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The critical points of the $\mathcal{W}_+$ functional introduced by M. Feldman, T. Ilmanen and L. Ni are the expanding Ricci solitons, which are special solutions of the Ricci flow. On compact manifolds, expanding solitons coincide with Einstein metrics. In this paper, we compute the first and second variations of the entropy functional of the $\mathcal{W}_+$ functional, and briefly discuss the linear stability of compact hyperbolic space forms.

1. Introduction

Perelman [2002] introduced two important functionals, denoted by $\mathcal{F}$ and $\mathcal{W}$. The corresponding entropy functionals λ and ν are monotone along the Ricci flow $\partial g_{ij}/\partial t = -2R_{ij}$ and are constant precisely on steady and shrinking solitons. H.-D. Cao, R. Hamilton and T. Ilmanen [Cao et al. 2004] presented the second variations of both entropy functionals and studied the linear stabilities of certain closed Einstein manifolds of nonnegative scalar curvature.

To find the corresponding variational structure for the expanding case, M. Feldman, T. Ilmanen and L. Ni [Feldman et al. 2005] introduced the functional $\mathcal{W}_+$. Let (M^n, g) be a compact Riemannian manifold, f a smooth function on M, and $\sigma > 0$. Define

$$\mathcal{W}_+(g, f, \sigma) = (4\pi\sigma)^{-n/2} \int_M e^{-f}\big(\sigma(|\nabla f|^2 + R) - f + n\big)\, dV,$$

$$\mu_+(g, \sigma) = \inf\left\{ W_+(g, f, \sigma) \;\middle|\; f \in C^\infty(M) \text{ with } (4\pi\sigma)^{-n/2}\int_M e^{-f}\, dV = 1 \right\},$$

$$\nu_+(g) = \sup_{\sigma > 0} \mu_+(g, \sigma).$$

Then ν_+ is nondecreasing along the Ricci flow and constant precisely on expanding solitons.

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In this note, analogous to [Cao et al. 2004], we present the first and second variations of the entropy ν_+. By computing the first variation of ν_+, one can see that the critical points are expanding solitons, which are actually negative Einstein manifolds (see [Cao and Zhu 2006], for example). Our main result is this:

Theorem 1.1. *Let (M^n, g) be a compact negative Einstein manifold. Let h be a symmetric 2-tensor. Consider the variation of metric $g(s) = g + sh$. Then the second variation of ν_+ is*

$$\frac{\mathrm{d}^2\nu_+(g(s))}{\mathrm{d}s^2}\Big|_{s=0} = \frac{\sigma}{\operatorname{Vol} g}\int_M \langle N_+h, h\rangle,$$

where

$$N_+h := \tfrac{1}{2}\Delta h + \operatorname{div}^* \operatorname{div} h + \tfrac{1}{2}\nabla^2 v_h + \operatorname{Rm}(h,\cdot) + \frac{g}{2n\sigma \operatorname{Vol} g}\int_M \operatorname{tr} h;$$

here tr *is the trace with respect to g and v_h is the unique solution of*

$$\Delta v_h - \frac{v_h}{2\sigma} = \operatorname{div}(\operatorname{div} h), \qquad \int_M v_h = 0.$$

In this case, we may still define the concept of linear stability. We say that an expanding soliton is *linearly stable* if $N_+ \le 0$; otherwise it is *linearly unstable*. Similar to [Cao et al. 2004], the N_+ operator is nonpositive definite if and only if the maximal eigenvalue of the Lichnerowicz Laplacian acting on the space of transverse traceless 2-tensors has a certain upper bound. Using the results in [Delay 2002; 2008] or [Lee 2006], one can then see that compact hyperbolic spaces are linearly stable. But unlike the positive Einstein case, it seems hard to find other examples of negative Einstein manifolds which are either linear stable or linear unstable.

2. The first variation of the expander entropy

Recall that in [Perelman 2002], the $\mathcal{F}$ functional is defined by

$$\mathcal{F}(f, g) = \int_M (|\nabla f|^2 + R)e^{-f}\,dV,$$

and its entropy $\lambda(g)$ is

$$\lambda(g) = \inf\left\{\mathcal{F}(f, g) \;\middle|\; f \in C^\infty(M) \text{ with } \int_M e^{-f} = 1\right\},$$

where R is the scalar curvature. By [Feldman et al. 2005, Theorem 1.7], we know that $\mu_+(g, \sigma)$ is attained by some function f. Moreover, if $\lambda(g) < 0$, then $\nu_+(g)$ can be attained by some positive number σ.

Lemma 2.1. *If $\nu_+(g)$ is realized by some f and σ, it is necessary that the pair (f, σ) solves the equations*

$$\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n + \nu_+ = 0 \tag{1}$$

and

$$(4\pi\sigma)^{-n/2} \int_M f e^{-f} \, dV = \frac{n}{2} - \nu_+. \tag{2}$$

Proof. For fixed $\sigma > 0$, suppose that $\mu_+(g, \sigma)$ is attained by some function f. Using the Lagrange multiplier method, consider the following functional

$$\begin{aligned} &L(g, f, \sigma, \lambda) \\ &= (4\pi\sigma)^{-n/2} \int_M e^{-f} \big(\sigma(|\nabla f|^2 + R) - f + n\big) \, dV + \lambda \bigg((4\pi\sigma)^{-n/2} \int_M e^{-f} \, dV - 1 \bigg). \end{aligned}$$

Denote by δf the variation of f. Then the variation of L is

$$\begin{aligned} 0 &= \delta L \\ &= (4\pi\sigma)^{-n/2} \int_M e^{-f} (-\delta f) \big(\sigma(|\nabla f|^2 + R) - f + n\big) \, dV \\ &\quad + (4\pi\sigma)^{-n/2} \int_M e^{-f} \big(2\sigma \nabla f \nabla(\delta f) - \delta f\big) \, dV - (4\pi\sigma)^{-n/2} \int_M \lambda (\delta f) e^{-f} \, dV \\ &= (4\pi\sigma)^{-n/2} \int_M e^{-f} (\delta f) \big(\sigma(-2\Delta f + |\nabla f|^2 - R)\big) \, dV \\ &\qquad + (4\pi\sigma)^{-n/2} \int_M e^{-f} (\delta f)(f - n - 1 - \lambda) \, dV \end{aligned}$$

Therefore,

$$\sigma(-2\Delta f + |\nabla f|^2 - R) + f - n - 1 - \lambda = 0.$$

Integrating both sides with respect to the measure $(4\pi\sigma)^{-n/2} e^{-f} dV$, we get

$$-\lambda - 1 = (4\pi\sigma)^{-n/2} \int_M e^{-f} \big(\sigma(|\nabla f|^2 + R) - f + n\big) \, dV = \mu_+(g, \sigma).$$

When σ and f realize $\nu_+(g)$, this is just Equation (1).

Now we consider the variations $\delta\sigma$ and δf of both σ and f. We have

$$\begin{aligned} 0 = (4\pi\sigma)^{-n/2} &\int_M e^{-f} \Big(-\frac{n}{2\sigma}\delta\sigma - \delta f\Big) \big(\sigma(|\nabla f|^2 + R) - f + n\big) \, dV \\ &+ (4\pi\sigma)^{-n/2} \int_M e^{-f} \big(\delta\sigma(|\nabla f|^2 + R) + 2\sigma \nabla f \nabla(\delta f) - \delta f\big) \, dV \end{aligned} \tag{3}$$

and

$$(4\pi\sigma)^{-n/2} \int_M e^{-f} \Big(-\frac{n}{2\sigma}\delta\sigma - \delta f\Big) \, dV = 0. \tag{4}$$

Using (1) and (4), we can write (3) as

$$\begin{aligned}0 &= (4\pi\sigma)^{-n/2}\int_M e^{-f}\big(\delta\sigma(|\nabla f|^2+R)-\delta f\big)\,dV\\ &= (4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(\frac{1}{\sigma}\delta\sigma(\nu_+ + f - n)+\frac{n}{2\sigma}\delta\sigma\Big)\,dV\\ &= (\delta\sigma)\frac{1}{\sigma}(4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(\nu_+ + f - \frac{n}{2}\Big)\,dV,\end{aligned}$$

which gives (2). □

Before computing the variations of the ν_+ functional, let's recall some variation formulas for curvatures. By direct computation, we have:

Lemma 2.2. *Suppose that h is a symmetric 2-tensor and $g(s)=g+sh$ is a variation of g. Then*

$$\frac{\partial R}{\partial s}\Big|_{s=0} = -h_{kl}R_{kl}+\nabla_p\nabla_k h_{pk}-\Delta\,\mathrm{tr}\,h \tag{5}$$

and

$$\begin{aligned}\frac{\partial^2 R}{\partial s^2}\Big|_{s=0} &= 2h_{kp}h_{pl}R_{kl}-2h_{kl}\frac{\partial R_{kl}}{\partial s}\Big|_{s=0}+g^{kl}\frac{\partial^2 R_{kl}}{\partial s^2}\Big|_{s=0}\\ &= 2h_{kp}h_{pl}R_{kl}-h_{kl}(2\nabla_p\nabla_k h_{pl}-\Delta h_{kl}-\nabla_k\nabla_l\,\mathrm{tr}\,h)\\ &\quad -\nabla_p\big(h_{pq}(2\nabla_k h_{kq}-\nabla_q\,\mathrm{tr}\,h)\big)+\nabla_k(h_{pq}\nabla_k h_{pq})\\ &\quad +\tfrac{1}{2}\nabla_p\,\mathrm{tr}\,h(2\nabla_k h_{kp}-\nabla_p\,\mathrm{tr}\,h)+\tfrac{1}{2}(\nabla_k h_{pq}\nabla_k h_{pq}-2\nabla_p h_{kq}\nabla_q h_{kp}),\end{aligned} \tag{6}$$

where ∇ is the Levi-Civita connection of g and $\mathrm{tr}\,h$ is the trace of h taken with respect to g.

Now we are ready to compute the first variation of $\nu_+(g)$.

Proposition 2.3. *Let (M^n, g) be a compact Riemannian manifold with $\lambda(g) < 0$. Let h be any symmetric covariant 2-tensor on M, and consider the variation*

$$g(s) = g + sh.$$

Then the first variation of $\nu_+(g(s))$ is

$$\frac{d\nu_+(g(s))}{ds}\Big|_{s=0} = (4\pi\sigma)^{-n/2}\int_M \sigma e^{-f}\Big(-R_{ij}-\nabla_i\nabla_j f-\frac{1}{2\sigma}g_{ij}\Big)h_{ij}\,dV,$$

where the smooth function f and $\sigma > 0$ realize $\nu_+(g)$.

Proof. By taking derivatives directly, we have

$$
\begin{aligned}
(7)\quad \frac{\partial \nu_+}{\partial s} &= (4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}g^{ij}h_{ij}\Big)\big(\sigma(|\nabla f|^2+R)\big)\,dV\\
&+(4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}g^{ij}h_{ij}\Big)(-f+n)\,dV\\
&+(4\pi\sigma)^{-n/2}\int_M e^{-f}\frac{\partial\sigma}{\partial s}(|\nabla f|^2+R)\,dV\\
&-(4\pi\sigma)^{-n/2}\int_M e^{-f}(\sigma g^{ip}g^{jq}h_{pq}\nabla_i f\nabla_j f)\,dV\\
&+(4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(\sigma\Big(2g^{ij}\nabla_i f\nabla_j\frac{\partial f}{\partial s}+\frac{\partial R}{\partial s}\Big)-\frac{\partial f}{\partial s}\Big)\,dV.
\end{aligned}
$$

Since $(4\pi\sigma)^{-n/2}\int_M e^{-f}\,dV=1$, we have

$$
(8)\qquad (4\pi\sigma)^{-n/2}\int_M\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}g^{ij}h_{ij}\Big)e^{-f}\,dV=0.
$$

Substituting (1), (2) and (8) in (7), we obtain

$$
\begin{aligned}
&\frac{\partial \nu_+(s)}{\partial s}\Big|_{s=0}\\
&=(4\pi\sigma)^{-n/2}\int_M\big(2\sigma(|\nabla f|^2-\Delta f)+\nu_+(0)\big)\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}g^{ij}h_{ij}\Big)e^{-f}dV\\
&\quad+(4\pi\sigma)^{-n/2}\int_M\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2+R)-\frac{\partial f}{\partial s}-\sigma h_{ij}\nabla_i f\nabla_j f\Big)e^{-f}dV\\
&\quad+(4\pi\sigma)^{-n/2}\int_M\sigma\Big(2\frac{\partial f}{\partial s}(|\nabla f|^2-\Delta f)+\nabla_i\nabla_j h_{ij}-\Delta\,\mathrm{tr}\,h-h_{ij}R_{ij}\Big)e^{-f}dV\\
&=(4\pi\sigma)^{-n/2}\int_M\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2+R)-\frac{\partial f}{\partial s}-\sigma(h_{ij}\nabla_i\nabla_j f+h_{ij}R_{ij})\Big)e^{-f}dV\\
&=(4\pi\sigma)^{-n/2}\int_M\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2+R)+\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}\Big)e^{-f}\,dV\\
&\qquad\qquad-(4\pi\sigma)^{-n/2}\int_M\sigma h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f+\frac{1}{2\sigma}g_{ij}\Big)e^{-f}dV\\
&=(4\pi\sigma)^{-n/2}\int_M\frac{1}{\sigma}\frac{\partial\sigma}{\partial s}\Big(f(0)-\frac{n}{2}+\nu_+(0)-2\sigma(|\nabla f|^2-\Delta f)\Big)e^{-f}dV\\
&\qquad\qquad-(4\pi\sigma)^{-n/2}\int_M\sigma h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f+\frac{1}{2\sigma}g_{ij}\Big)e^{-f}dV\\
&=-(4\pi\sigma)^{-n/2}\int_M\sigma h_{ij}\Big(R_{ij}+\nabla_i\nabla_j f+\frac{1}{2\sigma}g_{ij}\Big)e^{-f}dV.
\end{aligned}
$$

Hence, the first variation of ν_+ is

$$\frac{d\nu_+(g(s))}{ds}\Big|_{s=0} = (4\pi\sigma)^{-n/2}\int_M \sigma e^{-f}\Big(-R_{ij} - \nabla_i\nabla_j f - \frac{1}{2\sigma}g_{ij}\Big)h_{ij}dV. \quad \square$$

From the proposition, we can see that a critical point of $\nu_+(g)$ satisfies

$$\mathrm{Rc} + \nabla^2 f + \frac{1}{2\sigma}g = 0,$$

which means that (M, g) is a gradient expanding soliton.

3. The second variation

Now we compute the second variation of ν_+. Since any compact expanding soliton is Einstein (see [Cao and Zhu 2006], for example), f is a constant. After adding a constant to f we may assume that $f = n/2$.

In the following, as in [Cao et al. 2004], we set $\mathrm{Rm}(h, h) = R_{ijkl}h_{ik}h_{jl}$, $\operatorname{div}\omega = \nabla_i\omega_i$, $(\operatorname{div} h)_i = \nabla_j h_{ji}$, and $(\operatorname{div}^*\omega)_{ij} = -(\nabla_i\omega_j + \nabla_j\omega_i) = -\frac{1}{2}L_{\omega^\#}g_{ij}$, where h is a symmetric 2-tensor, ω is a 1-tensor, $\omega^\#$ is the dual vector field of ω, and $L_{\omega^\#}$ is the Lie derivative.

Proof of Theorem 1.1. Let (M, g) be a compact negative Einstein manifold with $f = n/2$ and $R_{ij} = -1/(2\sigma)g_{ij}$. For any symmetric 2-tensor h, consider the variation $g(s) = g + sh$. By Proposition 2.3, we know that $(d\nu_+/ds)|_{s=0} = 0$.

From (1) and (2), we get

$$\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}(0) - 2\sigma\Delta\frac{\partial f}{\partial s}(0) - \sigma\frac{\partial R}{\partial s}(0) + \frac{\partial f}{\partial s}(0) = 0, \tag{9}$$

and

$$(4\pi\sigma)^{-n/2}\int_M e^{-n/2}\bigg(\frac{n}{2}\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}(0) - \frac{\partial f}{\partial s}(0) + \frac{1}{2}\operatorname{tr} h\Big) + \frac{\partial f}{\partial s}(0)\bigg)\,dV = 0.$$

It follows by (8) that

$$(4\pi\sigma)^{-n/2}\int_M \frac{\partial f}{\partial s}(0)e^{-n/2}dV = 0 \tag{10}$$

and

$$\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}(0) = \frac{1}{\operatorname{Vol} g}\int_M \frac{1}{2}\operatorname{tr} h\,dV, \tag{11}$$

where $(4\pi\sigma)^{-n/2}e^{-n/2} = \frac{1}{\operatorname{Vol} g}$. Thus

$$\begin{aligned}
\frac{\mathrm{d}\nu_+}{\mathrm{d}s} &= (4\pi\sigma)^{-n/2}\int_M e^{-f}(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2}g^{ij}h_{ij})\big(\sigma(|\nabla f|^2 + R) - f + n\big)\,dV \\
&\quad + (4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2 + R) - \frac{\partial f}{\partial s}\Big)\,dV \\
&\quad + (4\pi\sigma)^{-n/2}\int_M \sigma e^{-f}\Big(-g^{ip}g^{jq}h_{pq}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j\frac{\partial f}{\partial s} + \frac{\partial R}{\partial s}\Big)\,dV \\
&= (4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2}g^{ij}h_{ij}\Big)\big(2\sigma(|\nabla f|^2 - \Delta f) + \nu_+\big)\,dV \\
&\quad + (4\pi\sigma)^{-n/2}\int_M e^{-f}\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2 + R) - \frac{\partial f}{\partial s}\Big)\,dV \\
&\quad + (4\pi\sigma)^{-n/2}\int_M \sigma e^{-f}\Big(-g^{ip}g^{jq}h_{pq}\nabla_i f\nabla_j f + 2g^{ij}\nabla_i f\nabla_j\frac{\partial f}{\partial s} + \frac{\partial R}{\partial s}\Big)\,dV \\
&= (4\pi\sigma)^{-n/2}\int_M \sigma e^{-f}g^{ij}h_{ij}(|\nabla f|^2 - \Delta f)\,dV \\
&\qquad + (4\pi\sigma)^{-n/2}\int_M e^{-f}\bigg(\sigma\Big(-g^{ip}g^{jq}h_{pq}\nabla_i f\nabla_j f + \frac{\partial R}{\partial s}\Big) - \frac{1}{2}g^{ij}h_{ij}\bigg)\,dV,
\end{aligned}$$

where we note that

$$\int_M 2\sigma e^{-f}g^{ij}\nabla_i f\nabla_j\frac{\partial f}{\partial s}\,dV = \int_M 2\sigma e^{-f}\frac{\partial f}{\partial s}(|\nabla f|^2 - \Delta f)\,dV$$

and

$$\begin{aligned}
&\int_M e^{-f}\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2 + R) - \frac{\partial f}{\partial s}\Big)\,dV \\
&\quad = \int_M e^{-f}\Big(\frac{\partial\sigma}{\partial s}(|\nabla f|^2 + R) + \frac{n}{2\sigma}\frac{\partial\sigma}{\partial s} - \frac{1}{2}g^{ij}h_{ij}\Big)\,dV \\
&\quad = \int_M e^{-f}\Big(\frac{1}{\sigma}\frac{\partial\sigma}{\partial s}\Big(\sigma(|\nabla f|^2 + R) + \frac{n}{2}\Big) - \frac{1}{2}g^{ij}h_{ij}\Big)\,dV \\
&\quad = \int_M e^{-f}\frac{1}{\sigma}\frac{\partial\sigma}{\partial s}\Big(\sigma(2|\nabla f|^2 - 2\Delta f) + f - \frac{n}{2} + \nu_+\Big)\,dV - \int_M e^{-f}\cdot\tfrac{1}{2}g^{ij}h_{ij}\,dV \\
&\quad = -\int_M e^{-f}\cdot\tfrac{1}{2}g^{ij}h_{ij}\,dV.
\end{aligned}$$

Since $f(0) = \frac{n}{2}$, we have

$$\frac{d^2\nu_+}{ds^2}\Big|_{s=0} = -\frac{1}{\operatorname{Vol} g}\int_M \sigma \operatorname{tr} h \Delta \frac{\partial f}{\partial s} dV + \frac{1}{\operatorname{Vol} g}\int_M \Big(-\frac{n}{2\sigma}\frac{\partial \sigma}{\partial s} - \frac{\partial f}{\partial s} + \frac{1}{2}\operatorname{tr} h\Big)\Big(\sigma \frac{\partial R}{\partial s} - \frac{1}{2}\operatorname{tr} h\Big) dV + \frac{1}{\operatorname{Vol} g}\int_M \Big(\frac{\partial \sigma}{\partial s}\frac{\partial R}{\partial s} + \sigma \frac{\partial^2 R}{\partial s^2} + \frac{1}{2}|h_{ij}|^2\Big) dV. \tag{12}$$

In the following, all quantities are evaluated at $s = 0$. First, we have

$$\begin{aligned} &\frac{1}{\operatorname{Vol} g}\int_M \sigma \frac{\partial^2 R}{\partial s^2} dV \\ &= \frac{\sigma}{\operatorname{Vol} g}\int_M \Big(-\frac{1}{\sigma}|h_{ij}|^2 - h_{kl}(2\nabla_p\nabla_k h_{pl} - \Delta h_{kl} - \nabla_k\nabla_l \operatorname{tr} h) \\ &\quad - \nabla_p\big(h_{pq}(2\nabla_k h_{kq} - \nabla_q \operatorname{tr} h)\big) + \nabla_k(h_{pq}\nabla_k h_{pq}) \\ &\quad + \tfrac{1}{2}\nabla_p \operatorname{tr} h(2\nabla_k h_{kp} - \nabla_p \operatorname{tr} h) + \tfrac{1}{2}(\nabla_k h_{pq}\nabla_k h_{pq} - 2\nabla_p h_{kq}\nabla_q h_{kp})\Big) dV \\ &= \frac{\sigma}{\operatorname{Vol} g}\int_M \Big(-\frac{1}{\sigma}|h_{ij}|^2 - h_{kl}\nabla_p\nabla_k h_{pl} - \frac{1}{2}|\nabla h|^2 - \frac{1}{2}|\nabla \operatorname{tr} h|^2\Big) dV \\ &= \frac{\sigma}{\operatorname{Vol} g}\int_M \Big(-\frac{1}{\sigma}|h_{ij}|^2 - \frac{1}{2}|\nabla h|^2 - \frac{1}{2}|\nabla \operatorname{tr} h|^2\Big) dV \\ &\qquad - \frac{\sigma}{\operatorname{Vol} g}\int_M h_{kl}(\nabla_k\nabla_p h_{pl} + R_{kq}h_{ql} + R_{pkql}h_{pq})\, dV \\ &= -\frac{1}{\operatorname{Vol} g}\int_M \frac{1}{2}|h_{ij}|^2 dV \\ &\qquad + \frac{\sigma}{\operatorname{Vol} g}\int_M \big(|\operatorname{div} h|^2 + \operatorname{Rm}(h,h) - \tfrac{1}{2}|\nabla h|^2 - \tfrac{1}{2}|\nabla \operatorname{tr} h|^2\big) dV. \end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned} \frac{1}{\operatorname{Vol} g}\int_M \frac{\partial \sigma}{\partial s}\frac{\partial R}{\partial s} dV &= \frac{\sigma}{n}\frac{1}{\operatorname{Vol} g}\int_M \operatorname{tr} h\, dV \frac{1}{\operatorname{Vol} g}\int_M \frac{\partial R}{\partial s} dV \\ &= \frac{1}{2n}\Big(\frac{1}{\operatorname{Vol} g}\int_M \operatorname{tr} h\, dV\Big)^2. \end{aligned} \tag{14}$$

Let v_h be the solution to the equation

$$\Delta v_h - \frac{v_h}{2\sigma} = \operatorname{div}\operatorname{div} h = \nabla_p\nabla_q h_{pq}, \qquad \int_M v_h = 0.$$

Then

$$\begin{aligned}
&\frac{1}{\operatorname{Vol} g}\int_M \Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big)\sigma\frac{\partial R}{\partial s}dV\\
&\quad=\frac{\sigma}{\operatorname{Vol} g}\int_M \Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big)\Big(\Delta v_h-\frac{v_h}{2\sigma}+\frac{1}{2\sigma}\operatorname{tr} h-\Delta\operatorname{tr} h\Big)\,dV\\
&\quad=-\Big(\frac{1}{\operatorname{Vol} g}\int_M\frac{1}{2}\operatorname{tr} h\,dV\Big)^2+\frac{\sigma}{\operatorname{Vol} g}\int_M v_h\Big(-\Delta\frac{\partial f}{\partial s}+\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV\\
&\qquad\qquad+\frac{\sigma}{\operatorname{Vol} g}\int_M \operatorname{tr} h\Big(\Delta\frac{\partial f}{\partial s}-\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV\\
&\qquad\qquad+\frac{\sigma}{\operatorname{Vol} g}\int_M \frac{1}{2}\operatorname{tr} h\Big(\Delta v_h-\frac{v_h}{2\sigma}+\frac{1}{2\sigma}\operatorname{tr} h-\Delta\operatorname{tr} h\Big)\,dV,
\end{aligned}$$

where we have used (11) to derive the first term in the last equality. Meanwhile,

$$\begin{aligned}
&-\frac{1}{\operatorname{Vol} g}\int_M \frac{1}{2}\operatorname{tr} h\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big)\\
&\qquad\qquad=-\frac{1}{\operatorname{Vol} g}\int_M \frac{1}{2}\operatorname{tr} h\Big(-2\sigma\Delta\frac{\partial f}{\partial s}-\sigma\frac{\partial R}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{1}{\operatorname{Vol} g}\int_M \Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big)\Big(\sigma\frac{\partial R}{\partial s}-\frac{1}{2}\operatorname{tr} h\Big)\,dV\\
&\quad=\frac{1}{\operatorname{Vol} g}\int_M \sigma\operatorname{tr} h\Delta\frac{\partial f}{\partial s}dV-\frac{1}{\operatorname{Vol} g}\int_M\frac{1}{4}(\operatorname{tr} h)^2dV\\
&\qquad-\Big(\frac{1}{\operatorname{Vol} g}\int_M\frac{1}{2}\operatorname{tr} h\,dV\Big)^2+\frac{\sigma}{\operatorname{Vol} g}\int_M v_h\Big(-\Delta\frac{\partial f}{\partial s}+\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV\\
&\qquad+\frac{\sigma}{\operatorname{Vol} g}\int_M \operatorname{tr} h\Big(\Delta\frac{\partial f}{\partial s}-\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV\\
&\qquad+\frac{\sigma}{\operatorname{Vol} g}\int_M \operatorname{tr} h\Big(\Delta v_h-\frac{v_h}{2\sigma}+\frac{1}{2\sigma}\operatorname{tr} h-\Delta\operatorname{tr} h\Big)\,dV.
\end{aligned}$$

Now since

$$\begin{aligned}
\frac{\sigma}{\operatorname{Vol} g}\int_M v_h\Big(-\Delta\frac{\partial f}{\partial s}+\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV&=\frac{\sigma}{\operatorname{Vol} g}\int_M v_h\Big(-\frac{n}{4\sigma^2}\frac{\partial\sigma}{\partial s}+\frac{1}{2}\frac{\partial R}{\partial s}\Big)\,dV\\
&=\frac{\sigma}{\operatorname{Vol} g}\int_M \frac{1}{2}v_h\Big(\Delta v_h-\frac{v_h}{2\sigma}+\frac{1}{2\sigma}\operatorname{tr} h-\Delta\operatorname{tr} h\Big)\,dV\\
&=\frac{\sigma}{\operatorname{Vol} g}\int_M -\frac{1}{2}|\nabla v_h|^2-\frac{v_h^2}{4\sigma}+\frac{v_h}{4\sigma}\operatorname{tr} h-\frac{1}{2}v_h\Delta\operatorname{tr} h\,dV
\end{aligned}$$

and

$$\frac{\sigma}{\operatorname{Vol} g}\int_M \operatorname{tr} h\Big(\Delta\frac{\partial f}{\partial s}-\frac{1}{2\sigma}\frac{\partial f}{\partial s}\Big)\,dV$$
$$=\frac{\sigma}{\operatorname{Vol} g}\int_M \operatorname{tr} h\Big(\frac{n}{4\sigma^2}\frac{\partial\sigma}{\partial s}-\frac{1}{2}\frac{\partial R}{\partial s}\Big)\,dV$$
$$=\Big(\frac{1}{\operatorname{Vol} g}\int_M\frac{1}{2}\operatorname{tr} h\;dV\Big)^2-\frac{\sigma}{\operatorname{Vol} g}\int_M\frac{1}{2}\operatorname{tr} h\Big(\Delta v_h-\frac{v_h}{2\sigma}+\frac{1}{2\sigma}\operatorname{tr} h-\Delta\operatorname{tr} h\Big)\,dV,$$

we have

$$(15)\quad \frac{1}{\operatorname{Vol} g}\int_M\Big(-\frac{n}{2\sigma}\frac{\partial\sigma}{\partial s}-\frac{\partial f}{\partial s}+\frac{1}{2}\operatorname{tr} h\Big)\Big(\sigma\frac{\partial R}{\partial s}-\frac{1}{2}\operatorname{tr} h\Big)\,dV$$
$$=\frac{1}{\operatorname{Vol} g}\int_M\sigma\operatorname{tr} h\Delta\frac{\partial f}{\partial s}dV+\frac{\sigma}{\operatorname{Vol} g}\int_M\Big(-\frac{1}{2}|\nabla v_h|^2-\frac{v_h^2}{4\sigma}+\frac{1}{2}|\nabla\operatorname{tr} h|^2\Big)\,dV.$$

Substituting (13), (14) and (15) in (12), we get

$$\frac{\mathrm{d}^2\nu_+}{\mathrm{d}s^2}\Big|_{s=0}=\frac{\sigma}{\operatorname{Vol} g}\left(\int_M\Big(|\operatorname{div} h|^2+\operatorname{Rm}(h,h)-\frac{1}{2}|\nabla h|^2-\frac{1}{2}|\nabla v_h|^2-\frac{v_h^2}{4\sigma}\Big)\,dV\right)+\frac{1}{2n}\left(\frac{1}{\operatorname{Vol} g}\int_M\operatorname{tr} h\,dV\right)^2$$
$$=\frac{\sigma}{\operatorname{Vol} g}\int_M\langle N_+h,h\rangle. \qquad \square$$

As a simple application, we discuss briefly the linear stability of negative Einstein manifolds. In analogy with [Cao et al. 2004], we say that a negative Einstein manifold is linearly stable if $N_+\le 0$, otherwise it is linearly unstable. As in that paper, decompose the space of symmetric 2-tensors as

$$\ker\operatorname{div}\oplus\operatorname{im}\operatorname{div}^*,$$

and further decompose $\ker\operatorname{div}$ as

$$(\ker\operatorname{div})_0\oplus\mathbb{R}g,$$

where $(\ker\operatorname{div})_0$ is the space of divergence free 2-tensors h with $\int_M\operatorname{tr} h=0$. It is easy to see that N_+ vanishes on $\operatorname{im}\operatorname{div}^*$, and on $(\ker\operatorname{div})_0$

$$N_+=\frac{1}{2}\Big(\Delta_L-\frac{1}{\sigma}\Big),$$

where $\Delta_L=\Delta+2\operatorname{Rm}(\cdot,\cdot)-2\operatorname{Rc}$ is the Lichnerowicz Laplacian on symmetric 2-tensors.

Moreover, we may write $(\ker \operatorname{div})_0$ as

$$(\ker \operatorname{div})_0 = S_0 \oplus S_1,$$

where S_0 is the subspace of trace free 2-tensors and

$$S_1 = \left\{ h \in (\ker \operatorname{div})_0 \,\middle|\, h_{ij} = \left(-\frac{1}{2\sigma}u + \Delta u\right) g_{ij} - \nabla_i \nabla_j u,\ u \in C^\infty(M) \text{ and } \int_M u = 0 \right\};$$

see [Buzzanca 1984], for example.

Define

$$Tu := \left(-\frac{1}{2\sigma}u + \Delta u\right) g_{ij} - \nabla_i \nabla_j u.$$

Since $\Delta_L(Tu) = T(\Delta u)$ for all smooth functions u and $\ker T = \{0\}$, we can see that the Lichnerowicz Laplacian and the Laplacian on function space have the same eigenvalues. Thus N_+ is always negative on S_1. Therefore, to study the linear stability of negative Einstein manifolds, it remains to look at the behavior of Δ_L acting on S_0 which is the space of transverse traceless 2-tensors.

Example. Suppose that M is an n dimensional compact real hyperbolic space with $n \geq 3$. By [Delay 2002] or [Lee 2006], the biggest eigenvalue of Δ_L on trace free symmetric 2-tensors on real hyperbolic space is $-\frac{1}{4}(n-1)(n-9)$. Since on M we have $\mathrm{Rc} = -(n-1)g$, we obtain

$$\frac{1}{\sigma} = 2(n-1).$$

Thus the biggest eigenvalue of N_+ on S_0 is not greater than $-\frac{1}{8}(n-1)^2$. This implies that M is linearly stable for $n \geq 3$.

Remarks. (1) When $n = 3$, D. Knopf and A. Young [2009] proved that closed 3-folds with constant negative curvature are geometrically stable under certain normalized Ricci flow. R. Ye [1993] had obtained a more powerful stability result earlier.

(2) For $n = 2$, R. Hamilton [1988] proved that when the average scalar curvature is negative, the solution of the normalized Ricci flow with any initial metric converges to a metric with constant negative curvature. In particular, they are linearly stable. On the other hand, in [Delay 2008] we see that the biggest eigenvalue of the Lichnerowicz Laplacian on trace free symmetric 2-tensors is 2. Thus N_+ is nonpositive definite on $(\ker \operatorname{div})_0$, which also implies the linear stability.

(3) For the noncompact case, V. Suneeta [2009] proved certain geometric stability of $\mathbb{H}^n$ using different methods.

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