mathfrak{X}(L)$ is complete if $\deg(C.L) \geq 2$. We say that a g^1_k on C arises from a multisecant if it is of the form $\mathfrak{X}(L)$ for some line L. We say the gonality of C can be *computed by multisecants* if there exists a line L such that $\mathfrak{X}(L)$ has degree $\mathrm{gon}(C)$.

3. Existence of 4-secant lines

The following statement is classical and well known, but it seems hard to find a reference.

Proposition 3.1. *Let C be a smooth irreducible curve of degree $d \geq 10$ in $\mathbb{P}^3$. Then C has an l-secant line L with $l \geq 4$. In particular, the gonality of C is at most $d-4$.*

Proof. The statement is clear if $\deg(C) \geq 4$ and C is contained in a plane or $\deg(C) \geq 7$ and C is contained in a quadric surface. If C is not contained in a quadric surface, we will show the Cayley number of 4-secants

$$\mathscr{C}(d, g) = \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{g(d^2-7d+13-g)}{2}$$

is positive. The existence of L then follows from intersection theory as explained in [Le Barz 1987] or in [Arbarello et al. 1985]. For fixed $d \geq 7$, the number $\mathscr{C}(d, g)$ is a decreasing function of g, because the partial derivative with respect to g is

$$g - \frac{d^2-7d+13}{2},$$

which is negative because $g \leq d^2/4 - d + 1$ when C is not contained in a plane.

But C is not even contained in a quadric surface; thus its genus is bounded above by $\frac{1}{6}d(d-3)+1$, and

$$\mathscr{C}(d, g) \geq \mathscr{C}\big(d, \tfrac{1}{6}d(d-3)+1\big) = \frac{d(d-3)(d-6)(d-9)}{72},$$

which is positive for $d \geq 10$. □

Remark 3.2. The result is sharp, because a smooth complete intersection of two cubic surfaces has degree 9 and no 4-secant line.

4. Nonexistence of 5-secant lines

Theorem 4.1. *Let $C \subset \mathbb{P}^3$ be a curve contained in an irreducible surface X of degree s. Suppose C is a Cartier divisor on X and*

$$H^0(\mathbb{P}^3, \mathscr{I}_C(s-2)) = 0, \quad H^1(\mathbb{P}^3, \mathscr{I}_C(m)) = 0 \text{ for } m = s-2, s-3, s-4.$$

If C is general in its linear system on X, then $\deg(C.L) \leq 4$ *for every line L not contained in X, and C has only finitely many 4-secant lines not contained in X.*

In particular, if X does not contain a line, then C does not have an l-secant line for any $l \geq 5$.

Proof. The statement is obvious if $s \leq 3$, so assume $s \geq 4$. The hypotheses imply $h^1\mathscr{O}(D) = 0$ for $D = C, C-H, C-2H$ because, by Serre duality,

$$h^1(\mathbb{P}^3, \mathscr{I}_C(m)) = h^1(X, \mathscr{O}_X(mH-C)) = h^1(X, \mathscr{O}_X(C+(s-4-m)H)).$$

Similarly, $H^2(\mathscr{O}_X(C-nH))$ is dual to

$$H^0(\mathscr{O}_X((s-4+n)H-C)) = H^0(X, \mathscr{I}_{C,X}(s-4+n)),$$

which by assumption is zero for $n \leq 2$. Thus we see that $h^0\mathscr{O}_X(D) = \chi\mathscr{O}_X(D)$ for $D = C, C-H, C-2H$.

Let L be a line not contained in X, and let V be the scheme theoretic intersection of X and L. Then V has degree s, and there is an exact sequence

$$0 \to \mathcal{O}_X(-2H) \to \mathcal{O}_X(-H)^{\oplus 2} \to \mathcal{I}_{V,X} \to 0.$$

Twisting by $\mathcal{O}_X(C)$ and taking cohomology we see that

$$h^0(\mathcal{I}_V(C)) = 2h^0(\mathcal{O}_X(C-H)) - h^0(\mathcal{O}_X(C-2H)).$$

Therefore

$$\begin{aligned} h^0(\mathcal{O}_X(C)) - h^0(\mathcal{I}_V(C)) &= h^0(\mathcal{O}_X(C)) - 2h^0(\mathcal{O}_X(C-H)) + h^0(\mathcal{O}_X(C-2H)) \\ &= \chi(\mathcal{O}_X(C)) - 2\chi(\mathcal{O}_X(C-H)) + \chi(\mathcal{O}_X(C-2H)) = s. \end{aligned}$$

This shows that the points of V impose independent conditions on the linear system $|C|$. It follows that the family of curves in $|C|$ meeting L in a scheme of length $l \le s$ has codimension l in $|C|$. This implies the statement because L varies in a four-dimensional family. $\square$

Corollary 4.2. *Let $C \subset \mathbb{P}^3$ be an ACM curve. Suppose that C is contained in a smooth surface $X \subset \mathbb{P}^3$ of degree $s = s_C$, and that C is general in its linear system on X. Then* $\deg(C.L) \le 4$ *for any line L not contained in X.*

In particular, if X does not contain a line, then C does not have an l-secant line for any $l \ge 5$.

Proof. The statement follows from Theorem 4.1 because C is ACM precisely when $H^1(\mathbb{P}^3, \mathcal{I}_C(m)) = 0$ for every m. $\square$

5. Gonality of curves on a smooth surface: Lazarsfeld's method

In this section we explain a construction due to Lazarsfeld [1986; 1997] that will be crucial in proving that every pencil of minimal degree on a general ACM curve arises from a multisecant.

When a curve C is contained in a smooth surface X, we associate a rank two vector bundle on X to a basepoint-free g^1_k on C as follows. The basepoint-free g^1_k is determined by a degree k line bundle $\mathcal{O}_C(Z)$ on C, and a surjective map of $\mathcal{O}_C$-modules

$$\beta : \mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(Z).$$

(Note that, since $k \ge 1$, the map $H^0(\beta) : H^0(\mathcal{O}_C^{\oplus 2}) \to H^0(\mathcal{O}_C(Z))$ is injective.)

Definition 5.1. Suppose C is an integral curve on the smooth projective surface X, and $\mathcal{Z}$ is a basepoint-free pencil on C defined by $\beta : \mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(Z)$. Let

$$\alpha : \mathcal{O}_X^{\oplus 2} \to \mathcal{O}_C(Z)$$

denote the map obtained composing β with the natural surjection $\mathcal{O}_X^{\oplus 2} \to \mathcal{O}_C^{\oplus 2}$. Then the kernel $\mathcal{E}$ of α is called the *bundle associated* to the pencil $\mathcal{Z}$.

Proposition 5.2. *Let $\mathscr{E}$ be the bundle associated to a pencil of degree k on C as in the previous definition. Then*

(a) *$\mathscr{E}$ is a rank two vector bundle on X.*

(b) $H^0(\mathscr{E}) = 0$.

(c) $c_1(\mathscr{E}) = \mathscr{O}_X(-C)$ *and* $c_2(\mathscr{E}) = \deg(Z)$, *so that*

$$\Delta(\mathscr{E}) \stackrel{\text{def}}{=} c_1^2(\mathscr{E}) - 4c_2(\mathscr{E}) = C^2 - 4k.$$

(Here we consider the first Chern class as an element of $A^1(X) \cong \operatorname{Pic}(X)$, while we view the c_1^2 and c_2 as integers, via the degree map for zero cycles.)

Proof. By definition of $\mathscr{E}$ there is an exact sequence:

$$0 \to \mathscr{E} \to \mathscr{O}_X^{\oplus 2} \to \mathscr{O}_C(Z) \to 0$$

Since $\mathscr{O}_C$ has rank zero and projective dimension 1 as an $\mathscr{O}_X$-module, $\mathscr{E}$ is a rank two vector bundle on X, whose Chern classes can be computed from the above sequence. If $H^0(\mathscr{E})$ were not zero, then $H^0(\alpha) : H^0(\mathscr{O}_C^{\oplus 2}) \to H^0(\mathscr{O}_C(Z))$ would not be injective, so α would induce a surjective map $\mathscr{O}_C \to \mathscr{O}_C(Z)$, contradicting $\deg Z = k \geq 1$. □

We recall the definition of Bogomolov instability for rank two vector bundles on a surface, and Bogomolov's theorem which gives a numerical condition for instability.

Definition 5.3. Let $\mathscr{E}$ be a rank two vector bundle on X. One says that $\mathscr{E}$ is *Bogomolov unstable* if there exist a finite subscheme $W \subset X$ (possibly empty) and divisors A and B on X sitting in an exact sequence

$$0 \to \mathscr{O}_X(A) \to \mathscr{E} \to \mathscr{I}_W \otimes \mathscr{O}_X(B) \to 0. \tag{5-1}$$

where $(A-B)^2 > 0$ and $(A-B).H > 0$ for some (hence every) ample divisor H. We say A is a *destabilizing divisor* of $\mathscr{E}$. It is unique up to linear equivalence.

Theorem 5.4 ([Bogomolov 1978]; compare [Huybrechts and Lehn 1997, 7.3.3] and [Lazarsfeld 1997, 4.2]). *Suppose the ground field $\mathbb{K}$ has characteristic zero. Let $\mathscr{E}$ be a rank two vector bundle on the smooth projective surface X, and let* $\Delta(\mathscr{E}) = c_1(\mathscr{E})^2 - 4c_2(\mathscr{E})$.

If $\Delta(\mathscr{E}) > 0$, *then $\mathscr{E}$ is Bogomolov unstable.*

Following Lazarsfeld's approach, we will show in Section 6 that the bundle associated to a pencil computing the gonality of a smooth ACM curve satisfies $\Delta(\mathscr{E}) > 0$, hence it is Bogomolov unstable, and there is a destabilizing divisor A. To work effectively we will need the following technical result that will be useful in two ways. First it immediately implies that, when $-A = H$ (plane section) or

$-A = H - L$ (plane section minus a line), the given pencil arises from a multisecant; later on the inequalities $A^2 \geq 0$ and $A.H < 0$ will be used to exclude all other possibilities for A.

Proposition 5.5. *Suppose X is a smooth projective surface, C is an integral curve on X, and $|Z|$ is a complete basepoint-free pencil on C. Let $\mathscr{E}$ be the rank 2 bundle on X associated to $|Z|$. Suppose there is an exact sequence*

$$0 \to \mathcal{O}_X(A) \xrightarrow{h} \mathscr{E} \to \mathscr{I}_W \otimes \mathcal{O}_X(B) \to 0 \tag{5-2}$$

with W zero-dimensional and B not effective. Then the linear system $|-A|$ on X contains two effective curves D_1 and D_2 with the following properties:

(a) *D_1 and D_2 meet properly in a 0-dimensional scheme V containing W.*

(b) *D_1 and D_2 meet C properly, and, if R is the base locus of the pencil cut out on C by $C.D_1$ and $C.D_2$, then*

$$\mathcal{O}_C(Z) \cong \mathcal{O}_X(-A) \otimes \mathcal{O}_C(-R);$$

that is, the pencil $|Z|$ is obtained by first restricting D_1 and D_2 to C and then removing the base locus R.

(c) *R is the residual scheme to W in V, that is, there is an exact sequence*

$$0 \to \mathcal{O}_W \to \mathcal{O}_V \to \mathcal{O}_R \to 0.$$

In particular $h^0\mathscr{I}_W(-A) \geq 2$, $A.H < 0$ for every ample divisor H, and $A^2 \geq 0$.

Remark 5.6. The proposition applies if $\mathscr{E}$ is Bogomolov unstable with destabilizing sequence (5-2). Indeed in this case, if H is an ample divisor on X, then $(A - B).H > 0$. Since $c_1(\mathscr{E}) = A + B = -C$ in $\operatorname{Pic}(X)$, we compute

$$-2B.H = (A - B).H + C.H > 0.$$

Therefore B is not effective.

Proof of Proposition 5.5. Dualizing $0 \to \mathscr{E} \to \mathcal{O}_X^{\oplus 2} \to \mathcal{O}_C(Z) \to 0$ we obtain an exact sequence

$$0 \to \mathcal{O}_X^{\oplus 2} \to \mathscr{E}(C) \to \mathcal{O}_C(C - Z) \to 0.$$

We now look at the composite map $g : \mathcal{O}_X^{\oplus 2} \to \mathscr{E}(C) \to \mathscr{I}_W(-A)$.

This map is nonzero, otherwise $\mathcal{O}_X^{\oplus 2}$ would map injectively into the kernel of $\mathscr{E}(C) \to \mathscr{I}_W(-A)$, which is $\mathcal{O}_X(C + A)$, absurd. Hence the image of g has rank one, and has the form $\mathscr{I}_Y(-A)$ for some proper subscheme $Y \subset X$ containing W. Then $\mathscr{I}_Y = \mathscr{I}_V(-D)$ where D is the divisorial part of Y, and V is zero dimensional. We obtain an exact sequence

$$0 \to \operatorname{Ker}(g) \to \mathcal{O}_X^{\oplus 2} \to \mathscr{I}_V(-A - D) \to 0.$$

It follows $\mathrm{Ker}(g) = \mathcal{O}_X(A + D)$ and $-A - D$ is effective. A diagram chase shows there is an exact sequence

$$0 \to \mathcal{O}_X(A + D) \to \mathcal{O}_X(C + A) \to \mathcal{O}_C(C - Z)$$

from which we see there is an effective curve C_0 linearly equivalent to $C - D$ contained in C. Since C is irreducible, this implies either $D = C$ or $D = 0$.

Now $-A - D$ is effective, so, if we had $D = C$, then $B = -A - C$ would be effective, contradicting the hypotheses. Hence the only possibility is $D = 0$.

Putting everything together we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(A) & \xrightarrow{g^\vee} & \mathcal{O}_X^{\oplus 2} & \xrightarrow{(s_1,s_2)} & \mathcal{I}_V(-A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{O}_X^{\oplus 2} & \longrightarrow & \mathcal{O}_C(Z) & \longrightarrow & 0 \end{array}$$

Now let D_1 and D_2 the divisors defined by the sections s_1 and s_2 of $\mathcal{O}_X(-A)$. The first row of the diagram shows D_1 and D_2 meet properly in the zero dimensional scheme V, which contains W by construction. The two sections remain independent in $H^0(\mathcal{O}_C(Z))$ because $H^0(\mathcal{E}) = 0$. Hence D_1 and D_2 meet C properly, and $D_1.C$ and $D_2.C$ span a pencil on C.

By the snake lemma, the kernel of the vertical map $\mathcal{I}_V(-A) \to \mathcal{O}_C(Z)$ is $\mathcal{I}_W(B) = \mathcal{I}_W(-A - C)$, hence a diagram chase produces an exact sequence

$$0 \to \mathcal{O}_C(Z) \to \mathcal{O}_X(-A) \otimes \mathcal{O}_C \to \mathcal{O}_V/\mathcal{O}_W \to 0$$

which proves the rest of the statement. □

Corollary 5.7. *Assume $X \subset \mathbb{P}^3$ is a smooth surface with plane section H, containing a smooth irreducible curve C. Suppose C is not contained in a plane. Let $|Z|$ be a complete basepoint-free pencil on C, and let $\mathcal{E}$ be the bundle on X associated to $|Z|$.*

(a) *If there is an exact sequence*

$$0 \to \mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{I}_W(B) \to 0$$

with W zero dimensional and $A + H$ effective, then there is a line L such that $|Z| = \mathfrak{L}(L)$ is the pencil cut out on C by planes through L. Furthermore, if X does not contain L, then $A = -H$ and W is the residual scheme to $C \cap L$ in $X \cap L$, while, if X contains L, then $A = L - H$ and W is empty.

(b) *Assume C is linearly normal and $|Z|$ is the pencil cut out on C by planes through a line L meeting C in a scheme of length at least 2. Then there exists an exact sequence as above with $A = -H$ if X does not contain L and $A = L - H$ if X contains L.*

Proof. (a) The divisor B is not effective; otherwise

$$B+(A+H)=(-A-C)+(A+H)=H-C$$

would be effective, which contradicts the assumption that C is not contained in plane.

Thus we may apply Proposition 5.5 to the given exact sequence to conclude the linear system $|-A|$ contains a pencil. By assumption $P=A+H$ is effective, and therefore in order that $|-A|=|H-P|$ may contain a pencil it is necessary that P be empty or a line.

If P is empty, by 5.5 the are two plane sections $D_1=H_1\cap X$ and $D_2=H_2\cap X$ of X meeting in a zero dimensional scheme V, hence the line $L=H_1\cap H_2$ is not contained in X. Proposition 5.5b shows $|Z|$ is obtained removing from the pencil spanned by $C\cap H_1$ and $C\cap H_2$ its base locus $C\cap L$, that is, $|Z|=\mathfrak{X}(L)$, and Proposition 5.5c shows W is the residual scheme to $C\cap L$ in $X\cap L$.

Finally, if P is a line, then D_1 and D_2 belong to $|H-P|$, hence their intersection $V=D_1\cap D_2$ is empty. It follows from Proposition 5.5 that $|Z|=\mathfrak{X}(P)$ and that and W is empty.

(b) By the definition of $\mathscr{E}$ there is an exact sequence

$$0\to\mathscr{E}\to\mathscr{O}_X^{\oplus 2}\to\mathscr{O}_C(Z)\to 0.$$

Comparing this sequence with

$$0\to\mathscr{O}_C\to\mathscr{O}_C(Z)\to\mathscr{O}_Z\to 0,$$

we obtain

$$0\to\mathscr{O}_X(-C)\to\mathscr{E}\to\mathscr{I}_{Z,X}\to 0.$$

Now twist by H and take cohomology to get a long exact sequence

$$0\to H^0(\mathscr{O}_X(H-C))\to H^0(\mathscr{E}(H))\to H^0(\mathscr{I}_{Z,X}(H))\to H^1(\mathscr{O}_X(H-C)).$$

Since Z is contained in a plane, $h^0(\mathscr{I}_{Z,X}(H))>0$, while $H^1(\mathscr{O}_X(H-C))=H^1(\mathscr{I}_C(H))=0$ because C is linearly normal. Hence $\mathscr{E}(H)$ has a section, and after removing torsion in the cokernel if necessary we find an exact sequence:

$$0\to\mathscr{O}_X(P-H)\to\mathscr{E}\to\mathscr{I}_W(H-P-C)\to 0,$$

with W zero dimensional and P effective. Now (b) follows from (a). □

6. ACM curves

In this section we show that, if C is an ACM curve of degree d having a pencil of minimal degree $k\le d-4$ on a smooth surface of degree $s=s_C$, then the bundle $\mathscr{E}$ associated to the given pencil satisfies $\Delta(\mathscr{E})>0$ (except for a small list of cases given in Proposition 6.10); hence, if the ground field has characteristic zero, it is Bogomolov unstable. The proof is based on the structure of the biliaison class of ACM curves which we now briefly recall. We also include some information about the minimal link Γ of a curve C, which we will need later.

Given a curve C in $\mathbb{P}^3$ its fundamental numerical invariants are, besides its degree d_C and its arithmetic genus $g(C) = 1 - \chi(\mathbb{O}_C)$:

- its index of speciality $e(C) = \max\{n : h^1\mathbb{O}_C(n) > 0\}$;
- the minimal degree s_C of a surface containing C;
- the integer $t_C = \min\{n : h^0(\mathcal{I}_C(n)) - h^0(\mathbb{O}_{\mathbb{P}^3}(n - s_C)) > 0)\}$. If C is integral or more generally if C lies on an integral surface of degree s_C, the integer t_C is the smallest n such that C is contained in a complete intersection of two surfaces of degree s_C and n.

When C is ACM, all its basic numerical invariants can be computed from the Hilbert function. It is convenient to express the Hilbert function through its second difference function, the so called *h-vector* h_C of C — see [Migliore 1998, §1.4] — because h_C is a finitely supported function. Thus one defines

$$h_C(n) = h^0(\mathbb{O}_C(n)) - 2h^0(\mathbb{O}_C(n-1)) + h^0(\mathbb{O}_C(n-2)).$$

If $s = s(C)$ and $e = e(C)$, the function h_C satisfies

$$\begin{cases} h(n) = n+1 \quad \text{if } \ 0 \le n \le s-1, \\ h(n) \ge h(n+1) \quad \text{if } \ n \ge s-1, \\ h(e+2) > 0 \quad \text{and} \quad h(n) = 0 \quad \text{for } n \ge e+3. \end{cases} \tag{6-1}$$

Thus we may write h as

$$h_C = \{1, 2, \ldots, s, h_C(s), \ldots, h_C(e+2)\}.$$

with $s = h_C(s-1) \ge h_C(s) \ge h_C(s+1) \ge \cdots \ge h_C(e+2)$.

We say that a finitely supported function $h : \mathbb{N} \to \mathbb{N}$ is an *h-vector* if it satisfies (6-1) for some $s \ge 1$. Every h-vector arises as the h-vector of an ACM curve in $\mathbb{P}^3$; see [Martin-Deschamps and Perrin 1990, Theorem V.1.3, p. 111] and Remark 7.7 below. It will be convenient to allow the identically zero function among h-vectors, and think of it as the h-vector of the empty curve. In terms of the h-vector, the fundamental invariants of C are:

Proposition 6.1. *For an ACM curve C in $\mathbb{P}^3$, with h-vector h_C, we have*

(1) $d_C = \sum h_C(n)$,

(2) $g(C) = 1 + \sum (n-1) h_C(n)$,

(3) $e(C) + 2 = \max\{n : h_C(n) > 0\}$,

(4) $s_C = \min\{n \geq 0 : h_C(n) < n+1\}$, *and*

(5) $t_C = \min\{n \geq 0 : h_C(n-1) > h_C(n)\}$.

Consistently with these formulas, for the empty curve we define $s = 0$, $d = 0$, $g = 1$, $e = -\infty$.

Remark 6.2. If C is an ACM curve with $s_C = s$, then

$$d_C = \sum h_C(n) \geq \sum_{n=0}^{s-1}(n+1) = \tfrac{1}{2}s(s+1).$$

The h-vectors of integral curves have a special form:

Definition 6.3 [Maggioni and Ragusa 1988]. An h-vector is of *decreasing type* if $h(a) > h(a+1)$ implies that for each $n \geq a$ either $h(n) > h(n+1)$ or $h(n) = 0$.

Remark 6.4. By a result from [Ellingsrud 1975] (see also [Martin-Deschamps and Perrin 1990, p. 5; corollaire 1.2 on p. 134; §1.7, p. 139]), the Hilbert scheme $A(h)$ of ACM curves in $\mathbb{P}^3$ with a given h-vector is smooth and irreducible, even when h is not of decreasing type.

Gruson and Peskine [1978] (see also [Maggioni and Ragusa 1988] and [Nollet 1998]) showed that, if C is an integral ACM curve, then h_C is of decreasing type, and conversely, if h is an h-vector of decreasing type, then there exists a smooth irreducible ACM curve C with $h_C = h$. Thus an h-vector h is of decreasing type if and only if the general curve C in $A(h)$ is smooth and irreducible.

If C is not irreducible, it may happen that every pair of surfaces X_1 and X_2 containing C of minimal degrees s_C and t_C have a common component. Nollet [1998, Proposition 1.5] generalized the result of Gruson and Peskine by showing that if C is contained in a complete intersection of type (s_C, t_C), then h_C is of decreasing type. We partially reproduce his argument here:

Lemma 6.5. (i) *Suppose an ACM curve D is contained in a complete intersection Y of type (s_D, t_D), and let Γ be the curve and linked to D by Y. Then*

$$e(\Gamma) + 3 < s_D.$$

(ii) *Let Γ be an ACM curve, and suppose $a \leq b$ are integers such that $a \geq e(\Gamma)+3$ and $b \geq e(\Gamma)+4$. Then the h-vector of a curve D linked to Γ by a complete intersection of type (a, b) is of decreasing type. If $a \geq e(\Gamma)+4$, then $s_D = a$ and $t_D = b$. If $a = e(\Gamma)+3$, then $s_D = a$ and $t_D = b-1$.*

Proof. If Γ and D are linked by a complete intersection Y of type (a, b), we have, by [Migliore 1998, 5.2.19],

$$h_\Gamma(n) = h_Y(n) - h_D(a+b-2-n) = h_Y(a+b-2-n) - h_D(a+b-2-n).$$

Suppose first $a = s_D$ and $b = t_D$. Then

$$h_\Gamma(s_D - 1) = h_Y(t_Y - 1) - h_D(t_D - 1) = s_Y - s_D = 0.$$

Therefore $e(\Gamma) + 3 \le s_D - 1$.

Next suppose $b \ge a \ge e(\Gamma) + 4$. Then $s_D \le a$ because $D \subseteq Y$, and

$$h_D(b-1) = h_Y(a-1) - h_\Gamma(a-1) = h_Y(a-1) = a$$

while

$$h_D(b) = h_Y(a-2) - h_\Gamma(a-2) \le h_Y(a-2) = a-1$$

hence $s_D = a$ and $t_D = b$.

If $a = e(\Gamma)+3$ and $b \ge e(\Gamma)+4$, then a similar calculation shows $h_D(b-2) = a$ and $h_D(b-1) < a$, so that $s_D = a$ and $t_D = b-1$.

It remains to show h_D is of decreasing type. Let $u = s(\Gamma)$. Then $u \le e(\Gamma)+3 \le a$ and $h_\Gamma(n) = h_Y(n) = n+1$ for $n \le u-1$; hence $h_D(n) = 0$ for $n \ge a+b-1-u$.

Since $h_\Gamma(n) \ge h_\Gamma(n+1)$ for $n \ge u-1$, we see that for $b-1 \le m \le a+b-2-u$

$$\begin{aligned} h_D(m) - h_D(m+1) &= h_Y(m) - h_Y(m+1) - h_\Gamma(a+b-2-m) + h_\Gamma(a+b-1-m) \\ &= 1 - \partial h_\Gamma(a+b-1-m) \ge 1, \end{aligned}$$

which shows that h_D is of decreasing type. □

Fix a smooth surface $X \subset \mathbb{P}^3$ of degree s. Two curves C and D on X are said to be *biliaison equivalent* if C is linearly equivalent to $D + nH$ for some integer n.

Definition 6.6. A curve C on a surface X is *minimal on* X if $C - H$ is not effective.

Proposition 6.7. *A curve C is minimal on a smooth surface X if and only if*

$$e(C) + 3 < \deg(X).$$

Proof. To say C is minimal is equivalent to saying $h^0(\mathcal{O}_X(C-H)) = 0$. By duality on X this is the same as $h^2(\mathcal{I}_C(s-3)) = 0$, where $s = \deg(X)$. On the other hand, $h^2(\mathcal{I}_C(s-3)) = h^1(\mathcal{O}_C(s-3))$, so the condition says $s-3 > e(C)$, or equivalently, $e(C) + 3 < s$. □

Definition 6.8. We say that an h-vector is s*-minimal* if the corresponding curve satisfies $e + 3 < s$. We say that an h-vector is s*-basic* if it is the h-vector of an integral curve C satisfying $s_C = t_C = s$. Thus the s-basic h-vectors are those h-vectors of decreasing type that begin with a string

$$\{1, 2, \ldots, s-1, s, m\}$$

with $m = h(s) \le s-1$.

Table 1 on the next page lists s-basic h-vectors for $s = 4$ and $s = 5$.

	d	g	h-vector	$C^2-4(d-4)$	$C^2-4(d-5)$	λ ($\Gamma = tH-C$)	$q(\lambda)$
$s=4$	10	11	1, 2, 3, 4	−4	0	1, 2, 3	20
	11	14	1, 2, 3, 4, 1	−2	2	2, 3	17
	12	17	1, 2, 3, 4, 2	0	4	1, 3	16
	13	20	1, 2, 3, 4, 3	2	6	1, 2	17
	13	21	1, 2, 3, 4, 2, 1	4	8	3	9
	14	24	1, 2, 3, 4, 3, 1	6	10	2	12
	15	28	1, 2, 3, 4, 3, 2	10	14	1	9
	16	33	1, 2, 3, 4, 3, 2, 1	16	20	∅	0
$s=5$	15	26	1, 2, 3, 4, 5	−9	−5	1, 2, 3, 4	50
	16	30	1, 2, 3, 4, 5, 1	−6	−2	2, 3, 4	46
	17	34	1, 2, 3, 4, 5, 2	−3	1	1, 3, 4	44
	18	38	1, 2, 3, 4, 5, 3	0	4	1, 2, 4	44
	18	39	1, 2, 3, 4, 5, 2, 1	2	6	3, 4	34
	19	42	1, 2, 3, 4, 5, 4	3	7	1, 2, 3	46
	19	43	1, 2, 3, 4, 5, 3, 1	5	9	2, 4	36
	20	47	1, 2, 3, 4, 5, 4, 1	8	12	2, 3	40
	20	48	1, 2, 3, 4, 5, 3, 2	10	14	1, 4	30
	21	52	1, 2, 3, 4, 5, 4, 2	13	17	1, 3	36
	21	54	1, 2, 3, 4, 5, 3, 2, 1	17	21	4	16
	22	57	1, 2, 3, 4, 5, 4, 3	18	22	1, 2	34
	22	58	1, 2, 3, 4, 5, 4, 2, 1	20	24	3	24
	23	63	1, 2, 3, 4, 5, 4, 3, 1	25	29	2	24
	24	69	1, 2, 3, 4, 5, 4, 3, 2	32	36	1	16
	25	76	1, 2, 3, 4, 5, 4, 3, 2, 1	41	45	∅	0
$s=6$	21	50	1, 2, 3, 4, 5, 6	−12	−8	1, 2, 3, 4, 5	105
	22	55	1, 2, 3, 4, 5, 6, 1	−8	−4	2, 3, 4, 5	100
	23	60	1, 2, 3, 4, 5, 6, 2	−4	0	1, 3, 4, 5	97
	24	65	1, 2, 3, 4, 5, 6, 3	0	4	1, 2, 4, 5	96
$s=7$	28	85	1, 2, 3, 4, 5, 6, 7	−12	−8	1, 2, 3, 4, 5, 6	196
	29	91	1, 2, 3, 4, 5, 6, 7, 1	−7	−3	2, 3, 4, 5, 6	190
	30	97	1, 2, 3, 4, 5, 6, 7, 2	−2	2	1, 3, 4, 5, 6	186
$s=8$	35	130	1, 2, 3, 4, 5, 6, 7, 4, 3	29	33	1, 2, 5, 6	154
	36	133	1, 2, 3, 4, 5, 6, 7, 8	−8	−4	1, 2, 3, 4, 5, 6, 7	336
	37	140	1, 2, 3, 4, 5, 6, 7, 8, 1	−2	2	2, 3, 4, 5, 6, 7	329
	45	196	1, 2, 3, 4, 5, 6, 7, 8, 9	1	5	1, 2, 3, 4, 5, 6, 7, 8	540

Table 1. s-basic h-vectors and s-minimal biliaison types.

Proposition 6.9. *Suppose C is an ACM curve contained in a smooth surface X of degree s_C. Let $s = s_C$, $t = t_C$ and $e = e(C)$. Then $e+3 \geq t \geq s$ and*

(a) *h_C is of decreasing type*;

(b) *if $\Gamma \in |tH - C|$, then $e(\Gamma)+3 < s$ and Γ is minimal on X*;

(c) *$C - mH$ is effective if and only if $m \leq e+4-s$*;

(d) *if $C_1 \in |C-(t-s)H|$, h_{C_1} is s-basic*;

(e) *if $C_2 \in |C-(t-s+1)H|$, h_{C_2} is of decreasing type.*

There is a one to one correspondence $h_\Gamma \mapsto h_{C_1}$ mapping s-minimal h-vectors to s-basic h-vectors.

Proof. Since C is ACM, the ideal sheaf $\mathcal{I}_{C,\mathbb{P}^3}$ is $(e+3)$-regular, hence $e+3 \geq t$. By definition of t, we have $t \geq s$, and C is contained in a surface F of degree t that does not contain X. Therefore C is contained in the complete intersection $X \cap F$ of type (s, t). Let $\Gamma \in |tH - C|$ be the curve linked to C by $X \cap F$: then $e(\Gamma_0)+3 < s$ and Γ is minimal (by either Lemma 6.5 or by definition of t).

Each of the curves C, C_1, C_2 is linked to a curve in the linear system $|\Gamma|$ by a complete intersection of type (s, t), (s, s), or $(s-1, s)$, respectively. By Lemma 6.5 the h-vectors of C, C_1 and C_2 are of decreasing type, and h_{C_1} is s-basic. □

There is a unique 1-basic h-vector, namely $h_0 = \{1\}$, the h-vector of a line. Every $(s-1)$-basic h-vector gives rise to two s-basic h vectors by performing a type A or type B transformation, defined as follows: (1) A type $A = A_s$ transformation consists of inserting an s to an $(s-1)$-basic h-vector $h = \{1, 2, \ldots, s-1, m, \ldots\}$ to transform it into the s-basic vector $h' = \{1, 2, \ldots, s-1, s, m \ldots\}$. Geometrically, if h is the h-vector of a curve C on a surface X of degree s, h' is the h-vector of the effective divisor $C + H$ on X. (2) A type B transformation consists of inserting a string $s, s-1$ to an $(s-1)$-basic h-vector $h = \{1, 2, \ldots, s-1, m, \ldots\}$ to transform it into the s-basic vector $h'' = \{1, 2, \ldots, s-1, s, s-1, m \ldots\}$. Geometrically, this operation breaks into two steps: suppose h is the h-vector of a curve C on a surface X_1 of degree $s-1$. Let $C_1 = C + H$ be obtained by adding to C a plane section of X_1, then pick a surface X_2 of degree s containing C_1, and finally let $C_2 = C_1 + H$ be obtained by adding to C_1 a plane section of X_2. Then h'' is the h-vector of C_2.

Conversely, any s-basic h-vector with $m = h(s) \leq s-2$ arises from a type A transformation of an $(s-1)$-basic h-vector, while any s-basic h-vector with $m = h(s) = s-1$ arises from a type B transformation of an $(s-1)$-basic h-vector. In particular, the number of s-basic h-vectors is 2^{s-1} (see Table 1).

Proposition 6.10. *Let C be an integral ACM curve in $\mathbb{P}^3$ with $s_C \geq 4$. Suppose C is contained in a smooth surface X of degree $s = s(C)$. Suppose C has a basepoint-free pencil of degree k, and let $\mathscr{E}$ be the bundle on X associated to such a pencil.*

(a) *If* $k \le d-5$, *then* $\Delta(\mathscr{E}) > 0$ *unless*

- $s = 4$ *and* $(d, g) = (10, 11)$, *or*
- $s = 5$ *and* $(d, g) = (15, 26),\ (16, 30)$, *or*
- $s = 6$ *and* $(d, g) = (21, 50),\ (22, 55),\ (23, 60)$, *or*
- $s = 7$ *and* $(d, g) = (28, 85),\ (29, 91)$, *or*
- $s = 8$ *and* $(d, g) = (36, 133)$.

(b) *If* $k = d-4$, *then* $\Delta(\mathscr{E}) > 0$ *unless*

- $s = 4$ *and* $(d, g) = (10, 11), (11, 14),\ (12, 17)$, *or*
- $s = 5$ *and* $(d, g) = (15, 26),\ (16, 30),\ (17, 34),\ (18, 38)$, *or*
- $s = 6$ *and* $(d, g) = (21, 50),\ (22, 55),\ (23, 60),\ (24, 65)$, *or*
- $s = 7$ *and* $(d, g) = (28, 85),\ (29, 91), (30, 97)$, *or*
- $s = 8$ *and* $(d, g) = (36, 133),\ (37, 140)$.

Proof. We can compute $\Delta(\mathscr{E})$ in terms of $d = d_C$ and $g = g(C)$:

$$\Delta(\mathscr{E}) = C^2 - 4k = 2g - 2 - (s-4)d - 4k = \delta_s(d, g) + 4(d-k),$$

where we have set $\delta_s(C) = \delta_s(d, g) = 2g - 2 - ds$. One can easily verify the following facts:

(1) Let $C \subseteq X_s$ be a curve on a surface X of degree s in $\mathbb{P}^3$, and consider the divisor $C + H$ on X_s. Then

$$\delta_s(C+H) - \delta_s(C) = 2d - 3s.$$

In particular, if $d \ge \frac{1}{2}s(s+1)$ and $s \ge 3$, $\delta_s(C+H) > \delta_s(C)$.

(2) Suppose $C \subseteq X_{s+1}$ is a curve on a surface X of degree $s+1$ in $\mathbb{P}^3$, and consider the divisor $C + H$ on X_{s+1} Then

$$\delta_{s+1}(C+H) - \delta_s(C) = d - 3(s+1).$$

In particular, if $d \ge \frac{1}{2}s(s+1)$ and $s \ge 6$, $\delta_{s+1}(C+H) \ge \delta_s(C)$, and the inequality is strict unless $s = 6$ and $d = 21$.

To prove the proposition, we have seen that $\Delta(\mathscr{E})$ can be computed in terms of d, g, s, k, which depend only on the h-vector and the choice of s, k. Therefore, using the two remarks (1), (2) just made and using biliaisons on each surface to reduce to s-basic h-vectors, and using the transformations of type A and B mentioned before the statement, it would be sufficient to prove that $\Delta > 0$ for all s-basic h-vectors with $s = 4$. Unfortunately this is not so, as $\Delta \le 0$ for the first three 4-basic h-vectors (see Table 1). Still the two remarks show that Δ becomes positive using the transformations of type A and B, with the only exceptions listed in the statement. Table 1 displays all h-vectors for which $\Delta \le 0$ for $k = d-4$ and $k = d-5$. $\square$

7. General ACM curves

We now generalize the results of [Gruson and Peskine 1978] by giving a description of a general ACM curve C with a given h-vector h, even when h is not of decreasing type. We show (Theorem 7.21) that C is a union of smooth ACM subcurves whose h-vectors are determined by that of C. The basic step is Proposition 7.18, which is a special case of [Davis 1985, Corollary 4.2], and says that C is the union of two ACM subcurves whenever h_C is not of decreasing type. As a corollary we show the existence of multisecant lines for ACM curves with h-vector of special types.

Definition 7.1. Let C_0 and C be two curves in $\mathbb{P}^3$.

(a) Following [Martin-Deschamps and Perrin 1990] we say that C is obtained by an *elementary biliaison* of height h from C_0 if there exists a surface X in $\mathbb{P}^3$ containing C_0 and C so that $\mathcal{I}_{C,X} \cong \mathcal{I}_{C_0,X}(-h)$. In the language of generalized divisors [Hartshorne 1994] this means C is linearly equivalent to $C_0 + hH$ on X, where H denotes the plane section.

(b) As a particular case, we say C is obtained by a *trivial* biliaison of height h if $\mathcal{I}_{C,X} = \mathcal{I}_{C_0,X}\mathcal{I}_{Y,X}$ where Y is a complete intersection of X and a surface of degree h. If Y meets C_0 properly, this means C is the union of C_0 and Y.

(c) By a *special biliaison of degree k* we mean an elementary biliaison of height one $C \sim C_0 + H$ on a surface of degree $k \geq e(C_0) + 4$. The condition $k \geq e(C_0) + 4$ guarantees $s_C = s_{C_0} + 1$ and $k = e(C) + 3$ by [Martin-Deschamps and Perrin 1990, p. 68].

Proposition 7.2 (Lazarsfeld–Rao property). *Suppose C is an ACM curve with index of speciality e. Then C can be obtained by a special biliaison of degree $k = e+3$ from some ACM curve C_0 satisfying $s_{C_0} = s_C - 1$.*

Proof. One knows — see for example [Strano 2004] — that an ACM curve C with index of speciality e can be obtained by an elementary biliaison of height 1 on a surface X of degree $e+3$ from an ACM curve C_0 satisfying

$$s_{C_0} = s_C - 1 \quad \text{and} \quad e(C_0) < e(C).$$

Since $\deg(X) = e + 3 \geq e(C_0) + 4$, this is a special biliaison. □

Remark 7.3. When $s_C = 1$, the curve C_0 above is the empty curve, which is therefore convenient to allow among ACM curves.

Corollary 7.4. *Let C be an ACM curve. Then there exist positive integers $k_1 < k_2 < \cdots < k_u$ such that C is obtained from the empty curve by a chain of u special biliaisons of degrees $k_1, \ldots, k_u$. The sequence $\lambda_C = (k_1, k_2, \ldots, k_u)$ is uniquely* *determined by C, and we will call it the biliaison type of C. Morever, we have*

$$d_C = \sum_{i=1}^{u} k_i, \quad g(C) = 1 + \frac{1}{2}\sum_{i=1}^{u} k_i(k_i - 3) + \sum_{i=1}^{u}(s_C - i)k_i,$$
$$s_C = u, \quad t_C - s_C + 1 = k_1, \quad e(C) + 3 = k_u.$$

Example 7.5. If $C \subset \mathbb{P}^3$ is ACM , then $d_C \geq \frac{1}{2}s_C(s_C+1)$, with equality if and only if $\lambda_C = (1, 2, 3, \dots, s_C - 1, s_C)$.

Remark 7.6. The biliaison type λ_C was introduced from a different point of view in [Green 1998], and it essentially the same thing as the numerical character $\{n_j\}$ of [Gruson and Peskine 1978]: the precise relationship, if $s = s_C$, is

$$n_j - j = k_{s-j} \qquad \text{for } j = 0, \dots, s-1.$$

The biliaison type (hence the numerical character) is equivalent to the h-vector of C. Indeed, h_C can be recovered from λ_C because one knows how h_C vector varies in an elementary biliaison, while λ_C can be computed out of h_C via the formula

$$k_i = \#\{n : h_C(n) \geq s_C + 1 - i\}.$$

One can visualize h_C and λ_C as follows. In the first quadrant of the (x, y) plane, draw a dot at (n, p) if n and p are integers satisfying $1 \leq p \leq h(n)$. Then $h(n)$ is the number of dots on the vertical line $x = n$, while k_i is the number of dots on the horizontal line $y = s - i + 1$. In particular, $k_1 = t_C - s_C + 1$ is the number of dots on the top horizontal line $y = s$, and $k_s = e(C) + 3$ is the number of dots on the bottom line $y = 1$.

Remark 7.7. The statement that every h-vector arises as the h-vector of an ACM curve in $\mathbb{P}^3$ is equivalent to the statement that every finite, strictly increasing sequence of positive integers $\lambda = (k_1, \dots, k_u)$ occurs as λ_C for some ACM curve $C \subset \mathbb{P}^3$. We can see this by induction on u. When $u = 1$, $\lambda = (k)$ is the biliaison type of a plane curve of degree k. If $u > 1$, by induction there is an ACM curve C_0 with $\lambda_{C_0} = (k_1, \dots, k_{u-1})$. Now $s_{C_0} \leq e(C_0) + 3 = k_{u-1} < k_u$. Therefore we can find a surface X of degree k_u containing C_0, and construct C from C_0 by a biliaison of height one on X. Since $e(C_0) + 3 < k_u$, the biliaison is special, hence λ_C equals the given λ. A refined version of this construction is in Theorem 7.21.

Definition 7.8. A sequence $\lambda = (k_1, k_2, \dots, k_u)$ *has a gap* at i if $k_{i+1} - k_i \geq 3$.

For example, the sequence λ_C of Figure 1 has a gap at $i = 2$.

Davis [1985] shows that a gap in λ_C forces C to break in the union of two ACM subcurves. We now give a more geometric proof of this result. For this we need some preliminary remarks. While in general the union C of two ACM curves B and D can fail to be ACM, it is certainly ACM if I_D/I_C is isomorphic to R_B up to

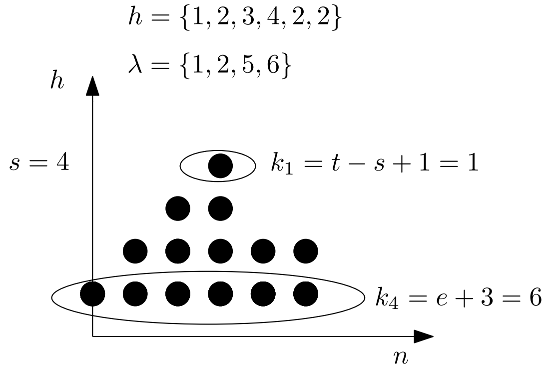

Figure 1. Biliaison type and h-vector.

a twist. This condition is satisfied when C is obtained from B by a trivial biliaison, and also when C is obtained from B by a chain of elementary biliaisons "trivial on B" (Lemma 7.16 below). Here are some preliminary examples.

Example 7.9. If C is obtained from a curve B by a trivial biliaison of height h on a surface X, "adding" to C the complete intersection Y of X with a surface of degree h, then

$$I_Y/I_C \cong \frac{I_Y/I_X}{I_C/I_X} \cong \frac{H^0_*(\mathcal{I}_{Y,X})}{H^0_*(\mathcal{I}_{C,X})} \cong \frac{H^0_*(\mathcal{O}_X(-h))}{H^0_*(\mathcal{I}_{B,X}(-h))} \cong R_B(-h)$$

Example 7.10. Let $D \subset \mathbb{P}^3$ be a curve, and L a line not contained in D. Set $C = D \cup L$, and let f be the degree of the scheme theoretic intersection $D \cap L$. Then $\mathcal{I}_{D,C} \cong \mathcal{I}_{D\cap L,L} \cong \mathcal{O}_L(-f)$. If D is ACM, it follows that $C = D \cup L$ is ACM if and only if $I_D/I_C \cong R_L(-f)$.

By the same argument, if B and D are two ACM curves meeting properly and $\mathcal{I}_{B\cap D,B} \cong \mathcal{O}_B(-f)$, then $C = B \cup D$ is ACM if and only if $I_D/I_C \cong R_B(-f)$.

From another point of view, suppose B and D are two ACM curves contained in a *smooth* surface X, and let $C = B + D$. Then

$$\mathcal{O}_B(-D) \overset{\text{def}}{=} \mathcal{O}_X(-D) \otimes \mathcal{O}_B \cong \mathcal{I}_{D,C}.$$

If $\mathcal{O}_B(-D) \cong \mathcal{O}_B(-f)$, then C is ACM if and only if $I_D/I_C \cong R_B(-f)$.

The condition $I_D/I_C \cong R_B(-f)$ implies that C is obtained by a "generalized liaison addition" of B and D in the sense of [Geramita and Migliore 1994]. The following proposition is essentially a special case of Theorem 1.3 of that reference.

Proposition 7.11. *Suppose that C contains two subcurves B and D, and that for some integer f there is an isomorphism of R_C-modules*:

$$I_D/I_C \cong R_B(-f). \tag{7-1}$$

(a) *There is a surface S of degree f containing D but not C, and the curve D is the scheme theoretic intersection of C and S. In particular, $f \geq s_D$.*

(b) *The degrees and genera of B, C and D are related by the formulas*

$$d_C = d_B + d_D, \quad g(C) = g(B) + g(D) + f d_B - 1.$$

If B and D have no common component, then C is the scheme-theoretic union of B and D, $\mathcal{I}_{B\cap D,B} \cong \mathcal{O}_B(-f)$, and $B.D = f d_B$.

If C is contained in a smooth surface X, then $C = B + D$ on X, and $\mathcal{O}_X(D) \otimes \mathcal{O}_B \cong \mathcal{O}_B(f)$. In particular, $B.D = f d_B$.

(c) *Suppose D is ACM. Then B is ACM if and only if C is ACM, in which case*

$$h_C(n) = h_B(n - f) + h_D(n)$$

(d) *Suppose B, C and D are ACM and $f = s_D$. If $\max\{\lambda_B\} < \min\{\lambda_D\}$ then*

$$\lambda_C = \lambda_B \cup \lambda_D.$$

Proof. The hypothesis $I_D/I_C \cong R_B(-f)$ is equivalent to there being a form $F \in H^0(\mathbb{P}^3, \mathcal{O}(f))$ such that the sequence

$$0 \to I_B/I_C(-f) \to R_C(-f) \xrightarrow{F} R_C \to R_D \to 0$$

is exact. In particular, $I_D = I_C + I_S$ where S is the surface of equation $F = 0$, hence D is the scheme theoretic union of C and S. Sheafifying the exact sequence

$$0 \to I_B(-f) \to I_C \to I_D/(F) \to 0$$

we obtain another exact sequence

$$0 \to H^1_*(\mathcal{I}_B)(-f) \to H^1_*(\mathcal{I}_C) \to H^1_*(\mathcal{I}_D).$$

It follows that, if D is ACM, then $H^1_*(\mathcal{I}_B)(-f) \cong H^1_*(\mathcal{I}_C)$, and B is ACM if and only if C is ACM.

If B and D are ACM, the relation between the h-vectors follows immediately from the exact sequence $0 \to R_B(-f) \to R_C \to R_D \to 0$.

The relation between the degrees and genera follows computing the Euler characteristics of the two sides of $\mathcal{I}_{D,C} \cong \mathcal{O}_B(-f)$.

Suppose B and D have no common components. The kernel of the natural surjective map

$$\mathcal{O}_B(-f) \cong \mathcal{I}_{D,C} \to \mathcal{I}_{B\cap D,B}$$

is supported on D and is a subsheaf of $\mathcal{O}_B$. Since B is locally Cohen–Macaulay and has no component in common with D, the kernel is zero, hence $\mathcal{O}_B(-f) \cong \mathcal{I}_{B\cap D,B}$.

Suppose C is contained in a smooth surface X. Since $D \subseteq C$, there is an effective divisor A on X such that $C = A + D$. Then

$$\mathcal{O}_B(-f) \cong \mathcal{I}_{D,C} \cong \mathcal{O}_X(-D) \otimes \mathcal{O}_A$$

from which we deduce $A = B$ and $\mathcal{O}_B(f) \cong \mathcal{O}_X(D) \otimes \mathcal{O}_B$, hence $B.D = f d_B$.

We deduce (d) from (c). By assumption

$$e(B) + 3 = \max\{\lambda_B\} < \min\{\lambda_D\} = t_D - s_D + 1.$$

On the other hand, $h_D(n) = s_D$ if and only if $s_D - 1 \le n \le t_D - 1$, and $h_B(n - s_D)$ is nonzero if and only if $s_D \le n \le s_D + e(B) + 2$. Since $t_D > s_D + e(B) + 2$, we see $h_D(n) = s_D$ whenever $h_B(n - s_D)$ is nonzero (h_B so to speak sits on the top of h_D, as in Figure 1). Now it follows from $h_C(n) = h_B(n - f) + h_D(n)$ that $\lambda_C = \lambda_B \cup \lambda_D$. □

Example 7.12. Figure 1 on page 289 shows the h-vector of a curve which is the union of a twisted cubic curve B and a divisor D of type (6, 5) on a smooth quadric surface. The biliaison types are $\lambda_B = \{1, 2\}$ and $\lambda_D = \{5, 6\}$.

Definition 7.13. Suppose $D_0 \subseteq C_0$ are curves in $\mathbb{P}^3$ contained in a surface X, and D is obtained from D_0 by an elementary biliaison of height h on X. The biliaison is defined by an injective morphism $v : \mathcal{I}_{D_0,X}(-h) \to \mathcal{O}_X$ whose image is $\mathcal{I}_{D,X}$. Then the image of the restriction of v to $\mathcal{I}_{C_0,X}(-h)$, is the ideal $\mathcal{I}_{C,X}$ of a curve $C \subset X$, obtained by biliaison from C_0. In this case, we say that the biliaison from C_0 to C is *induced* by the given biliaison from D_0 to D. Note that C contains D.

Remark 7.14. When D_0 is empty, a biliaison induced from D_0 is the same thing as a trivial biliaison. Indeed, in this case v is multiplication by a local equation of the complete intersection D in $\mathcal{O}_X$, and v maps $\mathcal{I}_{C_0,X}(-h)$ onto $\mathcal{I}_{C_0,X}\mathcal{I}_{D,X}$.

Remark 7.15. For an elementary biliaison from C_0 to C to be induced by a biliaison of D_0 it is enough that the corresponding morphism $u : \mathcal{I}_{C_0,X}(-h) \to \mathcal{O}_X$ lift to a morphism $\hat{u} : \mathcal{I}_{D_0,X}(-h) \to \mathcal{O}_X$. Indeed, $\hat{u}$ is automatically injective because its kernel $\mathcal{K}$ is isomorphic to a subsheaf of $\mathcal{I}_{D_0,C_0}(-h) \subseteq \mathcal{O}_{C_0}(-h)$, and at the same time is a subsheaf of $\mathcal{O}_X(-h)$; since $\mathcal{O}_X$ and $\mathcal{O}_{C_0}$ have no common associated points, we must have $\mathcal{K} = 0$.

Lemma 7.16. *Suppose C_0 contains B and D_0, and $I_{D_0}/I_{C_0} \cong R_B(-f)$. Suppose C is obtained by an elementary biliaison from C_0 induced by an elementary biliaison of height h from D_0 to D on a surface X. Then C contains D and B, and*

$$I_D/I_C \cong R_B(-f - h).$$

Proof. Since the biliaison from C_0 to C is induced by that from D_0 to D, C contains D, and

$$I_D/I_C \cong \frac{I_{D_0}/I_X(-h)}{I_{C_0}/I_X(-h)} \cong R_B(-f-h)$$

In particular, $R_B(-h-f)$ is an R_C-module, therefore $B \subseteq C$. □

Lemma 7.17. *Suppose C_0 contains B and D_0, and $I_{D_0}/I_{C_0} \cong R_B(-s_{D_0})$. If k is an integer such that*

$$k \geq \max(s_{D_0} + e(B) + 6, e(C_0) + 4),$$

then any height-one biliaison from C_0 to C on a surface of degree k is induced by a biliaison from D_0 to a curve D such that

$$I_D/I_C \cong R_B(-s_D)$$

Proof. The lemma generalizes [Martin-Deschamps and Perrin 1990, Remark 2.7c, p. 65], which treats the case $C_0 = B$ and $D_0 = \varnothing$. The statement in this case becomes: if $k \geq e(C_0) + 6$, then every height-one elementary biliaison from C_0 to C on a surface of degree k is trivial.

To prove the statement, let X be the degree k surface on which the biliaison from C_0 to C is defined, and apply $\mathrm{Hom}_{\mathcal{O}_X}(\,\cdot\,, \mathcal{O}_X)$ to the exact sequence

$$0 \to \mathcal{I}_{C_0,X}(-1) \to \mathcal{I}_{D_0,X}(-1) \to \mathcal{O}_B(-s_{D_0} - 1) \to 0$$

to see that $u : \mathcal{I}_{C_0,X}(-1) \to \mathcal{O}_X$ lifts to $\hat{u} : \mathcal{I}_{D_0,X}(-1) \to \mathcal{O}_X$ if and only if the image of u in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_B(-s_{D_0} - 1), \mathcal{O}_X)$ vanishes. Now by Serre duality on X the latter Ext group is dual to

$$H^1(X, \mathcal{O}_B(k - s_{D_0} - 5))$$

which is zero because $k \geq s_{D_0} + e(B) + 6$. Thus u lifts to give a height-one biliaison from D_0 to a curve D inducing the biliaison from C_0 to C. By Lemma 7.16 above $I_D/I_C \cong R_B(-s_{D_0} - 1)$. Finally, since $k \geq s_{D_0} + 1$, we have $s_D = s_{D_0} + 1$. □

The following proposition is a special case of [Davis 1985, Corollary 4.2].

Proposition 7.18. *Suppose the biliaison type $\lambda_C = (k_1, k_2, \ldots, k_s)$ of an ACM curve C has a gap at j. Then C contains ACM curves B and D such that*

$$\lambda_B = (k_1, k_2, \ldots, k_j), \quad \lambda_D = (k_{j+1}, k_{j+2}, \ldots, k_s), \quad \textit{and} \quad I_D/I_C \cong R_B(-s_D).$$

Furthermore, (B, D) is the unique pair of ACM curves with the above properties.

Proof. Note that $s = s_C$. Suppose first $j = s-1$, that is, $k_s \geq k_{s-1} + 3$. Since $k_s = e(C) + 3$, by Proposition 7.2 C is obtained by a special biliaison on a surface

X of degree k_s from an ACM curve B. By definition of biliaison type, $\lambda_B = (k_1, k_2, \ldots, k_{s-1})$. As $k_{s-1} = e(B) + 3$, we see

$$k_s \geq k_{s-1} + 3 = e(B) + 6.$$

By Lemma 7.17 the biliaison is trivial, so C contains a plane section D of X, and $I_D/I_C \cong R_B(-1)$. Since $\lambda_D = (\deg(X)) = (k_s)$, the statement holds when $j = s-1$

We now suppose $j < s-1$ and proceed by induction on $s-j$. By Proposition 7.2 C is obtained by a special biliaison on a surface X of degree k_s from an ACM curve C_0 whose biliaison type is $\lambda_0 := \lambda_{C_0} = (k_1, k_2, \ldots, k_{s-1})$. Thus λ_0 has a gap at j, and $s_{C_0} = s-1$, hence by induction C_0 contains ACM curves B and D_0 such that $\lambda_B = (k_1, k_2, \ldots, k_j)$, $\lambda_{D_0} = (k_{j+1}, k_{j+2}, \ldots, k_{s-1})$, and $I_{D_0}/I_{C_0} \cong R_B(-s_{D_0})$.

In particular, $s_{D_0} = s - j - 1$, so that

$$k_s \geq k_{j+1} + s - j - 1 \geq k_j + 3 + s_{D_0} = e(B) + 6 + s_{D_0}.$$

Since $k_s = e(C) + 3 \geq e(C_0) + 4$, by Lemma 7.17 the biliaison from C_0 to C is induced by a biliaison from D_0 to a curve D, and $I_D/I_C \cong R_B(-s_D)$. Finally, since D is obtained from D_0 by a special biliaison, D is ACM and $\lambda_D = \lambda_{D_0} \cup (k_s) = (k_{j+1}, k_{j+2}, \ldots, k_s)$.

It remains to prove uniqueness. Note that $s_D = s - j$ is determined by C, hence so is t_D because

$$t_D - s_d + 1 = \min(\lambda_D) = k_{j+1}.$$

By assumption $e(B) + 3 = k_j \leq k_{j+1} - 3 = t_D - s_D - 2$, hence from the exact sequence

$$0 \to \omega_D(m) \to \omega_C(m) \to \omega_B(s_D + m) \to 0$$

we see

$$H^0(\omega_D(m)) = H^0(\omega_C(m)) \quad \text{for every } m \leq 3 - t_D.$$

We will show that $\Omega_D = H^0_*(\omega_D)$ is generated over the polynomial ring $R = H^0_*(\mathbb{P}^3)$ by its elements of degree at most $3 - t_D$. Taking this for granted for the moment, it follows that Ω_D is the submodule of Ω_C generated by

$$\bigoplus_{m \leq 3 - t_D} H^0(\omega_C(m));$$

hence it is determined by C. But I_D is the annihilator of Ω_D, because R_D is Cohen–Macaulay with canonical module Ω_D, hence D is determined by C.

Since $t_D - s_D + 1 = k_{j+1} > 1$, the curve D is contained in a unique surface S of degree s_D, and therefore B is also determined, being the residual curve to $D = C \cap S$ in C.

To finish, we need to show $\Omega_D = H^0_*(\omega_D)$ is generated by its sections of degree at most $3 - t_D$. For this we choose a complete intersection Y of type (s_D, u)

containing D and let E be the curve linked to D by Y. As $\Omega_D \cong I_E/I_Y(-e_Y)$ and I_E is generated by its elements of degree at most $e(E)+3$, it is enough to show $e(Y)-t_D \geq e(E)$.

From $\omega_E(-e(Y)) \cong \mathcal{I}_D/\mathcal{I}_Y$ and $h^0(\mathcal{I}_D(t_D-1)) = h^0(\mathcal{I}_Y(t_D-1))$, we see that $h^0(\omega(t_D-1-e(Y))) = 0$; that is, $t_D - e(Y) \leq -e(E)$, as desired. □

Corollary 7.19. *Let $C \subset \mathbb{P}^3$ be an irreducible, reduced ACM curve that is contained in a smooth surface X of degree $s = s_C$. Let $t = t_C$ and $e = e(C)$.*

(a) *If $h_C(e+1) = 3$, $h_C(e+2) = 2$, then C has a unique $(e+3)$-secant line L, and every surface of degree at most $e+2$ containing C contains L as well.*

(b) *If $h_C(t) = s-2$, $h_C(t+1) = s-3$ (so that $s \geq 3$), then X contains a line L that is a $(t-s+1)$-secant of C.*

Remark 7.20. As a partial converse, we will see in the proof of Theorem 9.1 that, if, for every smooth C in the Hilbert scheme $A(h)$, the general surface of degree s containing C contains a line, then the h-vector of C satisfies either (a) or (b).

Proof of Corollary 7.19. Since X is smooth, by definition of t there is surface X_t of degree t containing C but not X. Thus C is contained in the complete intersection $Y = X \cap X_t$. Let Γ the curve linked to C by Y. Then on X

$$C \sim tH - \Gamma$$

where H denotes a plane section of X, and $\sim$ stands for linear equivalence. By [Migliore 1998, Corollary 5.2.19],

$$h_\Gamma(n) = h_Y(s+t-2-n) - h_C(s+t-2-n).$$

Case A: $h(e+1) = 3$ and $h(e+2) = 2$. The formula above implies

$$s_\Gamma = \min\{s, s+t-4-e\}.$$

But $t \leq e+3$ because $h_C(e+3) = 0$, hence $s_\Gamma = s+t-4-e$. The conditions on h_C then translate as follows:

$$h_\Gamma(s_\Gamma) = h_\Gamma(s_\Gamma+1) = s_\Gamma - 1.$$

If $s_\Gamma = 1$, this implies $\Gamma = L$ is a line. If $s_\Gamma \geq 2$, then the condition on h_Γ is equivalent to $\lambda_\Gamma = (1, k_2, \dots)$, with $k_2 \geq 4$ because $h_\Gamma(n) \geq s_\Gamma - 1$ at least for $n = s_\Gamma - 2, s_\Gamma - 1, s_\Gamma, s_\Gamma + 1$. By Proposition 7.18 Γ contains a line L and an ACM curve D with $I_D/I_\Gamma \cong R_L(1-s_\Gamma)$. We can treat the two cases simultaneously if we take D to be the empty curve when $s_\Gamma = 1$.

By Proposition 7.11, $\Gamma = L + D$ on X, and $L.D = s_\Gamma - 1$. Thus

$$C.L = (tH - L - D).L = t + s - 2 - s_\Gamma + 1 = s + t - s_\Gamma - 1 = e + 3.$$

In particular, every surface of degree at most $e+2$ containing C contains L as well. On the other hand, $C+L$ is an ACM curve, because it is linearly equivalent to $D+tH$. Therefore

$$I_C/I_{C+L} \cong R_L(-C.L) = R_L(-e-3).$$

It follows that $h_{C\cup L}(n)$ and $h_C(n)$ differ only for $n=e+3$, where their value is 1 and 0 respectively. In particular, $h_{C\cup L}(e+2) = h_C(e+2) = 2$ and $h_{C\cup L}(e+3) = 1$, so that by [Nollet 1998, Proposition 1.5] the homogeneous ideal of $C\cup L$ is generated by its forms of degree at most $e+2$, hence by the forms in I_C of degree at most $e+2$.

Suppose now M is an $(e+3)$-secant line of C. Then the homogeneous ideals of C and $C\cup M$ coincide in degrees at most $e+2$. It follows that the ideal of $C\cup L$ is contained in that of $C\cup M$, hence $C\cup L = C\cup M$ and $L = M$. Therefore L is the unique $(e+3)$-secant of C.

Case B: $h_C(t) = s-2$ and $h_C(t+1) = s-3$. Then $h_\Gamma(s-3) = h_\Gamma(s-2) = 1$ and $h_\Gamma(s-1) = 0$. This implies either $\lambda_\Gamma = (s-1)$, or $\lambda_\Gamma = (\dots, k_{u-1}, s-1)$ with $s-1-k_{u-1} \geq 3$. By Proposition 7.18, Γ contains a plane curve P of degree $s-1$ and an ACM curve B (possibly empty) such that $I_P/I_\Gamma \cong R_B(-1)$.

By Proposition 7.11, $\Gamma = B+P$ on X, and $B.P = d_B$. Let L be the line residual to P in the intersection of X with the plane of P. Then $B.L = B.H - B.P = 0$; hence

$$C.L = (tH-B-P).L = ((t-1)H-B+L).L = t-1+2-s = t-s+1. \quad \square$$

Given any sequence $\lambda = (k_1, k_2, , \dots, k_u)$ with $r-1$ gaps (for any $r \geq 1$), we can decompose λ uniquely as

$$\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_r, \tag{7-2}$$

where each λ_i has no gaps and, if a_i and b_i denote respectively the minimum and the maximum integer in λ_i, we have $a_{i+1} - b_i \geq 3$. We call (7-2) the *gap decomposition* of λ.

Theorem 7.21. *Let $A(\lambda)$ denote the Hilbert scheme parametrizing ACM curves having biliaison type λ. If C is general in $A(\lambda)$, then C is reduced and for every $f \geq e(C)+3$, there exists a smooth surface F of degree f containing C.*

Let $\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_r$ be the gap decomposition of λ. Then:

(a) *Every ACM curve $C \in A(\lambda)$ contains ACM subcurves D_i, $i = 1, 2, \dots, r$, such that $\lambda_{D_i} = \lambda_i$.*

(b) *If C is general in $A(\lambda)$, we have*

$$C = D_1 \cup D_2 \cup \cdots \cup D_r,$$

where the D_i are distinct smooth irreducible ACM curves satisfying $\lambda_{D_i} = \lambda_i$; for every $1 \le i_1 < i_2 < \cdots < i_h \le r$, the curve

$$D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_h}$$

is ACM and has biliaison type $\lambda_{i_1} \cup \lambda_{i_2} \cup \cdots \cup \lambda_{i_h}$.

Remark 7.22. The D_i in Theorem 7.21 (for $i \ge 2$) are not necessarily general in $A(\lambda_i)$: this is because they are forced to lie on surfaces containing D_j for $j < i$.

Proof of Theorem 7.21. Recall that by a theorem of Ellingsrud $A(\lambda)$ is irreducible (see Remark 6.4). By Proposition 7.18 and induction on the number of gaps we see that for each i, $1 \le i \le r$, there are ACM curves C_i and D_i with the following properties:

(1) $C_r = C$ and $C_1 = D_1$.

(2) If $2 \le i \le r$, C_i contains C_{i-1} and D_i, and $I_{D_i}/I_{C_i} = R_{C_{i-1}}(-s_{D_i})$.

(3) $\lambda_{D_i} = \lambda_i$ for every $1 \le i \le r$.

(4) $\lambda_{C_i} = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_i$ for every $1 \le i \le r$.

We claim that for every $1 \le i_1 < i_2 < \cdots < i_h \le r$ there are ACM curves $E_{i_1,i_2,\dots,i_h} \subseteq C_{i_h}$ such that

(1) if $h = 1$, $E_i = D_i$, and, if $h = r$, $E_{1,2,\dots,r} = C$;

(2) if $2 \le h \le r$, $E_{i_1,i_2,\dots,i_h}$ contains $E_{i_1,i_2,\dots,i_{h-1}}$ and D_{i_h}, and

$$I_{D_{i_h}}/E_{i_1,i_2,\dots,i_h} = R_{E_{i_1,i_2,\dots,i_{h-1}}}(-s_{D_{i_h}});$$

(3) $\lambda_{E_{i_1,i_2,\dots,i_h}} = \lambda_{i_1} \cup \lambda_{i_2} \cup \cdots \cup \lambda_{i_h}$.

We prove the statement by induction on h. When $h = 1$ there is nothing to prove. Suppose $h > 1$. By the induction hypothesis, there is a curve $A = E_{i_1,i_2,\dots,i_{h-1}} \subseteq C_{i_{h-1}}$ with the properties above. Let $B = C_{i_h - 1}$. By Lemma 7.23 below there exists a curve $C_0 \subseteq C_{i_h}$ containing B and D_{i_h} such that $I_{D_{i_h}}/I_{C_0} \cong R_A(-s_{D_{i_h}})$. Since A and D_{i_h} are ACM, it follows from Proposition 7.11 that C_0 is ACM as well. We define $E_{i_1,i_2,\dots,i_h}$ to be C_0. Then $E_{i_1,i_2,\dots,i_h}$ has the required properties (the formula for the biliaison type follows from part (d) of the same proposition).

To see the components D_i of a generic C are smooth, we follow the original proof of [Gruson and Peskine 1978, 2.5]. More precisely we show that, if

$$\lambda = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_r$$

is the gap decomposition of $\lambda = (k_1, \dots, k_s)$, there exists an ACM curve C with $\lambda_C = \lambda$ satisfying the following properties:

(1) C is contained in a smooth surface for every $f \ge k_s = e(C) + 3$.

(2) $C = D_1 \cup D_2 \cup \cdots \cup D_r$, where the D_i are smooth irreducible ACM curves satisfying $\lambda_{D_i} = \lambda_i$; in particular, C is reduced.

(3) $\omega_{D_r}(-e(D_r))$ has a section whose scheme of zeros is smooth (contains no multiple points).

We prove this statement by induction on s as in [Gruson and Peskine 1978, 2.5]. For $s = 1$, the statement is about plane curves and is well known (note that $e(C) + 3 = d_C$ for a plane curve C).

Assume now the statement is true for λ, fix a curve C with the properties above, and consider $\lambda^+ = \lambda \cup \{k_{s+1}\}$. We have two cases to consider:

Case 1: $k_{s+1} \le k_s + 3$. In this case λ^+ has a gap at s, and its gap decomposition is $\lambda^+ = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_r \cup \{k_{s+1}\}$.

By assumption, $k_{s+1} \ge k_s + 3 = e(C) + 6$; thus there exists a smooth surface X of degree k_{s+1} containing C. Let D_{r+1} be a general plane section of X, and let $C^+ = C \cup D_{r+1}$. Then D_{r+1} is smooth with $\lambda = (k_{s+1})$, thus C^+ satisfies (2) with respect to λ^+. It also satisfies (3) because $\omega_{D_{r+1}}(-e(D_{r+1})) \cong \mathbb{O}_{D_{r+1}}$. By construction C^+ lies on the smooth surface X of degree $k_{s+1} = e(C^+) + 3$. The fact that C^+ is contained in a smooth surface of degree f, for every $f > e(C^+)+3$, follows now from the fact that $\mathcal{I}_{C^+}(e(C^+)+3)$ is generated by its global sections; see, for example, [Peskine and Szpiro 1974] and [Nollet 1998, Corollary 2.9]. Thus C^+ also satisfies (1), and we are done in case 1.

Case 2: $k_{s+1} = k_s + 1$ or $k_s + 2$. In this case the gap decomposition of λ^+ is

$$\lambda^+ = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_{r-1} \cup \lambda_r^+$$

where $\lambda_r^+ = \lambda_r \cup \{k_{s+1}\}$.

We can still find a smooth surface X of degree k_{s+1} containing C because $k_{s+1} > e(C) + 3$. In particular, X contains D_r. The proof of [Gruson and Peskine 1978, 2.5] shows that the general curve D_r^+ in the linear system $D_r + H$ on X is smooth with $\lambda_{D_r^+} = \lambda_r^+$, and that $\omega_{D_r^+}(-e(D_r^+))$ has a section whose scheme of zeros is smooth. Thus

$$C^+ = D_1 \cup D_2 \cup \cdots \cup D_r^+$$

has the required properties (note that $e(C^+) + 3 = k_{s+1} = \deg(X)$). $\square$

Lemma 7.23. *Suppose $C \subset \mathbb{P}^3$ is a curve, with subcurves B, D such that*

$$I_D/I_C \overset{\beta}{\cong} R_B(-f).$$

If A is a subcurve of B, there exists a unique curve C_0 with the following properties:

(1) *C_0 is contained in C.*

(2) *C_0 contains A and D, and there is an isomorphism $I_D/I_{C_0} \overset{\alpha}{\cong} R_A(-f)$ which makes commutative the diagram*

$$\begin{array}{ccc} I_D/I_C & \xrightarrow[\cong]{\beta} & R_B(-f) \\ \downarrow & & \downarrow \\ I_D/I_{C_0} & \xrightarrow[\cong]{\alpha} & R_A(-f) \end{array}$$

where the vertical arrows are induced by the inclusions $C_0 \subseteq C$ and $A \subseteq B$.

If A and D have no common components, then $C_0 = A \cup D$.

Proof. The inclusion

$$I_A/I_B(-f) \hookrightarrow R_B(-f) \overset{\beta^{-1}}{\cong} I_D/I_C \hookrightarrow R_C$$

defines an ideal J in R_C. Uniqueness is clear, because if such a C_0 exists, we must have $I_{C_0}/I_C = J$. To show existence, let I be the inverse image of J in the polynomial ring $R = H^0_*(\mathbb{O}_{\mathbb{P}^3})$, so that $I/I_C \cong I_A/I_B(-f)$. The given isomorphism $I_D/I_C \overset{\beta}{\cong} R_B(-f)$ induces $I_D/I \overset{\alpha}{\cong} R_A(-f)$, hence an exact sequence

$$0 \to R_A(-f) \to R/I \to R_D \to 0.$$

From this exact sequence we see that R/I has depth at least one, hence I is the saturated ideal of a subscheme $C_0 \subset C$.

By construction I_{C_0}/I_C and $I_A/I_B(-f)$ are isomorphic, so that the given isomorphism $I_D/I_C \overset{\beta}{\cong} R_B(-f)$ induces another, $I_D/I_{C_0} \overset{\alpha}{\cong} R_A(-f)$, with the desired properties. Finally, we can check C_0 is a locally Cohen–Macaulay curve looking at the exact sequence

$$0 \to \mathbb{O}_A(-f) \to \mathbb{O}_{C_0} \to \mathbb{O}_D \to 0.$$

If A and D have no common components, then C_0 contains the union $A \cup D$. Since both C_0 and $A \cup D$ are locally Cohen–Macaulay curves of degree $d_A + d_D$, they must be equal. □

8. Bounds on the quadratic form $\phi(D, D)$

Let $X \subset \mathbb{P}^3$ be a smooth surface of degree $s \geq 2$. We will make use of the bilinear form on $\operatorname{Pic}(X)$:

$$\phi(D, E) = (D.H)\,(E.H) - s\,(D.E) = \det \begin{bmatrix} D.H & H^2 \\ D.E & E.H \end{bmatrix}.$$

This is essentially the positive definite product on $\mathrm{Pic}(X)/\mathbb{Z}H$ induced by the intersection product: by the algebraic Hodge index theorem, $\phi(D, D) \geq 0$ for any divisor D on X, and $\phi(D, D) = 0$ if and only if D is numerically (hence linearly) equivalent to a multiple of H.

In the proof of our main theorem it will be crucial to be able to bound $\phi(D, D)$ from below in terms of the degree d_D when D is an ACM curve on X. Note that if D is a curve on X, then

$$\phi(D, D) = d_D^2 + s(s-4)d_D - 2s(g(D)-1) \tag{8-1}$$

Thus, if we fix the degree d_D and s, then knowing $\phi(D, D)$ is the same as knowing the genus $g(D)$, and bounding $\phi(D, D)$ from below is the same as bounding $g(D)$ from above. In fact, the bounds of this section can be seen as a refinement of the bounds on the genus of an ACM curve of [Gruson and Peskine 1978]; see Remark 8.8. The form $\phi(D, D)$ has the advantage of being invariant if we replace D with $mH - D$ or $D + nH$, that is, it is invariant under liaison and biliaison on X. Thus one can compute $\phi(D, D)$ assuming D is a minimal curve on X.

To compute these bounds we note that, by (8-1), the form $\phi(D, D)$ for an ACM curve D depends only on the h-vector (or the biliaison type λ) of D and on s. Since it is enough to consider only minimal curves on X, and there only finitely many possible biliaison types λ of minimal curves for each s, our proof will proceed by a careful analysis of these λ.

We call a biliaison type λ *s-minimal* if it corresponds to a minimal ACM curve on a smooth surface X of degree s. Since minimal is equivalent to $e + 3 < s$ by Proposition 6.7, the s-minimal types λ are just those increasing sequences of positive integers $\lambda = (k_1, k_2, \dots, k_u)$ satisfying $k_u < s$. There are 2^{s-1} such possible sequences (including the empty one), and by Proposition 6.9 the corresponding curves are linked by a complete intersection (s, s) to curves with s-basic h-vectors. For any such λ, we let d, g, e be the corresponding invariants of the associated curve Γ, and we define

$$q(\lambda) = \phi(\Gamma, \Gamma) = d^2 + s(s-4)d - 2s(g-1). \tag{8-2}$$

Then one verifies the formula

$$q(\lambda) = \sum_{i=1}^{u} k_i(s-1)(s-k_i) \; - \; 2 \sum_{1 \leq i < j \leq u} k_i(s-k_j). \tag{8-3}$$

Table 1 on page 284 lists all the s-basic h-vectors and associated s-minimal biliaison types λ for $s = 4, 5$ and a few for $s = 6, 7, 8, 9$, together with the values q takes on them.

Definition 8.1. Suppose $\lambda = (k_1, k_2, \dots, k_u)$ is s-minimal. Then we define the *s-dual* λ' of λ to be

$$\lambda' = (s - k_u, s - k_{u-1}, \dots, s - k_1)$$

if $\lambda \neq \varnothing$. If $\lambda = \varnothing$, then $\lambda' = \varnothing$. Note that, if λ is the biliaison type of an ACM curve Γ, then λ' is the biliaison type of a curve linked to Γ by a complete intersection of two surfaces of degree $s_\Gamma = u_\lambda$ and s (see Section 6).

Proposition 8.2. *The invariants of λ' are $u_{\lambda'} = u_\lambda$, $d_{\lambda'} = u_\lambda s - d_\lambda$, $q(\lambda') = q(\lambda)$.*

Proof. The first two equalities are obvious. The equality $q(\lambda') = q(\lambda)$ follows from (8-3), or can be deduced from the invariance of $\phi(D, D)$ under liaison on X. □

We say that $\lambda_1 = (k_1, k_2, \dots, k_u)$ *precedes* $\lambda_2 = (l_1, l_2, \dots, l_v)$ and write $\lambda_1 < \lambda_2$ if $k_u < l_1$. In this case, if λ_2 is s-minimal, then

$$\lambda_1 \cup \lambda_2 = (k_1, k_2, \dots, k_u, l_1, \dots, l_v)$$

is also s-minimal. Note that $(\lambda \cup \mu)' = \mu' \cup \lambda'$.

Example 8.3. A plane curve of degree $k < s$ on a surface X of degree $s \geq 2$ is minimal. The corresponding λ sequence is $\lambda = (k)$, and $q(\lambda) = k(s-1)(s-k)$.

More generally if λ is the biliaison type of a complete intersection of two surfaces of degrees $a \leq b < s$ then $q(\lambda) = ab(s-a)(s-b)$.

Example 8.4. Let $\lambda = (1, 2, \dots, k-1, k)$ with $k < s$. Then $d_\lambda = \frac{1}{2}k(k+1)$ and

$$q(\lambda) = d_\lambda \left(s^2 - \tfrac{2}{3}s(2k+1) + d_\lambda\right)$$

The first statement of Proposition 8.5 below determines, once $q((k))$ is known, the function $q(\lambda)$ by induction on the number u_λ of elements of λ.

Proposition 8.5. *Suppose $\lambda < \mu$ are s-minimal.*

(a) $q(\lambda \cup \mu) = q(\lambda) + q(\mu) - 2d_\lambda d_{\mu'}$.

(b) *If $\lambda < (k)$ and $(k+1) < \mu$, then*

$$q(\lambda \cup (k+1) \cup \mu) - q(\lambda \cup (k) \cup \mu) = (s-1)(s-1-2k) - 2(d_{\mu'} - d_\lambda).$$

(c) *Suppose β is another s-minimal biliaison type, and h, k are two integers such that $\lambda < (h-1)$, $(h) < \beta < (k)$, and $(k+1) < \mu$. Let $\delta = \lambda \cup (h) \cup \beta \cup (k) \cup \mu$ and $\epsilon = \lambda \cup (h-1) \cup \beta \cup (k+1) \cup \mu$. Then*

$$q(\delta) - q(\epsilon) = 2s(k - h - u_\beta) \geq 2s > 0.$$

We next show that $q(\lambda)$ increases if one inserts a new integer in a sequence λ.

Corollary 8.6. *Let* $(k_1, k_2, \ldots, k_u)$ *be* s*-minimal.*

(a) *If* $k_u < k < s$, *then*

$$q(k_1, k_2, \ldots, k_u, k) \geq q(k_1, k_2, \ldots, k_u) + k(s-k)^2$$

In particular, $q(\lambda) \geq (s-1)^2$ *unless* $\lambda = \varnothing$.

(b) *If* $k_i < k < k_{i+1}$, *then*

$$q(k_1, k_2, \ldots, k_i, k, k_{i+1}, \ldots, k_u) \geq q(k_1, k_2, \ldots, k_r) + k(s-k).$$

Proof. Let $\lambda = (k_1, k_2, \ldots, k_u)$ By Proposition 8.5 we have

$$q(\lambda \cup (k)) = q(\lambda) + q(k) - 2d_\lambda(s-k) = q(\lambda) + (s-k)\,(k(s-1) - 2d_\lambda)\,.$$

Thus the first claim follows from

$$d_\lambda = \sum_1^r k_i \leq \frac{1}{2}k(k-1). \tag{8-4}$$

For the second claim, set $\lambda = (k_1, k_2, \ldots, k_i)$ and $\mu = (k_{i+1}, k_{i+2}, \ldots, k_u)$. Using Proposition 8.5 we compute

$$q(\lambda \cup ((k) \cup \mu)) - q(\lambda \cup \mu) = q((k) \cup \mu) - q(\mu) + 2d_\lambda(d_{\mu'} - d_{((k)\cup\mu)'})$$

Now $d_{\mu'} - d_{((k)\cup\mu)'} = -(s-k)$, while by duality and the first claim

$$q((k) \cup \mu) - q(\mu) = q(\mu' \cup (s-k)) - q(\mu') \geq (s-k)k^2.$$

Hence

$$\begin{aligned} q(\lambda \cup ((k) \cup \mu)) - q(\lambda \cup \mu) &\geq (s-k)k^2 - 2d_\lambda(s-k) \\ &= (s-k)(k^2 - 2d_\lambda) \geq k(s-k), \end{aligned}$$

where the last inequality follows from (8-4). □

We now prove a lower bound for $q(\lambda)$ in terms of the residue class of d_λ modulo s.

Proposition 8.7. *Let* λ *be* s*-minimal, of degree* d *congruent to* f *modulo* s, *with* $0 \leq f < s$. *Then*

(a) *If* $u_\lambda = 2$, *so that* $\lambda = (h, k)$ *with* $h + k \equiv f \pmod{s}$, *then*

$$q(\lambda) = \begin{cases} f(s-1)(s-f) + 2h(k-1)s & \text{if } h+k < s, \\ f(s-1)(s-f) + 2(s-k)(s-h-1)s & \text{if } h+k \geq s. \end{cases}$$

(b) *If* $u_\lambda \geq 3$ *and* $s \geq 5$, *we have*

$$q(\lambda) \geq 2s + m(f, s),$$

where $m(f, s)$ *denotes the minimum of* $q(\mu)$ *as* μ *varies among* s*-minimal* *biliaison types satisfying $u_\mu = 2$ and $d_\mu \equiv f$ or $d_\mu \equiv s-f \pmod{s}$. In fact,*

$$m(f,s) = \begin{cases} f(s-1)(s-f)+2s(f-2) & \text{if } 3 \le f \le s-f \text{ or } f = s-2, s-1, \\ f(s-1)(s-f)+2s(s-f-2) & \text{if } 3 \le s-f \le f \text{ or } f = 0,1,2. \end{cases}$$

This minimum is attained by $\lambda = (1, f-1)$ and $\lambda' = (s-f+1, s-1)$ when $3 \le f \le s-f$ or if $f = s-2, s-1$, and by $\lambda = (1, s-f-1)$ and $\lambda' = (f+1, s-1)$ when $3 \le s-f \le f$ or $f = 0, 1, 2$.

Proof. Part (a) is a simple computation. To prove part (b), note that the role of f and $s-f$ is symmetric, reflecting the fact that $q(\lambda) = q(\lambda')$. Thus we can replace λ with λ' whenever convenient. If $\lambda = (k_1, k_2, \dots, k_r)$ and there are two indices $i < j$ such that $k_i - 1 > k_{i-1}$ and $k_j + 1 < k_{j+1}$, we replace k_i by $k_i - 1$ and k_j by $k_j + 1$ to obtain a new increasing sequence λ_1 with the same degree as λ, hence the same f. Then $q(\lambda) \ge q(\lambda_1) + 2s$ by Proposition 8.5(c). When $u_\lambda = 2$, it follows that the minimum $m(f,s)$ is attained by sequences of the form $(1,k)$ or $(h, s-1)$, as in the statement. When $u_\lambda \ge 3$, iterating the procedure above and passing to the dual word if necessary, we may assume that λ is one of the following sequences:

$$\begin{array}{cl} (1,2,\dots,h) & 3 \le h < s \\ (1,2,\dots,h, s-m, s-(m-1), \dots, s-1) & 1 \le m \le h,\ 2 \le h \le s-m-2 \\ (1,2,\dots,h,k) & 2 \le h \le k-2 \\ (1,2,\dots,h,k,s-m,s-(m-1),\dots,s-1) & m \le h,\ 1 \le h \le k-2,\ k \le s-m-2 \end{array}$$

If $\lambda = (1, 2, \dots, s-1)$, we replace it with $(2, \dots, s-2)$, since

$$q(1,2,\dots,s-1) > q(2,\dots,s-2)$$

If $h \ge 2$, we define

$$\mu = (2,\dots,h-1,h+1,\dots)$$

to be the sequence obtained removing 1 and h from λ and adding $h+1$. If $h = 1$, then $\lambda = (1, k, s-1)$ with $3 \le k \le s-3$, in which case we define $\mu = (k+1, s-1)$.

Then $d_\mu = d_\lambda$, $u_\mu = u_\lambda - 1$, hence we will be done by induction on u_λ if we show $q(\lambda) \ge q(\mu) + 2s$. By Proposition 8.5(a) we can assume $\lambda = (1, 2, \dots, h)$ and $\mu = (2, \dots, h-1, h+1)$. Then one computes $q(\lambda) - q(\mu) = 2s$. □

Remark 8.8. One can show that the bound $q(\lambda) \ge f(s-1)(s-f)$ is equivalent to the bound in [Gruson and Peskine 1978] for the genus of an ACM curve of degree $d > s(s-1)$ not lying on a surface degree $s-1$. They also show that curves of maximal genus are linked to plane curves: in our notation this means $u_\lambda = 1$ if $q(\lambda)$ attains its minimal value $f(s-1)(s-f)$.

Corollary 8.9. *Let λ be s-minimal of degree d congruent to f modulo s, with $0 \le f < s$. If $u_\lambda \ge 2$, then*

$$q(\lambda) \ge \begin{cases} 2s(s-2) & \text{if } f = 0, \\ 3s^2 - 8s+1 & \text{if } f = 1 \text{ or } f = s-1, \\ 2s^2 - 4s + 4 & \text{if } f \notin \{0, 1, s-1\}. \end{cases}$$

Proof. We may assume $s \ge 5$ because the cases $s = 3, 4$ are easily checked; see Table 1. If $f = 0$, 1 or $s-1$, the statement follows immediately from Proposition 8.7. If $f \ne 0, 1, s-1$, again by the smae proposition we have

$$q(\lambda) \ge q(f) + 2s \ge q(2) + 2s = 2s^2 - 4s + 4. \qquad \square$$

Corollary 8.10. *Suppose $s \ge 5$ and let λ be s-minimal. Suppose $q(\lambda) \le (s+1)^2$. Then one of the following occurs*:

(1) $\lambda = \varnothing$ *and* $q(\lambda) = 0$.

(2) $\lambda = (1)$ *or* $\lambda = (s-1)$, *and* $q(\lambda) = (s-1)^2$.

(3) $5 \le s \le 7$ *and* $\lambda = (2)$ *or* $\lambda = (s-2)$, *so that* $q(\lambda) = 2(s-1)(s-2)$.

(4) $s = 6$ *and* $\lambda = (3)$, *so that* $q(\lambda) = 3(s-1)(s-3) = 45$.

(5) $s = 5$ *or* 6 *and* $\lambda = (1, s-1)$.

(6) $s = 5$ *and* $\lambda = (1, 3)$ *or* $\lambda = (2, 4)$, *in which case* $q(\lambda) = 36 = (s+1)^2$.

(7) $s = 5$ *and* $\lambda = (1, 2)$ *or* $\lambda = (3, 4)$, *in which case* $q(\lambda) = 34$.

Furthermore, if $q(\lambda) \le (s-1)^2$, *then either* (1) *or* (2) *occurs. If* $(s-1)^2 < q(\lambda) \le s^2$, *then either* $s = 4$ *and* $\lambda = (2)$ *or* $(1, 3)$, *or* $s = 5$ *and* $\lambda = (2)$ *or* (3).

Proof. Suppose first $\lambda = (f)$. Then $q(\lambda) = f(s-1)(s-f)$. One checks this is bigger than $(s+1)^2$ except in the cases listed in the statement.

Suppose now $u_\lambda \ge 2$. If $f = 0$, then $q(\lambda) \ge 2s(s-2)$ by Corollary 8.9, and this is bigger than $(s+1)^2$ unless $s \le 6$. When $s = 5$ or 6, one checks by hand the only possibility is $\lambda = (1, s-1)$.

If $f = 1$ or $s-1$, the lower bound for $q(\lambda)$ is

$$3s^2 - 8s+1,$$

which is bigger than $(s+1)^2$ unless $s \le 5$. When $s = 5$, one finds the two sequences $\lambda = (1, 3)$ or $\lambda = (2, 4)$.

If $f \ne 0, 1, s-1$, then $q(\lambda) \ge 2s^2 - 4s + 4$ which is bigger than $(s+1)^2$ unless $s \le 5$. When $s = 5$, one finds the two sequences $\lambda = (1, 2)$ or $\lambda = (3, 4)$ for which $q(\lambda) = 34$. $\square$

9. Gonality of a general ACM curve

In this section we give the proof of our main result.

Theorem 9.1. *Assume $\mathbb{K}$ has characteristic zero. Let $C \subset \mathbb{P}^3_{\mathbb{K}}$ be an irreducible, nonsingular ACM curve with h-vector h, and let $s = s_C$, $t = t_C$, $e = e(C)$ and $g = g(C)$. Assume that $s \geq 4$ and that (s, d, g) is not one of the following*: (4, 10, 11), (5, 15, 26), (5, 16, 30), (6, 21, 50), (6, 22, 55), (6, 23, 60), (7, 28, 85), (7, 29, 91), (8, 36, 133).

Suppose there is a smooth surface X of degree s containing C with the following properties:

(1) *The linear system $|tH - C|$ on X contains a reduced curve Γ, such that the irreducible components $D_1, \dots D_r$ are ACM curves, and*

$$\lambda_\Gamma = \lambda_{D_1} \cup \lambda_{D_2} \cup \cdots \cup \lambda_{D_r}$$

is the gap decomposition of λ_Γ.

(2) *The Picard group of X is* $\operatorname{Pic}(X) = \mathbb{Z}[H] \oplus \mathbb{Z}[D_1] \oplus \cdots \oplus \mathbb{Z}[D_r]$.

(3) *C is general in its linear system on X.*

Then

$$\operatorname{gon}(C) = d - l,$$

where $l = l(C)$ is the maximum order of a multisecant of C. Furthermore, with the possible exception of the values of (s, d, g) listed in Proposition 6.10(b), C has finitely many g^1_{d-l}; hence its Clifford index is

$$\operatorname{Cliff}(C) = \operatorname{gon}(C) - 2 = d - l - 2.$$

More precisely:

(a) *If $h(e+1) = 3$, $h(e+2) = 2$, then the gonality of C is $d-e-3$ and there is unique pencil of minimal degree, arising from the unique $(e+3)$-secant line of C* (compare Corollary 7.19).

(b) *if $h(t) = s-2$, $h(t+1) = s-3$, $t > s+3$, but the condition of case (a) above does not occur, then the gonality of C is $d - (t-s+1)$, and there is unique pencil of minimal degree, arising from the unique $(t-s+1)$-secant line of C.*

(c) *if neither case (a) nor (b) above occurs, then the gonality of C is $d-4$, and every g^1_{d-4} on C arises from a 4-secant line, unless either*

   (1) *(s, d, g) is in the list of Proposition 6.10(b), or*

   (2) *$s = 4$, $C \in |C_0 + bH|$ where $b \geq 2$ and C_0 has degree 4 and arithmetic genus 1; in this case $|\mathcal{O}_C(b)|$ is the unique g^1_{d-4} that does not arise from a 4-secant.*

Finally, if C has a complete basepoint-free pencil of degree $k < d-4$, then the pencil arises either from an $(e+3)$-secant line or from a $(t-s+1)$-secant line.

Remark 9.2. The conditions on h in (a) and (b) are not satisfied in any of the cases listed in Proposition 6.10(b).

Proof of Theorem 9.1. The gonality of C is at most $d-4$ by Proposition 3.1.

Suppose $\mathcal{Z}$ is a complete basepoint-free pencil of degree k on C, and assume $k \le d-4$, unless we are in one of the cases listed in Proposition 6.10(b), for which we assume $k \le d-5$. We will classify these pencils as follows. By the same proposition the bundle $\mathcal{E}$ associated to $\mathcal{Z}$ on X satisfies $\Delta(\mathcal{E}) > 0$, and then by Bogomolov's result (Theorem 5.4) it follows that $\mathcal{E}$ is Bogomolov unstable. Let $\mathcal{O}_X(A)$ be the line bundle that destabilizes $\mathcal{E}$. We will show that only the following cases can occur:

(1) for any h-vector, we can have $A = -H$; then by Corollary 5.7 the pencil $\mathcal{Z}$ arises from a multisecant line L that is not contained in X. Corollary 4.2 shows that $k = \deg \mathcal{Z} = d-4$ and that there is a finite set of such pencils.

(2) when $h(e+1) = 3$ and $h(e+2) = 2$, then C has a unique $(e+3)$-secant line L, and $\mathcal{Z} = \mathcal{Z}(L)$. In this case $L \subset X$ and $A = L - H$.

(3) if $t > s+3$, $h(t) = s-2$, $h(t+1) = s-3$, then C has a unique $(t-s+1)$-secant line L, and $\mathcal{Z} = \mathcal{Z}(L)$. In this case $L \subset X$ and $A = L - H$.

(4) $s = 4$, $C \in |C_0 + bH|$ where $b \ge 2$ and C_0 has degree 4 and arithmetic genus 1. In this case $\mathcal{Z} = |\mathcal{O}_C(b)|$ and $A = -C_0$. In particular, $\deg \mathcal{Z} = d-4$ and $\mathcal{Z}$ does not arise from a multisecant.

The statement of the theorem clearly follows from this classification. For the Clifford index, we use the fact, proved in [Coppens and Martens 1991], that $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ when C has a finite number of pencils of minimal degree.

We now proceed to classify the possible basepoint-free complete pencils $\mathcal{Z}$ of degree at most $d-4$. Let A be the divisor that destabilizes the bundle $\mathcal{E}$ associated to $\mathcal{Z}$. Recall that A sits in an exact sequence

$$0 \to \mathcal{O}_X(A) \to \mathcal{E} \to \mathcal{I}_{W,X}(B) \to 0$$

where W is zero-dimensional and $(A-B).H > 0$. From the exact sequence we see $A - B = 2A + C$ and

$$(2A+C)^2 = (A-B)^2 \ge \Delta(\mathcal{E}) = C^2 - 4k.$$

By Proposition 5.5 we also have $(-A).H > 0$ and $A^2 \ge 0$.

To be able to work effectively with the above inequalities, we write $x = A.H$ for the degree of A, and consider the bilinear form on $\mathrm{Pic}(X)$

$$\phi(D, E) = (D.H)\,(E.H) - s\,(D.E) = \det \begin{bmatrix} D.H & H^2 \\ D.E & E.H \end{bmatrix}.$$

We then obtain the following numerical constraints on x:

$$(9\text{-}1) \qquad -d < 2x < 0, \quad x^2 \geq \phi(A, A), \quad x^2 + dx + ks \geq \phi(A, A + C),$$

the last two inequalities being equivalent to $A^2 \geq 0$ and $(2A + C)^2 \geq C^2 - 4k$ respectively.

In $\mathrm{Pic}(X)$ we can write $A = \sum a_i D_i + cH$ with $a_i \in \mathbb{Z}$, $c \in \mathbb{Z}$. We wish to show

$$\phi(A, A + C) \geq 0.$$

We first prove $\phi(D_i, D_j) < 0$. Let $\lambda_\Gamma = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_r$ be the gap decomposition of λ_Γ, so that $\lambda_{D_i} = \lambda_i$. If $i < j$, $D_i + D_j$ is ACM with $\lambda_{D_i + D_j} = \lambda_i \cup \lambda_j$ by Theorem 7.21. Since $\phi(D, D) = q(\lambda_D)$ for an ACM curve D with $s_D < s$, by Proposition 8.5

$$(9\text{-}2) \qquad \phi(D_i, D_j) = -d_{\lambda_i} d_{\lambda'_j} < 0$$

(note that the formula $\phi(D_i, D_j) = -d_{\lambda_i} d_{\lambda'_j}$ is correct only for $i < j$).

To simplify notation we let $q_i = \phi(D_i, D_i)$ and $b_i = -\sum_{j \neq i} \phi(D_i, D_j)$. We claim that $q_i > 2b_i$ for every i. To prove this let $E_i = \sum_{j \neq i} D_j$. Then

$$\begin{aligned} \phi(\Gamma, \Gamma) &= \phi(D_i + E_i, D_i + E_i) = \phi(D_i, D_i) + \phi(E_i, E_i) + 2\phi(D_i, E_i) \\ &= \phi(E_i, E_i) + q_i - 2b_i; \end{aligned}$$

thus it is enough to show $\phi(\Gamma, \Gamma) > \phi(E_i, E_i)$, that is, $q(\lambda_\Gamma) > q(\lambda_{E_i})$. The latter inequality holds by Corollary 8.6; hence $q_i > 2b_i$.

We now compute

$$\begin{aligned} \phi(A, A) &= \sum_i a_i^2 \phi(D_i, D_i) + 2 \sum_{i<j} a_i a_j \phi(D_i, D_j) \\ &= \sum_i a_i^2 (q_i - b_i) - \sum_i a_i^2 \sum_{j \neq i} \phi(D_i, D_j) + 2 \sum_{i<j} a_i a_j \phi(D_i, D_j) \\ &= \sum_i a_i^2 (q_i - b_i) - \sum_{i<j} (a_i - a_j)^2 \phi(D_i, D_j), \end{aligned}$$

$$\begin{aligned} \phi(A, C) &= \phi\Big(\sum_i a_i D_i, t_C H - \sum_j D_j\Big) = \phi\Big(\sum_i a_i D_i, -\sum_j D_j\Big) \\ &= -\sum_{i,j} a_i \phi(D_i, D_j) = -\sum_i a_i (q_i - b_i). \end{aligned}$$

Therefore

$$\phi(A, A) = \sum_i a_i^2(q_i - b_i) - \sum_{i<j}(a_i - a_j)^2\phi(D_i, D_j), \tag{9-3}$$

$$\phi(A, C) = -\sum_i a_i(q_i - b_i), \tag{9-4}$$

$$\phi(A, A+C) = \sum_i (a_i^2 - a_i)(q_i - b_i) - \sum_{i<j}(a_i - a_j)^2\phi(D_i, D_j). \tag{9-5}$$

The last equality implies $\phi(A, A + C) \geq 0$ because the a_i are integers, $q_i > 2b_i \geq b_i$ and $\phi(D_i, D_j) < 0$.

We now show that $\phi(A, A+C) \geq 0$ implies $x \geq -s-1$.

By hypothesis $k \leq d - 4$; therefore

$$x^2 + dx + (d-4)s \geq x^2 + dx + ks \geq \phi(A, A+C) \geq 0.$$

Let δ be the discriminant of the equation $x^2 + dx + (d-4)s = 0$:

$$\delta = d^2 - 4sd + 16s = (d-2s)^2 - 4s(s-4).$$

Let $y = d-2s$. Since C is ACM and $s = s_C$, we have $d \geq \frac{1}{2}s(s+1)$ by Remark 6.2, hence

$$y - 2 = d - 2s - 2 \geq \tfrac{1}{2}(s^2 - 3s - 4) \geq \tfrac{1}{2}(s^2 - 4s).$$

In fact, we can have equality only if $s = 4$ and $d = 10$, while the hypotheses of the theorem when $s = 4$ require d to be at least 11. Thus $y - 2 > \frac{1}{2}s(s-4)$ and

$$\delta = y^2 - 4s(s-4) > y^2 - 8y + 16 = (y-4)^2.$$

Thus δ is positive, and the equation has two real roots, one smaller than $-d/2$, the other one, say $\bar{x}$, larger than $-d/2$. Since $-d/2 < x < 0$, we conclude $x \geq \bar{x}$. Furthermore, unless $s = 4$ and $d = 11$, we have $y - 4 \geq 0$ under the hypotheses of the theorem, hence

$$\bar{x} = -\frac{d}{2} + \frac{1}{2}\sqrt{\delta} > -\frac{d}{2} + \frac{1}{2}\sqrt{y^2 - 8y + 16} = -\frac{d}{2} + \frac{1}{2}(y-4) = -s-2.$$

The inequality $\bar{x} > -6$ holds also in case $s = 4$ and $d = 11$. Thus $x \geq -s-1$. Then from $x^2 \geq \phi(A, A)$ we see that

$$(s+1)^2 \geq \phi(A, A).$$

If all the a_i are zero, then $A = cH$ (this is the case if C is a complete intersection of X and another surface). Since $-s-1 \leq x = \deg A < 0$, we must have $A = -H$.

If not all the a_i are zero, let $1 \leq i_1 < \cdots < i_h \leq r$ be the indices for which $a_i \neq 0$. Formula (9-3) holds with this new set of indices, and shows that, if all the coefficients a_i are nonzero, then $\phi(A, A)$ attains its minimum when all the a_i are equal to 1. Thus

$$\phi(A, A) \geq \phi(D, D),$$

where $D = D_{i_1} + \cdots + D_{i_h}$ is the support of A.

Now D is ACM with biliaison type $\lambda_D = \lambda_{i_1} \cup \cdots \cup \lambda_{i_h}$ by Theorem 7.21. If λ_D is not one of the special cases listed in Corollary 8.10, then

$$\phi(D, D) = q(\lambda_D) > (s+1)^2,$$

contradicting $(s+1)^2 \geq \phi(A, A)$.

Suppose now λ_D is one of the special cases listed in Corollary 8.10. We still have $\phi(A, A) \geq (s-1)^2$ because λ_D is not empty. Before examining the various cases, let us remark that, if only one of the a_i is nonzero, so that

$$A = aD + cH$$

with D irreducible and $a \neq 0$, then either $a = 1$ or $a = -1$. This follows from

$$a^2 = \frac{\phi(A, A)}{\phi(D, D)} \leq \frac{(s+1)^2}{(s-1)^2} < 4.$$

Also note that D is irreducible precisely when λ_D has no gaps, that is, in all cases of Corollary 8.10 except when $s = 5$ or 6 and $\lambda = (1, s-1)$.

To complete the list of Corollary 8.10, observe from Table 1 that for $s = 4$ there are 7 possibilities for λ_D, because $\lambda \neq \varnothing$ and $u_\lambda < 4$, namely

$$(1),\ (2),\ (3),\ (1, 2),\ (1, 3),\ (2, 3),\ (1, 2, 3).$$

Case 1: $\lambda_D \neq (1)$, $\lambda_D \neq (s-1)$, and, when $s = 5$ or 6, $\lambda_D \neq (1, s-1)$.

Then $\phi(D, D) > (s-1)^2$ and λ_D has no gaps by Corollary 8.10. Thus D is irreducible, $A = aD + cH$ with $a = \pm 1$ and

$$(s+1)^2 \geq x^2 \geq \phi(A, A) = a^2 \phi(D, D) > (s-1)^2.$$

Hence $x = -s-1$ or $x = -s$.

<u>Case 1a:</u> $a = 1$, $x = -s-1$. In this case $d_D \equiv x \equiv -1 \pmod{s}$, and by Corollary 8.10 we must have $s \leq 5$. Furthermore by the last inequality in (9-1)

$$x^2 + dx + (d-4)s \geq 0,$$

that is

$$s^2 + 2s+1 - sd - d + (d-4)s \geq 0$$

so $d \leq s^2 - 2s+1$. This gives $d \leq 9$ if $s = 4$, and $d \leq 16$ if $s = 5$, while $d \geq \frac{1}{2}s(s+1)$ because C is an ACM curve $s_C = s$. Thus we must have $s = 5$, and examining the list in Corollary 8.10 we find $\lambda_D = (1, 3)$ is the only possibility. Then, for

$\Gamma = tH - C$, we know λ_Γ contains $\lambda_D = (1, 3)$ in its gap decomposition and $u_{\lambda_\Gamma} < 5$. This forces $\lambda_\Gamma = \lambda_D$, hence $D = \Gamma$ and therefore

$$d = st - \deg(\Gamma) \geq 25 - 4 = 21$$

a contradiction, so this case does not occur.

Case 1b: $a = 1$, $x = -s$. In this case $d_D \equiv x \equiv 0 \pmod{s}$ and $s^2 = x^2 \geq q(\lambda)$. By Corollary 8.10 the only possibility is $s = 4$ and $\lambda_D = (1, 3)$, which forces $D = \Gamma = tH - C$. Furthermore, we must have $\operatorname{gon}(C) = k = d - 4$ for the inequality $x^2 + dx + ks \geq \phi(A, A + C)$ of (9-1) to hold.

Since $x = -4 = \deg(D + cH)$, we see $c = -2$. Now pick an effective divisor $C_0 \in |-A| = |2H - D|$. Then C_0 is ACM with biliaison type $(1, 3)$, thus C_0 is up to a deformation with constant cohomology an elliptic quartic. By construction $C \in |C_0 + bH|$ with $b = t - 2 \geq 2$. (Note that $b = 2$ gives $(d, g) = (12, 17)$, which is in the list of Proposition 6.10(b).) For $b \geq 2$ the restriction of $|C_0|$ to C is $|\mathcal{O}_C(b)|$, and is a g^1_{d-4} on C that does not arise from a multisecant.

Case 1c: $a = -1$, $x = -s - 1$ or $-s$. In this case $A = -D + cH$, hence, if $D = D_i$,

$$\phi(A, A) + \phi(A, C) = 2\phi(D_i, D_i) + \sum_{j \neq i} \phi(-D_i, -D_j) = 2q_i - b_i \geq \tfrac{3}{2} q_i > \tfrac{3}{2}(s-1)^2.$$

Therefore

$$x^2 + dx + (d - 4)s \geq \tfrac{3}{2}(s-1)^2,$$

which contradicts both $x = -s - 1$ and $x = -s$, so this case does not occur.

Case 2: $\lambda_D = (1)$, so that D is a line $L \subset X$, and $A = cH + aL$ with $a = \pm 1$. In this case either $\Gamma = L$ and $\lambda_\Gamma = (1)$, or λ_Γ has a gap at the beginning:

$$\lambda_\Gamma = (1, 4, \dots)$$

In both cases $L = D_1$ is unique. The proof of Corollary 7.19 shows that the h-vector of C satisfies $h_C(e + 1) = 3$ and $h_C(e + 2) = 2$, and that $C.L = e + 3$. Thus in any case

$$\deg(Z) = \operatorname{gon}(C) \leq d - e - 3.$$

We wish to show that $A = L - H$ and $Z = \mathfrak{X}(L)$.

Recall that the degree x of A must satisfy the inequalities $-s - 1 < x < 0$ and

$$x^2 \geq a^2 \phi(L, L) = (s-1)^2.$$

We also know $x = cs + a$ with $a = \pm 1$. Therefore $c = -1$ and either $A = -H - L$ or $A = -H + L$.

Suppose first $A = -H - L$. Since $\deg(X) = s \geq 4$,

$$H^0 \mathcal{O}_X(H + L) \cong H^0 \mathcal{O}_X(H)$$

thus every curve B in the linear system $|-A| = |H+L|$ contains the line L. This contradicts Proposition 5.5, according to which we can find two effective divisors in $|-A|$ meeting properly. So $A = -H-L$ is impossible. Therefore $A = -H+L$, and $Z = \mathfrak{X}(L)$ by Corollary 5.7.

Case 3: $\lambda_D = (s-1)$, so that $D = H-L$ is a plane curve of degree $s-1$, residual to a line L in a plane section of X. Furthermore, $A = cH + aD = (c+a)H - aL$ with $a = \pm 1$.

In this case $D = D_r$, thus L is unique, and either $\Gamma = D_r$ or λ_Γ has a gap at the end. The proof of Corollary 7.19 shows that the h-vector of C satisfies $h_C(t) = s-2$, $h_C(t+1) = s-3$ and that L is a $(t-s+1)$-secant line for C. An argument analogous to the one of the previous case shows $A = -H+L$, so that $\mathfrak{X} = \mathfrak{X}(L)$.

Case 4: $\lambda_D = (1, s-1)$ with $s = 5$ or 6, hence $A = cH + a_1 L_1 + a_2 P$ where L_1 is a line, P is a plane curve of degree $s-1$, and a_1 and a_2 are nonzero. Note that $\phi(L_1, P) = -1$, therefore

$$\begin{aligned}\phi(A, A) &= (a_1^2 + a_2^2)(s-1)^2 - 2a_1a_2\\ &= (a_1^2 + a_2^2)(s^2 - 2s) + (a_1 - a_2)^2 \geq 2(s^2 - 2s) > s^2.\end{aligned}$$

On the other hand, $(s+1)^2 \geq x^2 \geq \phi(A, A)$. Therefore we must have $x = -s-1$ and $a_1^2 + a_2^2 < 3$, that is, a_1 and a_2 can only be 1 or -1.

Then

$$-s-1 = x = cs + a_1 + a_2(s-1),$$

from which we see $-1 \equiv a_1 - a_2 \pmod{s}$. This is impossible because $a_1 = \pm 1$ and $a_2 = \pm 1$.

This complete the list of possible cases, and proves the classification of complete basepoint-free pencils $\mathfrak{X}$ of degree at most $d-4$, hence the theorem $\square$

Remark 9.3. In the first of the cases excluded in the theorem, namely $s = 4$ and $(d, g) = (10, 11)$, we can prove $\mathrm{gon}(C) = 6 = d-4$ by the method of [Hartshorne 2002].

Theorem 9.4. *Assume the ground field is the complex numbers. Then the conclusions of Theorem 9.1 hold for the general ACM curve C in* $A(h)$.

Proof. Since the conclusions of Theorem 9.1 are semicontinuous on $A(h)$ (cf. [Arbarello and Cornalba 1981]), it is enough to show the existence of a single curve C for which the hypotheses of that theorem are satisfied. To check this, let h' denote the h-vector of a curve Γ linked by two surfaces of degrees s and $t = t_C$ to $C \in A(h)$. Note that h' may not be of decreasing type, but in any case $s_\Gamma \leq e_\Gamma + 3 < s$ by Lemma 6.5. By Theorem 7.21 a general curve Γ in $A(h')$ is reduced, its irreducible components are ACM, with biliaison type prescribed by λ_Γ; and, since $s > e_\Gamma + 3$, there exist smooth surfaces of any degree $\geq s-1$ containing Γ.

Now let h_2 be the h-vector of a curve C_2 linked to Γ by the complete intersection of two smooth surfaces of degree $s-1$ and s respectively. The flag Hilbert schemes parametrizing pairs (Γ, Y), where $\Gamma \in A(h')$ and Y is a complete intersection of type $(s-1, s)$, is irreducible [Martin-Deschamps and Perrin 1990, VII §3]. Thus a general Γ in $A(h')$ can be linked to a general $C_2 \in A(h_2)$. By Lemma 6.5 h_2 is of decreasing type, hence we may assume C_2 is smooth, and lies on smooth surfaces of degree $s-1$ and s. Since we are working over the complex numbers, we can use the Noether–Lefschetz type theorem of [Lopez 1991, II 3.1]. We apply this theorem to C_2 with $d = s$, $e = 1$, and T a smooth surface of degree $s-1$ through C_2 to conclude that, if X is a *very general* surface of degree s containing C_2, then $\mathrm{Pic}(X)$ is freely generated by the classes of a plane section H and of the irreducible components of Γ (here "very general" means, as usual, outside a countable union of proper subvarieties).

Now on X we can take for C a general curve in the linear system

$$|C_2 + (t-s+1)H| = |tH - \Gamma|.$$

The hypotheses of Theorem 9.1 are then satisfied for the smooth surface X and the curve C.

One can simplify the argument using a more recent result [Brevik and Nollet 2008, Theorem 1.1], which allows one to work directly with Γ rather than C_2. $\square$

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