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We prove triviality results for Einstein warped products with noncompact bases. These extend previous work by Kim and Kim. The proofs, from the viewpoint of quasi-Einstein manifolds introduced by Case, Shu and Wei, rely on maximum principles at infinity and Liouville-type theorems.

1. Introduction

The main purpose of this note is to prove the following triviality result for Einstein warped products, which extends a theorem by Kim and Kim [2003] to the case of noncompact bases.

Theorem 1. *Let $N^{n+m} = M^n \times_u F^m$, with $m > 1$, be a complete Einstein warped product with nonpositive scalar curvature ${}^N S \leq 0$, warping function $u(x) = e^{-f(x)/m}$ satisfying $\inf_M f = f_* > -\infty$ and complete Einstein fibre F. Then N is simply a Riemannian product if either of these conditions is satisfied*:

(a) *The function f has a local minimum.*

(b) *The base manifold M is complete and noncompact, the warping function satisfies $\int_M |f|^p e^{-f/m}\, d\mathrm{vol} < +\infty$, for some $1 < p < +\infty$, and $f(x_0) \leq 0$ for some point $x_0 \in M$.*

If M is compact, from (a) we recover the main result in [Kim and Kim 2003].

Our proof of Theorem 1 will rely on the link between Einstein warped product metrics and the *quasi-Einstein metrics* recently introduced by Case, Shu and Wei [2011]. In the spirit of [Pigola et al. 2011], that is, using methods from stochastic analysis and L^p-Liouville-type theorems, we shall prove scalar curvature estimates and triviality results for a complete quasi-Einstein manifold that largely extend previous theorems in [Case et al. 2011]. The main theorem will follow immediately.

In a final section, using similar techniques, we extend another triviality result for Einstein warped products obtained in [Case 2010]. A nonexistence result is also discussed.

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2. Quasi-Einstein manifolds

Consider the weighted manifold $(M^n, g_M, e^{-f} d\mathrm{vol})$, where M is a complete n-dimensional Riemannian manifold, f is a smooth real valued function on M and $d\mathrm{vol}$ is the Riemannian volume density on M. A natural extension of the Ricci tensor to weighted manifolds is the m-Bakry–Emery Ricci tensor

$$\mathrm{Ric}^m_f = \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{m} df \otimes df, \quad \text{for } 0 < m \leq \infty.$$

When f is constant this is the usual Ricci tensor, and when $m = \infty$ this is the Ricci Bakry–Emery tensor Ric_f. We call a metric m-quasi-Einstein if the m-Bakry–Emery Ricci tensor satisfies the equation

$$\mathrm{Ric}^m_f = \lambda g_M, \tag{1}$$

for some $\lambda \in \mathbb{R}$. This equation is especially interesting in that when $m = \infty$, it is exactly the gradient Ricci soliton equation. When f is constant, it gives the Einstein equation and we call the quasi-Einstein metric trivial. When m is a positive integer, it corresponds to warped product Einstein metrics.

Indeed, Case et al. [2011], elaborating on [Kim and Kim 2003], gave the following characterization of quasi-Einstein metrics.

Theorem 2. *Let $M^n \times_u F^m$ be an Einstein warped product with Einstein constant λ, warping function $u = e^{-f/m}$ and Einstein fibre F^m. Then the weighted manifold $(M^n, g_M, e^{-f} d\mathrm{vol})$ satisfies the quasi-Einstein equation* (1). *Furthermore the Einstein constant μ of the fibre satisfies*

$$\Delta f - |\nabla f|^2 = m\lambda - m\mu e^{(2/m)f}. \tag{2}$$

Conversely if the weighted manifold $(M^n, g_M, e^{-f} d\mathrm{vol})$ satisfies (1), *then f satisfies* (2) *for some constant $\mu \in \mathbb{R}$. Consider the warped product $N^{n+m} = M^n \times_u F^m$ with $u = e^{-f/m}$, and Einstein fibre F with ${}^F\mathrm{Ric} = \mu g_F$. Then N is Einstein with ${}^N\mathrm{Ric} = \lambda g_N$.*

3. Scalar curvature estimates

In this section, in the spirit of Theorem 3 of [Pigola et al. 2011], we generalize the scalar curvature estimates in Proposition 3.6 of [Case et al. 2011] to quasi-Einstein manifolds with non-constant scalar curvature. Possible rigidity at the endpoints is also discussed.

Theorem 3. *Let $(M^n, g_M, e^{-f} d\mathrm{vol})$ be a geodesically complete m-quasi-Einstein manifold, $1 < m < +\infty$, with scalar curvature S, and let $S_* = \inf_M S$.*

(a) *If* $\lambda > 0$, *then* M *is compact and*

$$\frac{n(n-1)}{m+n-1}\lambda < S_* \le n\lambda. \tag{3}$$

Moreover $S_* \neq n\lambda$ *unless* M *is Einstein.*

(b) *If* $\lambda = 0$ *and* $\inf_M f = f_* > -\infty$ *then* $S_* = 0$. *Moreover, either* $S > 0$ *or* $S(x) \equiv 0$. *In the latter case, either* f *is constant (and* M *is trivial), or* M *is isometric to the Riemannian product* $\mathbb{R} \times \Sigma$, *where* Σ *is a Ricci-flat, totally geodesic hypersurface.*

(c) *If* $\lambda < 0$ *and* $\inf_M f = f_* > -\infty$, *then*

$$n\lambda \le S_* \le \frac{n(n-1)}{m+n-1}\lambda \tag{4}$$

and $S(x) > n\lambda$ *unless* M *is Einstein.*

The proof of Theorem 3 will require the following formula obtained in [Case et al. 2011], which generalizes to the case $m < +\infty$ similar formulas for Ricci solitons ($m = +\infty$) obtained previously by Petersen and Wylie [2009]. Following the terminology introduced in [Petersen and Wylie 2010], the f-Laplacian on the weighted manifold $(M, g_M, e^{-f}d\mathrm{vol})$ is the diffusion-type operator defined by $\Delta_f u = e^f \mathrm{div}(e^{-f}\nabla u)$. It is clearly a symmetric operator on $L^2(M, e^{-f}d\mathrm{vol})$.

Lemma 4. *Let* $\mathrm{Ric}_f^m = \lambda g_M$, *for some* $\lambda \in \mathbb{R}$ *and* $m < +\infty$. *Set* $\tilde{f} = \frac{m+2}{m} f$. *Then*

$$\tfrac{1}{2}\Delta_{\tilde f} S = -\frac{m-1}{m}\Big|\mathrm{Ric} - \frac{1}{n} S g_M\Big|^2 - \frac{m+n-1}{mn}(S - n\lambda)\Big(S - \frac{n(n-1)}{m+n-1}\lambda\Big). \tag{5}$$

Proof of Theorem 3. First we show that $\inf_M S > -\infty$. According to Qian's version of Myers' theorem, this is obvious if $\lambda > 0$ because M is compact; see also the Appendix. In the general case, $\lambda \in \mathbb{R}$, we proceed as follows. Since

$$-\Big|\mathrm{Ric} - \frac{1}{n} S g_M\Big|^2 = -|\,\mathrm{Ric}\,|^2 + \frac{S^2}{n},$$

from (5) we obtain

$$\begin{aligned} \tfrac{1}{2}\Delta_{\tilde f} S &= -\frac{m-1}{m}|\,\mathrm{Ric}\,|^2 - \frac{1}{m}S^2 + \frac{m+2n-2}{m}\lambda S - \frac{n(n-1)}{m}\lambda^2 \\ &\le -\frac{1}{m}S^2 + \frac{m+2n-2}{m}\lambda S. \end{aligned} \tag{6}$$

Let $S_-(x) = \max\{-S(x), 0\}$. Then

$$\Delta_{\tilde f} S_- \ge \frac{2}{m}S_-^2 + \frac{2(m+2n-2)}{m}\lambda S_-. \tag{7}$$

Now, from Qian's estimates of weighted volumes [1997] (see also [Mari et al. 2010, Section 2 and references]), since $\mathrm{vol}_{\tilde f}(B_r) \le e^{-(2/m)f_*}\mathrm{vol}_f(B_r)$, we can apply the a priori estimate in [Pigola et al. 2011, Theorem 12] to inequality (7) on the complete weighted manifold $(M, g_M, e^{-\tilde{f}}\,d\mathrm{vol})$, and we obtain that S_- is bounded from above, or equivalently, $S_* = \inf_M S > -\infty$. Again from the volume estimates in [Qian 1997] and by Theorem 9 in [Pigola et al. 2011] applied to $(M, g_M, e^{-\tilde{f}}\,d\mathrm{vol})$, the weak maximum principle at infinity for the $\tilde{f}$-Laplacian holds on M. This produces a sequence $\{x_k\}$, such that $\Delta_{\tilde{f}} S(x_k) \geq -1/k$ and $S(x_k) \to S_*$. Taking the lim inf in (5) along $\{x_k\}$ shows that, for $m > 1$,

$$0 \leq -\frac{m+n-1}{mn}(S_* - n\lambda)\Big(S_* - \frac{n(n-1)}{m+n-1}\lambda\Big). \tag{8}$$

We now distinguish three cases.

(a) Assume $\lambda > 0$, so that M is compact. Equation (8) yields

$$\frac{n(n-1)}{m+n-1}\lambda \leq S_* \leq n\lambda.$$

Assume now that $S_* = n\lambda > 0$. Then $S \geq n\lambda \geq \dfrac{n(n-1)}{m+n-1}\lambda$, and from (5) we get

$$\tfrac{1}{2}\Delta_{\tilde{f}} S \leq -\frac{m+n-1}{mn}(S - n\lambda)\Big(S - \frac{n(n-1)}{m+n-1}\lambda\Big) \leq 0.$$

Since M is compact, S must be constant. Hence, $S = S_* = n\lambda$. Substituting in (5), we obtain that $\mathrm{Ric} = (1/n)Sg_M$, and thus that M is Einstein.

Now we show that $S_* > \big(n(n-1)/(m+n-1)\big)\lambda$. Suppose that S attains its minimum $\big(n(n-1)/(m+n-1)\big)\lambda$. Because the nonnegative function

$$v(x) = S(x) - \frac{n(n-1)}{m+n-1}\lambda$$

satisfies

$$\tfrac{1}{2}\Delta_{\tilde{f}} v \leq -\frac{m+n-1}{mn}\,v^2 + \lambda v \leq +\lambda v,$$

and v attains its minimum $v(x_0) = 0$, it follows from the minimum principle [Gilbarg and Trudinger 1983, page 35] that v vanishes identically. Hence,

$$S \equiv \frac{n(n-1)}{m+n-1}\lambda$$

is constant, and substituting in (5), we get that M is Einstein with

$$\mathrm{Ric} = \frac{n-1}{m+n-1}\lambda g_M.$$

Using this information with (1) we obtain that

$$\mathrm{Hess}(f) = \frac{1}{m}df \otimes df + \frac{m}{m+n-1}\lambda g_M > 0,$$

but this is clearly impossible because M is compact.

(b) Assume $\lambda = 0$. From (8) we conclude that $S_* = 0$. Note that, according to (5), $\Delta_{\tilde{f}} S \leq 0$. Therefore, by the minimum principle, either $S(x) > 0$ on M or $S(x) \equiv 0$. In the latter case, substituting in (5), we obtain that M is Ricci-flat and the m-quasi-Einstein equation reads $\mathrm{Hess}(f) - (1/m) df \otimes df = 0$. Therefore, either f is constant and M is Einstein, or the nonconstant function $u = e^{-f/m}$ satisfies Hess(u) $= 0$. A Cheeger–Gromoll-type argument now shows that M is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ along the Ricci-flat, totally geodesic hypersurface Σ of M.

(c) Assume $\lambda < 0$. From (8) we deduce that $n\lambda \leq S_* \leq (n(n-1)/(m+n-1))\lambda$. Suppose that $S(x_0) = n\lambda < 0$ for some $x_0 \in M$. Since the nonnegative function $w(x) = S(x) - n\lambda$ satisfies

$$\tfrac{1}{2}\Delta_{\tilde{f}} w \leq -\frac{m+n-1}{mn} w^2 - \lambda w \leq -\lambda w,$$

and w attains its minimum $w(x_0) = 0$, it follows from the minimum principle that w vanishes identically. Hence, $S \equiv n\lambda$ is constant, and substituting in (5) we get that M is Einstein. □

4. Triviality results under L^p conditions

It is well known that steady or expanding compact Ricci solitons are necessarily trivial. The same result is proven in [Kim and Kim 2003] for quasi-Einstein metrics on compact manifolds with finite m. For Ricci solitons, a generalization to the complete, noncompact setting is obtained in [Pigola et al. 2011].

In this section, using the scalar curvature estimates of Theorem 3, we get triviality for (not necessarily compact) quasi-Einstein metrics with $m < +\infty$, $\lambda \leq 0$.

Theorem 5. *Let* $(M^n, g_M, e^{-f} d\mathrm{vol})$ *be a geodesically complete noncompact* m*-quasi-Einstein manifold*, $1 < m < +\infty$. *If the quasi-Einstein constant* λ *is nonpositive and* f *satisfies, for some* $1 < p < +\infty$,

$$f \in L^p(M, e^{-f/m} d\mathrm{vol}), \tag{9}$$

and $\inf_M f = f_* > -\infty$, *then either* $f \equiv \mathrm{const} \leq 0$ *and* M *is Einstein or* $f > 0$.

Proof of Theorem 5. Tracing (1) and letting $\hat{f} = (1/m) f$, we have

$$\Delta_{\hat{f}} f = n\lambda - S. \tag{10}$$

Since $\lambda \leq 0$ and $f_* > -\infty$, from (4) of Theorem 3 we obtain that $\Delta_{\hat{f}} f \leq 0$. Applying [Pigola et al. 2011, Theorem 14] to $f_- = \max\{-f, 0\} \in L^p(M, e^{-\hat{f}} d\mathrm{vol})$, gives that f_- is constant. Hence, if there exists a point $x_0 \in M$, such that $f(x_0) \leq 0$, then $f \equiv f(x_0) \leq 0$. □

Remark 6. From the proof, it follows that if either M is compact or f attains its absolute minimum, then $f \equiv \text{const}$. Actually, it was pointed out to us by Dezhong Chen that the same conclusion holds if we merely assume that f attains a local minimum at some point $x_0 \in M$. The following proposition holds.

Proposition 7. *Let $(M, g_M, e^{-f} d\mathrm{vol})$ be a geodesically complete noncompact m-quasi-Einstein manifold, $1 < m < +\infty$. If the quasi-Einstein constant λ is nonpositive and f satisfies $f_* > -\infty$, then any local minimum of f is actually an absolute minimum.*

Proof. Assume that f attains a local minimum $x_0 \in M$. Evaluating (10) at x_0, we get

$$S(x_0) \le n\lambda.$$

Since $\lambda \le 0$, by Theorem 3, M is Einstein and S is identically $n\lambda$. Thus the quasi-Einstein equation (1) reads

$$\mathrm{Hess}(f) = (1/m) df \otimes df. \tag{11}$$

In particular, $\mathrm{Hess}(f)$ is positive semidefinite on M and this implies the thesis. □

5. Proof of the main theorem

Putting together the results of the previous sections, we easily obtain a proof of Theorem 1.

According to Theorem 2, M is quasi-Einstein. Statement (a) follows immediately from Remark 6 and Proposition 7. In case (b), since $(n+m)\lambda = {}^N S \le 0$, we get by Theorem 5 that f, and therefore u, is a constant function.

6. Other triviality results

Another triviality result for Einstein warped products was obtained by Case [2010].

Theorem 8 [Case 2010]. *Let $N^{n+m} = M^n \times_u F^m$ be a complete warped product with warping function $u(x) = e^{-f(x)/m}$, scalar curvature ${}^N S \ge 0$ and complete Einstein fibre F. Then N is simply a Riemannian product, provided the base manifold M is complete, and the scalar curvature of F satisfies ${}^F S \le 0$.*

In the following theorem, we obtain the same conclusion in case the fibers have nonnegative scalar curvature, assuming an integrability condition on the warping function u. We observe that nontrivial examples with ${}^N S \le 0$ and ${}^F S \ge 0$ are constructed in [Besse 1987, 9.118]. Thus, the integrabilty assumption is necessary.

Theorem 9. *Let $N^{n+m} = M^n \times_u F^m$ be a complete Einstein warped product with warping function $u(x) = e^{-f(x)/m}$, scalar curvature ${}^N S \le 0$, and complete Einstein fibre F. Then N is simply a Riemannian product, provided the base manifold M*

is complete, the warping function satisfies $\int_M e^{-((p+m)/m)f}\,d\mathrm{vol} < +\infty$ *for some* $1 < p < +\infty$, *and the scalar curvature of* F *satisfies* ${}^F S \geq 0$. *In this case* M *and* F *are Ricci-flat and* M *is compact.*

Combining Theorem 8 and Theorem 9 immediately gives the following.

Corollary 10. *Let* N *be a complete Ricci-flat warped product with complete Einstein fibre* F *and warping function* $u(x) = e^{-f(x)/m}$ *satisfying* $u \in L^p(M, e^{-f}d\mathrm{vol})$, *for some* $1 < p < +\infty$. *Then* N *is simply a Riemannian product.*

Proof of Theorem 9. Just observe that computing the f-Laplacian of u and using (2), one obtains

$$\Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{m^2}|\nabla f|^2. \tag{12}$$

Thus, in our assumptions, we obtain that $\Delta_f u \geq 0$. Since $0 < u \in L^p(M, e^{-f}d\mathrm{vol})$ [Pigola et al. 2011, Theorem 14], we obtain the constancy of u. Up to a rescaling of the metric of F, we can suppose $u = 1$.

Now, since the Riemannian product $M \times F$ is Einstein, both M and F are Einstein manifolds with the same Einstein constant. In particular, ${}^M S$ and ${}^F S$ have the same sign. By our assumption on the signs of ${}^N S$ and ${}^F S$, we thus obtain that both M and F are Ricci-flat. Finally, since u (and thus f) is constant, from the integrability condition, we obtain $\mathrm{vol}(M) < +\infty$. Thus, by a result of Calabi and Yau (see [Yau 1976]), we obtain that M is compact. □

We end this section with a nonexistence result. Recall that by the volume estimates in [Qian 1997] and by [Pigola et al. 2011, Theorem 9], the weak maximum principle for the f-Laplacian holds on $(M, g_M, e^{-f}d\mathrm{vol})$, provided $\mathrm{Ric}^m_f = \lambda g_M$ for some $\lambda \in \mathbb{R}$, $m < +\infty$.

Theorem 11. *There is no complete Einstein warped product* $N = M^n \times_u F^m$ *with warping function* $u = e^{-f/m} \in L^\infty(M)$, *scalar curvature* ${}^N S < 0$ *and Einstein fibre* F *with* ${}^F S \geq 0$.

Proof. Since $m\mu = {}^F S \geq 0$, from (12) we have

$$\Delta_f u \geq -\lambda u. \tag{13}$$

Since, by assumption, u satisfies $\sup_M u = u^* < +\infty$, by the weak maximum principle at infinity for the f-Laplacian, there exists a sequence $\{x_k\} \subset M$, along which $u(x_k) \geq u^* - 1/k$ and $\Delta_f u(x_k) \leq 1/k$. Thus evaluating (13) along $\{x_k\}$ and taking the limit as $k \to +\infty$, we obtain that $\lambda u^* \geq 0$, and since $u^* > 0$, we cannot have $\lambda < 0$. □

Appendix

An extension of Myers' theorem to weighted manifolds with a positive lower bound on the m-Bakry–Emery Ricci tensor (m finite) was obtained by Qian [1997]. For generalizations of Myers' theorem in a different direction, see [Morgan 2006].

In this section we extend Qian's theorem by allowing some negativity of the m-Bakry–Emery Ricci tensor. Our considerations begin with the following Bochner formula for the m-Bakry–Emery Ricci tensor; see, for example, [Setti 1998].

Let $u : M^n \to \mathbb{R}$ be a smooth function on a complete weighted manifold

$$(M^n, g_M, e^{-f} d\mathrm{vol}).$$

Then

$$\begin{aligned}(14)\quad & \tfrac{1}{2}\Delta_f |\nabla u|^2 \\ &= \big|\mathrm{Hess}(u)\big|^2 + g_M(\nabla u, \nabla\Delta_f u) + \mathrm{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m}\big|g_M(\nabla f, \nabla u)\big|^2.\end{aligned}$$

With this formula one obtains the following generalization of a well-known lemma, which estimates the integral of Ricci along minimizing geodesics. The proof is modeled on [Qian 1997].

Lemma 12. *Let* $(M^n, g_M, e^{-f} d\mathrm{vol})$ *be a complete weighted manifold, and consider the m-Bakry–Emery Ricci tensor* Ric_f^m *for m finite. Fix* $o \in M$ *and let* $r(x) = \mathrm{dist}(x, o)$. *For any point* $q \in M$, *let* $\gamma_q : [0, r(q)] \to M$ *be a minimizing geodesic from o to q, such that* $|\dot\gamma_q| = 1$. *If* $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$ *is such that* $h(0) = h(r(q)) = 0$, *then for every* $q \in M$, *it holds that*

$$0 \le \int_0^{r(q)} (m+n-1)(h')^2\, ds - \int_0^{r(q)} h^2\, \mathrm{Ric}_f^m(\dot\gamma_q, \dot\gamma_q)\, ds. \tag{15}$$

Proof. Fix a point $q \notin \mathrm{cut}(o)$. Straightforward computations show that

$$\frac{(\Delta_f r)^2}{m+n-1} \le \frac{(\Delta r)^2}{n-1} + \frac{|g_M(\nabla f, \nabla r)|^2}{m}, \tag{16}$$

$$|\,\mathrm{Hess}(r)|^2 \ge \frac{(\Delta r)^2}{n-1}. \tag{17}$$

Using (16) and (17), from the Bochner formula (14) applied to the distance function $r(x)$, we obtain that

$$0 \ge \frac{(\Delta_f r)^2}{m+n-1} + g_M(\nabla r, \nabla\Delta_f r) + \mathrm{Ric}_f^m(\nabla r, \nabla r).$$

Evaluating this along a minimizing geodesic γ_q, such that $|\dot\gamma_q| = 1$, we get

$$0 \ge \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} + \frac{d}{ds}(\Delta_f(r \circ \gamma_q)) + \mathrm{Ric}_f^m(\dot\gamma_q, \dot\gamma_q). \tag{18}$$

If $h \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, $h \geq 0$, and $h(0) = 0$, then multiplying (18) by h^2 and integrating on $[0, t]$, we obtain

$$0 \geq \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} ds + \int_0^t \frac{d}{ds}(\Delta_f r \circ \gamma_q) h^2 ds + \int_0^t h^2 \operatorname{Ric}_f^m(\dot{\gamma}_q, \dot{\gamma}_q) ds.$$

Since $(\Delta_f r \circ \gamma_q) h^2 \to 0$ as $r \to 0$, integrating by parts, we have

$$0 \geq \int_0^t h^2 \frac{(\Delta_f r \circ \gamma_q)^2}{m+n-1} ds + h^2(t)(\Delta_f r \circ \gamma_q)(t) - 2\int_0^t hh'(\Delta_f r \circ \gamma_q) ds + \int_0^t h^2 \operatorname{Ric}_f^m(\dot{\gamma}_q, \dot{\gamma}_q) ds. \tag{19}$$

Since

$$-2hh'(\Delta_f r \circ \gamma_q) \geq \frac{-h^2(\Delta_f r \circ \gamma_q)^2}{m+n-1} - (m+n-1)(h')^2,$$

we deduce that

$$0 \geq h^2(t)(\Delta_f r \circ \gamma_q) - \int_0^t (m+n-1)(h')^2 ds + \int_0^t \operatorname{Ric}_f^m(\dot{\gamma}_q, \dot{\gamma}_q) h^2 ds.$$

Thus, taking $t = r(q)$ and choosing h such that $h^2(r(q)) = 0$, we get (15) for $q \notin \mathrm{cut}(o)$. To treat the general case, one can use the Calabi trick. Namely, suppose that $q \in \mathrm{cut}(o)$. Translating the origin o to $o_\epsilon = \gamma_q(\epsilon)$ so that $q \notin \mathrm{cut}(o_\epsilon)$, using the triangle inequality, and finally, taking the limit as $\epsilon \to 0$, one checks that (15) also holds in this case. □

From Lemma 12, some Myers-type results can be proven. Here we generalize a theorem of Galloway [1979].

Theorem 13. *Let $(M^n, g_M, e^{-f} d\mathrm{vol})$ be a complete weighted manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from p to q parameterized by arc length. Suppose that there exist constants c and $G \geq 0$ such that for each pair of points p, q, it holds that*

$$\operatorname{Ric}_f^m(\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q})|_{\gamma_{p,q}(t)} \geq (m+n-1)\Big[c^2 + \frac{d}{dt}(g \circ \gamma_{p,q})\Big],$$

for some $C^1(M)$ function g satisfing $\sup_M |g| \leq G$, $m < +\infty$. Then M is compact and

$$\mathrm{diam}(M) \leq \frac{1}{c}\left[\frac{2G}{c} + \sqrt{\frac{4G^2}{c^2} + \pi^2}\,\right]. \tag{20}$$

Proof. Define L to be the length of $\gamma_{p,q}$ between p and q and set

$$h(t) := \sin\Big(\frac{\pi}{L} t\Big).$$

Compute

$$\int_0^L h^2(t)dt = \int_0^L \sin^2\Big(\frac{\pi}{L}t\Big)dt = \frac{L}{2}, \quad \int_0^L h'^2(t)dt = \frac{\pi^2}{L^2}\int_0^L \cos^2\Big(\frac{\pi}{L}t\Big)dt = \frac{\pi^2}{2L}.$$

Then, applying Lemma 12, we have

$$\begin{aligned}(21)\quad & \frac{\pi^2(m+n-1)}{2L} \\ & = \int_0^L (m+n-1)h'^2 \geq \int_0^L h^2 \operatorname{Ric}_f^m(\dot\gamma_{p,q}, \dot\gamma_{p,q})|_{\gamma_{p,q}} ds \\ & \geq c^2(m+n-1)\int_0^L h^2 + (m+n-1)\int_0^L h^2 \frac{d}{dt}(g\circ\gamma_{p,q}) \\ & = \frac{c^2(m+n-1)L}{2} + (m+n-1)h^2 g(\gamma_{p,q})\Big|_0^L \\ & \qquad - (m+n-1)\Big[\int_0^{L/2}\Big(\frac{d}{dt}h^2\Big)(g\circ\gamma_{p,q}) + \int_{L/2}^L \Big(\frac{d}{dt}h^2\Big)(g\circ\gamma_{p,q})\Big] \\ & \geq \frac{c^2(m+n-1)L}{2} - (m+n-1)G\Big[\int_0^{L/2}\Big(\frac{d}{dt}h^2\Big) + \int_{L/2}^L \Big|\frac{d}{dt}h^2\Big|\Big] \\ & \geq \frac{c^2(m+n-1)L}{2} - 2(m+n-1)G.\end{aligned}$$

Finally, this can be written as

$$c^2L^2 - 4GL - \pi^2 \leq 0,$$

which in turn implies (20), because p and q are arbitrary. □

Reasoning as in the classical case [Galloway 1982; Mastrolia et al. 2011], the validity of (15) and an integration by parts show that the compactness of M depends on the behavior, and on the position of the zeros, of the solution of the differential equation along minimizing geodesics

$$(22)\qquad -h''(t) - \frac{\operatorname{Ric}_f^m(\dot\gamma, \dot\gamma)}{m+n-1}h(t) = 0.$$

It remains to find sufficient conditions on Ric_f^m for which solutions of the differential equation (22) have a first zero at finite time. Minor changes to the proofs of the results contained in [Mastrolia et al. 2011] lead to similar compactness results in the weighted setting. In particular, we state the following theorem, in which a Myers-type conclusion is obtained assuming a nonpositive lower bound on Ric_f^m.

Theorem 14. *Let* $\operatorname{Ric}_f^m \geq -(m+n-1)B^2$, *for some constant* $B \geq 0$, $m < +\infty$. *Suppose there is a point* $q \in M$ *such that along each geodesic* $\gamma : [0, +\infty) \to M$

parameterized by arc length, with $\gamma(0) = q$, it holds that either

$$\int_a^b t \frac{\mathrm{Ric}_f^m(\dot\gamma, \dot\gamma)}{m+n-1}\, dt > B\Big\{b + a\frac{e^{2Ba}+1}{e^{2Ba}-1}\Big\} + \frac{1}{4}\log\Big(\frac{b}{a}\Big), \tag{23}$$

or

$$\int_a^b t^\alpha \frac{\mathrm{Ric}_f^m(\dot\gamma, \dot\gamma)}{m+n-1}(t)\, dt > B\Big\{b^\alpha + a^\alpha\frac{e^{2Ba}+1}{e^{2Ba}-1}\Big\} + \frac{\alpha^2}{4(1-\alpha)}\{a^{\alpha-1} - b^{\alpha-1}\}, \tag{24}$$

for some $0 < a < b$ and $\alpha \neq 1$. Then M is compact.

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