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Let $\mathbb{D}_m$ be the dihedral group of order $2m$ with $m = 4t$, $t \geq 3$. Given an algebraically closed field of characteristic 0, we classify all finite-dimensional Nichols algebras over $\mathbb{D}_m$ and all finite-dimensional pointed Hopf algebras whose group of group-likes is $\mathbb{D}_m$, by means of the lifting method. As a byproduct we obtain new examples of finite-dimensional pointed Hopf algebras.

Introduction

This paper is concerned with the classification of finite-dimensional Hopf algebras over an algebraically closed field $\Bbbk$ of characteristic 0. In particular, we study pointed Hopf algebras over dihedral groups $\mathbb{D}_m$, $m = 4t \geq 12$, using the lifting method, which leads to the study of finite-dimensional Nichols algebras in the category ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ of left Yetter–Drinfeld modules over $\mathbb{D}_m$. For more examples over $\mathbb{D}_p$, p an odd prime or 4, see [Andruskiewitsch and Graña 1999, Section 3.3].

Significant progress has been achieved in [Andruskiewitsch and Schneider 2010] in the case of pointed Hopf algebras with group-likes an abelian group. When the group of group-likes is not abelian, the solution is far from complete. Some hope is present in the lack of examples: in this situation, Nichols algebras tend to be infinite-dimensional; see, for example, [Andruskiewitsch and Zhang 2007; Andruskiewitsch and Fantino 2007b; Andruskiewitsch et al. 2011; 2009; Freyre et al. 2007; 2010]. Nevertheless, examples for which the Nichols algebras are finite-dimensional do exist. Over $\mathbb{S}_3$ and $\mathbb{S}_4$ these algebras were determined in [Andruskiewitsch et al. 2010b]. All of them arise from racks associated to a cocycle, and in [Andruskiewitsch et al. 2010b; García and García Iglesias 2011] the classification of pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ is completed, respectively.

Let G be a finite group and let A_0 be the group algebra of G. The main steps of the lifting method for the classification of all finite-dimensional pointed Hopf algebras with group G are as follows:

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(a) Determine all Yetter–Drinfeld modules V such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional.

(b) For such V, compute all Hopf algebras A such that $\operatorname{gr} A \simeq \mathfrak{B}(V) \# A_0$, the Radford–Majid product. We call A a *lifting* of $\mathfrak{B}(V)$ over G.

(c) Prove that any finite-dimensional pointed Hopf algebra with group G is generated by group-likes and skew-primitives.

Assume $G = \mathbb{D}_m$, $m = 4t$, $t \geq 3$. In Section 2 we complete step (a), that is, we determine all $V \in {}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ such that the Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional, and we describe explicitly these Nichols algebras. Then we prove steps (b) and (c) in Section 3, which are given by Theorem B and Theorem 3.2, respectively.

Summarizing, the main theorems are the following (for definitions see Definitions 2.6, 2.9 and 2.14).

Theorem A. *Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Then $\mathfrak{B}(M) \simeq \bigwedge M$, with M isomorphic either to M_I, M_L, or $M_{I,L}$, with $I \in \mathcal{I}$, $L \in \mathcal{L}$, or $(I, L) \in \mathcal{K}$, respectively.*

The proof of this theorem uses the classification of finite-dimensional Nichols algebras of diagonal type due to I Heckenberger [2009].

Although all finite-dimensional Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ turn out to be exterior algebras, which will be denoted by $\bigwedge M$, we write $\mathfrak{B}(M)$ to emphasize the Yetter–Drinfeld module structure. The following theorem gives all liftings of these families of Nichols algebras.

Theorem B. *Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{D}_m$. Then H is isomorphic to one of the following algebras*:

(a) $\mathfrak{B}(M_I) \# \Bbbk\mathbb{D}_m$ *with* $I = \{(i, k)\} \in \mathcal{I}$, $k \neq n$.

(b) $\mathfrak{B}(M_L) \# \Bbbk\mathbb{D}_m$ *with* $L \in \mathcal{L}$.

(c) $A_I(\lambda, \gamma)$ *with* $I \in \mathcal{I}$, $|I| > 1$ *or* $I = \{(i, n)\}$ *and* $\gamma \equiv 0$.

(d) $B_{I,L}(\lambda, \gamma, \theta, \mu)$ *with* $(I, L) \in \mathcal{K}$, $|I| > 0$ *and* $|L| > 0$.

Conversely, any Hopf algebra in this list is a lifting of a finite-dimensional Nichols algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$.

After the study of finite-dimensional pointed Hopf algebras over $\mathbb{S}_3 and \mathbb{S}_4$, this theorem is the first result that gives an infinite family of nonabelian groups where the classification of finite-dimensional pointed Hopf algebras with nontrivial examples is completed and, unlike the symmetric groups case, it provides for each dihedral group infinitely many nontrivial finite-dimensional pointed Hopf algebras.

The paper is organized as follows. In Section 1 we establish conventions and recall some basic facts about pointed Hopf algebras H such as the coradical filtration, the grading associated to it and the category ${}^{H_0}_{H_0}\mathcal{YD}$ of Yetter–Drinfeld modules over the coradical. If $G = G(H)$, the irreducible modules of ${}^{H_0}_{H_0}\mathcal{YD}$ are parameterized by pairs $(\mathbb{O}, \rho)$, where $\mathbb{O}$ is a conjugacy class of G and ρ is an irreducible representation of the centralizer of an element $\sigma \in \mathbb{O}$. At the end of this first section we recall the *type D* criterion [Andruskiewitsch et al. 2011, Theorem 3.6], which helps to determine when the Nichols algebra $\mathfrak{B}(\mathbb{O}, \rho)$ associated to $(\mathbb{O}, \rho)$ is infinite-dimensional, depending only on the rack structure of the conjugacy class $\mathbb{O}$. In Section 2 we work with Nichols algebras over the dihedral groups $\mathbb{D}_m$ with $m = 4t \geq 12$ and give the proof of Theorem A. We begin by determining which irreducible modules give rise to finite-dimensional Nichols algebras and then we extend our study to arbitrary modules. It turns out that all finite-dimensional Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ are exterior algebras of some irreducible modules or specific families of them. The last section of the paper is devoted to the classification of pointed Hopf algebras over $\mathbb{D}_m$, that is, to the proof of Theorem B. It consists mainly in the construction of the liftings of the finite-dimensional Nichols algebras given in Section 2. To do this, we show first in Theorem 3.2 that all finite-dimensional pointed Hopf algebras over $\mathbb{D}_m$, $m = 4t \geq 12$, are generated by group-likes and skew-primitive elements. Then we prove that if H is a pointed Hopf algebra over $\mathbb{D}_m$, some quadratic relations must hold and using these relations we define in Definitions 3.9 and 3.11 two families of quadratic algebras. Finally, using representation theory we prove that these algebras together with the bosonizations are all the possible liftings. We conclude the paper with the study of the isomorphism classes.

1. Preliminaries

1A. ***Conventions.*** We work over an algebraically closed field $\Bbbk$ of characteristic zero. Let H be a Hopf algebra over $\Bbbk$ with bijective antipode. We use Sweedler's [1969] notation $\Delta(h) = h_1 \otimes h_2$ for the comultiplication in H, but dropping the summation symbol.

The *coradical* H_0 of H is the sum of all simple subcoalgebras of H. In particular, if $G(H)$ denotes the group of *group-like elements* of H, we have $\Bbbk G(H) \subseteq H_0$. We say that a Hopf algebra is *pointed* if $H_0 = \Bbbk G(H)$. Denote by $\{H_i\}_{i \geq 0}$ the *coradical filtration* of H; if H_0 is a Hopf subalgebra of H, then $\operatorname{gr} H = \bigoplus_{n \geq 0} \operatorname{gr} H(n)$ is the associated graded Hopf algebra, with $\operatorname{gr} H(n) = H_n/H_{n-1}$ (set $H_{-1} = 0$). Letting $\pi : \operatorname{gr} H \to H_0$ be the homogeneous projection, $R = (\operatorname{gr} H)^{\operatorname{co}\pi}$ is the *diagram* of H; which is a braided Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of left Yetter–Drinfeld modules over H_0, and it is a graded subobject of $\operatorname{gr} H$. The linear space $R(1)$, with the braiding from ${}^{H_0}_{H_0}\mathcal{YD}$, is called the *infinitesimal braiding* of H and coincides with the subspace of primitive elements $P(R) = \{r \in R : \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$. It turns out that the Hopf algebra $\operatorname{gr} H$ is the Radford–Majid biproduct $\operatorname{gr} H \simeq R\#\Bbbk G(H)$ and the subalgebra of R generated by V is isomorphic to the Nichols algebra $\mathfrak{B}(V)$.

1B. ***Yetter–Drinfeld modules over $\Bbbk G$.*** Let G be a finite group. A left Yetter–Drinfeld module over $\Bbbk G$ is a left G-module and left $\Bbbk G$-comodule M such that

$$\delta(g\,.\,m) = ghg^{-1} \otimes g\,.\,m \quad \text{for all } m \in M_h,\ g, h \in G,$$

where $M_h = \{m \in M : \delta(m) = h \otimes m\}$; clearly, $M = \bigoplus_{h\in G} M_h$. The *support* of M is $\operatorname{supp} M = \{g \in G : M_g \neq 0\}$. Yetter–Drinfeld modules over G are completely reducible; see [Dijkgraaf et al. 1990; Cibils 1997, Andruskiewitsch and Graña 1999, Proposition 3.1.2]. Irreducible Yetter–Drinfeld modules over G are parameterized by pairs $(\mathbb{O}, \rho)$, where $\mathbb{O}$ is a conjugacy class and (ρ, V) is an irreducible representation of the centralizer $C_G(\sigma)$ of a fixed point $\sigma \in \mathbb{O}$. We denote the corresponding Yetter–Drinfeld module by $M(\mathbb{O}, \rho)$ and by $\mathfrak{B}(\mathbb{O}, \rho)$ its Nichols algebra.

Here is a precise description of the Yetter–Drinfeld module $M(\mathbb{O}, \rho)$. Let $\sigma_1 = \sigma, \dots, \sigma_n$ be a numeration of $\mathbb{O}$ and let $g_i \in G$ such that $g_i \sigma g_i^{-1} = \sigma_i$ for all $1 \le i \le n$. Then $M(\mathbb{O}, \rho) = \bigoplus_{1\le i\le n} g_i \otimes V$. Let $g_i v := g_i \otimes v$ be in $M(\mathbb{O}, \rho)$, $1 \le i \le n$, $v \in V$. If $v \in V$ and $1 \le i \le n$, then the action of $g \in G$ and the coaction are given by

$$g \cdot (g_i v) = g_j(\gamma \cdot v), \quad \delta(g_i v) = \sigma_i \otimes g_i v,$$

where $gg_i = g_j\gamma$ for some $1 \le j \le n$ and $\gamma \in C_G(\sigma)$. The explicit formula for the braiding is then given by

$$c(g_i v \otimes g_j w) = \sigma_i \cdot (g_j w) \otimes g_i v = g_h(\gamma \cdot v) \otimes g_i v \tag{1}$$

for any $1 \le i, j \le n$, $v, w \in V$, where $\sigma_i g_j = g_h \gamma$ for unique h, $1 \le h \le n$ and $\gamma \in C_G(\sigma)$. Since $\sigma \in Z(C_G(\sigma))$, Schur's Lemma says that σ acts by a scalar $q_{\sigma\sigma}$ on V.

The following are useful tools that, under certain conditions, allow us to determine if the dimension of a Nichols algebra is infinite. These results are about abelian and nonabelian subracks of a conjugacy class $\mathbb{O}$ of G, respectively.

Lemma 1.1 [Andruskiewitsch and Zhang 2007, Lemma 2.2]. *Let G be a finite group and $\mathbb{O}_\sigma$ be a conjugacy class in G. If $\mathbb{O}_\sigma$ is real* (*that is*, $\sigma^{-1} \in \mathbb{O}$) *and* $\dim \mathfrak{B}(\mathbb{O}_\sigma, \rho) < \infty$, *then $q_{\sigma\sigma} = -1$ and σ has even order.* □

We say that $\mathbb{O}$ is *of type D* if there exist $r, s \in \mathbb{O}$ such that $(rs)^2 \neq (sr)^2$ and r and s are not conjugate in some subgroup H of G containing r and s.

Lemma 1.2 [Andruskiewitsch et al. 2011, Theorem 3.6]. *If $\mathbb{O}$ is of type D, then $\mathfrak{B}(\mathbb{O}, \rho)$ is infinite-dimensional for all ρ.* □

Let A be a finite abelian group and $g \in \mathrm{Aut}(A)$. We denote by (A, g) the rack with underlying set A and rack multiplication $x \triangleright y := g(y) + (\mathrm{id} - g)(x)$, $x, y \in A$; this is a subrack of the group $A \rtimes \langle g \rangle$. Any rack isomorphic to some (A, g) is called *affine*.

For instance, consider the cyclic group $A = \mathbb{Z}/n$ and the automorphism g given by the inversion; the rack (A, g) is denoted $\mathcal{D}_n$ and called a dihedral rack. Thus, a family $(\mu_i)_{i \in \mathbb{Z}/n}$ of distinct elements of a rack X is isomorphic to $\mathcal{D}_n$ if $\mu_i \triangleright \mu_j = \mu_{2i-j}$ for all i, j.

Lemma 1.3 [Andruskiewitsch et al. 2010a, Lemma 2.1]. *If $m > 2$, then the dihedral rack $\mathcal{D}_{2m}$ is of type D.* □

2. Nichols algebras over $\mathbb{D}_m$ for $m = 4t \geq 12$

Let m be a positive integer, $m \geq 3$. The dihedral group of order $2m$ can be presented by generators and relations as

$$\mathbb{D}_m := \langle x, y : x^2 = 1 = y^m,\ xy = y^{-1}x \rangle.$$

From now on we assume that $m = 4t$ with $t \geq 3$ and set $n = m/2 = 2t$.

In this section we determine all finite-dimensional Nichols algebra over $\mathbb{D}_m$; see Theorem A and Table 2.

2A. ***Nichols algebras of irreducible Yetter–Drinfeld modules.*** The dimension of Nichols algebras of some irreducible Yetter–Drinfeld modules over $\mathbb{D}_m$ with m even was determined in [Andruskiewitsch and Fantino 2007a, Table 2]. Here we complete the study in the case $m = 4t \geq 12$, determining the dimension of the Nichols algebras of the irreducible Yetter–Drinfeld modules coming from the remaining two conjugacy classes $\mathcal{O}_x$ and $\mathcal{O}_{xy}$.

2A1. *The conjugacy class of y^n.* Since y^n is central, the conjugacy class and the centralizer of y^n in $\mathbb{D}_m$ are $\mathcal{O}_{y^n} = \{y^n\}$ and $C_{\mathbb{D}_m}(y^n) = \mathbb{D}_m$, respectively. The irreducible representations of $\mathbb{D}_m$ are well-known and they are of degree 1 or 2. Explicitly, there are:

(i) $n - 1 = (m-2)/2$ irreducible representations of degree 2. Letting ω be an m-th primitive root of 1, they are given by $\rho_\ell : \mathbb{D}_m \to \mathrm{GL}(2, \Bbbk)$,

$$\rho_\ell(x^a y^b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \begin{pmatrix} \omega^\ell & 0 \\ 0 & \omega^{-\ell} \end{pmatrix}^b, \quad 1 \leq \ell < n. \tag{2}$$

(ii) 4 irreducible representations of degree 1. They are given in Table 1.

Let ρ be an irreducible representation of $C_{\mathbb{D}_m}(y^n) = \mathbb{D}_m$. Since n is even, if $\deg \rho = 1$ or $\rho = \rho_\ell$ as in (2) with ℓ even, then $\dim \mathfrak{B}(\mathcal{O}_{y^n}, \rho) = \infty$. Indeed, here

σ	1	y^n	y^b, $1 \le b \le n-1$	x	xy
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^n$	$(-1)^b$	1	-1
χ_4	1	$(-1)^n$	$(-1)^b$	-1	1

Table 1. One-dimensional irreducible representations of $\mathbb{D}_m$ with $m = 2n$ even.

$q_{y^n y^n} \neq -1$ and Lemma 1.1 applies. In the cases when $\rho = \rho_\ell$ as in (2) with ℓ odd, we have that $q_{y^n y^n} = -1$. If $M_\ell = M(\mathcal{O}_{y^n}, \rho_\ell)$, then we have the following lemma, which is [Andruskiewitsch and Fantino 2007a, Theorem 3.1 (b) (i)].

Lemma 2.1. $\mathfrak{B}(\mathcal{O}_{y^n}, \rho_\ell) \simeq \bigwedge M_\ell$ *for all* ℓ *odd with* $1 \le \ell < n$. *In particular, the dimension* $\dim \mathfrak{B}(\mathcal{O}_{y^n}, \rho_\ell)$ *equals* 4. □

Notice that there are t irreducible Yetter–Drinfeld modules with support $\mathcal{O}_{y^n}$ such that its Nichols algebra is finite-dimensional.

2A2. *The conjugacy class of* y^i, $1 \le i \le n-1$. The conjugacy class and the centralizer of y^i in $\mathbb{D}_m$ are $\mathcal{O}_{y^i} = \{y^i, y^{-i}\}$ and $C_{\mathbb{D}_m}(y^i) = \langle y \rangle \simeq \mathbb{Z}/m$, respectively. The group of characters of $C_{\mathbb{D}_m}(y^i)$ is

$$\widehat{C_{\mathbb{D}_m}(y^i)} = \{\chi_{(k)} : 1 \le k \le m-1\},$$

where $\chi_{(k)}(y) := \omega^k$ with ω an m-th primitive root of 1. Let $M_{i,k} = M(y^i, \chi_{(k)})$. Since $\mathcal{O}_{y^i}$ is real, if $\chi_{(k)}(y^i) \neq -1$, then $\dim \mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)}) = \infty$, by Lemma 1.1. Assume that $\chi_{(k)}(y^i) = -1$; this amounts to saying there exists r with r odd and $1 \le r \le m-3$ such that $ik = rn$.

Let $\mathcal{N}_i = \{k : 0 \le k \le m-1,\ \chi_{(k)}(y^i) = -1\}$. Then, for every i with $1 \le i \le n-1$, there are $\operatorname{card} \mathcal{N}_i$ irreducible Yetter–Drinfeld modules with support $\mathcal{O}_{y^i}$ and dimension $\dim \mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)}) < \infty$. Let $\omega \in \Bbbk$ be a primitive m-th root of 1. We define $J = \{(i,k) : \omega^{ik} = -1, 1 \le i \le n-1, 1 \le k \le m-1\}$.

Remarks 2.2. Notice that if $(i, m) = 1$, then $\mathcal{N}_i = \{n\}$. Also,

- if $i = 2$, then $\mathcal{N}_2 = \{t, 3t\}$;
- if $i = 3$, then $\mathcal{N}_3 = \{n\}$ if $3 \nmid t$, whereas $\mathcal{N}_3 = \{2u, 6u, 10u\}$ if $t = 3u$;
- if $i = 4$, then $\mathcal{N}_4 = \varnothing$ if $2 \nmid t$, whereas $\mathcal{N}_4 = \{u, 3u, 5u, 7u\}$ if $t = 2u$.

The next lemma is [Andruskiewitsch and Fantino 2007a, Theorem 3.1 (b) (ii)].

Lemma 2.3. $\mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)}) \simeq \bigwedge M_{i,k}$ *for all* $(i,k) \in J$. *In particular, the dimension* $\dim \mathfrak{B}(\mathcal{O}_{y^i}, \chi_{(k)})$ *equals* 4. □

2A3. *The conjugacy classes of x and xy.* We show that these two conjugacy classes give rise to infinite-dimensional Nichols algebras.

Lemma 2.4. *The classes $\mathcal{O}_x$ and $\mathcal{O}_{xy}$ are of type D. Hence* $\dim \mathfrak{B}(\mathcal{O}_x, \rho)$ *and* $\dim \mathfrak{B}(\mathcal{O}_{xy}, \eta)$ *are infinite for all $\rho \in \widehat{C_{\mathbb{D}_m}(x)}$ and $\eta \in \widehat{C_{\mathbb{D}_m}(xy)}$.*

Proof. Since the classes $\mathcal{O}_x$ and $\mathcal{O}_{xy}$ are isomorphic as racks to the dihedral rack $\mathcal{D}_n$, the result follows from Lemma 1.3. □

Conjugacy class	Centralizer	Representation	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_m$	any	∞
$\mathcal{O}_{y^{m/2}} = \{y^{m/2}\}$, $\lvert\mathcal{O}_{y^{m/2}}\rvert = 1$	$\mathbb{D}_m$	$\chi_1, \chi_2, \chi_3, \chi_4$, ρ_ℓ, ℓ even	∞
		ρ_ℓ, ℓ odd	4
$\mathcal{O}_{y^i} = \{y^{\pm i}\}$, $i \neq 0, m/2$, $\lvert\mathcal{O}_{y^i}\rvert = 2$	$\mathbb{Z}/m \simeq \langle y \rangle$	$\chi_{(k)}$, $\omega_m^{ik} = -1$	4
		$\chi_{(k)}$, $\omega_m^{ik} \neq -1$	∞
$\mathcal{O}_x = \{xy^j : j \text{ even}\}$, $\lvert\mathcal{O}_x\rvert = m/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2 \simeq \langle x \rangle \oplus \langle y^{m/2} \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \varepsilon$	∞
$\mathcal{O}_{xy} = \{xy^j : j \text{ odd}\}$, $\lvert\mathcal{O}_{xy}\rvert = m/2$	$\mathbb{Z}/2 \times \mathbb{Z}/2 \simeq \langle xy \rangle \oplus \langle y^{m/2} \rangle$	$\varepsilon \otimes \varepsilon$, $\varepsilon \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \mathrm{sgn}$, $\mathrm{sgn} \otimes \varepsilon$	∞

Table 2. $\mathbb{D}_m$ for $m = 4t$ with $t \geq 3$.

2B. ***Nichols algebras of arbitrary Yetter–Drinfeld modules.*** In this section we determine all finite-dimensional Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Specifically, we prove that they are all exterior algebras over some Yetter–Drinfeld modules. For such a module, we write $\mathfrak{B}(M)$ instead of $\bigwedge M$ to emphasize the Yetter–Drinfeld module structure.

2B1. *Nichols algebras over the family $\{M_{i,k}\}$.* Recall $M_{i,k} = M(\mathcal{O}_{y^i}, \chi_{(k)})$ with $1 \leq i \leq n-1$, $0 \leq k \leq m-1$. We define an equivalence relation in J (see Section 2A2) by

$$(i,k) \sim (p,q) \text{ if } \omega^{iq+pk} = 1. \tag{3}$$

In such a case, one can prove that $\omega^{pk} = \omega^{iq} = -1$[1]. We denote by $[i,k] = \{(p,q) \in J : (p,q) \sim (i,k)\}$ the class of (i,k) under this equivalence.

[1]Write $n = (i,n)h$. Since $n|ik$, one has that $h|k$. As $n|iq+pk$, we have that $(i,n)|pk$ and thus $n|pk$. Then prove that $pk \equiv n \mod m$.

Proposition 2.5. *Let $M = M_{i_1,k_1} \oplus \cdots \oplus M_{i_r,k_r}$ with $(i_s, k_s) \in J$ for all $1 \le s \le r$. Then* $\dim \mathfrak{B}(M) < \infty$ *if and only if $(i_p, k_p) \sim (i_q, k_q)$ for all $1 \le p, q \le r$. In such a case,* $\mathfrak{B}(M) \simeq \bigwedge M$ *and* $\dim \mathfrak{B}(M) = 4^r$.

Proof. Assume first $r = 2$. Let $(i, k), (p, q) \in J$ and consider $M_{i,k}$ and $M_{p,q}$. Then $\mathbb{O}_{y^i} = \{\sigma_1 := y^i, \sigma_2 := y^{-i}\}$, $\mathbb{O}_{y^p} = \{\tau_1 := y^p, \tau_2 := y^{-p}\}$ and $\chi_{(k)}(y^i) = -1 = \chi_{(q)}(y^p)$. Set $g_1 = h_1 = 1$ and $g_2 = h_2 = x$. Then

$$g_1 y^i g_1^{-1} = \sigma_1, \quad g_2 y^i g_2^{-1} = \sigma_2, \quad h_1 y^p h_1^{-1} = \tau_1, \quad h_2 y^p h_2^{-1} = \tau_2.$$

Consider now the Yetter–Drinfeld module $M = M_{i,k} \oplus M_{p,q}$. As a vector space M is the $\Bbbk$-span of $\{g_1, g_2, h_1, h_2\}$. The braiding c in M is given by $c|_{M_{i,k}\oplus M_{i,k}} = c_{M_{i,k}}$, $c|_{M_{p,q}\otimes M_{p,q}} = c_{M_{p,q}}$, and

$$\begin{aligned}
c(g_1 \otimes h_1) &= \chi_{(q)}(y^i)\, h_1 \otimes g_1, & c(g_1 \otimes h_2) &= \chi_{(q)}(y^{-i})\, h_2 \otimes g_1,\\
c(g_2 \otimes h_1) &= \chi_{(q)}(y^{-i})\, h_1 \otimes g_2, & c(g_2 \otimes h_2) &= \chi_{(q)}(y^i)\, h_2 \otimes g_2,\\
c(h_1 \otimes g_1) &= \chi_{(k)}(y^p)\, g_1 \otimes h_1, & c(h_1 \otimes g_2) &= \chi_{(k)}(y^{-p})\, g_2 \otimes h_1,\\
c(h_2 \otimes g_1) &= \chi_{(k)}(y^{-p})\, g_1 \otimes h_2, & c(h_2 \otimes g_2) &= \chi_{(k)}(y^p)\, g_2 \otimes h_2.
\end{aligned}$$

Thus M is a diagonal vector space whose matrix of coefficients is

$$\mathfrak{Q} = \begin{pmatrix} -1 & -1 & \chi_{(q)}(y^i) & \chi_{(q)}(y^{-i}) \\ -1 & -1 & \chi_{(q)}(y^{-i}) & \chi_{(q)}(y^i) \\ \chi_{(k)}(y^p) & \chi_{(k)}(y^{-p}) & -1 & -1 \\ \chi_{(k)}(y^{-p}) & \chi_{(k)}(y^p) & -1 & -1 \end{pmatrix}.$$

Let $\lambda := \chi_{(q)}(y^i)\chi_{(k)}(y^p) = \omega^{iq+pk}$. If $\lambda \neq 1$, that is, $\omega^{iq+pk} \neq 1$ then $(i, k) \nsim (p, q)$ and $\dim \mathfrak{B}(M) = \infty$, by [Heckenberger 2009], since the generalized Dynkin diagram associated to M is given by Figure 1.

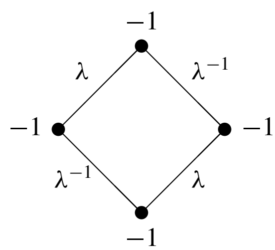

Figure 1

If $\lambda = 1$, that is, $\omega^{iq+pk} = 1$ then $(p, q) \sim (i, k)$. In such a case, $\omega^{pk} = \omega^{iq} = -1$ (see the paragraph after (3)), whence $\mathfrak{B}(M) = \bigwedge M$, since the braiding in M is $c = -$flips; in particular $\dim(M) = 16$.

Assume $r \ge 2$ and let $M = M_{i_1,k_1} \oplus \cdots \oplus M_{i_r,k_r}$ with $(i_s, k_s) \in J$ for all $1 \le s \le r$. In particular, $\omega^{i_s k_s} = -1$ for all $1 \le s \le r$. If there exist p, q, $1 \le p, q \le r$

such that $(i_p, k_p) \nsim (i_q, k_q)$, that is, $\chi_{(k_q)}(y^{i_p})\chi_{(k_p)}(y^{i_q}) = \omega^{i_p k_q + i_q k_p} \neq 1$, then $\dim \mathfrak{B}(M_{i_p,k_p} \oplus M_{i_q,k_q}) = \infty$ as above, which implies that $\dim \mathfrak{B}(M) = \infty$. Thus $(i_p, k_p) \sim (i_q, k_q)$ for all $1 \le p, q \le r$ and $\chi_{(k_q)}(y^{i_p})\chi_{(k_p)}(y^{i_q}) = \omega^{i_p k_q + i_q k_p} = 1$. As before, $\omega^{i_q k_p} = \omega^{i_p k_q} = -1$, which implies that $\mathfrak{B}(M) = \bigwedge M$, since the braiding in M is $c = -$flips; in particular $\dim \mathfrak{B}(M) = 4^r$. □

Definition 2.6. Let

$$\mathcal{I} = \left\{ I = \coprod_{s=1}^{r} \{(i_s, k_s)\} : (i_s, k_s) \in J \text{ and } (i_s, k_s) \sim (i_p, k_p),\ 1 \le s, p \le r \right\}.$$

For $I \in \mathcal{I}$, we define $M_I = \bigoplus_{(i,k)\in I} M_{i,k}$.

By Proposition 2.5, we have $\mathfrak{B}(M_I) \simeq \bigwedge M_I$ and $\dim \mathfrak{B}(M_I) = 4^{|I|}$.

Remark 2.7. Denote by $a_{i,k}, b_{i,k}$, $(i,k) \in I$ the primitive elements that generate $\mathfrak{B}(M_I)$. Then, the Yetter–Drinfeld module structure is given by

$$x \cdot a_{i,k} = b_{i,k}, \quad y \cdot a_{i,k} = \omega^k a_{i,k}, \quad \delta(a_{i,k}) = y^i \otimes a_{i,k}, \tag{4}$$

$$x \cdot b_{i,k} = a_{i,k}, \quad y \cdot b_{i,k} = \omega^{-k} b_{i,k}, \quad \delta(b_{i,k}) = y^{-i} \otimes b_{i,k}. \tag{5}$$

2B2. *Nichols algebras over the family* $\{M_\ell\}$. Recall that $M_\ell = M(\mathcal{O}_{y^n}, \rho_\ell)$. In this section we study Nichols algebras over sums of irreducible Yetter–Drinfeld modules isomorphic to M_ℓ with $1 \le \ell < n$, ℓ odd.

Proposition 2.8. *Let $M = M_{\ell_1} \oplus \cdots \oplus M_{\ell_r}$ with $1 \le \ell_s < n$ odd numbers. Then* $\mathfrak{B}(M) \simeq \bigwedge M$ *and* $\dim \mathfrak{B}(M) = 4^r$.

Proof. It suffices to show that the braiding c in M is $c = -$flips. Let $1 \le p, q \le r$ and denote by v_1, v_2 and w_1, w_2 the linear generators of M_{ℓ_p} and M_{ℓ_q}, respectively. Then $c = -$flips in $M_{\ell_p} \oplus M_{\ell_q}$. Indeed, we know that $c|_{M_{\ell_p}\otimes M_{\ell_p}} = -$flips and $c|_{M_{\ell_q}\otimes M_{\ell_q}} = -$flips, by Lemma 2.1, and by straightforward computations,

$$c(v_1 \otimes w_1) = y^n \cdot w_1 \otimes v_1 = \omega^{n\ell_q} w_1 \otimes v_1 = (-1)^{\ell_q} w_1 \otimes v_1 = -w_1 \otimes v_1,$$
$$c(v_1 \otimes w_2) = y^n \cdot w_2 \otimes v_1 = \omega^{-n\ell_q} w_2 \otimes v_1 = (-1)^{-\ell_q} w_2 \otimes v_1 = -w_2 \otimes v_1,$$
$$c(v_2 \otimes w_1) = y^n \cdot w_1 \otimes v_2 = \omega^{n\ell_q} w_1 \otimes v_2 = (-1)^{\ell_q} w_1 \otimes v_2 = -w_1 \otimes v_2,$$
$$c(v_2 \otimes w_2) = y^n \cdot w_2 \otimes v_2 = \omega^{-n\ell_q} w_2 \otimes v_2 = (-1)^{-\ell_q} w_2 \otimes v_2 = -w_2 \otimes v_2. \quad \square$$

Definition 2.9. Let

$$\mathcal{L} = \left\{ L = \coprod_{s=1}^{r} \{\ell_s\} : 1 \le \ell_1, \dots, \ell_r < n \text{ odd numbers} \right\}.$$

For $L \in \mathcal{L}$, we define $M_L = \bigoplus_{\ell\in L} M_\ell$.

By Proposition 2.8, we have $\mathfrak{B}(M_L) \simeq \bigwedge M_L$ and $\dim \mathfrak{B}(M_L) = 4^{|L|}$.

Remark 2.10. Denote by c_ℓ, d_ℓ with $\ell \in L$ the primitive elements that generate $\mathfrak{B}(M_L)$. Then, the Yetter–Drinfeld module structure is given by

$$x \cdot c_\ell = d_\ell, \quad y \cdot c_\ell = \omega^\ell c_\ell, \qquad \delta(c_\ell) = y^n \otimes c_\ell, \tag{6}$$

$$x \cdot d_\ell = c_\ell, \quad y \cdot d_\ell = \omega^{-\ell} d_\ell, \quad \delta(d_\ell) = y^n \otimes d_\ell. \tag{7}$$

2B3. *Nichols algebras over mixed families.*

Proposition 2.11. *Let $M_{i,k,\ell} = M_{i,k} \oplus M_\ell$ with $(i,k) \in J$ and $1 \le \ell < n$ be an odd number. Then* $\dim \mathfrak{B}(M_{i,k,\ell}) < \infty$ *if and only if k is odd and $(i,\ell) \in J$. In such a case,* $\mathfrak{B}(M_{i,k,\ell}) \simeq \bigwedge M_{i,k,\ell}$ *and* $\dim \mathfrak{B}(M_{i,k,\ell}) = 16$.

Proof. Let $c : M_{i,k,\ell} \otimes M_{i,k,\ell} \to M_{i,k,\ell} \otimes M_{i,k,\ell}$ be the braiding of $M_{i,k,\ell}$. As before, it suffices to show that $c = -$flips. Denote by $g_1 = 1$, $g_2 = x$ and v_1, v_2 the linear generators of $M_{i,k}$ and M_ℓ, respectively. Then by Lemmata 2.3 and 2.1, we have that $c|_{M_{i,k} \otimes M_{i,k}} = -$flips and $c|_{M_\ell \otimes M_\ell} = -$flips. Thus c is determined by the values

$$\begin{aligned}
&c(g_1 \otimes v_1) = y^i \cdot v_1 \otimes g_1 = \omega^{i\ell} v_1 \otimes g_1,\\
&c(v_1 \otimes g_1) = y^n \cdot g_1 \otimes v_1 = \chi_{(k)}(y^n) g_1 \otimes v_1 = \omega^{nk} g_1 \otimes v_1 = (-1)^k g_1 \otimes v_1,\\
&c(g_2 \otimes v_1) = y^{-i} \cdot v_1 \otimes g_2 = \omega^{-i\ell} v_1 \otimes g_2,\\
&c(v_1 \otimes g_2) = y^n \cdot g_2 \otimes v_1 = \chi_{(k)}(y^n) g_2 \otimes v_1 = \omega^{nk} g_2 \otimes v_1 = (-1)^k g_2 \otimes v_1,\\
&c(g_1 \otimes v_2) = y^i \cdot v_2 \otimes g_1 = \omega^{-i\ell} v_2 \otimes g_1,\\
&c(v_2 \otimes g_1) = y^n \cdot g_1 \otimes v_2 = \chi_{(k)}(y^n) g_1 \otimes v_2 = \omega^{nk} g_1 \otimes v_2 = (-1)^k g_1 \otimes v_2,\\
&c(g_2 \otimes v_2) = y^{-i} \cdot v_2 \otimes g_2 = \omega^{i\ell} v_2 \otimes g_2,\\
&c(v_2 \otimes g_2) = y^n \cdot g_2 \otimes v_2 = \chi_{(k)}(y^n) g_2 \otimes v_2 = \omega^{nk} g_2 \otimes v_2 = (-1)^k g_2 \otimes v_2.
\end{aligned}$$

This implies that M is a diagonal vector space with matrix of coefficients

$$\mathfrak{Q} = \begin{pmatrix} -1 & -1 & \omega^{i\ell} & \omega^{-i\ell} \\ -1 & -1 & \omega^{-i\ell} & \omega^{i\ell} \\ (-1)^k & (-1)^k & -1 & -1 \\ (-1)^k & (-1)^k & -1 & -1 \end{pmatrix}.$$

Let $\lambda := (-1)^k \omega^{i\ell}$. If $\lambda \neq 1$, then the generalized Dynkin diagram associated to $M_{i,k,\ell}$ is given by Figure 1, whence $\dim \mathfrak{B}(M) = \infty$, by [Heckenberger 2009]. Therefore, in order to have $\dim \mathfrak{B}(M_{i,k,\ell}) < \infty$ we must have that $\lambda = 1$, that is, $(-1)^k = \omega^{i\ell}$. By assumption $\omega^{ik} = -1$, thus $\omega^{ik\ell} = (-1)^\ell = -1$, because ℓ is odd. But $-1 = (\omega^{i\ell})^k = ((-1)^k)^k = (-1)^{k^2}$, thus k must be odd and $\omega^{i\ell} = -1$, that is, $(i,\ell) \in J$. In such a case, the braiding in $M_{i,k,\ell}$ is $c = -$flips and then $\mathfrak{B}(M_{i,k,\ell}) \simeq \bigwedge M_{i,k,\ell}$ and $\dim(M_{i,k,\ell}) = 16$. $\square$

For $I \in \mathcal{I}$ and $L \in \mathcal{L}$, define $M_{I,L} = \left(\bigoplus_{(i,k)\in I} M_{i,k}\right) \oplus \left(\bigoplus_{\ell \in L} M_\ell\right)$. The next result generalizes Proposition 2.11 for arbitrary finite sums.

Proposition 2.12. *Let $I \in \mathcal{I}$, $L \in \mathcal{L}$ and assume that k is odd for all $(i,k) \in I$. Then* $\dim \mathfrak{B}(M_{I,L}) < \infty$ *if and only if $(i,\ell) \in J$ for all $(i,k) \in I$, $\ell \in L$. In such a case,* $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ *and* $\dim \mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

Proof. Denote by $a_{i,k}, b_{i,k}$ and c_ℓ, d_ℓ the linear generators of $M_{i,k}$ and M_ℓ, respectively, for all $(i,k) \in I$, $\ell \in L$. Then by the proof of Propositions 2.5, 2.8 and 2.11, it follows that $\dim \mathfrak{B}(M_{I,L})$ is finite if and only k is odd for all $(i,k) \in I$ and $(i,\ell) \in J$ for all $(i,k) \in I$, $\ell \in L$. In such a case, the braiding in $M_{I,L}$ is given by $-$flips, whence $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$. □

Remark 2.13. Denote by $a_{i,k}, b_{i,k}, c_\ell, d_\ell$ with $(i,k) \in I$ and $\ell \in L$ the primitive elements that generate $\mathfrak{B}(M_{I,L})$. Then, the Yetter–Drinfeld module structure is determined by (4), (5), (6) and (7).

Definition 2.14. We define

$$\mathcal{K} = \{(I,L) : I \in \mathcal{I}, L \in \mathcal{L} \text{ and } k \text{ odd}, (i,\ell) \in J \text{ for all } (i,k) \in I, \ell \in L\}.$$

By Proposition 2.12, for all $(I,L) \in \mathcal{K}$, we have $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ and $\dim \mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

Proof of Theorem A. Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Since ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ is semisimple, M must be a finite direct sum of irreducible Yetter–Drinfeld modules. Then the result follows from Lemmata 2.1, 2.3 and Propositions 2.5, 2.8, 2.12. Clearly, Nichols algebras over distinct families are pairwise nonisomorphic, since they are generated by the set of primitive elements which are nonisomorphic as Yetter–Drinfeld modules. □

3. Liftings of Nichols algebras over dihedral groups

In this section we describe all finite-dimensional pointed Hopf algebras over dihedral groups $\mathbb{D}_m$, assuming that $m = 4t$, $t \geq 3$.

Let H be a Hopf algebra with bijective antipode and let $B \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. From B and H one can construct a new Hopf algebra $B \# H$, called the Radford–Majid product or bosonization, whose underlying vector space is $B \otimes H$ and the Hopf algebra structure is given by

$$(a \# h)(b \# k) = a(h_{(1)} \cdot b) \# h_{(2)} k,$$
$$\Delta(a \# h) = a^{(1)} \# (a^{(2)})_{(-1)} h_{(1)} \otimes (a^{(2)})_{(0)} \# h_{(2)},$$

for all $(a \# h), (b \# k) \in B \# H$, where $\Delta_B(a) = a^{(1)} \otimes a^{(2)}$ is the braided coproduct and $\delta_B(a) = a_{(-1)} \otimes a_{(0)}$ is the coaction of H on B.

Remark 3.1. Assume that A is a finite-dimensional pointed Hopf algebra with $A_0 = \Bbbk G(A)$ and let $\operatorname{gr} A = \bigoplus_{i>0} A_i/A_{i-1}$. Then $\operatorname{gr} A$ is a Hopf algebra which is isomorphic to the bosonization $R \# \Bbbk G(A)$, where $R = (\operatorname{gr} A)^{\operatorname{co} \pi}$. If $a \in R(1)$ is a homogeneous primitive element, that is, $\delta(a) = g \otimes a$, $g \in G(A)$, then $a \# 1 \in R \# \Bbbk G(A)$ is $(g, 1)$-primitive. Indeed, by the formula above we have $\Delta(a \# 1) = a^{(1)} \# (a^{(2)})_{(-1)} \otimes (a^{(2)})_{(0)} \# 1 = a \# 1 \otimes 1 \# 1 + 1 \# g \otimes a \# 1$. Consider now the projection, $\pi : A_1 \to A_1/A_0$, which is in particular a projection of Hopf $\Bbbk G(A)$-bimodules, and denote by σ a section of Hopf $\Bbbk G(A)$-bimodules. Since $A_1/A_0 = A_0 \oplus P(R) \# \Bbbk G(A)$, by [Andruskiewitsch and Schneider 1998, Lemma 2.4], we have that $a \# 1 \in A_1/A_0$, and $\sigma(a \# 1)$ is $(g, 1)$-primitive in A.

The following is a key step for the classification; see [Andruskiewitsch and Schneider 2000, Proposition 5.4; 2002, Theorem 7.6; Andruskiewitsch and Graña 2003, Theorem 2.1, Angiono and García Iglesias 2011, Theorem 2.6; García and García Iglesias 2011, Theorem 3.1]. It agrees with a well-known conjecture [Andruskiewitsch and Schneider 2000, Conjecture 1.4].

Theorem 3.2. *Let A be a finite-dimensional pointed Hopf algebra with $G(A) = \mathbb{D}_m$. Then A is generated by group-likes and skew-primitives.*

Proof. Since $\operatorname{gr} A = R \# \Bbbk\mathbb{D}_m$, with $R = \bigoplus_{n\geq 0} R(n)$ the diagram of A, it is enough to prove that R is a Nichols algebra, since then A would be generated by $G(A)$ and skew-primitive elements. Let S be the graded dual of R. By [Andruskiewitsch and Schneider 2000, Lemma 5.5], S is generated by $V = S(1)$ and R is a Nichols algebra if and only if $P(S) = S(1)$, that is, if S is itself a Nichols algebra.

Consider $\mathfrak{B}(V) \in {}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Since $V = R(1)^* = P(R)^*$ and $\mathfrak{B}(P(R))$ is finite-dimensional, we have by [Andruskiewitsch and Graña 1999, Proposition 3.2.30] that $\mathfrak{B}(V)$ is also finite-dimensional and by Theorem A, $\mathfrak{B}(V)$ is isomorphic to an exterior algebra $\mathfrak{B}(M_I)$, $\mathfrak{B}(M_L)$ or $\mathfrak{B}(M_{I,L})$, with $I \in \mathcal{I}$, $L \in \mathcal{L}$ and $(I, L) \in \mathcal{K}$, respectively. Moreover, a direct computation shows that the elements r in S representing the quadratic relations are primitive and since the braiding is $-$flips, they satisfy that $c(r \otimes r) = r \otimes r$. As $\dim S < \infty$, it must be $r = 0$ in S and hence there exists a projective algebra map $\mathfrak{B}(V) \twoheadrightarrow S$, which implies that $P(S) = S(1)$. $\square$

3A. ***Some liftings and quadratic relations.*** We begin this section with the following proposition that shows how to deform the relations in the Nichols algebras to get a lifting.

Let A be a finite-dimensional pointed Hopf algebra over $\Bbbk\mathbb{D}_m$. Then we have that $\operatorname{gr} A = \mathfrak{B}(M) \# \Bbbk\mathbb{D}_m$ by Theorem 3.2, and its infinitesimal braiding M is isomorphic either to M_I with $I \in \mathcal{I}$ and $|I| > 1$ or $I = \{(i, n)\}$, or M_L with $L \in \mathcal{L}$, or $M_{I,L}$ with $(I, L) \in \mathcal{K}$ and $|L| > 0$, $|I| > 0$; see Section 2B.

From now on, we denote by g, h the generators of $G(A) \simeq \mathbb{D}_m$ with $g^2 = 1 = h^m$ and $ghg = h^{-1}$.

For all $1 \leq r, s < m$, let $M_r^s = \{a \in M : \delta(a) = h^s \otimes a, h \cdot a = \omega^r a\}$. Then $M = \bigoplus_{r,s} M_r^s$. Following Remark 3.1, we write $x = \sigma(a \# 1)$ for the element in A defined by the Hopf $\Bbbk G(A)$-bimodule section σ. In particular, if $a \in M_r^s$, then x is $(h^s, 1)$-primitive and $hxh^{-1} = \omega^r x$.

Proposition 3.3. *Let A be a finite-dimensional pointed Hopf algebra that has $G(A) = \mathbb{D}_m$ and infinitesimal braiding M. Let $a \in M_r^s$, $b \in M_u^v$ with $1 \leq r, s, u, v < m$ and denote $x = \sigma(a \# 1)$, $y = \sigma(b \# 1)$. Then there exists $\lambda \in \Bbbk$ such that*

$$xy + yx = \delta_{u,m-r}\lambda(1 - h^{s+v}). \tag{8}$$

In particular, if $x = y$ we have that $x^2 = \delta_{r,n}\lambda'(1 - h^{2s})$ with $\lambda' = \lambda/2$.

Proof. By Theorem 3.2, M is isomorphic either to M_I with $I \in \mathcal{I}$ and $|I| > 1$ or $I = \{(i, n)\}$, or M_L with $L \in \mathcal{L}$, or $M_{I,L}$ with $(I, L) \in \mathcal{K}$ and $|L| > 0$, $|I| > 0$. As $a \in M_r^s, b \in M_u^v$, Propositions 2.5, 2.8 and 2.12 yield that $\omega^{sr} = -1 = \omega^{uv}$ and $(r, s) \sim (u, v)$, that is, $\omega^{rv+su} = 1$ and $\omega^{rv} = -1 = \omega^{su}$.

A straightforward computation yields that the element $\alpha = xy + yx$ is $(h^{s+v}, 1)$-primitive. Indeed,

$$\begin{aligned}
\Delta(\alpha) &= \Delta(xy + yx) \\
&= (x \otimes 1 + h^s \otimes x)(y \otimes 1 + h^v \otimes y) + (y \otimes 1 + h^v \otimes y)(x \otimes 1 + h^s \otimes x) \\
&= xy \otimes 1 + xh^v \otimes y + h^s y \otimes x + h^{s+v} \otimes xy + yx \otimes 1 \\
&\qquad\qquad + yh^s \otimes x + h^v x \otimes y + h^{s+v} \otimes yx \\
&= (xy + yx) \otimes 1 + h^{s+v} \otimes (xy + yx) + (xh^v + h^v x) \otimes y + (h^s y + yh^s) \otimes x \\
&= (xy + yx) \otimes 1 + h^{s+v} \otimes (xy + yx).
\end{aligned}$$

If $s + v \equiv 0 \bmod m$, then α is primitive. Since A is finite-dimensional, we must have that $\alpha = 0$. Suppose $s + v \not\equiv 0 \bmod m$. Then, by Theorem 3.2 there exist $(h^{s+v}, 1)$-primitive elements $x_{i,j} \in M_j^i$ with $i = s + v$ and $\lambda, \beta_{i,j} \in \Bbbk$ such that

$$\alpha = \lambda(1 - h^{s+v}) + \sum_{i=s+v,j} \beta_{i,j} x_{i,j}.$$

Conjugating on both sides by h gives

$$h\alpha h^{-1} = \omega^{r+u}\alpha = \lambda_{p,q,i,k}(1 - h^{s+v}) + \sum_{i=s+v,j} \beta_{i,j}\omega^j x_{i,j},$$

which implies that $\lambda = \lambda\omega^{r+u}$ and $\beta_{i,j}\omega^j = \beta_{i,j}\omega^{r+u}$ for all i, j. If $r+u \not\equiv 0 \bmod m$, then $\lambda_{p,q,i,k} = 0$. So to end the proof we must show $\beta_{i,j} = 0$ for all i, j. On the contrary, suppose $\beta_{i,j} \neq 0$ for some i, j. Then $j \equiv r + u \bmod m$. In such a case, as $i = s + v$,

$$-1 = \omega^{ij} = \omega^{(s+v)(r+u)} = \omega^{sr+vu}\omega^{su+vr} = 1,$$

a contradiction. In conclusion, we must have that $\alpha = \delta_{u,m-r}\lambda(1 - h^{s+v})$. $\square$

As a direct consequence of Proposition 3.3 we get the following corollaries. The first one shows that all pointed Hopf algebras over $\mathbb{D}_m$ whose infinitesimal braiding is isomorphic to M_I with $I=\{(i,k)\}\in\mathcal{I}$, $k\neq n$, or M_L with $L\in\mathcal{L}$, as in Sections 2B1 and 2B2, respectively, are isomorphic to bosonizations.

Corollary 3.4. *Let A be a finite-dimensional pointed Hopf algebra with $G(A)=\mathbb{D}_m$ such that its infinitesimal braiding M is isomorphic to M_I with $I=\{(i,k)\}\in\mathcal{I}$ and $k\neq n$, or M_L with $L\in\mathcal{L}$. Then $A\simeq \operatorname{gr} A\simeq \mathfrak{B}(M)\#\Bbbk\mathbb{D}_m$.*

Proof. Suppose $M\simeq M_I$ with $I=\{(i,k)\}\in\mathcal{I}$ and $k\neq n$ and denote $x=\sigma(a_{i,k}\#1)$, $y=\sigma(b_{i,k}\#1)$. By Proposition 3.3, we have that $x^2=0=y^2$ and $xy+yx=\delta_{i,m-i}\lambda(1-h^{i+m-i})=0$ for any $\lambda\in\Bbbk^\times$. Thus $A\simeq\operatorname{gr} A\simeq\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$. Assume now that $M\simeq M_L$ with $L\in\mathcal{L}$, and denote $x=\sigma(c_\ell\#1)$, $y=\sigma(c_{\ell'}\#1)$ with $\ell,\ell'\in L$, and $e_\ell, e_{\ell'}$ in the set of linear generators $\{c_\ell, d_\ell : \ell\in L\}$ of M. As x and y are $(h^n,1)$-primitive, again by Proposition 3.3 we have that $x^2=0=y^2$ and $xy+yx=\delta_{\ell',m-\ell}\lambda(1-h^{n+n})=0$ for any $\lambda\in\Bbbk^\times$. So $A\simeq\operatorname{gr} A\simeq\mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m$. □

The following two corollaries give the explicit relations that a lifting of a Nichols algebra over $\mathbb{D}_m$ must satisfy.

Corollary 3.5. *Let A be a finite-dimensional pointed Hopf algebra with $G(A)=\mathbb{D}_m$ such that its infinitesimal braiding is isomorphic to M_I with $I\in\mathcal{I}$ and $|I|>1$ or $I=\{(i,n)\}$. Denote $x_{p,q}=\sigma(a_{p,q}\#1)$ and $y_{p,q}=\sigma(b_{p,q}\#1)$ for all $(p,q)\in I$. Then there exist two families $\lambda=(\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$, and $\gamma=(\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$ of elements in $\Bbbk$ such that*

$$x_{p,q}x_{i,k}+x_{i,k}x_{p,q}=\delta_{q,m-k}\lambda_{p,q,i,k}(1-h^{p+i}),\tag{9}$$

$$y_{p,q}y_{i,k}+y_{i,k}y_{p,q}=\delta_{q,m-k}\lambda_{p,q,i,k}(1-h^{-p-i}),\tag{10}$$

$$x_{p,q}y_{i,k}+y_{i,k}x_{p,q}=\delta_{q,k}\gamma_{p,q,i,k}(1-h^{p-i}).\tag{11}$$

Remark 3.6. The symmetry of Equations (9) and (11) imply that the families $\lambda=(\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$ and $\gamma=(\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$ satisfy

$$\lambda_{p,m-k,i,k}=\lambda_{i,k,p,m-k}\quad\text{and}\quad\gamma_{p,k,i,k}=\gamma_{i,k,p,k}.\tag{12}$$

Corollary 3.7. *Let A be a finite-dimensional pointed Hopf algebra with $G(A)=\mathbb{D}_m$ such that its infinitesimal braiding is isomorphic to $M_{I,L}$ with $(I,L)\in\mathcal{K}$. Denote $x_{p,q}=\sigma(a_{p,q}\#1)$, $y_{p,q}=\sigma(b_{p,q}\#1)$, $z_\ell=\sigma(c_\ell\#1)$ and $w_\ell=\sigma(d_\ell\#1)$ for all $(p,q)\in I$, $\ell\in L$. Then there exist four families $\lambda=(\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\gamma=(\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\theta=(\theta_{p,q,\ell})_{(p,q)\in I,\ell\in L}$, and $\mu=(\mu_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ of* *elements in* $\Bbbk$ *such that the following relations in* A *hold*:

$$x_{p,q}^2 = 0 = y_{p,q}^2, \tag{13}$$

$$z_\ell w_{\ell'} + w_{\ell'} z_\ell = 0, \quad z_\ell z_{\ell'} + z_{\ell'} z_\ell = 0, \quad w_\ell w_{\ell'} + w_{\ell'} w_\ell = 0, \tag{14}$$

$$x_{p,q} x_{i,k} + x_{i,k} x_{p,q} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{p+i}), \tag{15}$$

$$y_{p,q} y_{i,k} + y_{i,k} y_{p,q} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{-p-i}), \tag{16}$$

$$x_{p,q} y_{i,k} + y_{i,k} x_{p,q} = \delta_{q,k} \gamma_{p,q,i,k} (1 - h^{p-i}), \tag{17}$$

$$x_{p,q} z_\ell + z_\ell x_{p,q} = \delta_{q,m-\ell}\, \theta_{p,q,\ell} (1 - h^{n+p}), \tag{18}$$

$$y_{p,q} w_\ell + w_\ell y_{p,q} = \delta_{q,m-\ell}\, \theta_{p,q,\ell} (1 - h^{n-p}), \tag{19}$$

$$x_{p,q} w_\ell + w_\ell x_{p,q} = \delta_{q,\ell}\, \mu_{p,q,\ell} (1 - h^{n+p}), \tag{20}$$

$$y_{p,q} z_\ell + z_\ell y_{p,q} = \delta_{q,\ell}\, \mu_{p,q,\ell} (1 - h^{n-p}). \tag{21}$$

Remark 3.8. As before, (15) and (17) imply the equalities in (12).

3B. ***Quadratic algebras.*** In this section we introduce two families of pointed Hopf algebras which are defined by quadratic relations. They are constructed by deforming the relations on two families of Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Moreover, we show that they are liftings of bosonizations of Nichols algebras and belong to the family of Hopf algebras that characterize pointed Hopf algebras over $\mathbb{D}_m$, $m = 4t$, $t \geq 3$.

The families of parameters. Let $I \in \mathcal{I}$ and $L \in \mathcal{L}$ be as in Definitions 2.6 and 2.9, respectively, and let $\lambda = (\lambda_{p,q,i,k})_{(p,q),(i,k) \in I}$, $\gamma = (\gamma_{p,q,i,k})_{(p,q),(i,k) \in I}$, $\theta = (\theta_{p,q,\ell})_{(p,q) \in I, \ell \in L}$, and $\mu = (\mu_{p,q,\ell})_{(p,q) \in I, \ell \in L}$ be families of elements in $\Bbbk$, satisfying the conditions

$$\lambda_{p,m-k,i,k} = \lambda_{i,k,p,m-k} \quad \text{and} \quad \gamma_{p,k,i,k} = \gamma_{i,k,p,k}. \tag{22}$$

In particular, θ and μ are families of free parameters in $\Bbbk$.

Definition 3.9. For $I \in \mathcal{I}$, denote by $A_I(\lambda, \gamma)$ the algebra generated by the elements $g, h, x_{p,q}, y_{p,q}$ with $(p,q) \in I$ satisfying the following relations:

$$g^2 = 1 = h^m, \quad ghg = h^{m-1}, \tag{23}$$

$$g x_{p,q} = y_{p,q} g, \quad h x_{p,q} = \omega^q x_{p,q} h, \quad h y_{p,q} = \omega^{-q} y_{p,q} h, \tag{24}$$

$$x_{p,q} x_{i,k} + x_{i,k} x_{p,q} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{p+i}), \tag{25}$$

$$x_{p,q} y_{i,k} + y_{i,k} x_{p,q} = \delta_{q,k} \gamma_{p,q,i,k} (1 - h^{p-i}). \tag{26}$$

It is a Hopf algebra with its structure determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(h) = h \otimes h,$$
$$\Delta(x_{p,q}) = x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, \quad \Delta(y_{p,q}) = y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q}.$$

Since it is generated by group-likes and skew-primitives, it is pointed by [Montgomery 1993, Lemma 5.5.1]. We will call the pair (λ, γ) a *lifting datum* for $\mathfrak{B}(M_I)$. We set $\gamma = 0$ if $|I| = 1$.

Example 3.10. If $I = \{(i, k)\}$ with $k \neq n$, then by Corollary 3.4, the Hopf algebra defined above is the bosonization $\mathfrak{B}(M_I) \# \Bbbk\mathbb{D}_m$. If $k = n$ we obtain the Hopf algebra $A_{i,n}(\lambda)$ generated by the elements g, h, x, y satisfying

$$g^2 = 1 = h^m, \qquad ghg = h^{m-1},$$
$$gx = yg, \qquad hx = -xh, \qquad hy = -yh,$$
$$x^2 = \lambda(1 - h^{2i}), \qquad y^2 = \lambda(1 - h^{-2i}), \quad xy + yx = 0.$$

It is a finite-dimensional pointed Hopf algebra with its structure given by

$$\Delta(g) = g \otimes g, \quad \Delta(h) = h \otimes h, \quad \Delta(x) = x \otimes 1 + h^i \otimes x, \quad \Delta(y) = y \otimes 1 + h^{-i} \otimes y.$$

Definition 3.11. For $(I, L) \in \mathcal{K}$, denote by $B_{I,L}(\lambda, \gamma, \theta, \mu)$ the algebra generated by $g, h, x_{p,q}, y_{p,q}, z_\ell, w_\ell$ satisfying the relations

$$g^2 = 1 = h^m, \quad ghg = h^{m-1}, \tag{27}$$
$$g x_{p,q} = y_{p,q} g, \quad h x_{p,q} = \omega^q x_{p,q} h, \tag{28}$$
$$g z_\ell = w_\ell g, \quad h z_\ell = \omega^\ell z_\ell h, \tag{29}$$
$$x_{p,q}^2 = 0 = y_{p,q}^2, \quad z_\ell w_{\ell'} + w_{\ell'} z_\ell = 0, \quad z_\ell z_{\ell'} + z_{\ell'} z_\ell = 0, \tag{30}$$
$$x_{p,q} x_{i,k} + x_{i,k} x_{p,q} = \delta_{q,m-k} \lambda_{p,q,i,k} (1 - h^{p+i}), \tag{31}$$
$$x_{p,q} y_{i,k} + y_{i,k} x_{p,q} = \delta_{q,k} \gamma_{p,q,i,k} (1 - h^{p-i}), \tag{32}$$
$$x_{p,q} z_\ell + z_\ell x_{p,q} = \delta_{q,m-\ell} \theta_{p,q,\ell} (1 - h^{n+p}), \tag{33}$$
$$x_{p,q} w_\ell + w_\ell x_{p,q} = \delta_{q,\ell} \mu_{p,q,\ell} (1 - h^{n+p}). \tag{34}$$

It is a Hopf algebra with its structure determined by

$$\Delta(g) = g \otimes g, \qquad \Delta(h) = h \otimes h,$$
$$\Delta(x_{p,q}) = x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, \qquad \Delta(y_{p,q}) = y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q},$$
$$\Delta(z_\ell) = z_\ell \otimes 1 + h^n \otimes z_\ell, \qquad \Delta(w_\ell) = w_\ell \otimes 1 + h^n \otimes w_\ell.$$

Since it is generated by group-likes and skew-primitive elements, it is pointed by [Montgomery 1993, Lemma 5.5.1]. We call the 4-tuple $(\lambda, \gamma, \theta, \mu)$ a *lifting datum* for $\mathfrak{B}(M_{I,L})$.

Example 3.12. If $I = \{(i, k)\}$ and $L = \{\ell\}$ with $1 \leq k, \ell < m$ odd numbers and $m - \ell \neq k$, then the Hopf algebra defined above is the bosonization $\mathfrak{B}(M_{I,L}) \# \Bbbk\mathbb{D}_m$. If $k = m - \ell$ we obtain the Hopf algebra $B_{I,L}(\theta, \mu)$ generated by the elements

g, h, x, y, z, w satisfying the relations

$$g^2 = 1 = h^m, \quad ghg = h^{m-1},$$
$$gx = yg, \quad hx = \omega^k xh, \quad gz = wg, \quad hz = \omega^{-k} zh,$$
$$x^2 = 0 = y^2, \quad z^2 = 0 = w^2, \quad xy + yx = 0,$$
$$zw + wz = 0, \quad xz + zx = \theta(1 - h^{n+i}), \quad xw + wx = \mu(1 - h^{n+i}).$$

As we have seen in Section 2, finite-dimensional Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ are exterior algebras; see Theorem A. In addition, the Hopf algebras $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$ defined above are quadratic algebras for all $I \in \mathcal{I}$ and $(I, L) \in \mathcal{K}$. Our next goal is to show that if H is a lifting of a finite-dimensional Nichols algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$, then H is isomorphic to a quadratic algebra defined above for some lifting data, and conversely, these Hopf algebras together with the bosonizations are all liftings of finite-dimensional Nichols algebras in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. First we work on quadratic algebras to obtain a bound on the dimensions of $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$. We follow [García and García Iglesias 2011] for our exposition.

Let W be a finite-dimensional vector space and let $T(W) = \bigoplus_{n\geq 0} W^{\otimes n}$ be the graded tensor algebra with the induced increasing filtration $F^i := \bigoplus_{j\leq i} W^{\otimes j}$. Let $R \subset W \otimes W$ be a subspace and denote by $J(R)$ the two-sided ideal of $T(W)$ generated by R. A (homogeneous) quadratic algebra $Q(W, R)$ is the quotient $T(W)/J(R)$. Analogously, for a subspace $P \subset F^2 = \Bbbk \oplus W \oplus W \otimes W$, we denote by $J(P)$ the two-sided ideal in $T(W)$ generated by P. A (nonhomogeneous) quadratic algebra $Q(W, P)$ is the quotient $T(W)/J(P)$.

Let $A = Q(W, P)$ be a nonhomogeneous quadratic algebra. It inherits an increasing filtration A_n from $T(W)$ given by $A_n = F^n/(J(P)\cap F^n)$ and the associated graded algebra is $\operatorname{gr} A = \bigoplus_{n\geq 0} A_n/A_{n-1}$, where $A_{-1} = 0$. Consider now the projection $\pi : F^2 \to W\otimes W$ with kernel F^1 and set $R = \pi(P) \subset W\otimes W$. Let $B = Q(W, R)$ be the homogeneous quadratic algebra defined by R. If $P \cap W = 0$, then we have an epimorphism $\rho : B \to \operatorname{gr} A$. Indeed, let $\rho' : T(W) \to \operatorname{gr} A$ be the graded algebra map induced by $W \hookrightarrow A_1 \twoheadrightarrow A_1/A_0$. Suppose $x \in R \subset W^{\otimes 2}$. Then there exist $x_0 \in \Bbbk$, $x_1 \in W$ such that $x - x_1 - x_0 \in P$ and therefore $x = x_1 + x_0 \in F^2/(J(P)\cap F^2) = A_2$. Thus $\rho'(x) = 0 \in A_2/A_1$, whence ρ' induces $\rho : B = T(W)/J(R) \twoheadrightarrow \operatorname{gr} A$.

Let G be a finite group and suppose that $\mathfrak{B}(V)$ is a finite-dimensional Nichols algebra in ${}^{\Bbbk G}_{\Bbbk G}\mathcal{YD}$ which is given by quadratic relations, that is, $\mathfrak{B}(V) = \widehat{\mathfrak{B}}_2(V)$. Denote by $p_i(x_{j_1}, \dots, x_{j_k}) = 0$ these quadratic relations, where $x_j \in P(V)$ are g_j-homogeneous elements with $g_j \in G$ and p_i, $i \in I$, are finitely many quadratic polynomials with coefficients in $\Bbbk$.

Assume H is a Hopf algebra containing $\Bbbk G$ as a Hopf subalgebra which is generated by $\Bbbk G$ and $P(V)$ such that x_j is $(g_j, 1)$-primitive for all $x_j \in P(V)$, $g x_j g^{-1} = g \cdot x_j$ for all $g \in G$, $x_j \in P(V)$ and $p_i(x_{j_1}, \dots, x_{j_k}) = \lambda_{i,j_1,\dots,j_k}(1 - g_{i,j_1,\dots,j_k})$ for some $\lambda_{i,j_1,\dots,j_k} \in \Bbbk$ and $g_{i,j_1,\dots,j_k} \in G$. The next lemma is a slight generalization of [García and García Iglesias 2011, Proposition 4.2].

Lemma 3.13. $\dim H \leq \dim \mathfrak{B}(V)|G|$.

Proof. H is the nonhomogeneous quadratic algebra $Q(W, P)$ defined by W and P, for $W = \Bbbk\{x_j, H_g : x_j \in P(V), g \in G\}$ and $P \subset \Bbbk \oplus W \oplus W \otimes W$ the subspace generated by

$$\big\{H_e - 1,\ H_g \otimes H_t - H_{gt},\ H_g \otimes x_j - g \cdot x_j \otimes H_g,\\ p_i(x_{j_1}, \dots, x_{j_k}) - \lambda_{i,j_1,\dots,j_k}(1 - g_{i,j_1,\dots,j_k})\big\}.$$

Let $R = \pi(P)$. Explicitly, $R \subset W \otimes W$ is the subspace generated by $\{H_g \otimes H_t, H_g \otimes x_j - g \cdot x_j \otimes H_g, p_i(x_{j_1}, \dots, x_{j_k})\}$. Let $B = Q(W, R)$ be the homogeneous quadratic algebra defined by W and R. Then $B \simeq \mathfrak{B}(V) \# Y_G$, where Y_G is the algebra linearly spanned by the set $\{1, y_g : g \in G\}$ with unit 1 and multiplication table given by $y_g y_t = 0$ for all $g, t \in G$ and # stands for the commutation relation $(1 \# y_g)(x_j \# 1) = g \cdot x_j \# y_g$, $(1 \# 1)(x_j \# 1) = x_j \# 1$ for all $g \in G$, $x_j \in P(V)$. Thus by the preceding discussion, there exists an epimorphism $\rho : \mathfrak{B}(V) \# Y_G \twoheadrightarrow \operatorname{gr} H$.

Since $P \cap F^1 = \Bbbk\{H_e - 1\}$, by [García and García Iglesias 2011, Lemma 4.1] we have $\rho(H_e) = 0$, whence there exists an epimorphism $\rho_e : B/By_eB \to \operatorname{gr} H$. The commutation relation and the fact that the elements $\{y_g\}_{g \in G}$ are pairwise orthogonal, give $By_eB = \mathfrak{B}(V)y_e \subset B$. This implies $\dim B^n - \dim(\mathfrak{B}(V)^n y_e) \geq \dim \operatorname{gr} H^n$ and since $\dim B^n = \dim \mathfrak{B}(V)^n(|G|+1)$, we have $\dim \mathfrak{B}(V)^n|G| \geq \dim \operatorname{gr} H^n$ and consequently $\dim H \leq \dim \mathfrak{B}(V)|G|$. ◻

The next corollary follows immediately.

Corollary 3.14. *For all $I \in \mathcal{I}$ and $(I, L) \in \mathcal{K}$ we have*

$$\dim A_I(\lambda, \gamma) \leq \dim \mathfrak{B}(M_I)|\mathbb{D}_m| = 4^{|I|}2m,$$
$$\dim B_{I,L}(\lambda, \gamma, \theta, \mu) \leq \dim \mathfrak{B}(M_{I,L})|\mathbb{D}_m| = 4^{|I|+|L|}2m.$$

3C. *Representation theory.* Let H be a finite-dimensional pointed Hopf algebra over $\Bbbk\mathbb{D}_m$. In this section we prove using representation theory that the quadratic algebras defined in Definitions 3.9 and 3.11 are liftings of finite-dimensional Nichols algebras over $\Bbbk\mathbb{D}_m$ for all lifting data (λ, γ) or $(\lambda, \gamma, \theta, \mu)$, and we end the section with the proof of Theorem B.

Let $H = A_I(\lambda, \gamma)$ with $I \in \mathcal{I}$ or $H = B_{I,L}(\lambda, \gamma, \theta, \mu)$ with $(I, L) \in \mathcal{K}$. By definition, the group $G(H)$ is a quotient of $\mathbb{D}_m$; in particular, any H-module is a $\Bbbk\mathbb{D}_m$-module. Denote by $\pi : \mathbb{D}_m \twoheadrightarrow G(H)$ this quotient. The following lemma is a key step to determine the dimension of H.

Lemma 3.15. *Let $\rho : H \to \mathrm{End}(V)$ be a representation of H such that*

(i) $\rho_{|G(H)} \circ \pi : \mathbb{D}_m \to \mathrm{End}(V)$ *is faithful*;

(ii) *if $H = A_I(\lambda, \gamma)$, then $\rho(x_{p,q}) \notin \Bbbk\rho(G(H))$ for all $(p, q) \in I$ and if $H = B_{I,L}(\lambda, \gamma, \theta, \mu)$, then $\rho(x_{p,q}), \rho(z_\ell) \notin \Bbbk\rho(G(H))$ for all $(p, q) \in I$ and $\ell \in L$.*

Then $\mathrm{gr}\, H = \mathfrak{B}(M) \# \Bbbk\mathbb{D}_m$ *and thus* $\dim H = \dim \mathfrak{B}(M)|\mathbb{D}_m|$.

Proof. Let $M = M_{I,L}$ and suppose that $H = B_{I,L}(\lambda, \gamma, \theta, \mu)$. Since $G(H)$ is a quotient of $\mathbb{D}_m$, from (i) it follows that $G(H) \simeq \mathbb{D}_m$. Thus H is a pointed Hopf algebra over $\mathbb{D}_m$ and by Theorem 3.2, $\mathrm{gr}\, H \simeq \mathfrak{B}(N) \# \Bbbk\mathbb{D}_m$ with $\mathfrak{B}(N)$ an exterior algebra; see Theorem A. Furthermore, by Lemma 3.13 we have that $\dim \mathfrak{B}(N) \leq \dim \mathfrak{B}(M)$. But by (ii) the map $\varphi : M \to H_1/H_0$, sending

$$a_{p,q} \mapsto \bar{x}_{p,q}, \quad b_{p,q} \mapsto \bar{y}_{p,q}, \quad c_\ell \mapsto \bar{z}_\ell, \quad d_\ell \mapsto \bar{w}_\ell,$$

induces an injective map $\phi : M \to N$ in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ which implies that $\dim \mathfrak{B}(N) \geq \dim \mathfrak{B}(M)$. The proof for $H = A_I(\lambda, \gamma)$ is completely analogous. $\square$

Proof of Theorem B. Let H be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$. Then $\mathrm{gr}\, H \simeq R \# \Bbbk\mathbb{D}_m$ and by Theorem 3.2 the diagram R is a Nichols algebra $\mathfrak{B}(M)$ for some $M \in {}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$ and consequently it is isomorphic to one of the Hopf algebras of Theorem A.

If $M \simeq M_I$ with $I = \{(i, k)\}$ and $k \neq n$ or $M \simeq M_L$ with $L \in \mathcal{L}$, then $H \simeq \mathfrak{B}(M) \# \Bbbk\mathbb{D}_m$ by Corollary 3.4. If $M \simeq M_I$ with $I \in \mathcal{I}$ and $|I| > 0$, then by Corollary 3.5 there exists a lifting datum (λ, γ) and an epimorphism of Hopf algebras $A_I(\lambda, \gamma) \twoheadrightarrow H$. Hence $\dim H \leq \dim A_I(\lambda, \gamma) \leq \dim \mathfrak{B}(M_I)|\mathbb{D}_m|$. This implies that $H \simeq A_I(\lambda, \gamma)$, since $\dim H = \dim \mathrm{gr}\, H = \dim \mathfrak{B}(M_I)|\mathbb{D}_m|$. If $M \simeq M_{I,L}$ with $(I, L) \in \mathcal{L}$, then using the same argument as before with Corollary 3.7 shows that $H \simeq B_{I,L}(\lambda, \gamma, \theta, \mu)$.

For the converse, it is clear that the algebras listed in items (a) and (b) are liftings of Nichols algebras over $\mathbb{D}_m$. Thus, we need to show only that the Hopf algebras $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$ are liftings for all $I \in \mathcal{I}$, $(I, L) \in \mathcal{K}$ and for all lifting data.

Assume first that $I \in \mathcal{I}$. Following Lemma 3.15, we give a representation for $A_I(\lambda, \gamma)$. Give I an order and write $I = ((i_1, k_1), \ldots, (i_r, k_r))$. Let V be a vector space with basis given by two families of vectors $\{u_\alpha\}$, $\{v_\alpha\}$, indexed by all possible ordered monomials in the letters $i_{s,1}, i_{s,2}$ for all $1 \leq s \leq r$ such that each letter appears at most once (set u_0, v_0 if no letter appears) and the order is given by $i_{s,p} < i_{t,p}$ for all $p = 1, 2$, if and only if $s < t$, $i_{s,1} < i_{s,2}$ for all $1 \leq s \leq r$ and $i_{s,2} < i_{t,1}$ if and only if $s < t$; for example, $v_{i_{1,1}i_{2,2}i_{3,1}i_{3,2}}$ is a basis element. In particular, $\dim V = 2 \dim \bigwedge M_I$.

For all $1 \leq j < n$, V bears an $A_I(\lambda,\gamma)$-module structure determined by

$$\begin{aligned} g\cdot u_0 &= v_0, \quad h\cdot u_0=\omega^j u_0, \quad x_{i_s,k_s}\cdot u_0 = u_{i_{s,1}}, \quad y_{i_s,k_s}\cdot u_0 = u_{i_{s,2}} \\ (x_{i_t,k_t}x_{i_s,k_s})\cdot u_0 &= -u_{i_{s,1}i_{t,1}} + \delta_{k_s,m-k_t}\lambda_{i_s,k_s,i_t,k_t}(1-\omega^{j(i_s+i_t)})u_0 \quad \text{if } t>s, \\ (y_{i_t,k_t}x_{i_s,k_s})\cdot u_0 &= -u_{i_{s,1}i_{t,2}} + \delta_{s,t}\gamma_{i_s,k_s,i_t,k_t}(1-\omega^{j(i_s-i_t)})u_0 \quad \text{if } t\geq s, \end{aligned}$$

because of the defining relations of $A_I(\lambda,\gamma)$; see Definition 3.9. Hence, the composition $\rho_{|G(A_I(\lambda,\gamma))}\circ\pi : \mathbb{D}_m \to \operatorname{End}(V)$ is faithful since $(\Bbbk\{u_0, v_0\}, \rho_{|G(A_I(\lambda,\gamma))}) \simeq (\Bbbk^2, \rho_j)$, and $\rho(x_{i_s,k_s}) \notin \Bbbk\rho(G(A_I(\lambda,\gamma))$ by definition. Therefore $\operatorname{gr} A_I(\lambda,\gamma) = \mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ and $A_I(\lambda,\gamma)$ is a lifting.

The proof for $B_{I,L}(\lambda,\gamma,\theta,\mu)$ is analogous. □

Isomorphism classes. In this last section we study the isomorphism classes of the families of Hopf algebras given by Theorem B.

Let H be a finite-dimensional pointed Hopf algebra over $\mathbb{D}_m$. Then H is isomorphic to a Hopf algebra listed in Theorem B; in particular, it is a lifting of a finite-dimensional Nichols algebra over $\mathbb{D}_m$.

It is clear that two algebras from different families are not isomorphic as Hopf algebras since their infinitesimal braidings are not isomorphic as Yetter–Drinfeld modules.

Thus, we have to show that two different members in the same family are not isomorphic. By the argument above, if $I=\{(i,k)\}$, $I'=\{(p,q)\}\in\mathcal{I}$ with $k,q\neq n$ and $(i,k)\neq(p,q)$, then $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m \not\simeq \mathfrak{B}(M_{I'})\#\Bbbk\mathbb{D}_m$, and if $L, L'\in\mathcal{L}$ with $L\neq L'$, then $\mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m \not\simeq \mathfrak{B}(M_{L'})\#\Bbbk\mathbb{D}_m$. We end the paper by showing the isomorphism classes of the families of the items (c) and (d).

Observe that $\mathbb{Z}/m$ acts on $\mathcal{I}$ with the action on each $I\in\mathcal{I}$ induced by

$$\ell\cdot(i_s,k_s) = \begin{cases} (\ell i_s, \ell^{-1}k_s) & \text{if } 1\leq \ell i_s < n \bmod m, \\ (m-\ell i_s, \ell^{-1}k_s) & \text{if } n \leq \ell i_s \bmod m. \end{cases}$$

Lemma 3.16. *Let $I, I'\in\mathcal{I}$. Then $A_I(\lambda,\gamma)\simeq A_{I'}(\lambda',\gamma')$ if and only if there exists $\ell\in\mathbb{Z}/m$ with $(\ell,m)=1$ such that $\ell\cdot I = I'$, and for all $(p,q),(i,k)\in I$,*

$$(35)\quad \begin{cases} \lambda_{p,q,i,m-q} = \lambda'_{\ell\cdot(p,q),\ell\cdot(i,m-q)} \\ \gamma_{p,q,i,q} = \gamma'_{\ell\cdot(p,q),\ell\cdot(i,q)} \end{cases} \quad \textit{if } p\ell,\ i\ell < n \textit{ or } n < p\ell,\ i\ell \bmod m,$$

$$(36)\quad \begin{cases} \delta_{q,m-k}\lambda_{p,q,i,k} = \delta_{k,q}\gamma'_{\ell\cdot(p,q),\ell\cdot(i,k)} \\ \delta_{q,k}\gamma_{p,q,i,q} = \delta_{q,m-k}\lambda'_{\ell\cdot(p,q),\ell\cdot(i,q)} \end{cases} \quad \textit{otherwise.}$$

In particular, $A_I(\lambda,\gamma)\simeq\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ if and only if $\lambda\equiv 0\equiv\gamma$.

Proof. Suppose $\varphi : A_I(\lambda,\gamma) \to A_{I'}(\lambda',\gamma')$ is a Hopf algebra isomorphism and denote by g, h, $x_{i,k}$, $y_{i,k}$ and g', h', $x'_{i,k}$, $y'_{i,k}$ the generators of $A_I(\lambda,\gamma)$ and $A_{I'}(\lambda',\gamma')$, respectively. Since both must have the same dimension, we have that $|I| = |I'|$. Moreover, $\varphi(g) = g'$, $\varphi(h) = h'^{\ell}$ for some $\ell \in \mathbb{Z}/m$ with $(m,\ell) = 1$, and $\varphi(x_{i,k})$ and $\varphi(y_{i,k})$ are $(h'^{i\ell}, 1)$-primitive and $(h'^{-i\ell}, 1)$-primitive in $A_{I'}(\lambda',\gamma')$ for all $(i,k) \in I$, respectively. Using that $\varphi(hx_{i,k}h^{-1}) = h'^{\ell}\varphi(x_{i,k})h'^{-\ell}$ we have that $\varphi(x_{i,k}) = a_{i,k,i\ell,k\ell^{-1}}x'_{i\ell,k\ell^{-1}}$ if $i\ell < n$ and $\varphi(x_{i,k}) = b_{i,k,m-i\ell,k\ell^{-1}}y'_{m-i\ell,k\ell^{-1}}$ if $n < i\ell < m$, for some $a_{i,k,p,q}, b_{i,k,p,q} \in \Bbbk^{\times}$. In particular, this implies that $I' = \ell \cdot I$. Clearly, we may assume that $a_{i,k,p,q}, b_{i,k,p,q} = 1$.

Let $(p,q), (i,k) \in I$ and suppose that $\ell p, \ell i < n$. Then applying φ_ℓ on both sides of (25) yields

$$x'_{\ell p,\ell^{-1}q}x'_{\ell i,\ell^{-1}k} + x'_{\ell i,\ell^{-1}k}x'_{\ell p,\ell^{-1}q} = \delta_{q,m-k}\lambda_{p,q,i,k}(1 - h'^{\ell(p+i)}).$$

But the left hand side equals $\delta_{\ell^{-1}q,m-\ell^{-1}k}\lambda'_{\ell p,\ell^{-1}q,\ell i,\ell^{-1}k}(1 - h'^{\ell p+\ell i})$, by the same relation in $A_{I'}(\lambda',\gamma')$. Hence $\lambda_{p,q,i,m-q} = \lambda'_{\ell\cdot(p,q),\ell\cdot(i,m-q)}$. On the other hand, applying φ_ℓ on both sides of (26) yields

$$x'_{\ell p,\ell^{-1}q}y'_{\ell i,\ell^{-1}k} + y'_{\ell i,\ell^{-1}k}x'_{\ell p,\ell^{-1}q} = \delta_{q,k}\gamma_{p,q,i,k}(1 - h'^{\ell(p-i)}).$$

But the left hand side equals $\delta_{\ell^{-1}q,\ell^{-1}k}\gamma'_{\ell p,\ell^{-1}q,\ell i,\ell^{-1}k}(1 - h'^{\ell p-\ell i})$, by the same relation in $A_{I'}(\lambda',\gamma')$. Hence $\gamma_{p,q,i,q} = \gamma'_{\ell\cdot(p,q),\ell\cdot(i,q)}$. The proof for the remaining cases is completely analogous.

Assume now that there exists $\ell \in \mathbb{Z}/m$ with $(\ell,m) = 1$, such that $\ell \cdot I = I'$, and equations (35), (36) hold for all $(p,q), (i,k) \in I$. Then we may define an algebra morphism by $\varphi_\ell(x_{p,q}) = x'_{\ell p,\ell^{-1}q}$, $\varphi_\ell(y_{p,q}) = y'_{\ell p,\ell^{-1}q}$ if $\ell p < n$ and $\varphi_\ell(x_{p,q}) = y'_{m-\ell p,\ell^{-1}q}$, $\varphi_\ell(y_{p,q}) = x'_{m-\ell p,\ell^{-1}q}$ if $n < \ell p \bmod m$. Equations (35), (36) and the fact that $\ell \cdot I = I'$, ensure that φ_ℓ is a well-defined surjective Hopf algebra map, which is indeed an isomorphism by Theorem B. □

Note also that $\mathbb{Z}/m$ also acts on $\mathcal{L}$ with the action induced by

$$\ell \cdot r = \begin{cases} \ell^{-1}r & \text{if } 1 \le \ell^{-1}r < n \bmod m, \\ m - \ell^{-1}r & \text{if } n \le \ell^{-1}r \bmod m. \end{cases}$$

Lemma 3.17. *Let $(I,L), (I',L') \in \mathcal{K}$. $B_{I,L}(\lambda,\gamma,\theta,\mu) \simeq B_{I',L'}(\lambda',\gamma',\theta',\mu')$ if and only if there exists $\ell \in \mathbb{Z}/m$ with $(\ell,m) = 1$, such that $\ell \cdot I = I'$, $\ell \cdot L = L'$, λ, λ' and γ, γ' satisfy conditions* (35) *and* (36), *and for all $(p,q) \in I$, $r \in L$,*

$$\begin{cases} \delta_{q,m-r}\theta_{p,q,r} = \delta_{q,m-r}\theta'_{\ell\cdot(p,q),\ell\cdot r} \\ \delta_{q,r}\mu_{p,q,r} = \delta_{q,r}\mu'_{\ell\cdot(p,q),\ell\cdot r} \end{cases} \qquad \textit{if } p\ell,\ r\ell^{-1} < n,$$

$$\begin{cases} \delta_{q,m-r}\theta_{p,q,r} = \delta_{q,r}\theta'_{\ell\cdot(p,q),\ell\cdot r} \\ \delta_{q,r}\mu_{p,q,r} = \delta_{q,m-r}\mu'_{\ell\cdot(p,q),\ell\cdot r} \end{cases} \qquad \textit{if } n < p\ell,\ r\ell^{-1} \bmod m,$$

$$\begin{cases} \delta_{q,m-r}\theta_{p,q,r} = \delta_{q,m-r}\mu'_{\ell\cdot(p,q),\ell\cdot r} \\ \delta_{q,r}\mu_{p,q,r} = \delta_{q,r}\theta'_{\ell\cdot(p,q),\ell\cdot r} \end{cases} \qquad \textit{if } p\ell < n < r\ell^{-1} \bmod m,$$

$$\begin{cases} \delta_{q,m-r}\theta_{p,q,r} = \delta_{q,r}\mu'_{\ell\cdot(p,q),\ell\cdot r} \\ \delta_{q,r}\mu_{p,q,r} = \delta_{q,m-r}\theta'_{\ell\cdot(p,q),\ell\cdot r} \end{cases} \qquad \textit{if } r\ell^{-1} < n < p\ell \bmod m.$$

Proof. The proof is completely analogous to the proof of Lemma 3.16. □

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