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The original Fujita approximation theorem states that the volume of a big divisor D on a projective variety X can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of X. One can also formulate it in terms of graded linear series as follows: Let $W_\bullet = \{W_k\}$ be the complete graded linear series associated to a big divisor D, where

$$W_k = H^0(X, \mathcal{O}_X(kD)).$$

For each fixed positive integer p, define $W_\bullet^{(p)}$ to be the graded linear subseries of $W_\bullet$ generated by W_p:

$$W_m^{(p)} = \begin{cases} 0 & \text{if } p \nmid m, \\ \mathrm{Image}(S^k W_p \to W_{kp}) & \text{if } m = kp. \end{cases}$$

Then the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. We will show that, under this formulation, the Fujita approximation theorem can be generalized to the case of multigraded linear series.

1. Introduction

Let X be an irreducible variety of dimension d over an algebraically closed field $\boldsymbol{K}$, and let D be a (Cartier) divisor on X. When X is projective, the following limit, which measures how fast the dimension of the section space $H^0(X, \mathcal{O}_X(mD))$ grows, is called the *volume* of D:

$$\mathrm{vol}(D) = \mathrm{vol}_X(D) = \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

One says that D is *big* if $\mathrm{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor [Lazarsfeld 2004a, Section 2.2.C], and it plays a key role in several recent works in birational geometry [Tsuji 2000; Boucksom et al. 2004; Hacon and McKernan 2006; Takayama 2006].

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When D is ample, one can show that $\operatorname{vol}(D) = D^d$, the self-intersection number of D. This is no longer true for a general big divisor D, since D^d may even be negative. However, Fujita [1994] showed that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of X. This theorem, known as *Fujita approximation*, has several implications for the properties of volumes, and is also a crucial ingredient in [Boucksom et al. 2004] (see [Lazarsfeld 2004b, Section 11.4] for more details).

Lazarsfeld and Mustaţă [2009] (henceforth [LM]) recently obtained, among other things, a generalization of Fujita approximation to *graded linear series*. Recall that a graded linear series $W_\bullet = \{W_k\}$ on a (not necessarily projective) variety X associated to a divisor D consists of finite dimensional vector subspaces

$$W_k \subseteq H^0(X, \mathbb{O}_X(kD))$$

for each $k \geq 0$, with $W_0 = \boldsymbol{K}$, such that

$$W_k \cdot W_\ell \subseteq W_{k+\ell}$$

for all $k, \ell \geq 0$. Here the product on the left denotes the image of $W_k \otimes W_\ell$ under the multiplication map $H^0(X, \mathbb{O}_X(kD)) \otimes H^0(X, \mathbb{O}_X(\ell D)) \to H^0(X, \mathbb{O}_X((k+\ell)D))$. In order to state the Fujita approximation for $W_\bullet$, they defined, for each fixed positive integer p, a graded linear series $W_\bullet^{(p)}$ which is the subgraded linear series of $W_\bullet$ generated by W_p:

$$W_m^{(p)} = \begin{cases} 0 & \text{if } p \nmid m, \\ \operatorname{Im}(S^k W_p \to W_{kp}) & \text{if } m = kp. \end{cases}$$

Then under mild hypotheses, they showed that the volume of $W_\bullet^{(p)}$ approaches the volume of $W_\bullet$ as $p \to \infty$. See [LM, Theorem 3.5] for the precise statement, as well as [LM, Remark 3.4] for how this is equivalent to the original statement of Fujita when X is projective and $W_\bullet$ is the complete graded linear series associated to a big divisor D (that is, $W_k = H^0(X, \mathbb{O}_X(kD))$ for all $k \geq 0$).

The goal of this note is to generalize the Fujita approximation theorem to *multigraded linear series*. We will adopt the following notation from [LM, Section 4.3]: Let $D_1, \dots, D_r$ be divisors on X. For $\vec{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$, write $\vec{m}D = \sum m_i D_i$, and put $|\vec{m}| = \sum |m_i|$.

Definition. A *multigraded linear series* $W_{\vec{\bullet}}$ on X associated to the D_i consists of finite-dimensional vector subspaces

$$W_{\vec{k}} \subseteq H^0(X, \mathbb{O}_X(\vec{k}D))$$

for each $\vec{k} \in \mathbb{N}^r$, with $W_{\vec{0}} = \boldsymbol{K}$, such that

$$W_{\vec{k}} \cdot W_{\vec{m}} \subseteq W_{\vec{k}+\vec{m}},$$

where the multiplication on the left denotes the image of $W_{\vec{k}} \otimes W_{\vec{m}}$ under the natural map

$$H^0(X, \mathcal{O}_X(\vec{k}D)) \otimes H^0(X, \mathcal{O}_X(\vec{m}D)) \to H^0(X, \mathcal{O}_X((\vec{k}+\vec{m})D)).$$

Given $\vec{a} \in \mathbb{N}^r$, denote by $W_{\vec{a},\bullet}$ the singly graded linear series associated to the divisor $\vec{a}D$ given by the subspaces $W_{k\vec{a}} \subseteq H^0(X, \mathcal{O}_X(k\vec{a}D))$. Then put

$$\mathrm{vol}_{W_{\vec{\bullet}}}(\vec{a}) = \mathrm{vol}(W_{\vec{a},\bullet})$$

(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\vec{a},\bullet}$ when $\vec{a} \in \mathbb{Q}^r_{\geq 0}$, given by

$$W_{\vec{a},k} = \begin{cases} W_{k\vec{a}} & \text{if } k\vec{a} \in \mathbb{N}^r, \\ 0 & \text{otherwise.} \end{cases}$$

Our multigraded Fujita approximation, similar to the singly graded version, is going to state that (under suitable conditions) the volume of $W_{\vec{\bullet}}$ can be approximated by the volume of the following finitely generated submultigraded linear series of $W_{\vec{\bullet}}$:

Definition. Given a multigraded linear series $W_{\vec{\bullet}}$ and a positive integer p, define $W_{\vec{\bullet}}^{(p)}$ to be the submultigraded linear series of $W_{\vec{\bullet}}$ generated by all $W_{\vec{m}_i}$ with $|\vec{m}_i| = p$, or concretely,

$$W_{\vec{m}}^{(p)} = \begin{cases} 0 & \text{if } p \nmid |\vec{m}|, \\ \displaystyle\sum_{\substack{|\vec{m}_i|=p \\ \vec{m}_1+\cdots+\vec{m}_k=\vec{m}}} W_{\vec{m}_1} \cdots W_{\vec{m}_k} & \text{if } |\vec{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when $W_{\vec{\bullet}}$ is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement. The Remark on page 335 points out what assumptions on $W_{\vec{\bullet}}$ are actually needed in the proof.

Theorem. *Let X be an irreducible projective variety of dimension d, and let D_1, $D_2, \ldots, D_r$ be big divisors on X. Let $W_{\vec{\bullet}}$ be the complete multigraded linear series associated to the D_i, namely*

$$W_{\vec{m}} = H^0(X, \mathcal{O}_X(\vec{m}D))$$

for each $\vec{m} \in \mathbb{N}^r$. *Then given any* $\varepsilon > 0$, *there exists an integer* $p_0 = p_0(\varepsilon)$ *having the property that if* $p \geq p_0$, *then*

$$\left|1 - \frac{\operatorname{vol}_{W_{\vec{\bullet}}^{(p)}}(\vec{a})}{\operatorname{vol}_{W_{\vec{\bullet}}}(\vec{a})}\right| < \varepsilon \tag{1}$$

for all $\vec{a} \in \mathbb{N}^r$.

2. Proof of the Theorem

The main tool in our proof is the theory of *Okounkov bodies* developed systematically in [Lazarsfeld and Mustaţă 2009]. Given a graded linear series $W_\bullet$ on a d-dimensional variety X, its Okounkov body $\Delta(W_\bullet)$ is a convex body in $\mathbb{R}^d$ that encodes many asymptotic invariants of $W_\bullet$, the most prominent one being the volume of $W_\bullet$, which is precisely $d!$ times the Euclidean volume of $\Delta(W_\bullet)$. The idea first appeared in Okounkov's papers [1996; 2003] in the case of complete linear series of ample line bundles on a projective variety. Later it was further developed and applied to much more general graded linear series by Lazarsfeld and Mustaţă [2009] and also independently by Kaveh and Khovanskii [2008; 2009].

Proof of the Theorem. Let $T = \{(a_1, \dots, a_r) \in \mathbb{R}^r_{\geq 0} \mid a_1 + \cdots + a_r = 1\}$, and let $T_{\mathbb{Q}}$ be the set of all points in T with rational coordinates. The fraction inside (1) is invariant under scaling of $\vec{a}$ due to homogeneity, hence it is enough to prove (1) for $\vec{a} \in T_{\mathbb{Q}}$.

Let $\Delta(W_{\vec{\bullet}}) \subseteq \mathbb{R}^d \times \mathbb{R}^r$ be the global Okounkov cone of $W_{\vec{\bullet}}$ as in [LM, Theorem 4.19], and let $\pi : \Delta(W_{\vec{\bullet}}) \to \mathbb{R}^r$ be the projection map. For each $\vec{a} \in T$, write $\Delta(W_{\vec{\bullet}})_{\vec{a}}$ for the fiber $\pi^{-1}(\vec{a})$. Define in a similar fashion the convex cone $\Delta(W_{\vec{\bullet}}^{(p)})$ and the convex bodies $\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}}$. By [LM, Theorem 4.19],

$$\Delta(W_{\vec{\bullet}})_{\vec{a}} = \Delta(W_{\vec{a},\bullet}) \quad \text{for all } \vec{a} \in T_{\mathbb{Q}}. \tag{2}$$

Although [LM, Theorem 4.19] requires $\vec{a}$ to be in the relative interior of T, here we know that (2) holds even for those $\vec{a}$ in the boundary of T because the big cone of X is open and $W_{\vec{\bullet}}$ was assumed to be the complete multigraded linear series. By the singly graded Fujita approximation, $\operatorname{vol}(W_{\vec{a},\bullet})$ can be approximated arbitrarily closely by $\operatorname{vol}(W_{\vec{a},\bullet}^{(p)})$ if p is sufficiently large. (Here by $W_{\vec{a},\bullet}^{(p)}$ we mean $W_{\vec{\bullet}}^{(p)}$ restricted to the $\vec{a}$ direction, which certainly contains $(W_{\vec{a},\bullet})^{(p)}$.) Hence given any finite subset $S \subset T_{\mathbb{Q}}$ and any $\varepsilon' > 0$, we have

$$\operatorname{vol}(\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}}) \geq \operatorname{vol}(\Delta(W_{\vec{\bullet}})_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S$$

as soon as p is sufficiently large.

Because the function $\vec{a} \mapsto \operatorname{vol}(\Delta(W_{\vec{\bullet}})_{\vec{a}})$ is uniformly continuous on T, given any $\varepsilon' > 0$, we can partition T into a union of polytopes with disjoint interiors

$T = \bigcup T_i$, in such a way that the vertices of each T_i all have rational coordinates, and on each T_i we have a constant M_i such that

$$M_i \le \operatorname{vol}(\Delta(W_{\vec{\bullet}})_{\vec{a}}) \le M_i + \varepsilon' \quad \text{for all } \vec{a} \in T_i. \tag{3}$$

Let S be the set of vertices of all the T_i. Then as we saw in the end of the previous paragraph, as soon as p is sufficiently large we have

$$\operatorname{vol}(\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}}) \ge \operatorname{vol}(\Delta(W_{\vec{\bullet}})_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S. \tag{4}$$

We claim that this implies

$$\operatorname{vol}(\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}}) \ge \operatorname{vol}(\Delta(W_{\vec{\bullet}})_{\vec{a}}) - 2\varepsilon' \quad \text{for all } \vec{a} \in T_{\mathbb{Q}}. \tag{5}$$

To show this, it suffices to verify it on each of the T_i. Let $\vec{v}_1, \ldots, \vec{v}_k$ be the vertices of T_i. Then each $\vec{a} \in T_i$ can be written as a convex combination of the vertices: $\vec{a} = \sum t_j \vec{v}_j$ where each $t_j \ge 0$ and $\sum t_j = 1$. Since $\Delta(W_{\vec{\bullet}}^{(p)})$ is convex, we have

$$\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}} \supseteq \sum t_j \, \Delta(W_{\vec{\bullet}}^{(p)})_{\vec{v}_j},$$

where the sum on the right means the Minkowski sum. By (3) and (4), the volume of each $\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{v}_j}$ is at least $M_i - \varepsilon'$, hence by the Brunn–Minkowski inequality [Kaveh and Khovanskii 2008, Theorem 5.4], we have

$$\operatorname{vol}(\Delta(W_{\vec{\bullet}}^{(p)})_{\vec{a}}) \ge M_i - \varepsilon' \quad \text{for all } \vec{a} \in T_i \cap T_{\mathbb{Q}}.$$

This combined with (3) shows that (5) is true on $T_i \cap T_{\mathbb{Q}}$, hence it is true on $T_{\mathbb{Q}}$ since the T_i cover T.

Since (1) follows from (5) by choosing a suitable ε', the proof is complete. □

Remark. In the statement of the Theorem we assume that $W_{\vec{\bullet}}$ is the complete multigraded linear series associated to big divisors. But in fact since the main tool we used in the proof is the theory of Okounkov bodies established in [Lazarsfeld and Mustață 2009], in particular [LM, Theorem 4.19], the really indispensable assumptions on $W_{\vec{\bullet}}$ are the same as those in [LM] (which they called Conditions (A′) and (B′), or (C′)). The only place in the proof where we invoke that we are working with a complete multigraded linear series is the sentence right after (2), where we want to say that (2) holds not only in the relative interior of T but also in its boundary. Hence if $W_{\vec{\bullet}}$ is only assumed to satisfy Conditions (A′) and (B′), or (C′), then given any $\varepsilon > 0$ and any compact set C contained in $T \cap \operatorname{int}(\operatorname{supp}(W_{\vec{\bullet}}))$, there exists an integer $p_0 = p_0(C, \varepsilon)$ such that if $p \ge p_0$ then

$$\operatorname{vol}_{W_{\vec{\bullet}}^{(p)}}(\vec{a}) > \operatorname{vol}_{W_{\vec{\bullet}}}(\vec{a}) - \varepsilon$$

for all $\vec{a} \in C \cap T_{\mathbb{Q}}$.

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References

[Boucksom et al. 2004] S. Boucksom, J.-P. Demailly, M. Paun, and T. Peternell, "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension", preprint, 2004. arXiv math/0405285

[Fujita 1994] T. Fujita, "Approximating Zariski decomposition of big line bundles", *Kodai Math. J.* **17**:1 (1994), 1–3. MR 95c:14053 Zbl 0814.14006

[Hacon and McKernan 2006] C. D. Hacon and J. McKernan, "Boundedness of pluricanonical maps of varieties of general type", *Invent. Math.* **166**:1 (2006), 1–25. MR 2007e:14022 Zbl 1121.14011

[Kaveh and Khovanskii 2008] K. Kaveh and A. Khovanskii, "Convex bodies and algebraic equations on affine varieties", preprint, 2008. arXiv 0804.4095

[Kaveh and Khovanskii 2009] K. Kaveh and A. Khovanskii, "Newton convex bodies, semigroups of integral points, graded algebras and intersection theory", preprint, 2009. arXiv 0904.3350

[Lazarsfeld 2004a] R. Lazarsfeld, *Positivity in algebraic geometry, I: Classical setting: line bundles and linear series*, Ergebnisse der Math. und ihrer Grenzgebiete (3) **48**, Springer, Berlin, 2004. MR 2005k:14001a Zbl 1093.14501

[Lazarsfeld 2004b] R. Lazarsfeld, *Positivity in algebraic geometry, II: Positivity for vector bundles, and multiplier ideals*, Ergebnisse der Math. und ihrer Grenzgebiete (3) **49**, Springer, Berlin, 2004. MR 2005k:14001b Zbl 1093.14500

[Lazarsfeld and Mustaţă 2009] R. Lazarsfeld and M. Mustaţă, "Convex bodies associated to linear series", *Ann. Sci. Éc. Norm. Supér.* (4) **42**:5 (2009), 783–835. MR 2011e:14012 Zbl 1182.14004

[Okounkov 1996] A. Okounkov, "Brunn–Minkowski inequality for multiplicities", *Invent. Math.* **125**:3 (1996), 405–411. MR 99a:58074 Zbl 0893.52004

[Okounkov 2003] A. Okounkov, "Why would multiplicities be log-concave?", pp. 329–347 in *The orbit method in geometry and physics* (Marseille, 2000), edited by C. Duval et al., Progr. Math. **213**, Birkhäuser, Boston, 2003. MR 2004j:20022 Zbl 1063.22024

[Takayama 2006] S. Takayama, "Pluricanonical systems on algebraic varieties of general type", *Invent. Math.* **165**:3 (2006), 551–587. MR 2007m:14014 Zbl 1108.14031

[Tsuji 2000] H. Tsuji, "Effective birationality of pluricanonical systems", preprint, 2000. arXiv math/0006166

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