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STABILITY OF THE KÄHLER–RICCI FLOW IN THE SPACE OF KÄHLER METRICS

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We prove that on a Fano manifold M admitting a Kähler–Ricci soliton (ω, X) , if the initial Kähler metric ω_{φ_0} is close to ω in a certain weak sense, then the weak Kähler–Ricci flow exists globally and converges in the sense of Cheeger and Gromov. In particular, φ_0 is not assumed to be K_X -invariant. The methods used are based on the metric geometry of the space of the Kähler metrics and are potentially applicable to other stability problems of geometric flows near the corresponding critical metrics.

1. Introduction

The Ricci flow was first introduced by Hamilton [1982] and now plays an important role in understanding the geometric and topological structure of the manifolds it lives on. We call the Ricci flow a *Kähler–Ricci flow* if the underlying manifold is a Kähler manifold. The *normalized Kähler–Ricci flow* is given by

$$(1-1) \quad \begin{cases} \frac{\partial}{\partial t} \omega = -\text{Ric} + \lambda \omega, \\ \omega(0) = \omega_{\varphi_0}, \end{cases}$$

where $\omega(0)$ stays in the canonical class $2\pi C_1(M)$ and λ is the sign of the first Chern class. Cao [1985] first showed that the Kähler–Ricci flow (1-1) has long-time existence. He also proved that the Kähler–Ricci flow converges to a Kähler–Einstein metric when the first Chern class is negative or zero. Now we restrict ourselves to the case where the first Chern class is positive. Since the Kähler–Ricci flow preserves the Kähler class, we can rewrite the Kähler–Ricci flow in terms of the Ricci potential:

$$(1-2) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_{\varphi}^n}{\omega^n} + \varphi - h_{\omega} + a(t), \\ \varphi(0) = \varphi_0, \end{cases}$$

where a is a function of t and the Ricci potential h_{ω} of the reference metric ω is

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defined by

$$(1-3) \quad \sqrt{-1} \partial \bar{\partial} h_\omega = \text{Ric}(\omega) - \omega \quad \text{and} \quad \int_M e^{h_\omega} \omega^n = \text{Vol}(M).$$

The convergence of the Kähler–Ricci flow has been studied by many authors. Chen and Tian [2002; 2006] proved it for Kähler–Einstein manifolds under the assumption of positivity of the Ricci curvature along the flow. Perelman (unpublished; see detailed proof in [Sesum and Tian 2008]) obtained an estimate of the Kähler–Ricci flow and proved that it converges to a Kähler–Einstein metric in the sense of Cheeger–Gromov, when one exists for any initial Kähler metric. Tian and Zhu [2007] extended this to the case of a Kähler–Ricci soliton for a K_X -invariant initial metric; a *Kähler–Ricci soliton* is a Kähler metric such that there is a holomorphic vector field X satisfying

$$(1-4) \quad L_X \omega = \text{Ric} - \omega.$$

Since the right side of (1-4) is real-valued, we obtain $L_{\text{Im } X} \omega = 0$ and $\text{Im } X$, the imaginary part of X , generates a one-parameter isometry group K_X . Without assuming that M admits a Kähler–Einstein metric or a Kähler–Ricci soliton, the analytic and geometric conditions of the convergence of the Kähler–Ricci flow are studied in [Phong and Sturm 2006; Phong et al. 2009, 2008; 2011, Tosatti 2010; Székelyhidi 2010; Munteanu 2009; Pali 2009; Chen and Wang 2010; Rubinstein 2009.

In order to study the asymptotic behavior of the Kähler–Ricci flow, we consider the flow’s stability problem. That is, on a Kähler manifold M admitting a Kähler–Ricci soliton (ω, X) , for what kind of neighborhood of ω does the Kähler–Ricci flow with initial datum in that neighborhood converge (in some sense — maybe exponentially) to the Kähler–Ricci soliton?

This stability problem has been investigated by many people; for references, see [Chen and Li 2009]. That work and [Tian and Zhu 2008] consider perturbing both the initial metric and the complex structure near a Kähler–Einstein metric.

In this paper, we focus on perturbing the initial metric near the Kähler–Ricci soliton without changing the complex structure. The main results of this paper are as follows, where $\mathcal{N}(\epsilon_0; B, p)$ is a small neighborhood of the zero function, to be specified in Section 6.

First we will give a direct proof, based on the geometry of the space of Kähler metrics, of long-time existence and convergence in the Cheeger–Gromov sense, within the frame of Donaldson’s program [2004]. The result is this:

Theorem 1.1. *If a Kähler manifold admits a Kähler–Ricci soliton (ω, X) , there exists a positive constant ϵ_0 such that, if the initial potential φ_0 stays in $\mathcal{N}(\epsilon_0; B, p)$, the weak Kähler–Ricci flow exists globally and converges in the Cheeger–Gromov*

sense. If, moreover, φ_0 is K_X -invariant, the weak modified Kähler–Ricci flow converges exponentially to a unique Kähler–Ricci soliton nearby.

When the Futaki invariant vanishes, it is obvious that the holomorphic vector fields X is zero and the Kähler–Ricci soliton is a Kähler–Einstein metric.

Theorem 1.2. *On a Kähler–Einstein manifold, there exists a positive constant ϵ_0 such that, if the initial potential φ_0 stays in $\mathcal{N}(\epsilon_0; B, p)$, the weak Kähler–Ricci flow exists globally and converges exponentially to a unique Kähler–Einstein metric nearby.*

Simon [1983] studied the asymptotic behavior of the gradient flow of the variation problem via the Łojasiewicz–Simon inequality, which compares the distance to the critical set with the norm of the gradient of the functional in the L^2 space under the condition that the functional should be analytic. The underlying idea is to reduce the infinite-dimensional problem to a finite-dimensional problem. Perelman [2002] introduced a new functional, called the μ functional, and pointed out that the Ricci flow is the gradient flow of the μ functional up to a diffeomorphism.

We will not apply the Łojasiewicz–Simon inequality to the μ functional directly. Instead, we provide a new approach to the study of the asymptotic behavior of the flow which is merely a pseudogradient flow of some functional, since in the Kähler setting, geometry gives us more information. To be precise, the critical set in the space of Kähler metrics is a finite-dimensional Riemannian symmetric space, which we will explain later.

Since the Kähler–Ricci flow is the pseudogradient flow of the K -energy, in order to make the mechanism of our proof more clear, we first prove Theorem 1.2 under the assumption that the $C^{2,\alpha}$ norm of φ_0 is small. Then we generalize our approach to the case of a Kähler–Ricci soliton (Theorem 1.1).

A sketch of the proofs goes as follows. We first prove that the Kähler–Ricci flow (1-2), after pullback by the corresponding holomorphic transformations, will always stay in a small neighborhood of the background Kähler–Einstein metric. When M has no nontrivial holomorphic vector field, it is not necessary to find the transformations; Section 3 gives a proof of this. However, in general, when M admits nontrivial holomorphic vector fields, we need a new method, developed in Section 4A, to pick up the appropriate transformations following the trace of the Kähler–Ricci flow in the space of normalized Kähler potential \mathcal{H}_0 ; see (2-2). It has been shown by Mabuchi [1987], Donaldson [1999] and Semmes [1992] independently that \mathcal{H}_0 is an infinite-dimensional symmetry space of negative curvature. Later, Chen [2000b] proved \mathcal{H}_0 is also a metric space. Since the space \mathcal{E}_0 of potentials of Kähler–Einstein metrics is a totally geodesic submanifold in \mathcal{H}_0 , the projection ρ minimizing the distance function from the Kähler–Ricci flow to \mathcal{E}_0 is uniquely determined. The Bando–Mabuchi uniqueness theorem [1987] on

the Kähler–Einstein metric implies that ω_ρ is different from the reference Kähler–Einstein metric by a holomorphic transformation. The projection Kähler–Einstein metric is exactly the new reference metric we’ve acquired.

Another way to derive a holomorphic transformation (Section 7) of $\varphi \in \mathcal{H}_0$ is to minimize in \mathcal{E}_0 the $I - J$ functional, introduced in [Bando and Mabuchi 1987] to prove the uniqueness of the Kähler–Einstein metric. However, this method cannot be applied in our case directly, since in general the hessian of the $I - J$ functional is not strictly positive; that is, the minimizer is not unique. Nevertheless, as we observed when the $C^{2,\alpha}$ norm of φ is small, the hessian of the $I - J$ functional is indeed strictly positive. Therefore, the holomorphic transformation is uniquely determined.

In Section 5, we prove stability of the Kähler–Ricci flow near a Kähler–Ricci soliton (ω, X) , similarly to the case of Kähler–Einstein metric. We use (ω, X) as the background metric. We first prove that the Kähler–Ricci flow (1-2) with small $C^{2,\alpha}$ initial Kähler potential will always stay in a small neighborhood of ω in the Cheeger–Gromov sense. The key idea is to use Perelman’s μ functional [2002] instead of the K -energy, since the hessian of the μ functional is nonnegative at a Kähler–Ricci soliton within the canonical class [Tian and Zhu 2008]. Furthermore, we reparametrize the Kähler–Ricci flow (1-1) by the automorphisms $\zeta(t)$ generated by the real part $\text{Re } X$ of X such that

$$(1-5) \quad \begin{cases} \frac{\partial}{\partial t} \omega_\phi = -\text{Ric}(\omega_\phi) + \omega_\phi + L_{\text{Re } X} \omega_\phi, \\ \omega_{\phi(0)} = \omega_{\phi_0}. \end{cases}$$

It is obvious that the Kähler–Ricci soliton is the stationary solution of the modified Kähler–Ricci flow (1-5). Since the Kähler–Ricci soliton (ω, X) is K_X -invariant and the Kähler–Ricci flow is also invariant under the holomorphic diffeomorphism, we assume without loss of generality that the initial datum is K_X -invariant. Then the exponential convergence of the modified Kähler–Ricci flow follows from [Phong et al. 2011].

Finally, in Section 6, at a fixed time, we show that the $C^{2,\alpha}$ norm of the potential is small when the initial value is small under certain weak conditions. The main idea is to use the estimate introduced in [Chen et al. 2008].

As a corollary of Theorem 1.1, we deduce that the limit metric of the Kähler–Ricci flow is unique. Let $\{\varphi(t_i)\}$ be a sequence of solutions of the Kähler–Ricci flow converging to a Kähler–Einstein metric or Kähler–Ricci soliton g_∞ , if one exists; then there exists some $\varphi \in \{\varphi(t_i)\}$ satisfying the stability condition of Theorem 1.1. According to that condition, the Kähler–Ricci flow with initial value φ converges exponentially to a Kähler–Einstein metric g_∞^1 or Kähler–Ricci soliton, respectively. Further, since we assume that $\{\varphi(t_i)\} \rightarrow g_\infty$, we must have $g_\infty^1 = g_\infty$.

We stress that the approach used to prove Theorem 1.1 is also applicable to the case of the general pseudogradient flow: neither the condition that the flow is the gradient flow of some functional, nor Perelman’s deep estimate, nor a prior long-time existence of the flow is required. It is possible that our method can be used to solve other, similar problems of geometric flow, such as the stability of the pseudo-Calabi flow near a constant scalar curvature Kähler metric in [Chen and Zheng 2010] and of the Calabi flow near a extremal metric in [Huang and Zheng 2010].

The paper is organized as follows: in Section 2 we review known results on the space of Kähler metrics and the well-posedness of the pseudo-Calabi flow obtained in [Chen and Zheng 2010] — see (2-10). In Sections 3 and 4 we prove Theorem 1.2 under the assumption that the $C^{2,\alpha}$ norm of the initial Kähler potential is small. Then we prove Theorem 1.1 under the same assumption in Section 5. Finally, in Section 6 we explain how to weaken the initial condition to the one stated in Theorem 1.2 and Theorem 1.1. In Section 7, we explain another method to choose the holomorphic transformation.

2. Notation and basic results

Let M be a compact Kähler manifold of complex dimension n with positive first Chern class $C_1(M)$ and let ω be a Kähler form representing the canonical class $2\pi C_1(M)$. In a local holomorphic coordinate z_1, z_2, \dots, z_n , the form ω is expressed by

$$\omega = \sqrt{-1} \sum_{i=1}^n g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.$$

The corresponding Riemannian metric is given by

$$g = \sum_{i=1}^n g_{i\bar{j}} dz^i \otimes dz^{\bar{j}}.$$

For a Kähler metric ω , the volume form is

$$dV = \omega^n = (\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge dz^{\bar{1}} \wedge \dots \wedge dz^n \wedge dz^{\bar{n}}.$$

The Ricci form

$$\text{Ric} = \sqrt{-1} \sum_{i=1}^n R_{i\bar{j}} dz^i \wedge dz^{\bar{j}} = -\sqrt{-1} \partial \bar{\partial} \log \det \omega^n$$

is a closed real $(1, 1)$ -form and belongs to $2\pi C_1(M)$. Accordingly, the scalar curvature satisfies

$$S\omega^n = n \text{Ric} \wedge \omega^{n-1}.$$

A direct calculation gives the average scalar curvature:

$$\underline{S} = \frac{1}{V} \int_M S dV = \frac{n}{V} \int_M \text{Ric} \wedge \omega^{n-1} = n.$$

Let \mathcal{K} be the set of Kähler forms on M representing $2\pi C_1(M)$ and let \mathcal{E} be the set of Kähler–Einstein metrics in \mathcal{K} . According to the $\partial\bar{\partial}$ lemma, for any Kähler metric ω' in \mathcal{K} there exists a smooth real-valued function φ such that $\omega' = \omega + \sqrt{-1} \partial\bar{\partial}\varphi$. Then the space of Kähler potentials of \mathcal{K} is given by

$$\mathcal{H} = \{ \varphi \in C^\infty(M, \mathbb{R}) \mid \omega + \sqrt{-1} \partial\bar{\partial}\varphi \in \mathcal{K} \}.$$

Apparently, we have an isomorphism

$$T\mathcal{H} \cong \mathcal{H} \times C^\infty(M, \mathbb{R}).$$

Mabuchi [1987], Donaldson [1999] and Semmes [1992] independently defined a Riemannian metric on \mathcal{H} by

$$\int_M f_1 f_2 \omega_\varphi^n,$$

for any $f_1, f_2 \in T_\varphi\mathcal{H}$. For any path $\varphi(t)$ ($0 \leq t \leq 1$) in \mathcal{H} , the length is given by

$$(2-1) \quad L(\varphi(t)) = \int_0^1 \sqrt{\int_M \varphi'(t)^2 \omega_{\varphi(t)}^n} dt,$$

and the geodesic equation is

$$\varphi''(t) - \frac{1}{2} |\nabla_t \varphi'(t)|_{\varphi(t)}^2 = 0,$$

where $'$ denotes differentiation in t and ∇_t denotes the covariant derivative for the metric $g_{\varphi(t)}$. The geodesic equation enables us to define the connection on the tangent bundle. For any tangent vector field $\psi(t)$ along the path $\varphi(t)$, the covariant derivative along $\varphi(t)$ is defined by

$$D_t \psi = \frac{\partial \psi}{\partial t} - \frac{1}{2} (\nabla_t \psi, \nabla_t \varphi')_{g_\varphi}.$$

Then the connection at φ is given by

$$G(X|Y)(\psi_1, \psi_2) = -\frac{1}{2} (\nabla \psi_1, \nabla \psi_2)_{g_\varphi},$$

for any ψ_1 and ψ_2 in $T_\varphi\mathcal{H}$. $G(X|Y)$ is torsion-free and metric-compatible.

Theorem 2.1 [Mabuchi 1987; Donaldson 1999; Semmes 1992]. *The Riemannian manifold \mathcal{H} is an infinite-dimensional symmetric space; it admits a Levi-Civita connection whose curvature is covariant constant. At a point $\varphi \in \mathcal{H}$ the curvature is given by*

$$R_\varphi(\delta_1\varphi, \delta_2\varphi)\delta_3\varphi = -\frac{1}{4} \{ \{ \delta_1\varphi, \delta_2\varphi \}_\varphi, \delta_3\varphi \}_\varphi,$$

where $\{ , \}_\varphi$ is the Poisson bracket on $C^\infty(M)$ of the symplectic form ω_φ .

Theorem 2.2 [Chen 2000b]. \mathcal{H} is a metric space, and is convex by $C^{1,1}$ geodesics.

Calabi and Chen [2002] proved \mathcal{H} is negatively curved in the sense of Alexandroff. We denote the space of normalized Kähler potentials by

$$(2-2) \quad \mathcal{H}_0 = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \text{ and } I(\varphi) = 0\},$$

where

$$I(\varphi) = \frac{1}{V} \sum_{p=0}^n \frac{1}{(p+1)!(n-p)!} \int_M \varphi \omega^{n-p} \wedge (\partial \bar{\partial} \varphi)^p.$$

In fact, \mathcal{H} can be naturally split as

$$\mathcal{H} = \mathcal{H}_0 \times \mathbb{R}.$$

This leads to a decomposition of the tangent space:

$$T_\varphi = \left\{ f \mid \int_M f \omega_\varphi^n = 0 \right\} \oplus \mathbb{R}.$$

On a Kähler-Einstein manifold (M, ω) , choose ω be the reference metric. It is clear from the definition (1-3) that $h_\omega = 0$. Substituting this into the potential equation (1-2) of the Kähler-Ricci flow, we obtain

$$(2-3) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi + a(t), \\ \varphi(0) = \varphi_0. \end{cases}$$

If we choose the normalization constant in (2-3) appropriately, namely,

$$(2-4) \quad a(t) = -\frac{1}{V} \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + \varphi \right) \omega_\varphi^n,$$

we see that

$$(2-5) \quad \partial_t I(\varphi) = \frac{1}{V} \int_M \partial_t \varphi \omega_\varphi^n = 0.$$

We first assume that $\varphi_0 \in \mathcal{H}_0$ satisfies $I(\varphi_0) = 0$; the general case will be treated in Section 6. Then (2-5) implies $I(\varphi) = 0$, which ensures that the solution φ of (2-3) always stays in \mathcal{H}_0 .

For any $\varphi \in \mathcal{H}$, Mabuchi [1986] defined the K -energy of (M, ω) as

$$(2-6) \quad v(\omega, \omega_\varphi) = -\frac{1}{V} \int_0^1 \int_M \dot{\varphi}(\tau) (S_{\varphi(\tau)} - \underline{S}) \omega_{\varphi(\tau)}^n d\tau,$$

where $\varphi(\tau)$ is an arbitrary piecewise smooth path from 0 to φ . An explicit expression of the K -energy is formulated in [Chen 2000a; Tian 2000] as

$$(2-7) \quad v_\omega(\varphi) = \frac{1}{V} \int_M \log \frac{\omega_\varphi^n}{\omega^n} \omega_\varphi^n + \frac{Sn!}{V} I(\varphi) - \frac{1}{V} \sum_{i=0}^{n-1} \frac{n!}{(i+1)!(n-i-1)!} \int_M \varphi \operatorname{Ric} \wedge \omega^{n-1-i} \wedge (\partial\bar{\partial}\varphi)^i.$$

In later sections we will simply write $v(\varphi)$ instead of $v_\omega(\varphi)$.

Theorem 2.3 [Mabuchi 1987]. *If ω is a critical point of $v(\varphi)$, the second variation of the K -energy satisfies*

$$\frac{d^2}{dt^2} v(\theta_t)|_{t=0} \geq 0$$

for every smooth path $\{\theta_t \mid -\epsilon \leq t \leq \epsilon\}$ in \mathcal{K} such $\theta_0 = \omega$.

Let $\operatorname{Aut}(M)$ be the group of holomorphic automorphisms of M and $\operatorname{Aut}_0(M)$ its identity component.

Theorem 2.4 [Bando and Mabuchi 1987; Bando 1987]. *Assume $\mathcal{E} \neq \emptyset$.*

- (i) *The K -energy is bounded from below on \mathcal{K} and takes its absolute minimum exactly on \mathcal{E} .*
- (ii) *\mathcal{E} consists a single $\operatorname{Aut}_0(M)$ -orbit.*

Indeed, the normalization constant $a(t)$ can be estimated by the K -energy.

Lemma 2.5. *Let φ be the solution of (2-3). The relation between $a(t)$ and the K -energy $v(\varphi)$ is given by*

$$(2-8) \quad a(t) + v(\varphi) = a(0) + v(\varphi_0).$$

Proof. We calculate the evolution of $a(t)$ along the Kähler–Ricci flow directly:

$$V \frac{d}{dt} a(t) = - \int_M (\Delta_\varphi + 1) \dot{\varphi} \omega_\varphi^n - \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + \varphi \right) \Delta_\varphi \dot{\varphi} \omega_\varphi^n.$$

By Stokes’ theorem and (2-5) the first term vanishes identically. Integration by parts and the use of (2-3) gives for the second term

$$\int_M (S_\varphi - n) \dot{\varphi} \omega_\varphi^n.$$

Since (2-6) implies

$$(2-9) \quad \frac{d}{dt} v(t) = - \frac{1}{V} \int_M (S_\varphi - n) \dot{\varphi} \omega_\varphi^n,$$

we obtain $\frac{d}{dt} a(t) = - \frac{d}{dt} v(t)$. We conclude by integrating both sides with respect to t . □

Since the K -energy is decreasing along the Kähler–Ricci flow, we immediately conclude the following according to Theorem 2.4.

Corollary 2.6. *On a Kähler–Einstein manifold, $a(t)$ is uniformly bounded along the Kähler–Ricci flow.*

Set

$$X = C^0([0, T], C^{2+\alpha}(M, g)) \cap C^1([0, T], C^\alpha(M, g)).$$

The following theorems asserting short-time existence, regularity and continuous dependence on initial data for the Kähler–Ricci flow were proved by Chen and the author, who defined a new second-order Monge–Ampère flow, called the pseudo-Calabi flow and coinciding with the Kähler–Ricci flow when the initial datum is restricted in the canonical Kähler class:

$$(2-10) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = -f(\varphi), \\ \Delta_\varphi f(\varphi) = S(\varphi) - \underline{S}. \end{cases}$$

Theorem 2.7 [Chen and Zheng 2010]. *Let $\varphi_0 \in C^{2,\alpha}(M, g)$ be such that*

$$\lambda\omega \leq \omega_{\varphi_0} \leq \Lambda\omega,$$

for two positive constants λ and Λ . Then the pseudo-Calabi flow has a unique solution $\varphi(x, t) \in X$, where T is the maximal existence time.

Theorem 2.8 [Chen and Zheng 2010]. *The solution of the pseudo-Calabi flow $\varphi \in X$ is smooth for any $t > 0$. More precisely, if $|\varphi(t)|_{C^{2,\alpha}} \leq A$ for any $0 \leq t \leq T$, there exists a constant C (depending on A, g, t_0 and k) such that $|\varphi(t)|_{C^{k,\alpha}} \leq C$ for any $T - t_0 \leq t \leq t_0 < T$.*

Theorem 2.9 [Chen and Zheng 2010]. *If ϕ is the solution of the pseudo-Calabi flow for an initial datum ϕ_0 on $[0, T]$, there is a neighborhood U of ϕ_0 such that the pseudo-Calabi flow has a solution $\varphi(t)$ on $[0, T]$ for any $\varphi_0 \in U$ and the mapping $\varphi_0 \mapsto \varphi(t)$ is C^k for $k = 0, 1, 2, \dots$.*

As a corollary of the continuous dependence on initial data we have:

Theorem 2.10 [Chen and Zheng 2010]. *Suppose M admits a constant scalar curvature Kähler metric ω . Let $\varphi_0 \in C^{2,\alpha}(M, g)$ be such that $\lambda\omega \leq \omega_{\varphi_0} \leq \Lambda\omega$ for positive constants λ and Λ . Then for any $T > 0$ there exists a positive constant $\epsilon_0(T)$ such that, if $|\varphi_0|_{C^{2,\alpha}(M,g)} \leq \epsilon_0(T)$, the pseudo-Calabi flow has a unique solution on $[0, T]$ and*

$$|\dot{\varphi}|_{C^\alpha(M,g)} + |\varphi|_{C^{2,\alpha}(M,g)} \leq C\epsilon_0(T) \quad \text{for all } t \in [0, T],$$

where C depends on M, g and T . As T goes to infinity, $\epsilon_0(T)$ goes to zero.

3. No nontrivial holomorphic vector fields

Let $\eta(M)$ be the set of all holomorphic vector fields on M . We start with the case $\eta(M) = \emptyset$. We shall prove the following proposition in this section.

Proposition 3.1 [Tian and Zhu 2008; Zhu 2009]. *Assume M admits a Kähler–Einstein metric ω and has no holomorphic vector fields. There exists a small positive constant ϵ_0 such that, if the initial datum satisfies*

$$|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0,$$

then the Kähler–Ricci flow g_φ converges smoothly to g .

Proof. We at first show that under the assumption of the proposition, the solution of (2-3) always stays in some small ϵ_1 -neighborhood of the zero function.

Lemma 3.2. *For any $\epsilon_1 > 0$, there exists a small positive constant ϵ_0 such that, if $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, then $|\varphi(t)|_{2,\alpha} \leq \epsilon_1$ for all $t \in [0, +\infty)$.*

Proof. Suppose that the conclusion fails; then there exists a sequence of initial data φ_s^0 such that

$$|\varphi_s^0|_{C^{2,\alpha}} \leq \frac{1}{s}.$$

By virtue of Theorem 2.10, we get a sequence of solutions $\varphi_s(t)$ satisfying the flow equations (2-3) with $\varphi_s(0) = \varphi_s^0$. Let T_s be the first time such that

$$(3-1) \quad |\varphi_s(T_s)|_{C^{2,\alpha}} = \epsilon_1 \quad \text{and} \quad |\varphi_s(t)|_{C^{2,\alpha}} < \epsilon_1 \quad \text{for } 0 \leq t < T_s.$$

According to Theorem 2.10 again, we have $T_s \geq T_1 > 0$. Moreover, we apply Theorem 2.8 to (2-3) on $[T_s - 2a, T_s]$ for fixed a such that $0 < a < T_s/2 - T_1/4$, then we obtain a uniform higher-order bound for the sequence of solutions:

$$|\varphi_s|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1, a) \quad \text{on } [T_s - a, T_s], \quad \text{for all } k \geq 0.$$

Consequently, there is a subsequence of $\phi_s = \varphi_s(T_s)$ converges smoothly to ϕ_∞ satisfying

$$(3-2) \quad |\phi_\infty|_{C^{2,\alpha}} = \epsilon_1.$$

It is obvious that g_{ϕ_∞} is still a Kähler metric. Since the K -energy is not only well defined for φ_s^0 by (2-7) but also decreasing along the Kähler–Ricci flow, Theorem 2.4 implies that

$$0 \leq v_\omega(\phi_s) \leq v_\omega(\varphi_s(0)) \leq \frac{C}{s}.$$

By passing the limit we obtain

$$\lim_{s \rightarrow \infty} v_\omega(\varphi_s) = v_\omega(\phi_\infty) = 0.$$

Using Theorem 2.4, we obtain that g_{ϕ_∞} is a Kähler–Einstein metric. From the same theorem we deduce that ϕ_∞ must be a constant. Furthermore the normalization condition $I(\phi_\infty) = 0$ leads to $\phi_\infty = 0$, which contradicts to (3-2). The lemma follows. \square

According to Theorem 2.8 and Lemma 3.2, we have $|\varphi(t)|_{C^k} \leq C_k$ for any $k \geq 3$ away from $t = 0$. It follows that for any sequence t_i there is a subsequence such that $\phi(t_i)$ converges smoothly to a limit function φ_∞ . Moreover, since the K -energy has a lower bound and it decays along the flow, ω_{φ_∞} must be a Kähler–Einstein metric. This, together with Theorem 2.4 and the normalization condition, implies that $\varphi_\infty = 0$. Because the t_i can be chosen arbitrarily, we conclude that the Kähler–Ricci flow converges smoothly to the original Kähler–Einstein metric. \square

4. M admits nontrivial holomorphic vector fields

4A. Choice and estimate of holomorphic transformations. When M admits holomorphic vector fields, we need to find an appropriate holomorphic transformation. Let $\mathcal{E}_0 \subset \mathcal{H}_0$ be the space of Kähler potentials of Kähler–Einstein metrics.

Let $\sigma_t^* \omega$ be any curve with $\sigma_0 = \text{id}$ in \mathcal{E}_0 . The tangent vector at ω is

$$\frac{d}{dt} \sigma_t^* \Big|_{t=0} \omega = L_X \omega.$$

Here $X = (\sigma_t)_*^{-1} \partial_t \sigma_t \Big|_{t=0}$ is the real part of some holomorphic vector field. Since $C_1(M) > 0$ implies that M is simply connected by [Kobayashi 1961], we obtain $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X$ for some function θ_X . Hence, the finite-dimensionality of the space of holomorphic vector fields implies that of \mathcal{E}_0 . Moreover, according to [Mabuchi 1987], \mathcal{E}_0 is also a totally geodesic submanifold of \mathcal{H}_0 . Then the point $\rho \in \mathcal{E}_0$ realizes the shortest distance between φ , and \mathcal{E}_0 is uniquely determined. In fact, according to Theorem 2.4, we obtain a holomorphic diffeomorphism $\sigma \in \text{Aut}_0(M)$ such that $\sigma^* \omega = \omega + \sqrt{-1} \partial \bar{\partial} \rho$. The K -energy is invariant under holomorphic transformations:

Lemma 4.1 [Mabuchi 1986]. $v(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) = v(\omega, \omega_\varphi) = v(\omega_\rho, \omega_\varphi)$.

Proof. Since ω and ω_ρ are both Kähler–Einstein metrics, Lemma (5.4.1) and Theorem (5.3) of [Mabuchi 1986] yield, respectively, the equalities

$$\begin{aligned} v(\omega, \omega_{(\sigma^{-1})^*(\varphi-\rho)}) &= v(\sigma^* \omega, \omega_\varphi) = v(\omega_\rho, \omega_\varphi) \\ &= v(\omega_\rho, \omega) + v(\omega, \omega_\varphi) = v(\omega, \omega_\varphi). \end{aligned} \quad \square$$

We next state two lemmas from [Chen and Zheng 2010] regarding the metric rephrased for economy geometry of the space of constant scalar curvature Kähler metrics. They show that when metrics stay close to ω , their projection metrics are uniformly bounded.

Lemma 4.2 [Chen and Zheng 2010]. *There exists a positive constant ϵ such that $|\rho|_{C^{3,\alpha}} \leq C_2\epsilon$ for any ρ satisfying $d(0, \rho) \leq \epsilon$.*

Proof. Since \mathcal{E}_0 is a finite-dimensional Riemannian symmetric space, a small ϵ neighborhood near $\rho = 0$ in this submanifold can be pulled back by the exponential map \exp_0 to the tangent space $T_0(\mathcal{E}_0)$ at 0. Set $\psi = \exp_0^{-1}(\rho)$. Then the length from ψ to 0 is ϵ . The norm induced by the distance on $T_0(\mathcal{E}_0)$ is equivalent to the $C^{2,\alpha}$ norm, since all norms on a finite-dimensional vector space are equivalent. Thus $|\exp_0^{-1}(\rho)|_{C^{2,\alpha}}$ is bounded by $C_1\epsilon$. Since the exponential map is a diffeomorphism in the ϵ neighborhood near $\rho = 0$, we obtain $|\rho|_{C^{2,\alpha}} \leq C_2\epsilon$ for some constant C_2 . The lemma follows by an appropriate choice of ϵ . \square

We can improve this conclusion for C^k for fixed $k \geq 0$, not only for $C^{3,\alpha}$ norm.

Lemma 4.3 [Chen and Zheng 2010]. *There exists a positive constant ϵ_1 such that $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$ implies*

$$|\rho|_{C^{3,\alpha}} \leq C_4 \quad \text{and} \quad |\sigma|_h \leq C_5,$$

where h is the left invariant metric in $\text{Aut}(M)$.

Proof. Choose a path $\gamma_t = t\varphi - I(t\varphi) \in \mathcal{H}_0$ for $0 \leq t \leq 1$. Denote $d(0, \varphi)$ the distance between 0 and φ . Using (2-1), we compute

$$\begin{aligned} d(0, \varphi) \leq L(\gamma_t) &= \int_0^1 \left(\int_M \left(\frac{\partial \gamma_t}{\partial t} \right)^2 \omega_{\gamma_t}^n \right)^{1/2} dt \\ &= \int_0^1 \left(\int_M (\varphi - \partial_t I(t\varphi))^2 \omega_{\gamma_t}^n \right)^{1/2} dt \leq C_3\epsilon_1 \end{aligned}$$

for $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$. Moreover, the choice of the ρ implies

$$d(0, \rho) \leq d(0, \varphi) + d(\varphi, \rho) \leq 2d(0, \varphi) \leq C_3\epsilon_1,$$

by the triangle inequality. From Lemma 4.2, it follows that $|\rho|_{C^{3,\alpha}} \leq C_4 = C_2C_3\epsilon_1$. Using [Chen and Tian 2006, Lemma 4.6], we derive $|\sigma|_h \leq C_5$ and the lemma follows. \square

Remark. Alternatively the holomorphic transformation can be derived by minimizing the $I - J$ functional, as in [Bando and Mabuchi 1987], which will be further discussed in Section 7. Those authors use this minimizer to prove the uniqueness of the Kähler–Einstein metric when the first Chern class is positive. The minimizer of the $I - J$ functional is not unique in general, since the second variation of this functional is not strictly positive. However, when the potential is small enough, the minimizer is unique. We mention also that Corollary 7.2 provides an estimate similar to Lemma 4.3.

4B. Long time existence and Cheeger–Gromov convergence. Set

$$\mathcal{S}(\epsilon_1, C(k, \epsilon_1)) = \{ \varphi \mid |\varphi|_{C^{2,\alpha}} \leq \epsilon_1; |\varphi|_{C^{k,\alpha}(M)} \leq C(k, \epsilon_1) \}.$$

It is obvious that $0 \in \mathcal{S}$. We will show that when the initial potential is small, the solution of (2-3) always stays in \mathcal{S} after pulling back by a sequence of holomorphic transformations.

Lemma 4.4. *For any $\epsilon > 0$, there is a small positive constant δ depends on ϵ and \mathcal{S} such that, for any $\varphi \in \mathcal{S}$, if $\nu_\omega(\varphi) \leq \delta$, then $|(\sigma^{-1})^*(\varphi - \rho)|_{C^{2,\alpha}} < \epsilon$.*

Proof. If the conclusion fails, we take a positive constant ϵ and a sequence of $\varphi_s \in \mathcal{S}$ satisfying

$$\nu_\omega(\varphi_s) \leq \frac{1}{s}$$

and such that

$$(4-1) \quad |(\sigma_s^{-1})^*(\varphi_s - \rho_s)|_{C^{2,\alpha}} \geq \epsilon.$$

Since $\varphi_s \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1))$, we obtain a subsequence φ_{s_j} of φ_s converging smoothly to φ_∞ . Let $\hat{\varphi}_s = (\sigma_s^{-1})^*(\varphi_s - \rho_s)$. Lemma 4.3 gives

$$|\rho_s|_{C^{3,\alpha}} \leq C_4 \quad \text{and} \quad |\sigma_s|_h \leq C_5,$$

which implies that there are, by the Arzelà–Ascoli theorem and the Bolzano–Weierstrass theorem respectively, subsequences of ρ_{s_j} and σ_{s_j} for which (using the same notation)

$$\rho_{s_j} \rightarrow \rho_\infty \text{ in } C^{3,\beta} \quad \text{for any } \beta < \alpha$$

and

$$\sigma_{s_j} \rightarrow \sigma_\infty \text{ in the left invariant metric.}$$

Combining with Lemma 4.1, which implies that

$$\nu_\omega(\varphi_\infty) = \nu_\omega(\hat{\varphi}_\infty) = 0,$$

we derive that $\hat{\varphi}_{s_j}$ converges to $\hat{\varphi}_\infty = (\sigma_\infty^{-1})^*(\varphi_\infty - \rho_\infty) \in \mathcal{E}_0$ in $C^{3,\beta}$ and $\sigma_\infty^* \omega = \omega + \partial\bar{\partial}\rho_\infty$. Moreover, according to Theorem 2.4, we have $\hat{\varphi}_\infty, \varphi_\infty \in \mathcal{E}_0$.

We claim that

$$d(\varphi_\infty, \rho_\infty) = 0.$$

Otherwise, for some sufficient large N , when $s_j > N$,

$$d(\varphi_{s_j}, \rho_{s_j}) = d(\varphi_{s_j}, \mathcal{E}_0)$$

has a strictly positive lower bound. Since the distance function is at least C^1 (see [Chen 2000b]), we have $d(\varphi_\infty, \mathcal{E}_0) > 0$, contradicting $\varphi_\infty \in \mathcal{E}_0$. Consequently, the claim holds. It follows that $\hat{\varphi}_\infty = 0$, in contradiction with the lower bound $|\hat{\varphi}_\infty|_{C^{2,\alpha}} \geq \epsilon$ of (4-1). □

Proposition 4.5. *Assume M admits a Kähler–Einstein metric ω and has nontrivial holomorphic vector fields. There is a small positive constant ϵ_0 such that, if $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, there is a unique solution $\varphi(t)$ and a corresponding holomorphic transformation $\varrho(t)$ such that the normalization potential of $\varrho(t)^*\omega(t)$ always stays in \mathcal{S} . Moreover, any sequence t_j has a subsequence (still denoted by t_j) such that $\varrho(t_j)^*\omega(t_j)$ converges smoothly to a Kähler–Einstein metric ω_∞ .*

Proof. We prove this proposition by contradiction. Let ϵ_1 be as in Lemma 4.3. Using Theorem 2.10, we assume there is a maximal time T such that

$$|\varphi|_{C^{2,\alpha}} < \epsilon_1 \text{ on } [0, T) \quad \text{and} \quad |\varphi(T)|_{C^{2,\alpha}} = \epsilon_1.$$

According to Theorem 2.8 we obtain $|\varphi(T)|_{C^{k,\alpha}} \leq C(k, \epsilon_1, t_0, g)$ on $[T - t_0, T]$ for a fixed $T/2 \leq t_0 \leq T$. Let the constant $C(k, \epsilon_1)$ be $C(k, \epsilon_1, t_0, g)$. So we get

$$\varphi(T) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1)).$$

There are two situations. If $\varphi(T)$ is a Kähler–Einstein metric, the flow will stop here and our theorem is proved. Otherwise, we will extend the flow as follows.

We first choose ϵ_0 small enough to guarantee

$$v_\omega(\varphi_0) \leq o\left(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1))\right),$$

where the constant $o(\epsilon_1/2, \mathcal{S}(\epsilon_1, C(k, \epsilon_1)))$ is determined in Lemma 4.4. Let the holomorphic transformation σ be the projection of $\varphi(T)$ in \mathcal{E}_0 with

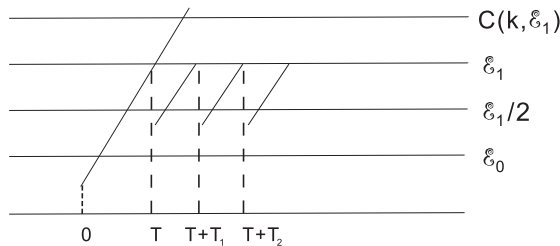
$$\sigma^*\omega = \omega + \sqrt{-1} \partial \bar{\partial} \rho.$$

We set φ_1^0 be the Kähler potential of the metric pulled back by σ , that is,

$$(\sigma^{-1})^*\omega_{\varphi(T)} = \omega + \sqrt{-1} \partial \bar{\partial} [(\sigma^{-1})^*(\varphi(T) - \rho)] = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_1^0.$$

Since the K -energy decreases along the Kähler–Ricci flow, Lemma 4.1 yields

$$(4-2) \quad v_\omega(\varphi_1^0) \leq o\left(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1))\right).$$



Idea of the proof of Proposition 4.5: solving the equation after pulling back.

Letting $\psi = (\sigma^{-1})^*(\varphi(T) - \rho)$, Lemma 4.4 implies that

$$(4-3) \quad |\psi|_{C^{2,\alpha}(g)} = |(\sigma^{-1})^*(\varphi(T) - \rho)|_{C^{2,\alpha}(g)} < \frac{\epsilon_1}{2}.$$

We next show that the Kähler-Ricci flow is invariant under the transformation. Let $\varphi_1 = (\sigma^{-1})^*(\varphi(t) - \rho)$. We compute

$$\begin{aligned} \frac{\partial}{\partial t} \varphi_1 &= (\sigma^{-1})^* \left[\log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \frac{1}{V} \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + \varphi \right) \omega_\varphi^n \right] \\ &= (\sigma^{-1})^* \left[\log \frac{\omega_\varphi^n}{\omega_\rho^n} + \varphi - \rho - \frac{1}{V} \int_M \left(\log \frac{\omega_\varphi^n}{\omega_\rho^n} + \varphi - \rho \right) \omega_\varphi^n \right] \\ &= \left[\log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1 - \frac{1}{V} \int_M \left(\log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1 \right) \omega_{\varphi_1}^n \right]. \end{aligned}$$

The second equality follows from the fact that ω_ρ is a Kähler-Einstein metric. We conclude that φ_1 is the solution of an equation of the form

$$(4-4) \quad \begin{cases} \frac{\partial}{\partial t} \varphi_1 &= \log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1 + a(t), \\ \varphi_1(0) &= \varphi_1^0 = (\sigma^{-1})^*(\varphi(T) - \rho), \end{cases}$$

where (4-3) and (4-2) hold. Again, Theorem 2.10 implies (4-4) has a solution on $[0, T_1]$ with $T_1 \geq T$ such that

$$|\varphi_1(T_1)|_{C^{2,\alpha}} = \epsilon_1.$$

According to Theorem 2.8, we also obtain

$$|\varphi(T_1)|_{C^{k,\alpha}} \leq C(k, \epsilon_1, t_0, g) \quad \text{on } [T_1 - t_0, T_1].$$

So we still have $\varphi(T_1) \in \mathcal{G}(\epsilon_1, C(k, \epsilon_1))$. Moreover, if we let

$$\varphi(t) = \sigma^* \varphi_1(t - T) + \rho \quad \text{on } [T, T + T_1],$$

the new $\varphi(t)$ is the solution of (2-3) on $[0, T + T_1]$.

We repeat the same steps inductively for

$$\varphi_{s-1}(T_{s-1}) \in \mathcal{G}(\epsilon_1, C(k, \epsilon_1)),$$

with $T_{s-1} \geq T$ obtained in Theorem 2.10, until φ_s becomes a Kähler-Einstein at time T_s , with $T_s < \infty$. If this does not happen, the Kähler-Ricci flow has long-time existence and the solution $\varphi(t)$ for all $t \geq 0$ is given by

$$\omega_{\varphi(t)} = \prod_{i=0}^{s-1} \sigma_i^* \omega_{\varphi_s(t)} \quad \text{on } \left[\sum_{i=0}^{s-1} T_i, \sum_{i=0}^s T_i \right).$$

Finally, we prove the convergence of the Kähler–Ricci flow. For any sequence $\{\varphi_{t_j}\}$, there is s such that $\sum_{i=0}^{s-1} T_i \leq t_j \leq \sum_{i=0}^s T_i$. Let

$$\varrho_j = \left(\prod_{i=0}^{s-1} \sigma_i \right)^{-1}.$$

We have

$$|\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^\alpha} \leq \epsilon_1 \quad \text{and} \quad |\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^k} \leq C(k, \epsilon_1).$$

Therefore all metrics are equivalent and their derivatives are bounded. We set

$$\omega_{\psi_{t_j}} = \varrho_j^* \omega_{\varphi_{t_j}}.$$

It follows that there is a subsequence of $\omega_{\psi_{t_j}}$ that converges to a limit metric ω_∞ (which depends on the choice of the subsequence). Since the K -energy is bounded below, we have $\lim_{s \rightarrow \infty} \nu(\omega, \omega_{\psi_{t_j}}) = 0$. It follows from Theorem 2.4 that g_∞ is a Kähler–Einstein metric. The proposition is proved. \square

Let $t_s = \sum_{i=0}^s T_i$. Following the argument in [Chen and Tian 2006], we can first connect each pair of points φ_{t_s} and $\varphi_{t_{s+1}}$ by a geodesic in the space of Kähler–Einstein metrics, so

$$\varrho(t) = \varrho(s) \exp((t - s)X_s) \quad \text{for all } t \in [s, s + 1],$$

with X_s uniformly bounded by Lemma 4.3. We then smooth the corner at each point t_s by replacing the broken line by a smooth curve in a small neighborhood of t_s without changing the value and the t derivative at the endpoints. Hence we have extended the holomorphic transformation to all t , while ensuring Lipschitz continuity in t .

Let $\omega_{\psi(t)} = \varrho(t)^* \omega_{\varphi(t)}$. We have already seen that the Kähler–Ricci flow converges to a Kähler–Einstein metric in Cheeger–Gromov sense; i.e., for any sequence $g(t_i)$, there is a subsequence $g(t_{i_j})$ and a holomorphic transformation $\varrho(t_{i_j})$ such that $\varrho(t_{i_j})^* g(t_{i_j})$ converges smoothly to a Kähler–Einstein metric g_∞ . So we have

$$\lim_{t \rightarrow \infty} \text{Ric}(g_{\psi_t}) - \omega_{\psi_t} = 0,$$

which leads to the convergence of the eigenvalue. To obtain the compactness of the sequence of holomorphic transformations $\varrho(t)$ and the exponential convergence of the Kähler–Ricci flow, we use an auxiliary result:

Theorem 4.6 [Phong et al. 2009, Theorem 2 and Remark (7)]. *If the Kähler–Ricci flow converges to a Kähler–Einstein metric in Cheeger–Gromov sense. Then the Kähler–Ricci flow must converge exponentially to a unique Kähler–Einstein metric nearby.*

5. Kähler–Ricci solitons

In this section we generalize our argument to the Kähler–Ricci solitons. According to [Fujiki 1978], the identity part of holomorphic transformation group $\text{Aut}_0(M)$ is meromorphically isomorphic to a linear algebraic group $L(M)$ and the quotient $\text{Aut}_0(M)/L(M)$ is a complex torus. Futaki and Mabuchi [1995] used the Chevalley decomposition to $L(M)$ to obtain a semidirect decomposition

$$\text{Aut}_0(M) = \text{Aut}_r(M) \ltimes R_u.$$

Here $\text{Aut}_r(M)$ is the reductive algebra group, which is the complexification of a maximal compact subgroup K , and R_u is the unipotent radical of $\text{Aut}_0(M)$. Let η_r be the Lie algebra of $\text{Aut}_r(M)$. Recall that a Kähler metric ω is called a *Kähler–Ricci soliton* if there is a holomorphic vector field X such that

$$(5-1) \quad L_X \omega = \text{Ric} - \omega.$$

Tian and Zhu [2000] proved the uniqueness of Kähler–Ricci solitons for a fixed X in the Lie algebra of $\text{Aut}_0(M)$.

Theorem 5.1 [Tian and Zhu 2000]. *If (ω, X) and (ω', X) are Kähler–Ricci solitons, there are holomorphic transformations $\sigma \in \text{Aut}_0(M)$ and $\tau \in \text{Aut}_r(M)$ such that $\sigma^* \omega = \tau^* \sigma^* \omega'$ and $\sigma^* X \in \eta_r$.*

Theorem 5.2 [Tian and Zhu 2002]. *If (ω, X) and (ω', X') are two Kähler–Ricci solitons, then there is a holomorphic transformation group $\sigma \in \text{Aut}_0(M)$ such that $\omega = \sigma^* \omega'$ and $X = \sigma_*^{-1} X'$.*

Since $L_{\text{Im } X} \omega = 0$, $\text{Im } X$ generates a one-parameter isometric group K_X . We further choose K such that $K_X \subseteq K$. According to Proposition 2.1 of [Tian and Zhu 2002], X lies in the center of η_r .

Now we fix a holomorphic vector field X . By the Hodge theory there is a real value function θ_X such that $L_X \omega = \sqrt{-1} \partial \bar{\partial} \theta_X$ with $\int_M e^{\theta_X} \omega^n = V$. Then the potential equation of the Kähler–Ricci flow (1-2) is

$$(5-2) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \theta_X + a(t), \\ \varphi(0) = \varphi_0. \end{cases}$$

We choose

$$a(t) = -\frac{1}{V} \int_M \left(\log \frac{\omega_\varphi^n}{\omega^n} + \varphi - \theta_X \right) \omega_\varphi^n;$$

moreover $I(\varphi_0) = 0$, so the Kähler–Ricci flow stays in \mathcal{H}_0 .

Recall the W -functional of [Perelman 2002], defined by

$$\mathring{W}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M [\tau(|\nabla f|^2 + S) + f - n] e^{-f} dV,$$

and invariant under diffeomorphisms σ and scaling C :

$$(5-3) \quad \mathring{W}(C\sigma^*g, \sigma^*f, C\tau) = \mathring{W}(g, f, \tau).$$

Recall also Perelman’s μ functional, defined by

$$\mu(g, \tau) = \inf \{ \mathring{W}(g, f, \tau) \mid (4\pi\tau)^{-n/2} \int_M e^{-f} dV = 1 \},$$

and also invariant under diffeomorphism. Its minimum is achieved by some smooth function f satisfying $\tau[(2\Delta f - |\nabla f|^2) + S] + f - n = \mu(g, \tau)$. The first variation of $\mu(g, \tau)$ at $g'_{ij} = v_{ij}$ for fixed τ is

$$\mu'(v_{ij}, \tau) = (4\pi\tau)^{-n/2} \int_M \left\{ -\tau \left(v_{ij}, \text{Ric} + D^2 f - \frac{1}{2\tau} g \right) \right\} e^{-f} dV_g.$$

So the (shrinking) Kähler–Ricci soliton is the critical point of $\mu(g, \tau = \frac{1}{2})$. The gradient flow of the μ functional equals to (1-1) with $\lambda = 1$ up to a diffeomorphism generated by ∇f . So the μ functional is nondecreasing along the Ricci flow. The second variation of this functional near a Kähler–Ricci soliton in the canonical class has been calculated:

Theorem 5.3 [Tian and Zhu 2008, Proposition 2.1]. *We have*

$$(5-4) \quad \frac{\partial^2}{\partial t^2} \mu(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)|_{t=0} \leq 0,$$

and equality holds if and only if $\dot{\varphi}(0)$ is the real part of the holomorphic potential of some holomorphic vector field.

So the only directions in which the Kähler–Ricci soliton ω in (5-4) vanishes are those tangent to the orbit of ω under the action of $\text{Aut}_0(M)$. We thus obtain the following local property of the μ functional:

Lemma 5.4. *A Kähler–Ricci soliton is a local maximum of $\mu(g)$ in \mathcal{H}_0 .*

Proof. Near a Kähler–Ricci soliton g , the tangent space $T_\omega(\mathcal{H}_0)$ splits as $\eta(M) \oplus N$, where N is the orthonormal part. Due to Theorem 5.3, $\mu(g') < \mu(g)$ along any direction in N . Moreover, since σ^*g is still a Kähler–Ricci soliton for any $\sigma \in \text{Aut}_0(m)$, we have $\mu(g') \equiv \mu(g)$ along any direction in $\eta(M)$. \square

As a result we deduce that a Kähler metric that achieves the maximum value of the $\mu(g)$ functional near a Kähler–Ricci soliton must be a Kähler–Ricci soliton.

Let $\mathcal{E}_0 \subset \mathcal{H}_0$ be the space of potentials of Kähler–Einstein solitons with respect to the holomorphic vector field X . Due to Theorem 5.1, \mathcal{E}_0 is a single orbit under the action of $\text{Aut}_r(M)$.

Lemma 5.5. \mathcal{E}_0 is a finite-dimensional totally geodesic submanifold of \mathcal{H}_0 .

Proof. Analogously to the case of the extremal metric in [Calabi 1985], Lemma A.2 and Theorem A of [Tian and Zhu 2000] imply that the identity component of the holomorphic isometric group of the Kähler–Ricci soliton (ω, X) is a maximal compact subgroup of $\text{Aut}_r(M)$ containing K_X . So $(\text{Aut}_r(M), K)$ is a Riemannian symmetric pair and \mathcal{E}_0 is $\text{Aut}_r(M)$ -equivariantly diffeomorphic to the Riemannian symmetric space $\text{Aut}_r(M)/K$. Then for any $\omega \in \mathcal{E}_0$, each geodesic starting at ω in \mathcal{E}_0 can be written in the form

$$\gamma(t) = \exp(t \operatorname{Re} Y)^* \omega,$$

for some nonzero Y whose imaginary part is a Killing vector field. Then Theorem 3.5 and Remark 3.3 in [Mabuchi 1987] show that $\gamma(t)$ is also a geodesic in \mathcal{H}_0 . \square

Now choose $\omega_\rho = \omega + \partial\bar{\partial}\rho$ such that ρ realizes the shortest distance between ψ and \mathcal{E}_0 . Clearly, ρ is uniquely determined. In fact, due to Theorem 5.1 we obtain a holomorphic diffeomorphism $\sigma \in \text{Aut}_r(M)$ such that

$$\sigma^* \omega = \omega_\rho = \omega + \sqrt{-1} \partial\bar{\partial}\rho,$$

with $\rho \in \mathcal{E}_0$. By an argument analogous to the one in Proposition 4.5, but using the μ functional instead of the K -energy, we obtain:

Lemma 5.6. For any $\epsilon > 0$, There is a small positive constant δ depends on ϵ and \mathcal{S} such that for any $\varphi \in \mathcal{S}$, if $\mu(\omega_\varphi) \geq \mu(\omega) - \delta$, then $|(\sigma^{-1})^*(\varphi - \rho)|_{C^{2,\alpha}} < \epsilon$.

Proof. If the conclusion fails, take a positive ϵ and a sequence of $\varphi_s \in \mathcal{S}$ satisfying

$$\mu(\omega_{\varphi_s}) \geq \mu(\omega) - \frac{1}{s} \quad \text{and} \quad |(\sigma_s^{-1})^*(\varphi_s - \rho_s)|_{C^{2,\alpha}} \geq \epsilon.$$

Since $\varphi_s \in \mathcal{S}$, we obtain a subsequence φ_{s_j} of φ_s converging smoothly to φ_∞ . Lemma 4.3 gives

$$|\rho_s|_{C^{3,\alpha}} \leq C_4 \quad \text{and} \quad |\sigma_s|_h \leq C_5,$$

which implies that $(\sigma_{s_j}^{-1})^*(\varphi_{s_j} - \rho_{s_j})$ converges in $C^{3,\beta}$ towards

$$\hat{\varphi}_\infty = (\sigma_\infty^{-1})^*(\varphi_\infty - \rho_\infty) \in \mathcal{E}_0, \quad \text{with } \sigma_\infty^* \omega = \omega + \partial\bar{\partial}\rho_\infty.$$

Then (5-3) implies that $\mu(\omega_{\varphi_\infty}) = \mu(\omega_{\hat{\varphi}_\infty}) = \mu(\omega)$. The rest of this proof is the same as for Lemma 4.4. \square

Proposition 5.7. Assume M admits a Kähler–Ricci soliton (ω, X) . There exists a small constant ϵ_0 such that, if $|\varphi_0|_{C^{2,\alpha}(M)} \leq \epsilon_0$, there is a unique solution $\varphi(t)$ and a corresponding holomorphic transformation $\varrho(t) \in \text{Aut}_r(M)$ such that the normalization potential of $\varrho(t)^* \omega_\varphi(t)$ always stays in \mathcal{S} . Moreover, for any sequence t_i , there is a subsequence t_{i_j} such that $\varrho(t_{i_j})^* g_{\varphi(t_{i_j})}$ converges smoothly to g_∞ .

Proof. The proof, by contradiction, is similar to that of Proposition 4.5. Let ϵ_1 be as in Lemma 4.3. Applying Theorem 2.10 to the potential equation (5-2), we assume there is a maximal time T such that

$$|\varphi|_{C^{2,\alpha}} < \epsilon_1 \text{ on } [0, T) \quad \text{and} \quad |\varphi(T)|_{C^{2,\alpha}} = \epsilon_1.$$

From Theorem 2.8 we obtain $|\varphi(T)|_{C^{k,\alpha}} \leq C(k, \epsilon_1, t_0, g)$ on $[T - t_0, T]$ for a fixed $T/2 \leq t_0 \leq T$. Let the constant $C(k, \epsilon_1)$ be $C(k, \epsilon_1, t_0, g)$. So we get

$$\varphi(T) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1)).$$

There are two situations. If $\varphi(T)$ is a Kähler–Ricci soliton, the flow will stop here and our theorem is proved. Otherwise, we will extend the flow as in the proof in Proposition 4.5.

We first choose ϵ_0 small enough to guarantee that

$$\mu(\omega_{\varphi_0}) \geq \mu(\omega) - o\left(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1))\right),$$

where the constant $o(\epsilon_1/2, \mathcal{S}(\epsilon_1, C(k, \epsilon_1)))$ is determined in Lemma 4.4. Let the holomorphic transformation σ be the projection of $\varphi(T)$ in \mathcal{E}_0 with

$$\sigma^*\omega = \omega + \sqrt{-1} \partial\bar{\partial}\rho.$$

Let φ_1^0 be the Kähler potential of the metric pulled back by σ , that is,

$$(\sigma^{-1})^*\omega_{\varphi(T)} = \omega + \sqrt{-1} \partial\bar{\partial}[(\sigma^{-1})^*(\varphi(T) - \rho)] = \omega + \sqrt{-1} \partial\bar{\partial}\varphi_1^0.$$

Since the μ functional is nondecreasing along the Kähler–Ricci flow, we obtain

$$(5-5) \quad \mu(\omega_{\varphi_1^0}) \geq \mu(\omega) - o\left(\frac{\epsilon_1}{2}, \mathcal{S}(\epsilon_1, C(k, \epsilon_1))\right).$$

Lemma 4.4 implies that

$$(5-6) \quad |(\sigma^{-1})^*(\varphi(T) - \rho)|_{C^{2,\alpha}(g)} < \frac{\epsilon_1}{2}.$$

Set $\varphi_1(t) = (\sigma^{-1})^*(\varphi(t) - \rho)$. Combining (5-2) and (5-1), we obtain that φ_1 is the solution of an equation of the form

$$(5-7) \quad \begin{cases} \frac{\partial}{\partial t}\varphi_1 = \log \frac{\omega_{\varphi_1}^n}{\omega^n} + \varphi_1 - \theta_X + a(t), \\ \varphi_1(0) = \varphi_1^0 = (\sigma^{-1})^*(\varphi(T) - \rho), \end{cases}$$

where (5-6) and (5-5) hold.

Again, Theorem 2.10 implies that (5-7) has a solution on $[0, T_1]$, with $T_1 \geq T$, such that

$$|\varphi_1(T_1)|_{C^{2,\alpha}} = \epsilon_1.$$

From Theorem 2.8, we also obtain $|\varphi(T_1)|_{C^{k,\alpha}} \leq C(k, \epsilon_1, t_0, g)$ on $[T_1 - t_0, T_1]$. So we still have

$$\varphi(T_1) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1)).$$

If we set $\varphi(t) = \sigma^* \varphi_1(t - T) + \rho$ on $[T, T + T_1]$, the new $\varphi(t)$ is the solution of (5-2) on $[0, T + T_1]$.

We repeat the same steps inductively for

$$\varphi_{s-1}(T_{s-1}) \in \mathcal{S}(\epsilon_1, C(k, \epsilon_1)),$$

with $T_{s-1} \geq T$ obtained in Theorem 2.10. We thus obtain a sequence of holomorphic transformations σ_i and the solution $\varphi(t)$ for all $t \geq 0$ given by

$$\omega_{\varphi(t)} = \prod_{i=0}^{s-1} \sigma_i^* \omega_{\varphi_s(t)} \quad \text{on} \quad \left[\sum_{i=0}^{s-1} T_i, \sum_{i=0}^s T_i \right).$$

Set $\varrho_j = \left(\prod_{i=0}^{s-1} \sigma_i \right)^{-1}$. We have

$$|\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^\alpha} \leq \epsilon_1 \quad \text{and} \quad |\varrho_j^* \omega_{\varphi_{t_j}} - \omega|_{C^k} \leq C(k, \epsilon_1).$$

It follows that there is a subsequence of $\varrho_j^* \omega_{\varphi_{t_j}}$ converging to a limit metric ω_∞ . According to Lemma 5.4, the μ functional is bounded above and ω_∞ is a Kähler–Ricci soliton. □

Assume ζ is generated by $\text{Re } X$:

$$\text{Re } X = (\zeta^{-1})_* \frac{\partial}{\partial t} \zeta.$$

Let ϱ and ϕ satisfy $\zeta^* \omega = \omega_\varrho$ and $\phi = \zeta^* \varphi + \varrho$. We obtain the modified Kähler–Ricci flow of the form

$$\begin{cases} \frac{\partial}{\partial t} \omega_\phi = -\text{Ric}(\omega_\phi) + \omega_\phi + L_{\text{Re } X} \omega_\phi, \\ \omega_{\phi(0)} = \omega_{\varphi_0}. \end{cases}$$

We apply [Phong et al. 2011, Theorem 1] to obtain:

Theorem 5.8. *If the Kähler–Ricci flow converges to a Kähler–Ricci soliton in the Cheeger–Gromov sense and the initial Kähler potential is K_X -invariant, then the modified Kähler–Ricci flow converges exponentially to a unique Kähler–Ricci soliton nearby.*

Zhu [2009] also discussed the stability of Kähler–Ricci flow near a Kähler–Ricci soliton by using Perelman’s estimate (unpublished) and Chen and Tian’s energy method [2002; 2006].

6. Weak flow

In this section we weaken the initial condition. Let $a(t) = 0$ in (2-3); the potential equation then reads

$$(6-1) \quad \begin{cases} \frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi, \\ \varphi(0) = \varphi_0. \end{cases}$$

We defined φ_0 is the limit of $\varphi_s \in \text{PSH}(M, \omega) \cap L^\infty(M)$ in L^∞ norm. Meanwhile, $\omega_{\varphi_0} \geq 0$ in the sense of currents. Let the weak solution be a limit of a sequence of approximation solutions by

$$\varphi(t) = \lim_{s \rightarrow 0} \varphi(s, t).$$

The Kähler–Ricci flow with weak initial data was studied in [Chen and Ding 2007; Chen and Tian 2008; 2008]. We also have:

Theorem 6.1 [Song and Tian 2009, Proposition 3.2]. *If φ_0 is defined above with $|\varphi_0|_{L^\infty} \leq A$ and $|\omega_{\varphi_0}^n / \omega^n|_{L^p(M, \omega)} \leq B$ for $p > 1$, there is a unique smooth solution $g_\varphi(t)$ of (1-1) for $t > 0$ such that*

$$\lim_{t \rightarrow 0^+} \varphi(t) = \varphi_0.$$

The estimate in Song and Tian’s proof is that

$$(6-2) \quad |\varphi(t)|_{C^k} \leq C(t, T, k, A, B) \quad \text{on } (0, T].$$

For fixed B and p , introduce the space

$$\mathcal{N}(\epsilon_0; B, p) = \left\{ \varphi \mid |\varphi|_{L^\infty} \leq \epsilon_0, \left| \frac{\omega_\varphi^n}{\omega^n} \right|_{L^p(M, \omega)} \leq B \text{ for some } p > 1 \right\},$$

Here B and p should be chosen such that $\mathcal{N}(\epsilon_0; B, p)$ is not the empty set. Clearly, if $|\varphi_0|_{C^{1,1}} \leq \epsilon_0$, then $\varphi_0 \in \mathcal{N}(\epsilon_0, 1 + (2^n - 1)\epsilon_0, \infty)$. Actually, we have:

Lemma 6.2. *Fix $t_0 \in (0, T]$. For any $\epsilon_1 > 0$ there is a small ϵ_0 such that for any $\varphi_0 \in \mathcal{N}(\epsilon_0; B, p)$ we have $|\varphi(t_0)|_{C^{2,\alpha}} \leq \epsilon_1$.*

Proof. If the conclusion fails, choose a sequence of φ_s such that

$$|\varphi_s|_{L^\infty} \leq \frac{1}{s} \quad \text{and} \quad \left| \frac{\omega_{\varphi_s}^n}{\omega^n} \right|_{L^p(M, \omega)} \leq B.$$

For each corresponding solution $\varphi_s(t)$ constructed by Theorem 6.1, we have

$$(6-3) \quad |\varphi_s(t_0)|_{C^{2,\alpha}} > \epsilon_1.$$

Setting $g_{a\varphi i\bar{j}} = \int_0^t (g_{i\bar{j}} + a\varphi_{i\bar{j}}) da > 0$, we rewrite (6-1) as follows

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \Delta_{g_{a\varphi}} \varphi + \varphi, \\ \varphi_s(0) = \varphi_s. \end{cases}$$

From the maximum principle we obtain

$$(6-4) \quad \sup_M |\varphi_s(t_0)| \leq e^{t_0} \sup_M |\varphi_s|.$$

By (6-2), we can pass a subsequence of $\varphi_{s_i}(t_0)$ such that $\lim_{i \rightarrow \infty} \varphi_{s_i}(t_0) = \varphi_\infty(t_0)$ in C^k for $k \geq 0$. Let $s = s_i$ in (6-4). Then the limit approaches $\sup_M |\varphi_\infty(t_0)| \leq 0$, which contradicts (6-3). \square

Now we have a $C^{2,\alpha}$ small initial datum $\varphi(t_0)$; we normalize it to be $\varphi_0 - I(\varphi_0)$ which is also $C^{2,\alpha}$ small. Then we can solve Equation (2-3) with this initial datum. Combining Propositions 3.1 and 4.5, Theorem 4.6, and Lemma 6.2, we obtain Theorem 1.2. Analogously, we apply Proposition 5.7, Theorem 5.8 and Lemma 6.2 to obtain Theorem 1.1.

7. Another choice of holomorphic transformations

In this section, we follow the arguments in [Bando and Mabuchi 1987; Chen and Tian 2002] to find a good holomorphic transformation. The I and J functionals are defined as

$$\begin{aligned} I(\omega, \omega_\varphi) &= \frac{1}{V} \int_M \varphi(\omega^n - \omega_\varphi^n), \\ J(\omega, \omega_\varphi) &= \frac{1}{V} \sum_{i=0}^{n-1} \int_M \frac{i+1}{n+1} \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}. \end{aligned}$$

From [Aubin 1998] we know that I and J are both semipositive functionals and satisfy

$$(7-1) \quad 0 \leq I(\omega, \omega_\varphi) \leq (n+1)(I(\omega, \omega_\varphi) - J(\omega, \omega_\varphi)) \leq nI(\omega, \omega_\varphi) \quad \text{for } \varphi \in \mathcal{H}.$$

Fix $\varphi \in \mathcal{H}_0$. Consider the functional

$$\Psi(\sigma) = (I - J)(\omega_\varphi, \sigma^* \omega) = (I - J)(\omega_\varphi, \omega_\rho),$$

which is defined for any σ in the reductive subgroup $\text{Aut}(M)$ with $\sigma^* \omega = \omega + \partial \bar{\partial} \rho$. Since ω_ρ is a Kähler-Einstein metric, it satisfies

$$(7-2) \quad \log \frac{\omega_\rho^n}{\omega^n} + \rho = 0 \quad \text{and} \quad I(\rho) = 0.$$

If ω_ρ is the minimal point of Ψ , for any $u \in \Lambda_1(\omega_\rho)$, we have

$$(7-3) \quad \int_M (\rho - \varphi)u\omega_\rho^n = 0.$$

It is known that $\eta(M) \cong \Lambda_1(\omega)$ for any Kähler–Einstein metric ω [Matsushima 1957]. To prove that the minimizer of Ψ is always attained, it suffices to prove:

Proposition 7.1 [Bando and Mabuchi 1987]. *For all*

$$\rho \in \{ \rho \mid \sigma^* \omega = \omega_\rho, \sigma \in \text{Aut}_r(M), \Psi(\sigma) \leq r \},$$

we have

$$|\varphi - \rho|_{C^{2,\alpha}(g_\varphi)} \leq C(|\varphi|_{C^{4,\alpha}}).$$

Proof. Clearly,

$$-\Delta_\varphi(\rho - \varphi) < n \quad \text{and} \quad -\Delta_\rho(\rho - \varphi) > -n.$$

A lower bound for the Green function is given by

$$(7-4) \quad G_\varphi \geq -\gamma \frac{D_\varphi^2}{\text{Vol}_\varphi} \doteq -A_\varphi,$$

since the volume is constant in a fixed Kähler class and the diameter of g_φ is bounded by $C \text{diam}(g)$ when $|\varphi|_{C^2} \leq C$. Using Green’s formula and (7-4), we obtain

$$(7-5) \quad \begin{aligned} \sup_M(\rho - \varphi) &= \frac{1}{V} \int_M (\rho - \varphi)\omega_\varphi^n - \frac{1}{V} \int_M \Delta_\varphi(\rho - \varphi)(y)(G_\varphi(x, y) + A_\varphi)\omega_\varphi^n(y) \\ &\leq \frac{1}{V} \int_M (\rho - \varphi)\omega_\varphi^n + nA_\varphi. \end{aligned}$$

Similarly, we deduce that

$$(7-6) \quad \begin{aligned} \inf_M(\rho - \varphi) &= \frac{1}{V} \int_M (\rho - \varphi)\omega_\rho^n - \frac{1}{V} \int_M \Delta_\rho(\rho - \varphi)(y)(G_\rho(x, y) + A_\rho)\omega_\rho^n(y) \\ &\geq \frac{1}{V} \int_M (\rho - \varphi)\omega_\rho^n - nA_\rho. \end{aligned}$$

Because $\text{Ric}(\rho) = \omega_\rho$, we have $\text{diam}(g_\rho) \leq \sqrt{2n-1}\pi$ Myers’ theorem. Combining (7-5) and (7-6) we get

$$(7-7) \quad \text{Osc}_M(\rho - \varphi) \geq \frac{1}{V} \int_M (\rho - \varphi)(\omega_\varphi^n - \omega_\rho^n) + C(|\varphi|_{C^2}).$$

From (7-1) we obtain

$$\frac{1}{V} \int_M (\rho - \varphi)(\omega_\varphi^n - \omega_\rho^n) = I(\omega_\varphi, \omega_\rho) \leq (n + 1)(I - J)(\omega_\varphi, \omega_\rho) \leq (n + 1)r.$$

Since ω_ρ is a Kähler–Einstein metric, we have

$$(7-8) \quad (\omega_\varphi + \sqrt{-1} \partial\bar{\partial}(\rho - \varphi))^n = e^{-(\rho - \varphi) + h_\varphi} \omega_\varphi^n,$$

with

$$\sqrt{-1} \partial\bar{\partial}h_\varphi = \text{Ric}(\omega_\varphi) - \omega_\varphi \quad \text{and} \quad \int_M e^{h_\varphi} \omega_\varphi^n = \text{Vol}(M).$$

By using the second-order estimate in [Yau 1978], we get

$$\begin{aligned} n + \Delta_\varphi(\rho - \varphi) &\leq e^{C \text{Osc}_M(\rho - \varphi)} C \left(\sup_M (\inf_{i \neq k} |R_{\varphi i \bar{i} k \bar{k}}|), \inf_M S_\varphi, \sup_M h_\varphi \right) \\ &\leq e^{C \text{Osc}_M(\rho - \varphi)} C(|\varphi|_{C^4}). \end{aligned}$$

Then the Krylov estimate shows that $\rho - \varphi$ has $C^{2,\alpha}$ bound. □

Thus we also obtain a uniform bound for the gauge ρ . Our previous discussion implies:

Corollary 7.2. *If $|\varphi|_{C^{4,\alpha}}$ is bounded and ρ is the minimizer of Ψ , then $|\varphi - \rho|_{C^{2,\alpha}}$ and $|\rho|_{C^{2,\alpha}}$ are both bounded.*

This implies that g_ρ is equivalent to g .

We now turn to the uniqueness of the critical points of the functional Ψ when φ is small. The second variation of Ψ at ρ is given by

$$(7-9) \quad D^2\Psi_\rho(u, v) = \frac{1}{V} \int_M \left(1 + \frac{1}{2} \Delta_\rho \rho\right) uv \omega_\rho^n.$$

Lemma 7.3. *For all $|\varphi|_{C^{2,\alpha}} \leq \epsilon_1$ and $u \in \Lambda_1(\omega_\rho)$, the bilinear form $D^2\Psi_\rho(u, u)$ is positive definite. Hence ρ is unique.*

Proof. Note that (7-8) can be rewritten as

$$(7-10) \quad (\omega_\rho + \sqrt{-1} \partial\bar{\partial}(\varphi - \rho))^n = e^{-(\varphi - \rho) - h_\varphi} \omega_\rho^n.$$

By definition, h_φ is given by

$$h_\varphi = -\log \frac{\omega_\varphi^n}{\omega_\rho^n} - \varphi - \log \left(\frac{1}{V} \int_M e^{-\varphi} \right) \omega_\rho^n.$$

We conclude that

$$(7-11) \quad |h_\varphi|_{C^{2,\alpha}(g_\rho)} \leq C\epsilon_1 \leq \delta,$$

by the assumption on φ . Let

$$C_{\perp}^{2,\alpha}(M) = \left\{ \varphi \in C^{2,\alpha}(M) \mid \int_M \varphi u \omega_{\rho}^n \text{ for all } u \in \Lambda_1(\omega_{\rho}) \right\}.$$

Define the operator of (7-10) by

$$\Phi(a, b) = \log \frac{(\omega_{\rho} + \sqrt{-1} \partial \bar{\partial} a)^n}{\omega_{\rho}^n} + a + b, \quad C_{\perp}^{2,\alpha}(M) \times C^{\alpha}(M) \rightarrow C^{\alpha}(M).$$

It is clear from (7-10) that $\Phi(\varphi - \rho, h_{\varphi}) = 0$. The linearized operator of (7-10) at $(a, b) = (0, 0)$ is given by

$$\delta_a \Phi(v) = \Delta_{\rho} v + v.$$

We infer that $\delta_a \Phi$ is invertible from $C_{\perp}^{2,\alpha}(M)$ to $C_{\perp}^{\alpha}(M)$. The implicit function theorem implies that there is a small δ neighborhood of 0 in $C^{\alpha}(M)$ such that when $|h_{\varphi}|_{C^{2,\alpha}(g_{\rho})} \leq \delta$, we have from (7-3) that

$$|\varphi - \rho|_{C^{2,\alpha}(g_{\rho})} \leq C\delta.$$

Then we use Corollary 7.2 and (7-11) to obtain

$$|\rho|_{C^{2,\alpha}} \leq |\varphi - \rho|_{C^{2,\alpha}} + |\varphi|_{C^{2,\alpha}} \leq C\epsilon_1.$$

Hence $|\rho|_{C^{2,\alpha}}$ is small if we choose appropriate ϵ_1 and the bilinear form $D^2\Psi_{\rho}(u, u)$ is positive definite. \square

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