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In this paper, we present some new Wirtinger-type inequalities on time scales. As special cases, some new continuous and discrete Wirtinger-type inequalities are given. The obtained inequalities are applied to a certain class of half-linear dynamic equations on time scales in order to establish sufficient conditions for disconjugacy.

1. Introduction

Wirtinger-type inequalities are studied in the literature in various modifications, both in the continuous and in the discrete settings. In principle, it is an integral or sum estimate between the function and its derivative or difference, respectively. These types of inequalities have received a lot of attention because of applications, for example, in number theory, especially in studies concerning the distribution of the zeros of the Riemann-zeta function [Hall 2002a; 2002b]. In this paper, we are interested in deriving some new Wirtinger-type inequalities on time scales and in employing them to study nonoscillation of the solutions of a certain class of half-linear dynamic equations.

The study of dynamic equations on time scales, which goes back to its founder, Stefan Hilger [1990], is an area of mathematics that has recently received a lot of attention. It was created in order to unify the studies of differential and difference equations. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice — once for differential equations and again for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a time scale $\mathbb{T}$, which may be an arbitrary closed subset of the reals. This way, results are obtained not only related to the set of real numbers or integers but those pertaining to more general time scales.

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The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus [Kac and Cheung 2002], that is, when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$, and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$, where $q>1$. Dynamic equations on a time scale have an enormous potential for applications. In population dynamics, they can model insect populations that are continuous while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population [Bohner and Peterson 2001]. There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A cover article in the *New Scientist* [Spedding 2003] discusses several possible applications. Since then, several authors have expounded on various aspects of this new theory. The books on the subject of time scales by Bohner and Peterson [2001; 2003] summarize and organize much of time scale calculus. An overview of various inequalities on time scales is given in [Agarwal et al. 2001].

The classical Wirtinger inequality, [Hardy et al. 1988, Theorem 257], is given by

$$\int_a^b (y'(t))^2 dt \geq \int_a^b y^2(t)\,dt \tag{1-1}$$

for any $y\in \mathrm{C}^1([a,b])$ satisfying $y(a)=y(b)=0$. Beesack [1961] extended the inequality (1-1) and proved that

$$\int_a^b (y'(t))^4 dt \geq \frac{3}{4}\int_a^b y^4(t)\,dt \tag{1-2}$$

for any $y\in \mathrm{C}^2([a,b])$ such that $y(a)=y(b)=0$. Beesack [1961, page 59] proved that

$$\int_0^\pi (y'(t))^{2k} dt \geq \frac{2k-1}{\left(k\sin\dfrac{\pi}{2k}\right)^{2k}}\int_0^\pi y^{2k}(t)\,dt, \qquad \text{for } k\geq 1, \tag{1-3}$$

where $y\in \mathrm{C}^1([0,\pi])$ with $y(0)=y(\pi)=0$. Agarwal and Pang [1995] proved that

$$\int_0^\pi (y'(t))^{2k} dt \geq \frac{2\Gamma(2k+1)}{\pi^{2k}\Gamma^2((2k+1)/2)}\int_0^\pi y^{2k}(t)\,dt, \quad \text{for } k\geq 1, \tag{1-4}$$

for any $y\in \mathrm{C}^1([0,\pi])$ such that $y(0)=y(\pi)=0$. Brnetić and Pečarić [1998] proved that

$$\int_0^\pi (y'(t))^{2k} dt \geq \frac{1}{\pi^{2k} I(k)}\int_0^\pi y^{2k}(t)\,dt, \qquad \text{for } k\geq 1, \tag{1-5}$$

where $y \in \mathrm{C}^1([0,\pi])$ with $y(0) = y(\pi) = 0$, and

$$I(k) = \int_0^1 \frac{1}{(t^{1-2k} + (1-t)^{1-2k})}\,dt.$$

Boyd [1969] proved the inequality

$$\int_0^1 |y'(t)|^{2k} dt \ge \frac{1}{C^{2k}(k)} \int_0^1 |y(t)|^{2k} dt, \quad \text{where } y(0) = y(1) = 0, \tag{1-6}$$

with best constant (by applied variational techniques, the determination of the best constant depends on a nonlinear eigenvalue problem for an integral operator)

$$C(k) := \left(\frac{2k}{2k-1}\right)^{\frac{1}{2k}} (2k)^{(2k-1)/(2k)} \left(\Gamma\left(\frac{1}{2k}\right)\Gamma\left(\frac{2k-1}{2k}\right)\right)^{-1}. \tag{1-7}$$

Note that inequality (1-6) is the best one in the literature, and we hope that it can be extended to time scales.

Hinton and Lewis [1975] extended the inequality (1-1) and proved by using the Schwarz inequality that

$$\int_a^b \frac{M^2(t)}{|M'(t)|} (y'(t))^2 dt \ge \frac{1}{4} \int_a^b |M'(t)| y^2(t)\,dt \tag{1-8}$$

for any positive $M \in \mathrm{C}^1([a,b])$ with $M'(t) \neq 0$, $y \in \mathrm{C}^2([a,b])$, and $y(a) = y(b) = 0$. Peña [1999] established the discrete analogue of (1-8) and proved the following result: For a positive sequence $\{M_n\}_{0 \le n \le N+1}$ satisfying either $\Delta M > 0$, or $\Delta M < 0$ on $[0, N] \cap \mathbb{Z}$,

$$\sum_{n=0}^{N} \frac{M_n M_{n+1}}{|\Delta M_n|} (\Delta y_n)^2 \ge \frac{1}{\psi} \sum_{n=0}^{N} |\Delta M_n| y_{n+1}^2 \tag{1-9}$$

holds for any sequence $\{y_n\}_{0 \le n \le N+1}$ with $y_0 = y_{N+1} = 0$, where

$$\psi = \left(\sup_{0 \le n \le N} \frac{M_n}{M_{n+1}}\right) \left[1 + \left(\sup_{0 \le n \le N} \frac{|\Delta M_n|}{|\Delta M_{n+1}|}\right)^{1/2}\right]^2.$$

Hilscher [2002] proved a Wirtinger-type inequality on time scales, which gives a unification of (1-8) and (1-9); see also [Agarwal et al. 2008]. Hilscher proved that

$$\int_a^b \frac{M(t) M(\sigma(t))}{|M^{\Delta}(t)|} (y^{\Delta}(t))^2 \Delta t \ge \frac{1}{\psi^2} \int_a^b |M^{\Delta}(t)| y^2(\sigma(t)) \Delta t \tag{1-10}$$

holds for a positive function $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ with either $M^{\Delta} > 0$ or $M^{\Delta} < 0$ on $\mathcal{I}^{\kappa}$, $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ with $y(a) = y(b) = 0$, $\mathcal{I} = [a,b]_{\mathbb{T}} \subset \mathbb{T}$, and

$$\psi = \left(\sup_{t \in \mathcal{I}^{\kappa}} \frac{M(t)}{M(\sigma(t))}\right)^{1/2} + \left[\left(\sup_{t \in \mathcal{I}^{\kappa}} \frac{\mu(t)|M^{\Delta}(t)|}{M(\sigma(t))}\right) + \left(\sup_{t \in \mathcal{I}^{\kappa}} \frac{M(t)}{M(\sigma(t))}\right)\right]^{1/2}.$$

Our aim in Sections 3 and 4 is to derive some new Wirtinger-type inequalities on time scales, which are different from the inequality (1-10), where we will use $y^{\gamma+1}$ instead of the term $(y^\sigma)^{\gamma+1}$ (due to the fact that y^σ is not differentiable on an arbitrary time scale). In particular, we obtain (1-8) as a consequence of our results. In Sections 5 and 6, we obtain, as special cases, new Wirtinger-type inequalities in the continuous and the discrete case, respectively. In Section 7, we apply our results to a certain class of half-linear dynamic equations on time scales in order to derive sufficient conditions for nonoscillation.

2. Preliminaries

Before we state and prove the main results, for completeness, we recall the following concepts related to the notion of time scales. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points, and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $\mathrm{C_{rd}}(\mathbb{T})$. Rd-continuous functions possess antiderivatives. (For the definition of derivative, refer to [Bohner and Peterson 2001; 2003]; here we only note that $f^\Delta = f'$ if $\mathbb{T} = \mathbb{R}$ and $f^\Delta = \Delta f$ if $\mathbb{T} = \mathbb{Z}$.) Integrals are defined in terms of antiderivatives (so integrals are usual integrals for $\mathbb{T} = \mathbb{R}$ and sums for $\mathbb{T} = \mathbb{Z}$). We define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$, the notation f^σ is short for $f \circ \sigma$, where $\sigma : \mathbb{T} \to \mathbb{R}$ is the forward jump operator defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g:

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \quad \text{and} \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

The integration by parts formula is given by

$$\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t.$$

To prove the main result, we will use the following lemma.

Lemma 2.1 [Bohner and Peterson 2001, Theorem 6.13, Hölder's Inequality]. *Let $a, b \in \mathbb{T}$. For two functions $f, g \in \mathrm{C_{rd}}([a, b], \mathbb{R})$, we have*

$$\int_a^b |f(t)g(t)|\Delta t \le \left[\int_a^b |f(t)|^p \Delta t\right]^{1/p} \left[\int_a^b |g(t)|^q \Delta t\right]^{1/q},$$

where $p > 1$ and $1/p + 1/q = 1$.

3. Main result

Throughout this paper, we assume that the function $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathscr{I})$ is positive, where $\mathscr{I} = [a, b]_{\mathbb{T}}$, and define

$$\alpha := \sup_{t\in\mathscr{I}^\kappa}\left(\frac{M(\sigma(t))}{M(t)}\right)^{\frac{\gamma}{\gamma+1}} \quad \text{and} \quad \beta := \sup_{t\in\mathscr{I}^\kappa}\left(\frac{\mu(t)|M^\Delta(t)|}{M(t)}\right)^{\gamma}. \tag{3-1}$$

Now, we are in a position to state and prove the main result.

Theorem 3.1. *Suppose $\gamma \geq 1$ is an odd integer. For a positive $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathscr{I})$ satisfying either $M^\Delta > 0$, or $M^\Delta < 0$ on $\mathscr{I}^\kappa$, we have*

$$\int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geq \frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b |M^\Delta(t)|y^{\gamma+1}(t)\Delta t \tag{3-2}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathscr{I})$ with $y(a) = y(b) = 0$, where $\Psi(\alpha, \beta, \gamma)$ is the largest root of

$$x^{\gamma+1} - 2^{\gamma-1}(\gamma+1)\alpha x^\gamma - 2^{\gamma-1}\beta = 0. \tag{3-3}$$

Proof. Let y and M be defined as above, and denote

$$A := \int_a^b |M^\Delta(t)|y^{\gamma+1}(t)\Delta t \quad \text{and} \quad B := \int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t.$$

Using the integration by parts formula and the fact that $y(a) = y(b) = 0$, we have

$$\begin{aligned} A &= \int_a^b |M^\Delta(t)|y^{\gamma+1}(t)\Delta t \\ &= \{\mathrm{sgn}(M^\Delta(a))\}\int_a^b M^\Delta(t)y^{\gamma+1}(t)\Delta t \\ &= \{\mathrm{sgn}(M^\Delta(a))\}\left\{[M(t)y^{\gamma+1}(t)]_a^b - \int_a^b M(\sigma(t))(y^{\gamma+1})^\Delta(t)\Delta t\right\} \\ &= -\{\mathrm{sgn}(M^\Delta(a))\}\int_a^b M(\sigma(t))(y^{\gamma+1})^\Delta(t)\Delta t \\ &\leq \int_a^b M(\sigma(t))|(y^{\gamma+1})^\Delta(t)|\Delta t. \end{aligned} \tag{3-4}$$

By the Pötzsche chain rule [Bohner and Peterson 2001, Theorem 1.90] we obtain

$$\begin{aligned} |(y^{\gamma+1})^\Delta(t)| &= (\gamma+1)\left|\int_0^1 [y(t) + \mu(t)hy^\Delta(t)]^\gamma dh\right||y^\Delta(t)| \\ &\leq (\gamma+1)|y^\Delta(t)|\int_0^1 |y(t)+\mu(t)hy^\Delta(t)|^\gamma dh. \end{aligned} \tag{3-5}$$

Applying the inequality [Mitrinović et al. 1993, page 500]

$$|u+v|^\gamma \le 2^{\gamma-1}(|u|^\gamma + |v|^\gamma), \quad \text{for } u, v \in \mathbb{R},$$

with $u = y(t)$ and $v = \mu(t)hy^\Delta(t)$, we have from (3-5) that

$$\begin{aligned} |(y^{\gamma+1})^\Delta(t)| &\le (\gamma+1)|y^\Delta(t)| \int_0^1 |y(t)+\mu(t)hy^\Delta(t)|^\gamma\, dh \\ &\le 2^{\gamma-1}(\gamma+1)|y^\Delta(t)| \left\{ \int_0^1 |y(t)|^\gamma\, dh + \int_0^1 |\mu(t)hy^\Delta(t)|^\gamma\, dh \right\} \\ &= 2^{\gamma-1}(\gamma+1)|y^\Delta(t)||y(t)|^\gamma + 2^{\gamma-1}|y^\Delta(t)||\mu(t)y^\Delta(t)|^\gamma. \end{aligned} \tag{3-6}$$

Substituting (3-6) into (3-4), we have

$$\begin{aligned} A &\le 2^{\gamma-1}(\gamma+1)\int_a^b M(\sigma(t))|y^\Delta(t)||y(t)|^\gamma\,\Delta t \\ &\qquad + 2^{\gamma-1}\int_a^b M(\sigma(t))\mu^\gamma(t)|y^\Delta(t)|^{\gamma+1}\,\Delta t \\ &= 2^{\gamma-1}(\gamma+1)\int_a^b \left\{ \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right)^{\frac{1}{\gamma+1}} \left(\frac{M^\sigma|M^\Delta|}{M}|y|^{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}} \right\}(t)\,\Delta t \\ &\qquad + 2^{\gamma-1}\int_a^b \left\{ \left(\frac{\mu|M^\Delta|}{M}\right)^\gamma \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right) \right\}(t)\,\Delta t. \end{aligned} \tag{3-7}$$

Applying the Hölder inequality (Lemma 2.1) with $p = \gamma+1,\ q = \dfrac{\gamma+1}{\gamma}$,

$$f = \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right)^{\frac{1}{\gamma+1}} \quad \text{and} \quad g = \left(\frac{M^\sigma|M^\Delta|}{M}|y|^{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}},$$

we obtain from (3-7) the upper bound for A as

$$\begin{aligned} &2^{\gamma-1}(\gamma+1)\left\{\int_a^b \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right)(t)\,\Delta t\right\}^{\frac{1}{\gamma+1}} \left\{\int_a^b \left(\frac{M^\sigma|M^\Delta|}{M}|y|^{\gamma+1}\right)(t)\,\Delta t\right\}^{\frac{\gamma}{\gamma+1}} \\ &\qquad + 2^{\gamma-1}\int_a^b \left\{ \left(\frac{\mu|M^\Delta|}{M}\right)^\gamma \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right) \right\}(t)\,\Delta t \\ &\le 2^{\gamma-1}(\gamma+1)\alpha\left\{\int_a^b \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right)(t)\,\Delta t\right\}^{\frac{1}{\gamma+1}} \left\{\int_a^b (|M^\Delta||y|^{\gamma+1})(t)\,\Delta t\right\}^{\frac{\gamma}{\gamma+1}} \\ &\qquad + 2^{\gamma-1}\beta\int_a^b \left(\frac{M^\sigma M^\gamma}{|M^\Delta|^\gamma}|y^\Delta|^{\gamma+1}\right)(t)\,\Delta t, \end{aligned}$$

that is,

$$A \le 2^{\gamma-1}\big((\gamma+1)\alpha B^{1/(\gamma+1)}A^{\gamma/(\gamma+1)} + \beta B\big). \tag{3-8}$$

Dividing both sides of (3-8) by $B^{1/(\gamma+1)}A^{\gamma/(\gamma+1)}$, we obtain

$$\frac{A^{1/(\gamma+1)}}{B^{1/(\gamma+1)}} \le 2^{\gamma-1}\Big((\gamma+1)\alpha + \beta\frac{B^{\gamma/(\gamma+1)}}{A^{\gamma/(\gamma+1)}}\Big),$$

and putting

$$C := \frac{A^{1/(\gamma+1)}}{B^{1/(\gamma+1)}} > 0,$$

we get

$$C \le 2^{\gamma-1}\Big((\gamma+1)\alpha + \frac{\beta}{C^\gamma}\Big),$$

which gives

$$C^{\gamma+1} - 2^{\gamma-1}(\gamma+1)\alpha C^\gamma - 2^{\gamma-1}\beta \le 0, \quad \text{with } C = \frac{A^{1/(\gamma+1)}}{B^{1/(\gamma+1)}}. \tag{3-9}$$

By (3-9), we have

$$C \le \Psi(\alpha, \beta, \gamma),$$

where $\Psi(\alpha, \beta, \gamma)$ is the largest root of Equation (3-3), and thus, replacing C in terms of A and B as in (3-9), we have

$$A^{1/(\gamma+1)} \le \Psi(\alpha, \beta, \gamma)B^{1/(\gamma+1)},$$

that is,

$$A \le \Psi^{\gamma+1}(\alpha, \beta, \gamma)B,$$

which is the desired inequality (3-2). □

Using Theorem 3.1 with $\gamma = 1$, we obtain the following result.

Corollary 3.2. *For a positive function $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ satisfying either $M^\Delta > 0$, or $M^\Delta < 0$ on $\mathcal{I}^\kappa$, we have*

$$\int_a^b \frac{M(t)M(\sigma(t))}{|M^\Delta(t)|}(y^\Delta(t))^2\Delta t \ge \frac{1}{\varphi^2}\int_a^b |M^\Delta(t)|y^2(t)\Delta t \tag{3-10}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ with $y(a) = y(b) = 0$, where

$$\varphi := \sqrt{\sup_{t\in\mathcal{I}^\kappa}\frac{M(\sigma(t))}{M(t)}} + \sqrt{\sup_{t\in\mathcal{I}^\kappa}\frac{M(\sigma(t))}{M(t)} + \sup_{t\in\mathcal{I}^\kappa}\frac{\mu(t)|M^\Delta(t)|}{M(t)}}.$$

Proof. When $\gamma = 1$, we see that the largest root $\Psi(\alpha, \beta, 1)$ of the quadratic equation $x^2 - 2x\alpha - \beta = 0$ is given by

$$\Psi(\alpha, \beta, 1) = \alpha + \sqrt{\alpha^2 + \beta} = \varphi.$$

Therefore, (3-10) follows from Theorem 3.1. □

Example 3.3. Note that in the case where $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$, $\mu(t)=0$, and $y^{\Delta}=y'$. Then $\varphi=2$ in Corollary 3.2, and in this case, (3-10) becomes the classical Wirtinger-type inequality (1-8) obtained by Hinton and Lewis.

In the case where $\mathbb{T}=\mathbb{Z}$ and $\mathcal{I}=[0,N+1]\cap\mathbb{Z}$, Corollary 3.2 gives a new result, which is stated as follows.

Corollary 3.4. *For any positive sequence $\{M_n\}_{0\le n\le N+1}$ satisfying either $\Delta M>0$, or $\Delta M<0$ on $[0,N]\cap\mathbb{Z}$, we have*

$$\sum_{n=0}^{N}\frac{M_nM_{n+1}}{|\Delta M_n|}(\Delta y_n)^2\geq\frac{1}{\psi^2}\sum_{n=0}^{N}|\Delta M_n|y_n^2$$

for any sequence $\{y_n\}_{0\le n\le N+1}$ with $y_0=y_{N+1}=0$, where

$$\psi:=\sqrt{\sup_{0\le n\le N}\frac{M_{n+1}}{M_n}}+\sqrt{\sup_{0\le n\le N}\frac{M_{n+1}}{M_n}+\sup_{0\le n\le N}\frac{|\Delta M_n|}{M_n}}.$$

Proof. When $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$, $\mu(t)=1$, and $y^{\Delta}=\Delta y$. Then $\varphi=\psi$ in Corollary 3.2, and the claim follows by using this in Corollary 3.2. □

4. Further dynamic Wirtinger inequalities

We now apply different algebraic inequalities to establish some new Wirtinger-type inequalities. We use the inequality [Mitrinović et al. 1993, page 518]

$$(u+v)^{\gamma+1}\leq u^{\gamma+1}+(\gamma+1)u^{\gamma}v+L_{\gamma}vu^{\gamma},\quad \text{where } u,v>0. \tag{4-1}$$

Note that $L_{\gamma}>1$ in (4-1) is a constant which does not depend on u or v.

Theorem 4.1. *Suppose $\gamma\geq 1$ is an odd integer. For a positive $M\in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ satisfying either $M^{\Delta}>0$, or $M^{\Delta}<0$ on $\mathcal{I}^{\kappa}$, we have*

$$\int_a^b\frac{M^{\gamma}(t)M(\sigma(t))}{|M^{\Delta}(t)|^{\gamma}}(y^{\Delta}(t))^{\gamma+1}\Delta t\geq\frac{1}{\Phi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b|M^{\Delta}(t)|y^{\gamma+1}(t)\Delta t \tag{4-2}$$

for any $y\in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ with $y(a)=y(b)=0$, where $\Phi(\alpha,\beta,\gamma)$ is the largest root of

$$(\gamma+1)x+\frac{(\gamma+1)\gamma}{2}x^2-(\gamma+1)x^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)x^{\gamma}-2^{\gamma-1}\beta=0. \tag{4-3}$$

Proof. We proceed exactly as in the proof of Theorem 3.1 to get (3-9), that is,

$$C^{\gamma+1}\leq 2^{\gamma-1}(\gamma+1)\alpha C^{\gamma}+2^{\gamma-1}\beta,\quad \text{with } C=\frac{A^{1/(\gamma+1)}}{B^{1/(\gamma+1)}}. \tag{4-4}$$

Applying the inequality (4-1) with $u=C$, $v=1$, and $L_{\gamma}=2^{\gamma-1}(\gamma+1)>1$, we have

$$C^{\gamma+1}\geq(C+1)^{\gamma+1}-(\gamma+1)C^{\gamma}-2^{\gamma-1}(\gamma+1)C^{\gamma},$$

which together with (4-4) implies that

$$(C+1)^{\gamma+1}-(\gamma+1)C^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)C^{\gamma}-2^{\gamma-1}\beta\leq 0. \tag{4-5}$$

Using the inequality (which follows easily from the binomial formula)

$$(1+u)^{\gamma+1}\geq(\gamma+1)u+\frac{(\gamma+1)\gamma}{2}u^2, \quad \text{where } u>0, \tag{4-6}$$

for $u=C$, we have from (4-5) that

$$(\gamma+1)C+\frac{(\gamma+1)\gamma}{2}C^2-(\gamma+1)C^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)C^{\gamma}-2^{\gamma-1}\beta\leq 0.$$

Therefore,

$$C\leq\Phi(\alpha,\beta,\gamma),$$

where $\Phi(\alpha,\beta,\gamma)$ is the largest root of Equation (4-3), and thus, replacing C in terms of A and B as in (4-4), we have

$$A^{1/(\gamma+1)}\leq\Phi(\alpha,\beta,\gamma)B^{1/(\gamma+1)},$$

that is,

$$B\geq\frac{1}{\Phi^{\gamma+1}(\alpha,\beta,\gamma)}A,$$

which is the desired inequality (4-2). □

Using the inequality

$$(1+u)^{\gamma+1}\geq\frac{(\gamma+1)\gamma}{2}u^2, \quad \text{where } u>0,$$

instead of the inequality (4-6) that was used in the proof of Theorem 4.1, we have the following new result.

Theorem 4.2. *Suppose $\gamma\geq 1$ is an odd integer. For a positive $M\in C^1_{\mathrm{rd}}(\mathcal{J})$ satisfying either $M^{\Delta}>0$, or $M^{\Delta}<0$ on $\mathcal{J}^{\kappa}$, we have*

$$\int_a^b\frac{M^{\gamma}(t)M(\sigma(t))}{|M^{\Delta}(t)|^{\gamma}}(y^{\Delta}(t))^{\gamma+1}\Delta t\;\geq\;\frac{1}{\Omega^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b|M^{\Delta}(t)|y^{\gamma+1}(t)\Delta t \tag{4-7}$$

for any $y\in C^1_{\mathrm{rd}}(\mathcal{J})$ with $y(a)=y(b)=0$, where $\Omega(\alpha,\beta,\gamma)$ is the largest root of

$$\frac{(\gamma+1)\gamma}{2}x^2-(\gamma+1)x^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)x^{\gamma}-2^{\gamma-1}\beta=0. \tag{4-8}$$

In the following, we apply the inequality [Hardy et al. 1988, Theorem 41]

$$u^{\gamma+1}+\gamma v^{\gamma+1}-(\gamma+1)uv^{\gamma}\geq 0, \quad \text{where } u,v\geq 0, \tag{4-9}$$

and derive a new Wirtinger-type inequality on time scales.

Theorem 4.3. *Suppose $\gamma \geq 1$ is an odd integer. For a positive $M \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ satisfying either $M^\Delta > 0$, or $M^\Delta < 0$ on $\mathcal{I}^\kappa$, we have*

$$\int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \geq \frac{1}{\Lambda^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b |M^\Delta(t)|y^{\gamma+1}(t)\Delta t \tag{4-10}$$

for any $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I})$ with $y(a) = y(b) = 0$, where $\Lambda(\alpha, \beta, \gamma)$ is the largest root of

$$(\gamma+1)(x+1) - \gamma - (\gamma+1)x^\gamma - 2^{\gamma-1}(\gamma+1)(1+\alpha)x^\gamma - 2^{\gamma-1}\beta = 0. \tag{4-11}$$

Proof. We proceed exactly as in the proof of Theorem 4.1 to get (4-5), that is,

$$(C+1)^{\gamma+1} - (\gamma+1)C^\gamma - 2^{\gamma-1}(\gamma+1)C^\gamma \leq 2^{\gamma-1}(\gamma+1)\alpha C^\gamma + 2^{\gamma-1}\beta. \tag{4-12}$$

Applying the inequality (4-9) with $u = C+1$ and $v = 1$, we have from (4-12) that

$$(\gamma+1)(C+1) - \gamma - (\gamma+1)C^\gamma - 2^{\gamma-1}(\gamma+1)(1+\alpha)C^\gamma - 2^{\gamma-1}\beta \leq 0.$$

Therefore $C \leq \Lambda(\alpha, \beta, \gamma)$, where $\Lambda(\alpha, \beta, \gamma)$ is the largest root of (4-11). This implies, from the definition of C, that

$$A^{1/(\gamma+1)} \leq \Lambda(\alpha, \beta, \gamma)B^{1/(\gamma+1)}.$$

Thus $A \leq \Lambda^{\gamma+1}(\alpha, \beta, \gamma)B$, which is the desired inequality (4-10). □

5. Applications: Continuous Wirtinger inequalities

As special cases of the time scales results presented in Sections 3 and 4, we now give some new Wirtinger-type inequalities in the continuous case. Note that in the case where $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$ and $\mu(t) = 0$, so that

$$\alpha = 1 \quad \text{and} \quad \beta = 0. \tag{5-1}$$

Then we have, from Theorems 3.1, 4.1, 4.2, and 4.3, the following results.

Theorem 5.1. *Let $\gamma \geq 1$ be an odd integer. For a positive $M \in \mathrm{C}^1([a,b])$ satisfying either $M' > 0$, or $M' < 0$ on $[a,b]$, we have*

$$\int_a^b \frac{M^{\gamma+1}(t)}{|M'(t)|^\gamma}(y'(t))^{\gamma+1}dt \geq \frac{1}{2^{\gamma^2-1}(\gamma+1)^{\gamma+1}}\int_a^b |M'(t)|y^{\gamma+1}(t)\,dt$$

for any $y \in \mathrm{C}^1([a,b])$ with $y(a) = y(b) = 0$.

Proof. Using (5-1), Equation (3-3) becomes

$$x^{\gamma+1} - 2^{\gamma-1}(\gamma+1)x^{\gamma} = 0.$$

Hence, $\Psi(1, 0, \gamma)$ from Theorem 3.1 is given by

$$\Psi(1, 0, \gamma) = 2^{\gamma-1}(\gamma+1),$$

and the proof is completed by applying Theorem 3.1. □

Theorem 5.2. *Let $\gamma \geq 1$ be an odd integer. For a positive $M \in \mathrm{C}^1([a, b])$ satisfying either $M' > 0$, or $M' < 0$ on $[a, b]$, we have*

$$\int_a^b \frac{M^{\gamma+1}(t)}{|M'(t)|^{\gamma}}(y'(t))^{\gamma+1}dt \geq \frac{1}{\Phi^{\gamma+1}(1, 0, \gamma)} \int_a^b |M'(t)|y^{\gamma+1}(t)\,dt$$

for any $y \in \mathrm{C}^1([a, b])$ with $y(a) = y(b) = 0$, where $\Phi(1, 0, \gamma)$ is the largest root of

$$(\gamma+1)x + \frac{(\gamma+1)\gamma}{2}x^2 - (\gamma+1)x^{\gamma} - 2^{\gamma}(\gamma+1)x^{\gamma} = 0.$$

Proof. The claim follows from (5-1) and Theorem 4.1 with $\mathbb{T} = \mathbb{R}$. □

Theorem 5.3. *Let $\gamma \geq 1$ be an odd integer. For a positive $M \in \mathrm{C}^1([a, b])$ satisfying either $M' > 0$, or $M' < 0$ on $[a, b]$, we have*

$$\int_a^b \frac{M^{\gamma+1}(t)}{|M'(t)|^{\gamma}}(y'(t))^{\gamma+1}dt \geq \frac{1}{\Omega^{\gamma+1}(1, 0, \gamma)} \int_a^b |M'(t)|y^{\gamma+1}(t)\,dt$$

for any $y \in \mathrm{C}^1([a, b])$ with $y(a) = y(b) = 0$, where $\Omega(1, 0, \gamma)$ is the largest root of

$$\frac{(\gamma+1)\gamma}{2}x^2 - (\gamma+1)x^{\gamma} - 2^{\gamma}(\gamma+1)x^{\gamma} = 0.$$

Proof. The claim follows from (5-1) and Theorem 4.2 with $\mathbb{T} = \mathbb{R}$. □

Theorem 5.4. *Let $\gamma \geq 1$ be an odd integer. For a positive $M \in \mathrm{C}^1([a, b])$ satisfying either $M' > 0$ or $M' < 0$, on $[a, b]$, we have*

$$\int_a^b \frac{M^{\gamma+1}(t)}{|M'(t)|^{\gamma}}(y'(t))^{\gamma+1}dt \geq \frac{1}{\Lambda^{\gamma+1}(1, 0, \gamma)} \int_a^b |M'(t)|y^{\gamma+1}(t)dt$$

for any $y \in \mathrm{C}^1([a, b])$ with $y(a) = y(b) = 0$, where $\Lambda(1, 0, \gamma)$ is the largest root of

$$(\gamma+1)(x+1) - \gamma - (\gamma+1)x^{\gamma} - 2^{\gamma}(\gamma+1)x^{\gamma} = 0.$$

Proof. The claim follows from (5-1) and Theorem 4.3 with $\mathbb{T} = \mathbb{R}$. □

6. Applications: Discrete Wirtinger inequalities

As special cases of the time scales results presented in Sections 3 and 4, we now give some new Wirtinger-type inequalities in the discrete case. Note that in the case where $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1$ and $\mu(t)=1$, so that

$$\alpha=\sup_{0\le n\le N}\left(\frac{M_{n+1}}{M_n}\right)^{\frac{\gamma}{\gamma+1}} \quad \text{and} \quad \beta=\sup_{0\le n\le N}\left(\frac{|\Delta M_n|}{M_n}\right)^{\gamma}. \tag{6-1}$$

Then we have, from Theorems 3.1, 4.1, 4.2, and 4.3, the following results.

Theorem 6.1. *Let $\gamma\ge 1$ be an odd integer. For a positive sequence $\{M_n\}_{0\le n\le N+1}$ satisfying either $\Delta M>0$, or $\Delta M<0$ on $[0,N]\cap\mathbb{Z}$, we have*

$$\sum_{n=0}^{N}\frac{M_n^{\gamma}M_{n+1}}{|\Delta M_n|^{\gamma}}(\Delta y_n)^{\gamma+1}\ge\frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\sum_{n=0}^{N}|\Delta M_n|y_n^{\gamma+1}$$

for any sequence $\{y_n\}_{0\le n\le N+1}$ with $y_0=y_{N+1}=0$, where $\Psi(\alpha,\beta,\gamma)$ is the largest root of

$$x^{\gamma+1}-2^{\gamma-1}(\gamma+1)\alpha x^{\gamma}-2^{\gamma-1}\beta=0.$$

Theorem 6.2. *Let $\gamma\ge 1$ be an odd integer. For a positive sequence $\{M_n\}_{0\le n\le N+1}$ satisfying either $\Delta M>0$ or, $\Delta M<0$ on $[0,N]\cap\mathbb{Z}$, we have*

$$\sum_{n=0}^{N}\frac{M_n^{\gamma}M_{n+1}}{|\Delta M_n|^{\gamma}}(\Delta y_n)^{\gamma+1}\ge\frac{1}{\Phi^{\gamma+1}(\alpha,\beta,\gamma)}\sum_{n=0}^{N}|\Delta M_n|y_n^{\gamma+1}$$

for any sequence $\{y_n\}_{0\le n\le N+1}$ with $y_0=y_{N+1}=0$, where $\Phi(\alpha,\beta,\gamma)$ is the largest root of

$$(\gamma+1)x+\frac{(\gamma+1)\gamma}{2}x^2-(\gamma+1)x^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)x^{\gamma}-2^{\gamma-1}\beta=0.$$

Theorem 6.3. *Let $\gamma\ge 1$ be an odd integer. For a positive sequence $\{M_n\}_{0\le n\le N+1}$ satisfying either $\Delta M>0$, or $\Delta M<0$ on $[0,N]\cap\mathbb{Z}$, we have*

$$\sum_{n=0}^{N}\frac{M_n^{\gamma}M_{n+1}}{|\Delta M_n|^{\gamma}}(\Delta y_n)^{\gamma+1}\ge\frac{1}{\Omega^{\gamma+1}(\alpha,\beta,\gamma)}\sum_{n=0}^{N}|\Delta M_n|y_n^{\gamma+1}$$

for any sequence $\{y_n\}_{0\le n\le N+1}$ with $y_0=y_{N+1}=0$, where $\Omega(\alpha,\beta,\gamma)$ is the largest root of

$$\frac{(\gamma+1)\gamma}{2}x^2-(\gamma+1)x^{\gamma}-2^{\gamma-1}(\gamma+1)(1+\alpha)x^{\gamma}-2^{\gamma-1}\beta=0.$$

Theorem 6.4. *Let $\gamma \geq 1$ be an odd integer. For a positive sequence $\{M_n\}_{0\leq n\leq N+1}$ satisfying either $\Delta M > 0$, or $\Delta M < 0$ on $[0, N] \cap \mathbb{Z}$, we have*

$$\sum_{n=0}^{N} \frac{M_n^{\gamma} M_{n+1}}{|\Delta M_n|^{\gamma}} (\Delta y_n)^{\gamma+1} \geq \frac{1}{\Lambda^{\gamma+1}(\alpha, \beta, \gamma)} \sum_{n=0}^{N} |\Delta M_n| y_n^{\gamma+1}$$

for any sequence $\{y_n\}_{0\leq n\leq N+1}$ with $y_0 = y_{N+1} = 0$, where $\Lambda(\alpha, \beta, \gamma)$ is the largest root of

$$(\gamma+1)(x+1) - \gamma - (\gamma+1)x^{\gamma} - 2^{\gamma-1}(\gamma+1)(1+\alpha)x^{\gamma} - 2^{\gamma-1}\beta = 0.$$

Remark 6.5. Notice that the results can also be applied to other discrete time scales such as $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, $\mathbb{T} = \mathbb{N}^2$, $\mathbb{T} = \sqrt{\mathbb{N}_0}$, $\mathbb{T} = \sqrt[3]{\mathbb{N}_0}$, etc., where the integration on discrete time scales is given by

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)\cap\mathbb{T}} \mu(t) f(t) \quad \text{for } a < b.$$

7. Applications: Half-linear dynamic equations

In this section, we apply the result from Section 3 to the half-linear dynamic equation

$$(r|y^{\Delta}|^{\gamma})^{\Delta}(t) + p(t)|y(t)|^{\gamma} = 0 \tag{7-1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma \geq 1$ is an odd positive integer, and r and p are real-valued rd-continuous functions defined on $\mathbb{T}$ with $r(t) \neq 0$ for all $t \in \mathbb{T}$. We will establish a sufficient condition for disconjugacy. We say that a solution y of (7-1) has a generalized zero at t if $y(t) = 0$. We say y has a generalized zero in $(t, \sigma(t))$ in case $r(t)y(t)y(\sigma(t)) < 0$. Equation (7-1) is called disconjugate on the interval $[a, b]_{\mathbb{T}}$ if there is no nontrivial solution of (7-1) with two (or more) generalized zeros in $[a, b]_{\mathbb{T}}$. Equation (7-1) is said to be nonoscillatory on $[a, \infty)_{\mathbb{T}}$ if there exists $c \in [a, \infty)_{\mathbb{T}}$, such that (7-1) is disconjugate on $[c, d]_{\mathbb{T}}$ for every $d > c$. In the opposite case, (7-1) is said to be oscillatory on $[a, \infty)_{\mathbb{T}}$. The oscillation of solutions of Equation (7-1) may equivalently be defined as follows: A nontrivial solution of (7-1) is called oscillatory if it has infinitely many (isolated) generalized zeros in $[a, \infty)_{\mathbb{T}}$; otherwise it is called nonoscillatory. Equation (7-1) is said to be oscillatory if all its solutions are oscillatory. By the Sturm separation theorem, we see that oscillation is an interval property, that is, if there exists a sequence of subintervals $[a_i, b_i]_{\mathbb{T}}$ of $[a, \infty)_{\mathbb{T}}$, as $i \to \infty$, such that for every i there exists a solution of (7-1) that has at least two generalized zeros in $[a_i, b_i]_{\mathbb{T}}$, then every solution of (7-1) is oscillatory in $[a, \infty)_{\mathbb{T}}$. Hence, we can speak about oscillation and nonoscillation of (7-1). We define a class $U = U(a, b)$ of so-called admissible functions by

$$U(a,b) := \{y \in \mathrm{C}^1_{\mathrm{rd}}(\mathcal{I},\mathbb{R}) :\ y(a) = y(b) = 0\},$$

and define the functional $\mathcal{F}$ on $U(a,b)$ by

$$\mathcal{F}(y) := \int_a^b \left\{ r(t)|y^{\Delta}(t)|^{\gamma+1} - p(t)|y(t)|^{\gamma+1} \right\} \Delta t.$$

We say that $\mathcal{F}$ is positive definite on $U(a,b)$ provided $\mathcal{F}(y) \geq 0$ for all $y \in U(a,b)$, and $\mathcal{F}(y) = 0$ if and only if $y = 0$. Now we turn our attention to the roundabout theorem for (7-1); see [Agarwal et al. 2003, Theorem 5.1] or [Řehák 2002; 2005].

Theorem 7.1. *Suppose that the functions r and p are rd-continuous and $r(t) \neq 0$ for all $t \in \mathbb{T}$. Then Equation (7-1) is disconjugate on a time scale interval $[a,b]_{\mathbb{T}}$ if and only if the functional $\mathcal{F}$ is positive definite on $U(a,b)$.*

In view of Theorem 7.1, we will prove the disconjugacy of a certain class of half-linear dynamic equations (7-1) in terms of the positivity of the functional $\mathcal{F}$. In the following, we apply only Theorem 3.1 since the other theorems can be applied similarly. In what follows, we will assume that $\Psi(\alpha,\beta,\gamma)$ is the largest root of

$$x^{\gamma+1} - 2^{\gamma-1}(\gamma+1)\alpha x^{\gamma} - 2^{\gamma-1}\beta = 0.$$

We consider the following example as an application of Theorem 3.1.

Example 7.2. Let $\mathbb{T} \subset (0,\infty)$ be an arbitrary time scale. Consider the equation

$$\left((\sigma(t))^{\gamma-1}(y^{\Delta}(t))^{\gamma}\right)^{\Delta} + \frac{\lambda}{t\sigma(t)} y^{\gamma}(t) = 0, \quad t \in [a,b]_{\mathbb{T}}, \tag{7-2}$$

where $\lambda > 0$. Equation (7-2) is of the form (7-1) with

$$r(t) = (\sigma(t))^{\gamma-1} \quad \text{and} \quad p(t) = \frac{\lambda}{t\sigma(t)}.$$

We apply Theorem 3.1 with $M(t) = \dfrac{1}{t}$. We see by using the quotient rule from Section 2 that

$$\frac{M^{\gamma}(t)M(\sigma(t))}{|M^{\Delta}(t)|^{\gamma}} = \frac{(1/t^{\gamma})\cdot(1/\sigma(t))}{|-1/(t\sigma(t))|^{\gamma}} = (\sigma(t))^{\gamma-1} = r(t).$$

Now let $y \in U(a,b)$ be nontrivial. Then $y \in \mathrm{C}^1_{\mathrm{rd}}(\mathscr{I})$ and $y(a) = y(b) = 0$. By Theorem 3.1, we obtain

$$\begin{aligned}\int_a^b r(t)(y^\Delta(t))^{\gamma+1}\Delta t &= \int_a^b \frac{M^\gamma(t)M(\sigma(t))}{|M^\Delta(t)|^\gamma}(y^\Delta(t))^{\gamma+1}\Delta t \\ &\geq \frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b |M^\Delta(t)|y^{\gamma+1}(t)\Delta t \\ &= \frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b \frac{y^{\gamma+1}(t)}{t\sigma(t)}\Delta t \\ &= \frac{1}{\lambda\Psi^{\gamma+1}(\alpha,\beta,\gamma)}\int_a^b p(t)y^{\gamma+1}(t)\Delta t \\ &> \int_a^b p(t)y^{\gamma+1}(t)\Delta t,\end{aligned}$$

provided

$$0 < \lambda < \frac{1}{\Psi^{\gamma+1}(\alpha,\beta,\gamma)}. \tag{7-3}$$

Thus, assuming (7-3), $\mathscr{F}$ is positive definite on $U(a,b)$. Therefore, by Theorem 7.1, Equation (7-2) is disconjugate on $[a,b]_{\mathbb{T}}$ provided (7-3) holds.

Example 7.3. Let $\gamma = 1$ and $\mathbb{T} = \mathbb{R}$. In this case, Equation (7-2) becomes

$$y''(t) + \frac{\lambda}{t^2}y(t) = 0, \qquad t \in [a,b]. \tag{7-4}$$

From Example 7.2 (see also the calculations $\alpha = 1$, $\beta = 0$, $\Psi(1,0,1) = 2$ in Example 3.3), we see that if $\lambda < \frac{1}{4}$, then Equation (7-4) is disconjugate. In fact, if $\lambda = \frac{1}{5}$, then Equation (7-4) has a nonoscillatory solution $y(t) = t^{(\sqrt{5}-1)/(2\sqrt{5})}$ which satisfies $y(0) = 0$ and $\lim_{t\to\infty} M(t)y^2(t) = 0$.

Example 7.4. Let $\mathbb{T} = (0,\infty)$. Consider the second-order half-linear differential equation

$$((y')^\gamma)'(t) + \frac{\lambda}{t^{\gamma+1}}y^\gamma(t) = 0, \qquad t \in [a,b], \tag{7-5}$$

where $\lambda > 0$. Equation (7-5) is of the form (7-1) with

$$r(t) = 1 \quad \text{and} \quad p(t) = \frac{\lambda}{t^{\gamma+1}}.$$

We apply Theorem 5.1 with

$$M(t) = \frac{\gamma^\gamma}{t^\gamma}.$$

In this case, we see that

$$\frac{M^{\gamma+1}(t)}{|M'(t)|^{\gamma}} = \frac{(\gamma^{\gamma}/t^{\gamma})^{\gamma+1}}{|-\gamma^{\gamma+1}/t^{\gamma+1}|^{\gamma}} = 1 = r(t).$$

Now let $y \in U(a,b)$ be nontrivial. Then $y \in \mathrm{C}^1([a,b])$, and $y(a) = y(b) = 0$. By Theorem 5.1, we obtain

$$\begin{aligned}\int_a^b r(t)(y'(t))^{\gamma+1}dt &= \int_a^b \frac{M^{\gamma+1}(t)}{|M'(t)|^{\gamma}}(y'(t))^{\gamma+1}dt \\ &\geq \frac{1}{2^{\gamma^2-1}(\gamma+1)^{\gamma+1}}\int_a^b |M'(t)|y^{\gamma+1}(t)\,dt \\ &= \frac{1}{2^{\gamma^2-1}(\gamma+1)^{\gamma+1}}\int_a^b \frac{\gamma^{\gamma+1}y^{\gamma+1}(t)}{t^{\gamma+1}}\,dt \\ &= \frac{\gamma^{\gamma+1}}{\lambda 2^{\gamma^2-1}(\gamma+1)^{\gamma+1}}\int_a^b p(t)y^{\gamma+1}(t)\,dt \\ &> \int_a^b p(t)y^{\gamma+1}(t)dt,\end{aligned}$$

provided

$$0 < \lambda < \frac{\gamma^{\gamma+1}}{2^{\gamma^2-1}(\gamma+1)^{\gamma+1}}. \tag{7-6}$$

Thus, assuming (7-6), $\mathcal{F}$ is positive definite on $U(a,b)$. Therefore, by Theorem 7.1, Equation (7-5) is disconjugate on $[a,b]$ provided (7-6) holds. Note that the oscillation constant of (7-5) is $\gamma^{\gamma+1}/(\gamma+1)^{\gamma+1}$ [Saker 2010, page 223].

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