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We classify exact indecomposable module categories over the representation category of all nontrivial Hopf algebras with coradical $\mathbb{S}_3$ and $\mathbb{S}_4$. As a byproduct, we compute all its Hopf–Galois extensions and we show that these Hopf algebras are cocycle deformations of their graded versions.

1. Introduction

Given a tensor category $\mathscr{C}$, an *exact module category* [Etingof and Ostrik 2004a] over $\mathscr{C}$ is an abelian category $\mathscr{M}$ equipped with a biexact functor $\otimes : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$ subject to natural associativity and unit axioms, such that, for any projective object $P \in \mathscr{C}$ and any $M \in \mathscr{M}$, the object $P \otimes M$ is again projective.

Exact module categories, or *representations* of $\mathscr{C}$, are interesting objects to consider. They are implicitly present in many areas of mathematics and mathematical physics, such as subfactor theory [Böckenhauer et al. 2000], affine Hecke algebras [Bezrukavnikov and Ostrik 2004], extensions of vertex algebras [Kirillov and Ostrik 2002; Huang and Kong 2004], Calabi–Yau algebras [Ginzburg 2007], and conformal field theory, see for example [Barmeier et al. 2010; Fuchs and Schweigert 2003; Coquereaux and Schieber 2007; Coquereaux and Schieber 2008]. Module categories have been used in the study of fusion categories [Etingof et al. 2005], and in the theory of (weak) Hopf algebras [Ostrik 2003b; Mombelli 2010; Nikshych 2008].

The classification of exact module categories over a fixed finite tensor category $\mathscr{C}$ has been undertaken by several authors:

- when $\mathscr{C}$ is the semisimple quotient of $U_q(\mathfrak{sl}_2)$, by [Kirillov and Ostrik 2002; Etingof and Ostrik 2004b];
- over the tensor categories of representations of finite supergroups, by [Etingof and Ostrik 2004a];

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- over $\operatorname{Rep}(D(G))$, where $D(G)$ is the Drinfeld double of a finite group G, by [Ostrik 2003a];
- over the tensor category of representations of Lusztig's small quantum group $u_q(\mathfrak{sl}_2)$, by [Mombelli 2010];
- and more generally over $\operatorname{Rep}(H)$, where H is a lifting of a quantum linear space, by [Mombelli 2011].

The main goal of this paper is the classification of exact module categories over the representation category of any nontrivial (that is, different from the group algebra) finite-dimensional Hopf algebra with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$.

Finite-dimensional Hopf algebras with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$ were classified in [Andruskiewitsch et al. 2010] and [García and García Iglesias $\geq$ 2011], respectively. For all these Hopf algebras, the associated graded Hopf algebras $\operatorname{gr} H$ is of the form $\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n$ for $n = 3$ or 4, where X is a finite set equipped with a map $\triangleright : X \times X \to X$ satisfying certain axioms that make it into a *rack*, and where $q : X \times X \to \Bbbk^{\times}$ is a 2-cocycle. We obtain the result:

Let $n = 3$ or 4, and let $\mathcal{M}$ be an exact indecomposable module category over $\operatorname{Rep}(\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n)$. There exist

- a subgroup $F < \mathbb{S}_n$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^{\times})$,
- a subset $Y \subseteq X$ invariant under the action of F,
- a family of scalars $\{\xi_C\}$ compatible (see Definition 7.1) with (F, ψ, Y),

such that $\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$, where $\mathcal{B}(Y, F, \psi, \xi)$ is a left $\mathfrak{B}(X, q) \# \Bbbk\mathbb{S}_n$-comodule algebra constructed from the data (Y, F, ψ, ξ).

We also show that, if H is a finite-dimensional Hopf algebra with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$, then H and $\operatorname{gr} H$ are cocycle deformations of each other. This implies that there is a bijective correspondence between module categories over $\operatorname{Rep}(H)$ and $\operatorname{Rep}(\operatorname{gr} H)$.

The content of the paper is as follows. In Section 3 we recall the basic results on module categories over finite-dimensional Hopf algebras. We recall the main result of [Mombelli 2011] that gives an isomorphism between Loewy-graded comodule algebras, and a semidirect product of a twisted group algebra and a homogeneous coideal subalgebra inside the Nichols algebra.

In Section 4 we show how to distinguish Morita equivariant classes of comodule algebras over pointed Hopf algebras.

In Section 5 we recall the definition of a rack X and a ql-datum $\mathfrak{Q}$, and how to construct (quadratic approximations to) Nichols algebras $\widehat{\mathfrak{B}}_2(X, q)$ and pointed Hopf algebras $\mathcal{H}(\mathfrak{Q})$ from them. In particular, we recall a presentation of all finite-dimensional Hopf algebras with coradical $\Bbbk\mathbb{S}_3$ or $\Bbbk\mathbb{S}_4$.

In Section 6, we give a classification of connected homogeneous left coideal subalgebras of $\widehat{\mathfrak{B}}_2(X, q)$ and also a presentation by generators and relations.

In Section 7 we introduce a family of comodule algebras large enough to classify module categories. We give an explicit Hopf–biGalois extension over $\widehat{\mathfrak{B}}_2(X, q)\#\Bbbk\mathbb{S}_n$, $n \in \mathbb{N}$, and a lifting $\mathcal{H}(\mathcal{Q})$, proving that there is a bijective correspondence between module categories over $\operatorname{Rep}(\widehat{\mathfrak{B}}_2(X, q)\#\Bbbk\mathbb{S}_n)$ and $\operatorname{Rep}(\mathcal{H}(\mathcal{Q}))$, $n = 3, 4$. In particular, we obtain that any pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$ is a cocycle deformation of its associated graded algebra, a result analogous to a theorem of Masuoka for abelian groups [Masuoka 2008]. Finally, the classification of module categories over $\operatorname{Rep}(\widehat{\mathfrak{B}}_2(X, q)\#\Bbbk\mathbb{S}_n)$ is presented in this section and, as a consequence, all Hopf–Galois objects over $\widehat{\mathfrak{B}}_2(X, q)\#\Bbbk\mathbb{S}_n$ are described.

2. Preliminaries and notation

We will denote by $\Bbbk$ an algebraically closed field of characteristic zero. The tensor product over the field $\Bbbk$ will be denoted by $\otimes$. All vector spaces, algebras and categories will be considered over $\Bbbk$. For any algebra A, ${}_A\mathcal{M}$ will denote the category of finite-dimensional left A-modules.

The symmetric group on n letters is denoted $\mathbb{S}_n$, and the conjugacy class of all j-cycles in $\mathbb{S}_n$ is denoted $\mathcal{O}^n_j$. For any group G, any 2-cocycle $\psi \in Z^2(G, \Bbbk^\times)$, and any $h \in G$, we will denote $\psi^h(x, y) = \psi(h^{-1}xh,\ h^{-1}yh)$ for all $x, y \in G$.

If H is a Hopf algebra, a 2-cocycle σ in H is a convolution-invertible linear map $\sigma : H \times H \to \Bbbk$ such that

$$\sigma(x_{(1)}, y_{(1)})\,\sigma(x_{(2)}\,y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\,\sigma(x, y_{(2)}\,z_{(2)}) \tag{2-1}$$

and $\sigma(x, 1) = \sigma(1, x) = \varepsilon(x)$, for every $x, y, z \in H$. The set of 2-cocycles in H is denoted by $Z^2(H)$.

If A is an H-comodule algebra via $\lambda : A \to H \otimes A$, we will say that a (right) ideal J is H-costable if $\lambda(J) \subseteq H \otimes J$. We will say that A is (right) H-simple if there is no nontrivial (right) ideal H-costable in A.

If $H = \bigoplus H(i)$ is a coradically graded Hopf algebra, we will say that a left coideal subalgebra $K \subseteq H$ is *homogeneous* if $K = \bigoplus K(i)$ is graded as an algebra and, for any n, $K(n) \subseteq H(n)$ and $\Delta(K(n)) \subseteq \bigoplus_{i=0}^n H(i) \otimes K(n-i)$. K is said to be *connected* if $\mathcal{K} \cap H(0) = \Bbbk$.

If $H = \mathfrak{B}(V)\#\Bbbk G$, where V is a Yetter–Drinfeld module over G and $K \subseteq H$ is a coideal subalgebra, we will denote by $\operatorname{Stab} K$ the biggest subgroup of G such that the adjoint action of $\operatorname{Stab} K$ leaves K invariant.

If H is a finite-dimensional Hopf algebra, then $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$ will denote the coradical filtration. When $H_0 \subseteq H$ is a Hopf subalgebra, the associated graded algebra $\operatorname{gr} H$ is a coradically graded Hopf algebra. If (A, λ) is a left

H-comodule algebra, the coradical filtration on H induces a filtration on A, given by $A_n = \lambda^{-1}(H_n \otimes A)$. This filtration is called the *Loewy series* on A.

The associated graded algebra $\operatorname{gr} A$ is a left $\operatorname{gr} H$-comodule algebra. The algebra A is right H-simple if and only if $\operatorname{gr} A$ is right $\operatorname{gr} H$-simple; see [Mombelli 2010, Section 4].

3. Representations of tensor categories

Given a tensor category $\mathscr{C} = (\mathscr{C}, \otimes, a, \mathbf{1})$, a *module category* over $\mathscr{C}$ (or a *representation* of $\mathscr{C}$) is an abelian category $\mathscr{M}$ equipped with an exact bifunctor

$$\overline{\otimes} : \mathscr{C} \times \mathscr{M} \to \mathscr{M}$$

and natural associativity and unit isomorphisms

$$m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$$

and $\ell_M : \mathbf{1} \otimes M \to M$, satisfying natural associativity and unit axioms; see [Etingof and Ostrik 2004a; Ostrik 2003b]. We assume, as in the first of these papers, that all module categories have only finitely many isomorphism classes of simple objects.

A module category is *indecomposable* if it is not equivalent to a direct sum of two nontrivial module categories. A module category $\mathscr{M}$ over a finite tensor category $\mathscr{C}$ is *exact* [Etingof and Ostrik 2004a] if, for any projective $P \in \mathscr{C}$ and any $M \in \mathscr{M}$, the object $P \otimes M$ is again projective in $\mathscr{M}$.

If $\mathscr{M}$ is an exact module category over $\mathscr{C}$, then the dual category $\mathscr{C}^*_{\mathscr{M}}$ (see [Etingof and Ostrik 2004a]) is a finite tensor category. There is a bijective correspondence between the set of equivalence classes of exact module categories over $\mathscr{C}$ and over $\mathscr{C}^*_{\mathscr{M}}$; see [Etingof and Ostrik 2004a, Theorem 3.33]. This implies that, for any finite-dimensional Hopf algebra, there is a bijective correspondence between the set of equivalence classes of exact module categories over $\operatorname{Rep}(H)$ and $\operatorname{Rep}(H^*)$.

3A. ***Module categories over pointed Hopf algebras.*** We are interested in exact indecomposable module categories over the representation category of finite-dimensional Hopf algebras. If H is a Hopf algebra and $\lambda : \mathscr{A} \to H \otimes \mathscr{A}$ is a left H-comodule algebra, the category ${}^H\mathscr{M}_{\mathscr{A}}$ is the category of finite-dimensional right $\mathscr{A}$-modules left H-comodules, where the comodule structure is a $\mathscr{A}$-module morphism. If $\mathscr{A}'$ is another left H-comodule algebra the category ${}^H_{\mathscr{A}}\mathscr{M}_{\mathscr{A}'}$ is defined analogously.

The category of finite-dimensional left $\mathscr{A}$-modules ${}_{\mathscr{A}}\mathscr{M}$ is a representation of $\operatorname{Rep}(H)$. The action $\overline{\otimes} : \operatorname{Rep}(H) \times {}_{\mathscr{A}}\mathscr{M} \to {}_{\mathscr{A}}\mathscr{M}$ is given by $V \,\overline{\otimes}\, M = V \otimes M$ for all $V \in \operatorname{Rep}(H)$ and $M \in {}_{\mathscr{A}}\mathscr{M}$. The left $\mathscr{A}$-module structure on $V \otimes M$ is given by the coaction λ.

If $\mathcal{M}$ is an exact indecomposable module over $\operatorname{Rep}(H)$, then there exists a left H-comodule algebra $\mathcal{A}$ right H-simple with trivial coinvariants such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ as modules over $\operatorname{Rep}(H)$; see [Andruskiewitsch and Mombelli 2007, Theorem 3.3].

If $\mathcal{A}$ and $\mathcal{A}'$ are two right H-simple left H-comodule algebras such that the categories ${}_{\mathcal{A}}\mathcal{M}$ and ${}_{\mathcal{A}'}\mathcal{M}$ are equivalent as representations over $\operatorname{Rep}(H)$, then there exists an equivariant Morita context (P, Q, f, g); that is, $P \in {}^{H}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$, $Q \in {}^{H}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}'}$, $f : P \otimes_{\mathcal{A}} Q \to \mathcal{A}'$ and $g : Q \otimes_{\mathcal{A}'} P \to \mathcal{A}$, such that the latter are bimodule isomorphisms. Moreover, it holds that $\mathcal{A}' \simeq \operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\operatorname{End}_{\mathcal{A}}(P)$ is given by $\lambda(T) = T_{(-1)} \otimes T_{(0)}$, where

$$\langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha,\ T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)}, \tag{3-1}$$

for any $\alpha \in H^*$, $T \in \operatorname{End}_{\mathcal{A}}(P)$ and $p \in P$. See [Andruskiewitsch and Mombelli 2007] for more details.

From the previous paragraph, we can see that the categories ${}^{H}\mathcal{M}_{\mathcal{A}}$ play a central role in the theory. The following theorem will be of great use in the next section.

Theorem 3.1 [Skryabin 2007]. *Let H be a Hopf algebra and $\mathcal{A}$ a left H-comodule algebra, both finite-dimensional.*

(i) *If $\mathcal{A}$ is H-simple and $M \in {}^{H}\mathcal{M}_{\mathcal{A}}$, then there exists $t \in \mathbb{N}$ such that M^t, the direct sum of t copies of M, is a free $\mathcal{A}$-module.*

(ii) *$M \in {}^{H}\mathcal{M}_{\mathcal{A}}$ is free as an A-module if and only if there exists a maximal ideal $J \subset \mathcal{A}$ such that $M/M \cdot J$ is free as a $\mathcal{A}/J$-module.* □

Part (i) of this theorem is present in the proof of Theorem 3.5 of [Skryabin 2007]. Part (ii), which is Theorem 4.2 of the same paper, will be particularly useful when the ideal J is such that $\mathcal{A}/J = \Bbbk$, since in this case $M/M \cdot J$ is automatically free.

Theorem 3.2 [Mombelli 2011, Theorem 3.3]. *Let G be a finite group and let H be a finite-dimensional pointed Hopf algebra with coradical $\Bbbk G$. Assume there exists $V \in {}^{G}_{G}\mathcal{YD}$ such that $\operatorname{gr} H = U = \mathfrak{B}(V) \# \Bbbk G$. Let $\mathcal{A}$ be a left H-comodule algebra right H-simple with trivial coinvariants. There exist*

(1) *a subgroup $F \subseteq G$,*

(2) *a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,*

(3) *a homogeneous left coideal subalgebra $\mathcal{K} = \bigoplus_{i=0}^{m} \mathcal{K}(i) \subseteq \mathfrak{B}(V)$ where $\mathcal{K}(1) \subseteq V$ is a $\Bbbk G$-subcomodule invariant under the action of F,*

such that $\operatorname{gr} \mathcal{A} \simeq \mathcal{K} \# \Bbbk_\psi F$ as left U-comodule algebras. □

The algebra structure and the left U-comodule structure of $\mathcal{K} \# \Bbbk_\psi F$ is given as follows. If $x, y \in \mathcal{K}$, $f, g \in F$ then

$$(x \# g)(y \# f) = x(g \cdot y) \# \psi(g, f)\, gf,$$

$$\lambda(x \# g) = (x_{(1)} \# g) \otimes (x_{(2)} \# g),$$

where the action of F on $\mathcal{K}$ is the restriction of the action of G on $\mathfrak{B}(V)$ as an object in ${}^{G}_{G}\mathcal{YD}$. Observe that F is necessarily a subgroup of $\operatorname{Stab}\mathcal{K}$.

4. Equivariant equivalence classes of comodule algebras

In this section we will show how to distinguish equivalence classes of some comodule algebras over pointed Hopf algebras, and then we will apply this result to our cases. Many of the ideas here are already contained in [Mombelli 2010; Mombelli 2011], although with less generality.

Let Γ be a finite group and H be a finite-dimensional pointed Hopf algebra with coradical $\Bbbk\Gamma$ and with coradical filtration $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_m = H$. Assume there is $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $\operatorname{gr} H = U = \mathfrak{B}(V) \# \Bbbk\Gamma$.

Lemma 4.1. *Take* Γ, U *as above, and let* $\sigma \in Z^2(\Gamma, \Bbbk^\times)$ *be a* 2*-cocycle. There exists a* 2*-cocycle* $\varsigma \in Z^2(U)$ *such that* $\varsigma|_{\Gamma\times\Gamma} = \sigma$.

Proof. Consider the linear map $\varsigma : U \times U \to \Bbbk$ defined on homogeneous elements $x, y \in U$ by

$$\varsigma(x, y) = \begin{cases} \sigma(x, y) & \text{if } x, y \in U(0); \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\varsigma(x, 1) = \varsigma(1, x) = \varepsilon(x)$ by definition. We have to check that (2-1) holds for $x \in U(m)$, $y \in U(n)$, $z \in U(k)$, and $m, n, k \in \mathbb{N}$. If $k > 0$, the left-hand side of (2-1) is zero. Set $\Delta(z) = \sum_{i=0}^{k} z^i \otimes z^{k-i}$, with $z^s \in U(s)$, $s = 0, \dots, k$. Analogously, set $\Delta(y) = \sum_{j=0}^{n} y^j \otimes y^{n-j}$, with $y^t \in U(t)$, $t = 0, \dots, n$. Then the right-hand side is

$$\sum_{i=0}^{k} \sum_{j=0}^{n} \varsigma(x, y^{n-j} z^{k-i})\, \varsigma(y^j, z^i) = \varsigma(x, y^n z^k) = 0,$$

and thus (2-1) holds. Both sides of this equation are similarly seen to be zero if $m > 0$ or $n > 0$, while the case $m = n = k = 0$ holds by definition of ς. This map is convolution invertible, and its inverse ς^{-1} is defined in an analogous manner, using σ^{-1}. □

Let $\mathcal{A}, \mathcal{A}'$ be two right H-simple left H-comodule algebras. Let $F, F' \subseteq \Gamma$ be subgroups and let $\psi \in Z^2(F, \Bbbk^\times)$, $\psi' \in Z^2(F', \Bbbk^\times)$ be two cocycles such that $\mathcal{A}_0 = \Bbbk_\psi F$ and $\mathcal{A}'_0 = \Bbbk_{\psi'} F'$. Let $K, K' \in \mathfrak{B}(V)$ be two homogeneous coideal subalgebras such that $\operatorname{gr}\mathcal{A} = K \# \Bbbk_\psi F$ and $\operatorname{gr}\mathcal{A}' = K' \# \Bbbk_{\psi'} F'$.

The main result of this section is this:

Theorem 4.2. *The categories* ${}_{\mathcal{A}}\mathcal{M}$ *and* ${}_{\mathcal{A}'}\mathcal{M}$ *are equivalent as modules over* $\operatorname{Rep}(H)$ *if and only if there exists an element* $g \in \Gamma$ *such that* $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ *as comodule algebras.*

Proof. Let us assume that ${}_{\mathcal{A}}\mathcal{M} \cong {}_{\mathcal{A}'}\mathcal{M}$ as Rep(H)-modules. By [Andruskiewitsch and Mombelli 2007, Proposition 1.24] there exists an equivariant Morita context (P, Q, f, h); that is, $P \in {}^{H}_{\mathcal{A}'}\mathcal{M}_{\mathcal{A}}$, $Q \in {}^{H}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}'}$, $f : P \otimes_{\mathcal{A}} Q \to \mathcal{A}'$ and $h : Q \otimes_{\mathcal{A}'} P \to \mathcal{A}$, where the latter are bimodule isomorphisms, and $\mathcal{A}' \simeq \operatorname{End}_{\mathcal{A}}(P)$ as comodule algebras. The comodule structure on $\operatorname{End}_{\mathcal{A}}(P)$ is given by $\lambda : \operatorname{End}_{\mathcal{A}}(P) \to H \otimes \operatorname{End}_{\mathcal{A}}(P)$ with $\lambda(T) = T_{(-1)} \otimes T_{(0)}$, where

$$\langle \alpha, T_{(-1)} \rangle T_{(0)}(p) = \langle \alpha,\ T(p_{(0)})_{(-1)} \mathcal{S}^{-1}(p_{(-1)}) \rangle T(p_{(0)})_{(0)}, \tag{4-1}$$

for any $\alpha \in H^*$, $T \in \operatorname{End}_{\mathcal{A}}(P)$, and $p \in P$.

For any $i = 0, \dots, m$, define $P(i) = P_i / P_{i-1}$, where $P_{-1} = 0$. The graded vector space $\operatorname{gr} P = \bigoplus_{i=0}^{m} P(i)$ has an obvious structure that makes it into an object in the category ${}^{U}\mathcal{M}_{K\#\Bbbk_\psi F}$. Denote the coaction by $\overline{\delta} : \operatorname{gr} P \to U \otimes \operatorname{gr} P$. In particular, $\operatorname{gr} P \in {}^{U}\mathcal{M}_{K}$; thus, by Theorem 3.1(2), we have $\operatorname{gr} P \simeq M \otimes K$, where $M = \operatorname{gr} P / (\operatorname{gr} P \cdot K^+)$ since $K/K^+ = \Bbbk$.

We have $\overline{\delta}(\operatorname{gr} P \cdot K^+) \subset (U \otimes \operatorname{gr} P)(K^+ \otimes 1 + U \otimes K^+)$, since $K = \Bbbk \oplus K^+$ and thus the map $\overline{\delta}$ induces a new map $\widehat{\delta} : M \to U' \otimes M$, where $U' = U / U K^+ U$. Notice that U' is a pointed Hopf algebra with coradical $\Bbbk\Gamma$, since U is coradically graded and the ideal $U K^+ U$ is homogeneous and does not intersect U_0. M is also a $\Bbbk_\psi F$-module with $\overline{m} \cdot f = \overline{m \cdot f}$, for $f \in F$, $\overline{m} \in M$. This action is easily seen to be well defined and, moreover, $M \in {}^{U'}\mathcal{M}_{\Bbbk_\psi F}$.

Let $\Psi \in Z^2(\Gamma, \Bbbk^*)$ be a 2-cocycle such that $\Psi|_{F \times F} = \psi$; see [Brown 1982, Proposition III (9.5)]. Let $\zeta \in Z^2(U')$ be such that $\zeta|_{\Gamma \times \Gamma} = \Psi^{-1}$, as in Lemma 4.1. By [Mombelli 2010, Lemma 2.1], there exists and equivalence of categories

$$ {}^{U'^{\zeta}}\mathcal{M}_{\Bbbk F} \simeq {}^{U'}\mathcal{M}_{(\Bbbk F)_\Psi}. $$

By Theorem 3.1(2), any object in ${}^{U'^{\zeta}}\mathcal{M}_{\Bbbk F}$ is a free $\Bbbk F$-module. Thus, there is an object N in ${}^{U/U(\Bbbk F)^+}\mathcal{M}$ such that $\operatorname{gr} P \simeq N \otimes K \otimes \Bbbk_\psi F$. Therefore, $\dim P = (\dim N)(\dim \mathcal{A})$. Similarly, we can assume that there is an $s \in \mathbb{N}$ such that $\dim Q = s \dim \mathcal{A}'$.

Using Theorem 3.1(1), there exists $t \in \mathbb{N}$ such that P^t is a free right $\mathcal{A}$-module; that is, there is a vector space T such that $P^t \simeq T \otimes \mathcal{A}$, and hence

$$ t \dim N = \dim T. \tag{4-2}$$

Since $P \otimes_{\mathcal{A}} Q \simeq \mathcal{A}'$, we have $P^t \otimes_{\mathcal{A}} Q \simeq T \otimes Q \simeq \mathcal{A}'^t$ and so $s \dim T \dim \mathcal{A}' = t \dim \mathcal{A}'$. Using (4-2), we see that $s \dim N = 1$, so $\dim N = 1$, so $\dim P = \dim \mathcal{A}$.

Claim 4.1. *If $n \in P_0$, then $P = n \cdot \mathcal{A}$.*

Notice that $P_0 \neq 0$. In fact, if $P_0 = 0$ and $k \in \mathbb{N}$ is minimal with $P_k \neq 0$, then $\lambda(P_k) \subset \sum_{j=0}^{k} H_{k-j} \otimes P_j = H_0 \otimes P_k$, which is a contradiction. Let $g \in \Gamma$ be such that $\lambda(n) = g \otimes n$. Now, if $J = \{a \in \mathcal{A} : n \cdot a = 0\}$, then J is a right ideal of $\mathcal{A}$. We will prove that $J = 0$. Let $a \in J$ and write $\lambda(a) = \sum_{i=1}^{n} a^i \otimes a_i$ in such way that the set $\{a^i : i = 1, \dots, n\} \subset H$ is linearly independent. Now, $\{ga^i : i = 1, \dots, n\} \subset H$ is also linearly independent, and we have $0 = \lambda(n \cdot a) = \sum_{i=1}^{n} ga^i \otimes n \cdot a_i$. Thus, $n \cdot a_i = 0$ for all $i = 1, \dots, n$; that is, $\lambda(a) \in H \otimes J$ and J is H-costable. As $\mathcal{A}$ is right H-simple, we have $J = 0$. Therefore, the action $\cdot : N \otimes \mathcal{A} \to P$ is injective and, since $\dim P = \dim N \dim \mathcal{A}$, the claim follows.

It is not difficult to prove that the linear map $\phi : g\mathcal{A}g^{-1} \to \operatorname{End}_{\mathcal{A}}(P)$ given by $\phi(gag^{-1})(n \cdot b) = n \cdot ab$ is an isomorphism of H-comodule algebras.

Conversely, if $\mathcal{A}' \simeq g\mathcal{A}g^{-1}$ as comodule algebras and $M \in {}_{\mathcal{A}}\mathcal{M}$, then the set gMg^{-1} has a natural structure of $\mathcal{A}'$-module in such way that the functor $F : {}_{\mathcal{A}}\mathcal{M} \to {}_{\mathcal{A}'}\mathcal{M}$ with $M \mapsto gMg^{-1}$ is an equivalence of $\operatorname{Rep}(H)$-modules. □

5. Pointed Hopf algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$

In this section we describe all pointed Hopf algebras whose coradical is the group algebra of the groups $\mathbb{S}_3$ and $\mathbb{S}_4$. These were classified in [Andruskiewitsch et al. 2010] and [García and García Iglesias ≥ 2011], respectively.

Recall that a *rack* is a pair $(X, \triangleright)$, where X is a nonempty set and $\triangleright : X \times X \to X$ is a function, such that, for all $i \in X$, $\phi_i = i \triangleright (\cdot) : X \to X$ is a bijection, and satisfies $i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k)$ for all $i, j, k \in X$. See [Andruskiewitsch and Graña 2003a] for detailed information on racks.

Let $(X, \triangleright)$ be a rack. A 2-cocycle $q : X \times X \to \Bbbk^\times$, denoted by $(i, j) \mapsto q_{ij}$, is a function such that, for all $i, j, k \in X$,

$$q_{i, j \triangleright k}\, q_{j,k} = q_{i \triangleright j, i \triangleright k}\, q_{i,k}.$$

In this case, it is possible to generate a braiding c^q in the vector space $\Bbbk X$ with basis $\{x_i\}_{i \in X}$ by setting $c^q(x_i \otimes x_j) = q_{ij} x_{i \triangleright j} \otimes x_i$ for all $i, j \in X$. We denote by $\mathfrak{B}(X, q)$ the Nichols algebra of this braided vector space.

5A. *Quadratic approximations to Nichols algebras.*

Let

$$\mathcal{J} = \bigoplus_{r \geq 2} \mathcal{J}^r$$

be the defining ideal of the Nichols algebra $\mathfrak{B}(X, q)$. We give a description of the space $\mathcal{J}^2$ of quadratic relations.

Let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i, j) \sim (i \triangleright j, i)$. Let $C \in \mathcal{R}$ and $(i, j) \in C$. Take $i_1 = j$, $i_2 = i$ and, recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. Set $n(C) = \#C$ and

$$\mathcal{R}' = \left\{ C \in \mathcal{R} : \prod_{h=1}^{n(C)} q_{i_{h+1}, i_h} = (-1)^{n(C)} \right\}.$$

Let $\mathcal{T}$ be the free associative algebra in the variables $\{T_l\}_{l\in X}$. If $C \in \mathcal{R}'$, consider the quadratic polynomial

$$\phi_C = \sum_{h=1}^{n(C)} \eta_h(C)\, T_{i_{h+1}} T_{i_h} \in \mathcal{T}, \tag{5-1}$$

where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1} q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_h i_{h-1}}$ with $h \geq 2$. Then, a basis of the space $\mathcal{J}^2$ is given by

$$\phi_C\big(\{x_i\}_{i\in X}\big) \quad \text{for } C \in \mathcal{R}'. \tag{5-2}$$

We denote by $\widehat{\mathfrak{B}}_2(X, q)$ the quadratic approximation of $\mathfrak{B}(X, q)$, that is, the algebra defined by relations $\langle \mathcal{J}^2 \rangle$. For more details, see [García and García Iglesias $\geq$ 2011, Lemma 2.2].

Let G be a finite group. A *principal YD-realization* of (X, q) over G [Andruskiewitsch and Graña 2003a, def 3.2] is a way to realize this braided vector space $(\Bbbk X, c^q)$ as a Yetter–Drinfeld module over G. Explicitly, it is a collection $(\,\cdot\,, g, (\chi_i)_{i\in X})$, where

- $\cdot$ is an action of G on X,
- $g : X \to G$ is a function such that $g_{h\cdot i} = h g_i h^{-1}$ and $g_i \cdot j = i \triangleright j$,
- the family $(\chi_i)_{i\in X}$, where $\chi_i : G \to \Bbbk^*$ is a 1-cocycle (that is,

$$\chi_i(ht) = \chi_i(t)\, \chi_{t\cdot i}(h),$$

for all $i \in X$ and $h, t \in G$) satisfies $\chi_i(g_j) = q_{ji}$.

If $(\,\cdot\,, g, (\chi_i)_{i\in X})$ is a principal YD-realization of (X, q) over G, then $\Bbbk X \in {}^G_G\mathcal{YD}$, as follows. The action and coaction of G are determined by

$$\delta(x_i) = g_i \otimes x_i, \quad h \cdot x_i = \chi_i(h)\, x_{h\cdot i} \quad \text{for } i \in X, h \in G.$$

Lemma 5.1. *If $(i \triangleright j) \triangleright i = j$ for any $i, j \in X$, then*

$$\chi_i(f)\, q_{f\cdot i \triangleright f\cdot j,\, f\cdot i} = \chi_j(f)\, q_{i\triangleright j, i} \quad \textit{for any } f \in G \textit{ and } i, j \in X. \tag{5-3}$$

5B. *Nichols algebras over $\mathbb{S}_n$.* Let X be $\mathcal{O}_2^n$ or $\mathcal{O}_4^4$, considered as racks with the map $\triangleright$ given by conjugation. Consider the maps

$$\operatorname{sgn} : \mathbb{S}_n \times X \to \Bbbk^*,\ (\sigma, i) \mapsto \operatorname{sgn}(\sigma),$$

$$\chi : \mathbb{S}_n \times \mathcal{O}_2^n \to \Bbbk^*, (\sigma, i) \mapsto \chi_i(\sigma) = \begin{cases} 1 & \text{if } i = (a, b) \text{ and } \sigma(a) < \sigma(b), \\ -1 & \text{if } i = (a, b) \text{ and } \sigma(a) > \sigma(b). \end{cases}$$

We will deal with the cocycles

$$-1 : X \times X \to \Bbbk^*, \qquad (j,i) \mapsto \operatorname{sgn}(j) = -1, \qquad i, j \in X;$$
$$\chi : \mathcal{O}_2^n \times \mathcal{O}_2^n \to \Bbbk^*, \qquad (j,i) \mapsto \chi_i(j) \qquad i, j \in \mathcal{O}_2^n.$$

The quadratic approximations of the corresponding Nichols algebras are

$$\begin{aligned}\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1) = \Bbbk\big\langle x_{(lm)},\ 1 \le l < m \le n \,\big|\, & x_{(ab)}^2,\ x_{(ab)}x_{(ef)} + x_{(ef)}x_{(ab)},\\ & x_{(ab)}x_{(bc)} + x_{(bc)}x_{(ac)} + x_{(ac)}x_{(ab)},\\ & 1 \le a < b < c \le n,\ 1 \le e < f \le n,\ \{a,b\} \cap \{e,f\} = \varnothing\big\rangle,\\ \widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, \chi) = \Bbbk\big\langle x_{(lm)},\ 1 \le l < m \le n \,\big|\, & x_{(ab)}^2,\ x_{(ab)}x_{(ef)} - x_{(ef)}x_{(ab)},\\ & x_{(ab)}x_{(bc)} - x_{(bc)}x_{(ac)} - x_{(ac)}x_{(ab)},\\ & x_{(bc)}x_{(ab)} - x_{(ac)}x_{(bc)} - x_{(ab)}x_{(ac)},\\ & 1 \le a < b < c \le n,\ 1 \le e < f \le n,\ \{a,b\} \cap \{e,f\} = \varnothing\big\rangle,\\ \widehat{\mathfrak{B}}_2(\mathcal{O}_4^4, -1) = \Bbbk\big\langle x_i,\ i \in \mathcal{O}_4^4 \,\big|\, & x_i^2,\ x_i x_{i^{-1}} + x_{i^{-1}} x_i,\\ & x_i x_j + x_k x_i + x_j x_k \text{ if } ij = ki \text{ and } j \ne i \ne k \in \mathcal{O}_4^4\big\rangle.\end{aligned}$$

Example 5.2. A principal YD-realization of $(\mathcal{O}_2^n, -1)$ or $(\mathcal{O}_2^n, \chi)$, respectively of $(X, q) = (\mathcal{O}_4^4, -1)$, over $\mathbb{S}_n$, respectively over $\mathbb{S}_4$, is given by the inclusion $X \hookrightarrow \mathbb{S}_n$, and the action $\cdot$ is the conjugation. The family $\{\chi_i\}$ is determined by the cocycle. In either case, g is injective. For $n = 3, 4, 5$, this is in fact the only possible realization over $\mathbb{S}_n$.

Remark 5.3. Notice that all $(\mathcal{O}_2^n, -1)$ and $(\mathcal{O}_2^n, \chi)$ for any n, and $(\mathcal{O}_4^4, -1)$ satisfy that $\mathcal{R} = \mathcal{R}'$. When $n = 3, 4, 5$, we have from [Andruskiewitsch and Graña 2003b; García and García Iglesias ≥ 2011]

$$\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, -1) = \mathfrak{B}(\mathcal{O}_2^n, -1),$$
$$\widehat{\mathfrak{B}}_2(\mathcal{O}_2^n, \chi) = \mathfrak{B}(\mathcal{O}_2^n, \chi),$$
$$\dim \mathfrak{B}(\mathcal{O}_2^n, -1), \dim \mathfrak{B}(\mathcal{O}_2^n, \chi) < \infty.$$

5C. ***Pointed Hopf algebras constructed from racks.*** A *quadratic lifting datum* (or ql-datum) [García and García Iglesias ≥ 2011, definition 3.5] is a collection $\mathcal{Q} = \big(X, q, G, \big(\cdot, g, (\chi_l)_{l \in X}\big), (\gamma_C)_{C \in \mathcal{R}'}\big)$ consisting of a rack X, a 2-cocycle q, a finite group G, a principal YD-realization $\big(\cdot, g, (\chi_l)_{l \in X}\big)$ of (X, q) over G such that $g_i \ne g_j g_k$ for all $i, j, k \in X$, and a collection $(\gamma_C)_{C \in \mathcal{R}'} \in \Bbbk$ satisfying that, for each $C = \{(i_2, i_1), \dots, (i_n, i_{n-1})\} \in \mathcal{R}'$ and $k \in X$, we have

$$\gamma_C = 0 \qquad \text{if } g_{i_2} g_{i_1} = 1, \tag{5-4}$$
$$\gamma_C = q_{k i_2} q_{k i_1} \gamma_{k \triangleright C} \qquad \text{if } k \triangleright C = \{k \triangleright (i_2, i_1), \dots, k \triangleright (i_n, i_{n-1})\}. \tag{5-5}$$

To each ql-datum $\mathcal{Q}$ is attached a pointed Hopf algebra $\mathcal{H}(\mathcal{Q})$, generated as an algebra by $\{a_l, H_t : l \in X,\ t \in G\}$ subject to the relations

(5-6) $\qquad H_e = 1, \quad H_t H_s = H_{ts} \qquad$ for $t, s \in G$;

(5-7) $\qquad H_t\, a_l = \chi_l(t)\, a_{t\cdot l}\, H_t \qquad$ for $t \in G,\ l \in X$;

(5-8) $\qquad \phi_C(\{a_l\}_{l\in X}) = \gamma_C(1 - H_{g_i g_j}) \quad$ for $C \in \mathcal{R}',\ (i, j) \in C$.

Here, ϕ_C is as in (5-1) above. The algebra $\mathcal{H}(\mathcal{Q})$ has a structure of pointed Hopf algebra by setting

$$\Delta(H_t) = H_t \otimes H_t, \quad \Delta(a_i) = g_i \otimes a_i + a_i \otimes 1 \qquad \text{for } t \in G, i \in X.$$

See [García and García Iglesias ≥ 2011] for further details.

5D. ***Pointed Hopf algebras over*** $\mathbb{S}_n$**.** The following ql-data provide examples of (possibly infinite-dimensional) pointed Hopf algebras over $\mathbb{S}_n$: for $\alpha, \beta, \lambda \in \Bbbk$ and $t = (\alpha, \beta)$,

(1) $\mathcal{Q}_n^{-1}[t] = (\mathbb{S}_n, \mathcal{O}_2^n, -1, \cdot, \iota, \{0, \alpha, \beta\})$,

(2) $\mathcal{Q}_n^{\chi}[\lambda] = (\mathbb{S}_n, \mathcal{O}_2^n, \chi, \cdot, \iota, \{0, 0, \alpha\})$, and

(3) $\mathcal{D}[t] = (\mathbb{S}_4, \mathcal{O}_4^4, -1, \cdot, \iota, \{\alpha, 0, \beta\})$.

We will present explicitly the algebras $\mathcal{H}(\mathcal{Q})$ associated to these data. It will follow that relations (5-8) for each $C \in \mathcal{R}'$ with the same cardinality are $\mathbb{S}_n$-conjugated. Thus, for each C with a given number of elements, it is enough to consider a single relation.

Example 5.4. $\mathcal{H}(\mathcal{Q}_n^{-1}[t])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ with relations

$$H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;$$
$$H_j a_i = -a_{jij} H_j \quad \text{for } i, j \in \mathcal{O}_2^n;$$
$$a_{(12)}^2 = 0;$$
$$a_{(12)}a_{(34)} + a_{(34)}a_{(12)} = \alpha(1 - H_{(12)}H_{(34)});$$
$$a_{(12)}a_{(23)} + a_{(23)}a_{(13)} + a_{(13)}a_{(12)} = \beta(1 - H_{(12)}H_{(23)}).$$

Example 5.5. $\mathcal{H}(\mathcal{Q}_n^{\chi}[\lambda])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_2^n, r \in \mathbb{S}_n\}$ with relations

$$H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;$$
$$H_j a_i = \chi_i(j)\, a_{jij} H_j \quad \text{for } i, j \in \mathcal{O}_2^n;$$
$$a_{(12)}^2 = 0;$$
$$a_{(12)}a_{(34)} - a_{(34)}a_{(12)} = 0;$$
$$a_{(12)}a_{(23)} - a_{(23)}a_{(13)} - a_{(13)}a_{(12)} = \alpha(1 - H_{(12)}H_{(23)}).$$

Example 5.6. $\mathcal{H}(\mathfrak{D}[t])$ is the algebra generated by $\{a_i, H_r : i \in \mathcal{O}_4^4, r \in \mathbb{S}_4\}$ with relations

$$H_e = 1, \quad H_r H_s = H_{rs} \quad \text{for } r, s \in \mathbb{S}_n;$$
$$H_j a_i = -a_{jij} H_j \quad \text{for } i \in \mathcal{O}_4^4,\ j \in \mathcal{O}_2^4;$$
$$a_{(1234)}^2 = \alpha(1 - H_{(13)} H_{(24)});$$
$$a_{(1234)} a_{(1432)} + a_{(1432)} a_{(1234)} = 0;$$
$$a_{(1234)} a_{(1243)} + a_{(1243)} a_{(1423)} + a_{(1423)} a_{(1234)} = \beta(1 - H_{(12)} H_{(13)}).$$

These Hopf algebras have been defined in [Andruskiewitsch and Graña 2003b, def 3.7], [García and García Iglesias ≥ 2011, def 3.9] and [García and García Iglesias ≥ 2011, def 3.10], respectively. Each of these $\mathcal{H}(\mathfrak{Q})$ satisfies $\operatorname{gr} \mathcal{H}(\mathfrak{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$ for $G = \mathbb{S}_n$, with n as appropriate [García and García Iglesias ≥ 2011, propositions 5.4, 5.5, 5.6].

Remark 5.7. We have the following classification results:

(1) $\mathcal{H}(\mathfrak{Q}_3^{-1}[t])$, with $t = (0, 0)$ or $t = (0, 1)$, are all the nontrivial finite-dimensional pointed Hopf algebras over $\mathbb{S}_3$ [Andruskiewitsch et al. 2010].

(2) $\mathcal{H}(\mathfrak{Q}_4^{-1}[t])$, $\mathcal{H}(\mathfrak{Q}_4^{\chi}[\zeta])$ and $\mathcal{H}(\mathfrak{D}[t])$, with $t \in \mathbb{P}_{\Bbbk}^1 \cup \{(0, 0)\}$ and $\zeta \in \{0, 1\}$, is a complete list of the nontrivial finite-dimensional pointed Hopf algebras over $\mathbb{S}_4$ [García and García Iglesias ≥ 2011].

We will classify module categories over the category of representations of any pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$, that is, of the algebras listed in Remark 5.7.

6. Coideal subalgebras of quadratic Nichols algebras

A fundamental piece of information to determine simple comodule algebras is the computation of homogeneous coideal subalgebras inside the Nichols algebra. This is part of Theorem 3.2. The study of coideal subalgebras is an active field of research in the theory of Hopf algebras and quantum groups, see for example [Heckenberger and Kolb 2011; Heckenberger and Schneider ≥ 2011; Kharchenko ≥ 2011; Kharchenko and Sagahon 2008].

In this section we present a description of all homogeneous left coideal subalgebras in the quadratic approximations of the Nichols algebras constructed from racks.

Fix $n \in \mathbb{N}$, let $X = \{i_1, \dots, i_n\}$ be a rack of n elements and $q : X \times X \to \Bbbk^*$ a 2-cocycle. Let $\mathcal{R}$ be as in Section 5A. Assume that, for any equivalence class C in $\mathcal{R}$ and $i, j, k \in X$, we have

$$(i, j), (i, k) \in C \Rightarrow j = k \quad \text{and} \quad (i, j), (k, i) \in C \Rightarrow k = i \triangleright j. \tag{6-1}$$

Let G be a finite group and let $(\,\cdot\,, g, (\chi_i)_{i\in X})$ be a principal YD-realization of (X, q) over G. We further assume that

$$\text{(6-2)} \qquad g \text{ is injective} \quad \text{and} \quad \mathcal{R} = \mathcal{R}'.$$

For each subset $Y \subseteq X$, with $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$, denote by $\mathcal{K}_Y$ the subalgebra of $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk 1$ generated by $x_{j_1}, \dots, x_{j_r}$. Set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$.

Proposition 6.1. *For each set $Y = \{i_{j_1}, \dots, i_{j_r}\} \subseteq X$ the algebra $\mathcal{K}_Y$ is an homogeneous coideal subalgebra of $\mathcal{H}$. For each such selection, if $S = \{g_{i_{j_1}}, \dots, g_{i_{j_r}}\}$, then*

$$\operatorname{Stab} \mathcal{K}_Y = S_Y^G = \{h \in G : h S_Y h^{-1} = S_Y\}.$$

Moreover, if $\mathcal{K}$ is a homogeneous coideal subalgebra of $\mathcal{H}$ generated in degree one, then there exists a unique $Y \subseteq X$ such that

$$\mathcal{K} = \mathcal{K}_Y.$$

In particular, the set of homogeneous coideal subalgebras of $\mathcal{H}$ generated in degree one inside $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk 1$ is in bijective correspondence with the set 2^X of parts of X.

Proof. It is clear that $\mathcal{K} = \mathcal{K}_Y$ is a homogeneous coideal subalgebra. To describe $\operatorname{Stab} \mathcal{K}$ it is enough to compute the stabilizer of the vector space $\Bbbk\{x_{j_1}, \dots, x_{j_r}\}$. But $h \cdot x_{j_k} = \chi_{j_k}(h)\, x_{h \cdot j_k}$, $k = 1, \dots, r$, and $x_{h\cdot j_k} \in \{x_{j_1}, \dots, x_{j_r}\}$ if and only if $h \cdot j_k \in \{j_1, \dots, j_r\}$, if and only if $g_{h\cdot j_k} = g_{j_l}$ for some $l = 1, \dots, r$. And the first part of the proposition follows since $g_{h\cdot j_k} = h g_{j_k} h^{-1}$.

Let $\mathcal{K}$ be a homogeneous coideal subalgebra of $\mathcal{H}$ generated in degree one. If $\mathcal{K} = \Bbbk$ the result is trivial, so assume that $\mathcal{K} \neq \Bbbk$. Since $\mathcal{K}$ is homogeneous, $\mathcal{K}(1) \neq 0$. Let $0 \neq y = \sum_i \lambda_i x_i \in \mathcal{K}(1)$; then

$$\Delta(y) = y \otimes 1 + \sum_i \lambda_i H_{g_i} \otimes x_i \quad \text{which implies} \quad \sum_i \lambda_i H_{g_i} \otimes x_i \in \mathcal{H}_0 \otimes \mathcal{K}(1).$$

Let $\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{t\in G} H_t \otimes \kappa_t$, where $\kappa_t = \sum_{j\in X} \eta_{tj} x_j \in \mathcal{K}(1)$ with $\eta_{tj} \in \Bbbk$, for all t, j.

From the assumption (6-2) we know that $H_t = H_{g_j}$ if and only if $t = g_j$, and $g_i = g_j$ if and only if $i = j$, where $i, j \in X$ and $t \in G$. Hence $\eta_{tk} = 0$ if $t \neq g_k$ for some $k \in X$. Set $\eta_{ij} = \eta_{g_i j}$. Thus,

$$\sum_i \lambda_i H_{g_i} \otimes x_i = \sum_{i,j} \eta_{ij} H_{g_i} \otimes x_j.$$

Therefore, $\lambda_i \neq 0$ implies $\eta_{ij} = \delta_{i,j}\lambda_i$, and so $\kappa_i = x_i$. Thus, $\{x_i \mid \lambda_i \neq 0\} \subset \mathcal{K}$ and $\mathcal{K}(1) = \bigoplus_{x_i \in \mathcal{K}(1)} \Bbbk\{x_i\}$. Therefore, if $Y = \{i \in X : x_i \in \mathcal{K}(1)\}$, then $\mathcal{K} = \mathcal{K}_Y$. Finally, if $Y \neq Y'$, then it follows from the injectivity of g that $\mathcal{K}_Y \ncong \mathcal{K}_{Y'}$ as coideal subalgebras. □

The next general lemma will be useful in Section 6A to prove that certain subalgebras are generated in degree one. Given a rack X, let us recall the notion of derivations δ_i associated to every element of the canonical basis $\{e_i\}_{i\in X}$. If $\{e^i\}_{i\in X}$ denotes the dual basis of $\{e_i\}_{i\in X}$, then $\delta_i = (\mathrm{id}\otimes e^i)\Delta$. For $i\in X$, we denote by X_i the set $X\setminus\{i\}$, and thus $\Bbbk X_i = \Bbbk\{x_j \mid j\in X_i\}$. Let us assume, furthermore, that

$$q_{ii} = -1 \quad \text{for all } i\in X. \tag{6-3}$$

By (6-2), this condition is satisfied if, for example, $\dim\widehat{\mathfrak{B}}_2(X,q) < \infty$ or X is such that $i\triangleright i = i$ for all i.

Lemma 6.2. *Let $\mathcal{K}\subset\widehat{\mathfrak{B}}_2(X,q)\,\#\,\Bbbk 1$ be a homogeneous coideal subalgebra of $\mathcal{H}$, and let $i\in X$. If there is an $\omega\in\mathcal{K}$ such that $\delta_i(\omega)\neq 0$, then $x_i\in\mathcal{K}(1)$.*

Proof. Let $\mathcal{K}=\bigoplus_s\mathcal{K}(s)$, $\omega\in T(\Bbbk X)$, and $i\in X$. In $\mathcal{H}$,

$$\omega = \alpha_i(\omega)+\beta_i(\omega)x_i \quad \text{with } \alpha_i(\omega),\,\beta_i(\omega)\in\mathcal{K}_{X_i}.$$

It suffices to see this for a homogeneous monomial ω. We see it by induction on $\ell=\ell(\omega)\in\mathbb{N}$ such that $\omega\in T^{\ell}(\Bbbk X)$. If $\ell=0$ or $\ell=1$, this is clear. Let us assume it holds for $\ell=n-1$, for some $n\in\mathbb{N}$. If $\ell(\omega)=n$ and $\omega=x_{j_1}\dots x_{j_n}$, two possibilities hold: $j_1\neq i$ or $j_1=i$. In the first case, let $\omega'=x_{j_2}\dots x_{j_n}$. Then, $\ell(\omega')\le n-1$ and therefore there exist $\alpha_i(\omega'),\,\beta_i(\omega')\in\mathcal{K}_{X_i}$ such that $\omega'=\alpha_i(\omega')+\beta_i(\omega')x_i$. As $x_{j_1}\alpha_i(\omega'),\ x_{j_1}\beta_i(\omega')\in\mathcal{K}_{X_i}$, in this case the claim follows.

In the second case, let $j=j_2$ and let us note that $j\neq i$, by (6-3). By (6-2), we can consider the relation

$$x_ix_j = q_{ij}x_{i\triangleright j}x_i - q_{ij}q_{i\triangleright j\,i}x_jx_{i\triangleright j}.$$

Thus, if $\omega''=x_{j_3}\dots x_{j_n}$, then $\omega=q_{ij}x_{i\triangleright j}x_i\omega''-q_{ij}q_{i\triangleright j\,i}x_jx_{i\triangleright j}\omega''$ and both members of this sum belong to $\mathcal{K}_{X_i}+\mathcal{K}_{X_i}x_i$ because of the previous case; the claim follows.

Let $\pi:\bigoplus_{s=0}^m\mathcal{H}(s)\otimes\mathcal{K}(m-s)\to\mathcal{H}(m-1)\otimes\mathcal{K}(1)$ be the canonical linear projection. Let $\omega\in T(\Bbbk X)$, $i\in X$ and $\alpha_i(\omega)$, $\beta_i(\omega)$ as above. Then,

$$\pi\,\Delta(\omega)\ \in\ \beta_i(\omega)\otimes x_i+\bigoplus_{j\neq i}\mathcal{H}\otimes x_j.$$

Notice that $\delta_i(\omega)=\beta_i(\omega)$, and therefore, if $\delta_i(\omega)\neq 0$, it follows that $x_i\in\mathcal{K}(1)$, by using (6-2) as in the proof of Proposition 6.1. □

In this part we will assume that X is one of the racks $\mathcal{O}_2^n$, $n\in\mathbb{N}$, or $\mathcal{O}_4^4$, with q one of the cocycles in Section 5B. Notice that (6-1) is satisfied in these cases. Using the previous results, we will describe explicitly all connected homogeneous coideal subalgebras of the bosonization of the quadratic approximations to Nichols algebras described in Section 5B.

We first introduce some notation. Let $Y \subset X$ be a subset, and define

$$\mathcal{R}_1^Y = \{C \in \mathcal{R} : C \subseteq Y \times Y\},$$
$$\mathcal{R}_2^Y = \{C \in \mathcal{R} : |C \cap Y \times Y| = 1\},$$
$$\mathcal{R}_3^Y = \{C \in \mathcal{R} : C \cap Y \times Y = \varnothing\}.$$

Remark 6.3. For the ql-data in Section 5D, we have $\mathcal{R} = \mathcal{R}_1^Y \cup \mathcal{R}_2^Y \cup \mathcal{R}_3^Y$ for any subset Y. If $f \in \operatorname{Stab} \mathcal{K}_Y$ then $f \cdot \mathcal{R}_s^Y \subseteq \mathcal{R}_s^Y$ for any $s = 1, 2, 3$. Also, (6-3) holds.

Definition 6.4. Take the free associative algebra $\mathcal{T}$ in the variables $\{T_l\}_{l \in Y}$. According to this, we set $\vartheta_{C,Y}(\{T_l\}_{l \in Y})$ in $\mathcal{T}$ as

$$\vartheta_{C,Y}(\{T_l\}_{l\in Y}) = \begin{cases} \phi_C(\{T_l\}_{l\in X}) & \text{if } C \in \mathcal{R}_1^Y; \\ T_i T_j T_i - q_{i\triangleright j,i}\, T_j T_i T_j & \text{if } C \in \mathcal{R}_2^Y,\ (i,j) \in C \cap Y \times Y; \\ 0 & \text{if } C \in \mathcal{R}_3^Y. \end{cases} \tag{6-4}$$

We define the algebra $\mathcal{L}_Y$ as

$$\mathcal{L}_Y = \Bbbk\langle \{y_i\}_{i \in Y}\rangle / \langle \vartheta_{C,Y}(\{y_l\}_{l\in Y}) : C \in \mathcal{R}\rangle. \tag{6-5}$$

If $Y = X$, then $\mathcal{L}_X \cong \mathfrak{B}(X, q)$. For simplicity, we sometimes write ϑ_C for $\vartheta_{C,Y}$.

Take $\mathfrak{B}$ to be one of the quadratic (Nichols) algebras $\widehat{\mathfrak{B}}_2(\mathbb{O}_2^n, -1)$, $\widehat{\mathfrak{B}}_2(\mathbb{O}_2^n, \chi)$, or $\mathfrak{B}(\mathbb{O}_4^4, -1)$. Accordingly, set $X = \mathbb{O}_2^n$, $q = -1$, χ or $(X, q) = (\mathbb{O}_4^4, -1)$. Consider a YD-realization for (X, q) such that (6-2) is satisfied (for instance, one from Example 5.2). Set $\mathcal{H} = \mathfrak{B} \# \Bbbk G$.

Theorem 6.5. *Let $Y \subset X$. $\mathcal{L}_Y$ is an $\mathcal{H}$-comodule algebra with coaction*

$$\delta(y_i) = g_i \otimes y_i + x_i \otimes 1, \quad i \in Y.$$

The map $y_i \mapsto x_i$, $i \in Y$, defines an epimorphism of $\mathcal{H}$-comodule algebras $\mathcal{L}_Y \twoheadrightarrow \mathcal{K}_Y$. Moreover, if $n = 3$, it is an isomorphism and $\mathcal{L}_Y \cong \mathcal{K}_Y$.

Proof. The relations that define $\mathcal{L}_Y$ are satisfied in $\mathfrak{B}$. In fact, it suffices to check this in the case $C \in \mathcal{R}_2^Y$, since in the other ones we have $\vartheta_C = 0$ or $\vartheta_C = \phi_C$, and $\phi_C = 0$ in $\mathfrak{B}$; see (5-2). Now, if $C \in \mathcal{R}_2^Y$ and $(i, j) \in C \cap Y \times Y$, let $k = i \triangleright j$. By the definition of $\mathcal{R}_2^Y$, we necessarily have $k \neq i, j$. Then, if we multiply the relation $x_i x_j - q_{ij} x_{i\triangleright j} x_i + q_{ij} q_{i\triangleright j\, i} x_j x_{i\triangleright j} = 0$ by x_i on the right, and apply these relations to the outcome, we get

$$\begin{aligned} 0 &= x_i x_j x_i + q_{ij} q_{i\triangleright j\, i}\, x_j x_{i\triangleright j} x_i = x_i x_j x_i + q_{i\triangleright j\, i} x_j (x_i x_j + q_{ij} q_{i\triangleright j\, i}\, x_j x_{i\triangleright j}) \\ &= x_i x_j x_i + q_{i\triangleright j\, i}\, x_j x_i x_j. \end{aligned}$$

Thus, we have an algebra projection $\pi : \mathcal{L}_Y \twoheadrightarrow \mathcal{K}_Y$. It is straightforward to see that

$$\delta(\vartheta_{C,Y}(\{y_l\}_{l\in Y})) = \vartheta_{C,Y}(\{x_l\}_{l\in Y}) \otimes 1 + g_{C,Y} \otimes \vartheta_{C,Y}(\{y_l\}_{l\in Y}) \quad \text{for every } C \in \mathcal{R},$$

where

$$g_{C,Y} = \begin{cases} g_i g_j & \text{if } C \in \mathcal{R}_1^Y,\ (i,j) \in C, \\ g_i g_j g_i & \text{if } C \in \mathcal{R}_2^Y,\ (i,j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases}$$

Therefore, δ provides $\mathcal{L}_Y$ with a structure of $\mathcal{H}$-comodule in such a way that π becomes a homomorphism.

We analyze now the particular case $n = 3$. If $|Y| = 1$, the result is clear. Let us suppose then that $Y = \{i, j\} \subset \mathbb{O}_2^3$. Notice that the map π is homogeneous. If $\gamma \in \ker(\pi)$, then $\pi(\gamma) = 0$ in $\mathfrak{B}(\mathbb{O}_2^3, -1)$. By (5-2), we necessarily have $\deg\gamma \geq 3$. Now, if $\deg\gamma = 3$, then

$$\gamma = \alpha y_i y_j y_i + \beta y_j y_i y_j = (\alpha+\beta) y_j y_i y_j.$$

for $\alpha, \beta \in \Bbbk$. Then, $\pi(\gamma) = 0$ implies that $\alpha = -\beta$ and $\gamma = 0$. Finally, we can see that there are no elements $\gamma \in \mathcal{L}_Y$ with $\deg\gamma \geq 4$. In fact, an element γ with $\deg\gamma = 4$ would be of the form

$$\gamma = \alpha y_i y_j y_i y_j + \beta y_j y_i y_j y_i = \alpha y_i y_i y_j y_i + \beta y_j y_j y_i y_j = 0.$$

This also shows there are no elements of greater degree. Therefore, $\mathcal{L}_Y = \mathcal{K}_Y$. □

Remark 6.6. If $n \neq 3$, then in general $\mathcal{L}_Y \neq \mathcal{K}_Y$. In fact, when $n = 4$, $q = -1$ and we take $Y = \{(13), (23), (34)\} \subseteq \mathbb{O}_2^4$, we have

$$\mathcal{L}_Y \cong \Bbbk\langle x, y, z : x^2, y^2, z^2, xyx - yxy, yzy - zyz, xzx - zxz\rangle.$$

Now, in the subalgebra of $\mathfrak{B}(\mathbb{O}_2^4, -1)$ generated by $x = x_{(23)}$, $y = x_{(34)}$ and $z = x_{(13)}$, we have the relation

$$\begin{aligned} (xyz)^2 &= x_{(23)}x_{(34)}x_{(13)}x_{(23)}x_{(34)}x_{(13)} \\ &= -x_{(23)}x_{(34)}(x_{(23)}x_{(12)} + x_{(12)}x_{(13)})x_{(34)}x_{(13)} \\ &= x_{(23)}x_{(34)}x_{(23)}x_{(34)}x_{(12)}x_{(13)} + x_{(23)}x_{(12)}x_{(34)}x_{(13)}x_{(34)}x_{(13)} \\ &= x_{(23)}x_{(23)}x_{(34)}x_{(23)}x_{(12)}x_{(13)} + x_{(23)}x_{(12)}x_{(34)}x_{(34)}x_{(13)}x_{(34)} \\ &= 0. \end{aligned}$$

But $(xyz)^2 \neq 0$ in $\mathcal{L}_Y$. We will prove this using GAP [2008] with the package GBNP [Cohen and Gijsbers $\geq$ 2011]. See Proposition 6.9(6) for a description of $\mathcal{K}_Y$ in this case.

6A. ***Coideal subalgebras of Hopf algebras over*** $\mathbb{S}_n$. Set $n = 3$ or 4, let $\mathfrak{B}$ be a finite-dimensional Nichols algebra over $\mathbb{S}_n$, and $\mathcal{H} = \mathfrak{B} \# \Bbbk\mathbb{S}_n$. Recall that these Nichols algebras coincide with their quadratic approximations. We will describe all the coideal subalgebras of $\mathcal{H}$. We will also calculate their stabilizer subgroups.

We start out by proving that in this case these coideal subalgebras are generated in degree one.

Theorem 6.7. *If $\mathcal{K}$ is a homogeneous left coideal subalgebra of $\mathcal{H}$, then $\mathcal{K}$ is generated in degree one. In particular, $\mathcal{K} = \mathcal{K}_Y$ for a unique $Y \subseteq X$.*

Proof. We will see that, given $\omega \in \mathcal{K}$, we have $\omega \in \langle x_i \,:\, \delta_i\omega \neq 0\rangle$. Then, by Lemma 6.2, it will follow that $\omega \in \langle \mathcal{K}(1)\rangle$. Let $I = \{i \in X \,:\, \delta_i\omega = 0\}$ and let us assume I has m elements. We will proceed case by case, for $m = 0, \dots, 6$.

The cases when $m = 0$ (that is, $x_i \in \mathcal{K}(1)$ for all $i \in X$), $m = 6$, and in general $m = n$ (since then $\omega = 0$, see [Andruskiewitsch and Graña 2003b, Section 6]) are clear. The case $m = 1$ is Lemma 6.2, which also holds for any $n \in \mathbb{N}$.

Consider the case $m = 2$, for any $n \in \mathbb{N}$. Let $I = \{i, p\}$. We know that there is an expression of ω without, say, x_i. Let us see that we can write ω without x_i nor x_p. Let $j \in X$ such that $p \triangleright j = i$. Using relations as in Lemma 6.2, and using that $x_l x_r x_l = -q_{l\triangleright r l}\, x_r x_l x_r$ and $x_r x_l x_r x_l = 0$ for all $l, r \in X$, we can assume that ω can be written as

$$\omega = \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_j$$

with γ^0, γ^1, γ^2 not containing x_i- or x_p-factors in their expressions.

In more detail, we can assume that $\omega \in T^\ell(\Bbbk X)$ is a homogeneous monomial. For each appearance of a factor $x_p x_l$ with $l \neq j$, we replace it by $q_{pl}\, x_l x_{p\triangleright l} + q_{pl}\, q_{p\triangleright l p}\, x_{p\triangleright l} x_p$. That is, we replace by an expression in which x_p is located more to the right, and an expression that does not contain x_i or x_p (in the position where we had an x_p). If we have a factor of the form $x_p x_j$, we move it to the right until we get to $x_p x_j x_p$, but we can replace this expression by $-q_{p\triangleright j p}\, x_j x_p x_j$.

Now, $0 = \delta_p\omega = \gamma^1 g_p + \gamma^2 g_p x_j = (\gamma^1 + q_{pj}\gamma^2 x_i) g_p$, and therefore we have

$$\omega = \gamma^0 + \gamma^1 x_p + q_{pj}\gamma^2 x_i x_p + q_{pj} q_{ip}\gamma^2 x_j x_i \;=\; \gamma^0 + q_{pj} q_{ip}\gamma^2 x_j x_i.$$

But then, $0 = \delta_i\omega = q_{pj} q_{ip}\gamma^2 x_j g_i$, and therefore $\omega = \gamma^0$ can be written without x_i or x_p.

This finishes the case $n = 3$, since in this case $|X| = 3$. We now set $n = 4$, and deal with the cases $m = 3, 4, 5$.

Consider the case $m = 3$. Fix $I = \{i_1, i_2, p\}$. There are three possibilities:

(6-6) $$I = \{i, j, i \triangleright j\};$$

(6-7) $$I = \{i, j, k\} \quad \text{such that } i \triangleright k = k \text{ or } j \triangleright k = k;$$

(6-8) $$I = \{i, j, l\} \quad \text{(the remaining case)}.$$

Let $j_1, j_2 \in X$ be such that $p \triangleright j_s = i_s$ for $s = 1, 2$. We can assume that ω is written without x_{i_s} for $s = 1, 2$. Notice that j_1, j_2 do not always exist. For instance, in (6-6) there are no j_1 or j_2, and in (6-7) j_1 or j_2 do not exist. We analyze the three cases separately.

In (6-6), as there are no j_1, j_2, we can write ω in the form $\omega=\gamma^0+\gamma^1 x_p$, with γ^0, γ^1 without factors x_j, $j\in I$. But from $\delta_p\omega=0$ it follows that $\omega=\gamma^0$ and, therefore, we can write ω without factors x_j, $j\in I$.

Case (6-7) is similar. Assume, for example, that $i_2\triangleright p=p$. Then, we have no j_2. Accordingly, we can assume that ω is of the form

$$\omega=\gamma^0+\gamma^1x_p+\gamma^2x_px_{j_1}=\gamma^0+\gamma^1x_p+q_{pj_1}\gamma^2x_{i_1}x_p+q_{pj_1}q_{j_1i_1}\gamma^2x_{j_1}x_{i_1}$$

with $\gamma^1, \gamma^2, \gamma^3$ without factors x_j, $j\in I$. Now, $0=\delta_p\omega=(\gamma^1p+q_{pj_1}\gamma^2x_{i_1})g_p$ and thus $\omega=\gamma^0+q_{pj_1}q_{j_1i_1}\gamma^2x_{j_1}x_{i_1}$ but, as $\delta_{i_1}\omega=0$, it follows $\omega=\gamma^0$ and therefore ω is written without factors x_j, $j\in I$.

It remains to see (6-8). The existence of j_1, j_2 makes this case more subtle than the previous ones. Let us analyze the set $I=\{i_1,i_2,p\}$. We have $k=i_1\triangleright i_2=i_2\triangleright i_1\notin I$ but, moreover, we have $X=\{i_1,i_2,p,k,j_2,j_1\}$. In fact, we can have neither $i_1\triangleright i_2=j_1$ (since this implies $i_2=p$) nor $i_1\triangleright i_2=j_2$ (since this implies $i_1=p$). More, we have $i_2\triangleright j_1=j_1$, and therefore $x_{i_2}x_{j_1}=q_{i_2j_1}x_{j_1}x_{i_2}$. Set

$$\begin{aligned} a&=x_p, & b&=x_{j_1}, & c&=x_{j_2},\\ d&=x_{i_1}, & e&=x_{i_2}, & f&=x_k. \end{aligned}$$

We analyze which are the longest words that we can write with the "conflicting" factors a, b and c, starting with a. Recall that $aba=\pm bab$ and $abb=0$. Starting with ab, we can preliminary form the words $abca$ and $abcb$. Now, $abcac=\pm babca$, and thus we discard it. Consider $abcb$. Since $abcabc=0$, we are left with $abcaba$. As $abcabab=0$, we reach $abcabac$. As $abcabaca=abcabacb=0$, we keep this word. In the case of $abcb$, arguing similarly, we reach $abcbacb$. If we start with acb, as $acbc=\pm abcb$, we consider those words starting with $acba$. The longest one is $acbacab$, but this is $\pm abcbacb$. So the longest word we can form that was not considered before is $acbac$.

In consequence, we can assume there exist $\gamma^i\in\mathcal{K}$, $i=0,\dots,15$, without factors x_j, $j\in I$, such that ω is of the form

$$\begin{aligned}\omega=\gamma^0+\gamma^1a+\gamma^2ab+\gamma^3abc+\gamma^4abca+\gamma^5abcab+\gamma^6abcaba\\ +\gamma^7abcabac+\gamma^8abcb+\gamma^9abcba+\gamma^{10}abcbac+\gamma^{11}abcbacb\\ +\gamma^{12}ac+\gamma^{13}acb+\gamma^{14}acba+\gamma^{15}acbac.\end{aligned}$$

Using the relations and the fact that $\delta_s\omega=0$ for $s=p,i_1,i_2$, we will show that we can write ω without factors x_s, $s=p,i_1,i_2$. When using the relations, by abuse of notation, we will omit the scalars $q_{..}$ that may appear, including those in the (new) factors γ^i. When needed, we will denote by $\gamma^{i'}, \gamma^{i''}, \gamma^{i'''}\in\mathcal{K}$ some of these scalar multiples of the γ^i.

As $\delta_p\omega = 0$, we can rewrite ω as

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bdc + \gamma^{3'} dce + \gamma^5 bdcbd + \gamma^{5'} dcebd + \gamma^7 abcabce\\ + \gamma^8 bdcb + \gamma^{8'} dceb + \gamma^{8''} debd + \gamma^{10} abcbce + \gamma^{11} abcbebd\\ + \gamma^{11'} abcbceb + \gamma^{12} ce + \gamma^{13} ebd + \gamma^{13'} ceb + \gamma^{15} acbea + \gamma^{15'} acbce.\end{aligned}$$

Using that $\delta_{i_1}\omega = 0$ together with the relations $dc = \pm cd$, $be = \pm eb$, $bcb = \pm cbc$ and $abcabc = bcbc = 0$, we see that

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bcd + \gamma^{3'} cde + \gamma^5 bdcbd + \gamma^{5'} dcebd\\ + \gamma^7 abcdeae + \gamma^8 bdcb + \gamma^{8'} dcbe + \gamma^{8''} dbed + \gamma^{11} abcbeda\\ + \gamma^{12} ce + \gamma^{13} bed + \gamma^{13'} cbe + \gamma^{15} acbce + \gamma^{15'} edaea.\end{aligned}$$

Using that $\delta_{i_2}\omega = 0$ together with the relations, we get to

$$\begin{aligned}\omega = \gamma^0 + \gamma^2 bd + \gamma^3 bcd + \gamma^5 bcbad + \gamma^{5'} cbafe + \gamma^{5''} cbaed\\ + \gamma^8 bcad + \gamma^{8'} bcba + \gamma^{8''} baed + \gamma^{8'''} bafe + \gamma^{11} abcbebd\\ + \gamma^{11'} abcbeab + \gamma^{11''} abcbfea + \gamma^{11'''} abcbfac + \gamma^{13} bed + \gamma^{13'} bfe.\end{aligned}$$

Using now that $\delta_{i_1}\omega = 0$,

$$\begin{aligned}\omega = \gamma^0 + \gamma^5 cbafe + \gamma^8 bcba + \gamma^{8'} bafe + \gamma^{11} abcbacb\\ + \gamma^{11'} abcbfce + \gamma^{11''} abcbfac + \gamma^{13} bfe.\end{aligned}$$

Using again that $\delta_{i_2}\omega = 0$,

$$\begin{aligned}\omega &= \gamma^0 + \gamma^8 bcba + \gamma^{11} abcbacb + \gamma^{11'} abcbafc\\ &= \beta^0 + \beta^1 a + \beta^2 abcbacb + \beta^3 abcbabf\end{aligned}$$

for $\beta^i \in \mathcal{K}$, $i = 0, \dots, 3$, without factors x_j, $j \in I$. Using that $\delta_p\omega = 0$,

$$\omega = \beta^0 + \beta^2 dedaeda + \beta^3 dedadaf = \beta^0,$$

since $edaeda = dada = 0$. That is, we can write ω without any factors x_j, $j \in I$.

In the case $m = 4$, we look at the different subsets of three elements of I. If we have a subset of three elements that corresponds to the case (6-8), it follows that ω can be written without the factors x_j with j in that subset, and then ω is in an algebra isomorphic to $\mathfrak{B}(\mathbb{O}_2^3, -1)$, for which we have already proved the result. If we have a subset as in the case (6-6), when we add to this subset a fourth element we obtain another subset as in the case (6-8). If our subset corresponds to the case (6-7), in order to get to a case different from (6-8), we necessarily have to add a fourth element such that I is

$$I = \{i, j, k, l\} \quad \text{with } i \triangleright k = k \text{ and } j \triangleright l = l.$$

We analyze this case. If $p \in I$ is fixed and ω is written without factors x_j with $j \in I \setminus \{p\} = \{i_1, i_2, i_3\}$, notice that if $p \triangleright i_3 = i_3$ there is no other j_3 such that $p \triangleright j_3 = i_3$ and, moreover, if j_1, j_2 are such that $p \triangleright j_s = i_s$ for $s = 1, 2$, then $x_{j_1} x_{j_2} = \pm x_{j_2} x_{j_1}$. Therefore, we can assume that there are γ^i for $i = 0, \dots, 4$, such that they do not contain factors x_j for $j \in I$, and such that ω can be written as

$$\begin{aligned}
\omega &= \gamma^0 + \gamma^1 x_p + \gamma^2 x_p x_{j_1} + \gamma^3 x_p x_{j_1} x_{j_2} + \gamma^4 x_p x_{j_1} x_{j_2} x_p \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} + \gamma^4 x_{i_1} x_{i_2} x_p && (\text{since } \delta_p \omega = 0) \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} + \gamma^{3'} x_{i_1} x_{j_2} x_{i_2} && (\text{since } \delta_p \omega = 0) \\
&= \gamma^0 + \gamma^2 x_{j_1} x_{i_1} + \gamma^3 x_{j_1} x_{i_1} x_{j_2} && (\text{since } \delta_{i_2} \omega = 0) \\
&= \gamma^0 + \gamma^3 x_{j_1} x_{j_2} x_{i_2} && (\text{since } \delta_{i_1} \omega = 0) \\
&= \gamma^0 && (\text{since } \delta_{i_2} \omega = 0).
\end{aligned}$$

Then, we can write ω without x_j for $j \in I$. In the case when $m = 5$, ω necessarily belongs to an algebra isomorphic to $\mathfrak{B}(\mathbb{O}_2^3, -1)$. □

Now, we apply Theorems 6.5 and 6.7 to calculate the coideal subalgebras and stabilizer subgroups of $\mathcal{H} = \mathfrak{B}(\mathbb{O}_2^3) \# \Bbbk\mathbb{S}_3$.

Corollary 6.8. *The following are all the proper homogeneous left coideal subalgebras of* $\mathfrak{B}(\mathbb{O}_2^3, -1) \# \Bbbk\mathbb{S}_3$:

(1) $\mathcal{K}_i = \langle x_i \rangle \cong \Bbbk[x]/\langle x^2 \rangle$ *for* $i \in \mathbb{O}_2^3$,

(2) $\mathcal{K}_{i,j} = \langle x_i, x_j \rangle \cong \Bbbk\langle x, y \rangle / \langle x^2, y^2, xyx - yxy \rangle$ *for* $i, j \in \mathbb{O}_2^3$.

The nontrivial stabilizer subgroups of $\mathbb{S}_3$ *are, respectively, case*

(1) $\operatorname{Stab} \mathcal{K}_i = \mathbb{Z}_2 \cong \langle i \rangle \subset \mathbb{S}_3$,

(2) $\operatorname{Stab} \mathcal{K}_{i,j} = \mathbb{Z}_2 \cong \langle k \rangle \subset \mathbb{S}_3$ *for* $k \neq i, j$. □

Next, we use the computer program [GAP 2008], together with the package [Cohen and Gijsbers ≥ 2011], to compute the coideal subalgebras of the finite-dimensional Nichols algebras over $\mathbb{S}_4$ associated to the rack of transpositions $\mathbb{O}_2^4$. In the same way can be computed the coideal subalgebras of the Nichols algebra $\mathfrak{B}(\mathbb{O}_4^4, -1)$ associated to the rack of 4-cycles. The presentation of these algebras may not be minimal, in the sense that there may be redundant relations. Moreover, in the general case, non-redundant relations in a coideal subalgebra $\mathcal{K}$ may become redundant when computing the bosonization with a subgroup $F \leq \operatorname{Stab} \mathcal{K}$.

First, we need to establish some notation and conventions. Let $\Bbbk\langle x, y, z \rangle$ be the free algebra in the variables x, y, z. We set the ideals

$$R^{\pm}(x, y, z) = \langle x^2, y^2, z^2, xy + yz \pm zx \rangle \subset \Bbbk\langle x, y, z \rangle.$$

Set $\mathfrak{B}_4^+ = \mathfrak{B}(\mathbb{O}_2^4, -1)$ and $\mathfrak{B}_4^- = \mathfrak{B}(\mathbb{O}_2^4, \chi)$. Recall that Y stands for a subset of $\mathbb{O}_2^4$.

Proposition 6.9. *Let $\varepsilon = \pm$. Any homogeneous proper coideal subalgebra $\mathcal{K}^\varepsilon$ of $\mathfrak{B}_4^\varepsilon \# \Bbbk 1$ is isomorphic to one of the algebras in the following list*:

- $\dim \mathcal{K}^\varepsilon(1) = 1$:

  (1) $Y = \{i\}$, $\mathcal{K}^\varepsilon = \Bbbk[x]/\langle x^2\rangle$, *and* $\dim \mathcal{K}^\varepsilon = 2$.

- $\dim \mathcal{K}^\varepsilon(1) = 2$:

  (2) $Y = \{i, j\}$, $i \triangleright j = j$, $\mathcal{K}^\varepsilon = \Bbbk\langle x, z\rangle/\langle x^2, z^2, xz + \varepsilon zx\rangle$, *and* $\dim \mathcal{K}^\varepsilon = 4$.

  (3) $Y = \{i, j\}$, $i \triangleright j \neq j$, $\mathcal{K}^\varepsilon = \Bbbk\langle x, y\rangle/\langle x^2, y^2, xyx - \varepsilon yxy\rangle$, *and* $\dim \mathcal{K}^\varepsilon = 6$.

- $\dim \mathcal{K}^\varepsilon(1) = 3$:

  (4) $Y = \{i, j, k\}$, $i \triangleright j = k$, $\mathcal{K}^\varepsilon = \Bbbk\langle x, y, z\rangle/\langle R^\varepsilon(x, y, z)\rangle$, *and* $\dim \mathcal{K}^\varepsilon = 12$.

  (5) $Y = \{i, j, k\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$,
  $\mathcal{K}^\varepsilon_{i,j,k} := \Bbbk\langle x, y, z\rangle/\langle x^2, y^2, z^2, xyx - \varepsilon yxy, zyz - \varepsilon yzy, xz + \varepsilon zx\rangle$,
  and $\dim \mathcal{K}^\varepsilon = 24$.

  (6) $Y = \{i, j, k\}$, $i \triangleright j$, $j \triangleright k$, $i \triangleright k \notin \{i, j, k\}$,
  $$\begin{aligned}\mathcal{K}^\varepsilon_Y = \Bbbk\langle x, y, z \ : \ & x^2, y^2, z^2,\\ & yxy - \varepsilon xyx,\ zxz - \varepsilon xzx,\ zyz - \varepsilon yzy,\\ & zxyz + yzxy + xyzx,\ zyxz + yxzy + xzyx\\ & zxyxzx + \varepsilon yzxyxz, zxyxzy + \varepsilon xzxyxz\rangle,\end{aligned}$$
  and $\dim \mathcal{K}^\varepsilon = 48$.

- $\dim \mathcal{K}^\varepsilon(1) = 4$:

  (7) $Y = \{i, j, k, l\}, i \triangleright j = k, i \triangleright l = l$,
  $$\begin{aligned}\mathcal{K}^\varepsilon_Y = \Bbbk\langle x, y, z, w \ : \ & x^2,\ y^2,\ z^2,\ w^2,\\ & zx + \varepsilon yz + \varepsilon xy,\ zy + yx + \varepsilon xz,\ wz + \varepsilon zw,\\ & yxy - \varepsilon xyx,\ wxw - \varepsilon xwx,\ wyw - \varepsilon ywy,\\ & wyx + \varepsilon wxz - \varepsilon zwy,\ wyz + wxy - zwx\\ & wxyz - zwxz,\ wxzw + xwxz,\\ & wxyw + ywxy + xywx,\ wxyxz - \varepsilon zwxyx,\\ & wxyxwx + \varepsilon ywxyxw,\ wxyxwy + \varepsilon xwxyxw\rangle,\end{aligned}$$
  and $\dim \mathcal{K}^\varepsilon = 96$.

  (8) $Y = \{i, j, k, l\}$, $i \triangleright j \neq j, k$, $i \triangleright k = k$, $j \triangleright l = l$,

$$\begin{aligned}\mathcal{K}_Y^\varepsilon = \Bbbk\langle x, y, z, w \ : \ & x^2,\ y^2,\ z^2,\ w^2,\ zy+\varepsilon yz,\ wx+\varepsilon xw,\\ & yxy-\varepsilon xyx,\ zxz-\varepsilon xzx,\ wyw-\varepsilon ywy,\\ & wzw-\varepsilon zwz,\ zxyx+yzxy,\ zxyz+\varepsilon xzxy,\\ & wyx-\varepsilon zwy-yxz+\varepsilon xzw,\\ & wzx-\varepsilon zxy-ywz+\varepsilon xyw,\\ & wyzxy-\varepsilon ywyzx-xyzwy+xyxzw,\\ & wyzxw+zxywz-yxzwy-xwyzx,\\ & wyzw-\varepsilon zxwz-yzxw+yxwy+\varepsilon xwyz-\varepsilon xyzx\rangle,\end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 144$.

– $\dim \mathcal{K}^\varepsilon(1) = 5$:

(9) $Y = \{i, j, k, l, m\}$,

$i \triangleright j = k,\ i \triangleright l = m,\ j \triangleright l \neq l,\ k \triangleright m \neq m,\ j \triangleright m = m,\ k \triangleright l = l,$

$$\begin{aligned}\mathcal{K}^\varepsilon = \Bbbk\langle x, y, z, w, u \ : \ & x^2,\ y^2,\ z^2,\ w^2,\ u^2,\ wz+\varepsilon zw,\ uy+\varepsilon yu,\\ & zx+\varepsilon yz+\varepsilon xy,\ zy+yx+\varepsilon xz,\\ & ux+\varepsilon wu+\varepsilon xw,\ uw+wx+\varepsilon xu,\\ & yxy-\varepsilon xyx,\ wxw-\varepsilon xwx,\\ & wyw-\varepsilon ywy,\ uzu-\varepsilon zuz,\\ & wyx+\varepsilon wxz-\varepsilon zwy,\ wyz+wxy-zwx,\\ & uzw-\varepsilon wxz-xuz,\ wxyz-zwxz,\\ & wxyw+ywxy+xywx,\\ & wxyxz-\varepsilon zwxyx,\ wxzw+xwxz,\\ & wxyxwx+\varepsilon ywxyxw,\ wxyxwy+\varepsilon xwxyxw\rangle,\end{aligned}$$

and $\dim \mathcal{K}^\varepsilon = 288$.

The stabilizers subgroups of $\mathbb{S}_4$ *are*, *respectively*,

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle g_i, g_j\rangle \subset \mathbb{S}_4$ *with* $i \triangleright j = j$;

(2) $D_4 \cong \langle g_i, \sigma\rangle \subset \mathbb{S}_4$ (*if*, *for example*, $g_i = (12)$ *and* $\sigma = (1324)$);

(3) $\mathbb{Z}_2 \cong \langle g_k\rangle \subset \mathbb{S}_4,\ k = i \triangleright j$.

(4) $\mathbb{S}_3 \cong \langle g_i, g_j, g_k\rangle \subset \mathbb{S}_4,\ i \triangleright j = k$;

(5) $\mathbb{Z}_2 \cong \langle g_j g_l\rangle,\ j \neq l,\ j \triangleright l = l$;

(6) $\mathbb{S}_3 \cong \langle g_{i\triangleright j}, g_{j\triangleright k}, g_{k\triangleright i}\rangle \subset \mathbb{S}_4$;

(7) *If* $\mathcal{K}^\varepsilon$ *belongs to items* (7) *or* (8), *then* $\operatorname{Stab} \mathcal{K}^\varepsilon = 1$. □

Examples 6.10. We give, as an illustration, an example of a subset $Y \subseteq \mathbb{O}_2^4$ for each case in the previous proposition. Note that, for any comodule algebra $\mathcal{K}_{Y'}$, if Y' is not on the following list, then $\mathcal{K}_{Y'}$ is $\mathbb{S}_4$-conjugated to another algebra $\mathcal{K}_Y$ with Y on the list.

(1) $Y = \{(12)\}$,

(2) $Y = \{(12), (34)\}$,

(3) $Y = \{(12), (13)\}$,

(4) $Y = \{(12), (13), (23)\}$,

(5) $Y = \{(12), (13), (34)\}$,

(6) $Y = \{(12), (13), (14)\}$,

(7) $Y = \{(12), (13), (23), (14)\}$,

(8) $Y = \{(12), (13), (24), (34)\}$,

(9) $Y = \{(12), (13), (23), (14), (24)\}$. □

Remark 6.11. Let $Y \subset \mathbb{O}_2^4$ and let $Z \subset \mathbb{O}_2^4$ be such that $\mathbb{O}_2^4 = Y \sqcup Z$, as sets. Denote by Y_j one of the subsets of item (j) of Proposition 6.9, and by Z_j the corresponding complement. Notice that we have the following bijections

$$Z_1 \cong Y_9, \quad Z_2 \cong Y_8, \quad Z_3 \cong Y_7, \quad Z_4 \cong Y_6, \quad Z_5 \cong Y_5.$$

Therefore, $\dim \mathcal{K}_Y \dim \mathcal{K}_Z = \dim \mathfrak{B}^{\varepsilon}$ for every Y. An analogous statement holds when $X = \mathbb{O}_4^4$.

7. Representations of Rep($\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$)

In this section, we take $\mathfrak{Q} = \big(X, q, G, \big(\cdot, g, (\chi_l)_{l \in X}\big), (\lambda_C)_{C \in \mathcal{R}'}\big)$ as one of the ql-data from Section 5D. Note that in this case the set $C_i = \{(i, i)\}$ belongs to $\mathcal{R} = \mathcal{R}'$ and $(i \triangleright j) \triangleright i = j$ for any $i, j \in X$. Let $\mathcal{H}(\mathfrak{Q})$ be the corresponding Hopf algebra defined in Section 5C, and set $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$. We will assume that $\dim \widehat{\mathfrak{B}}_2(X, q) < \infty$ (and thus $\dim \mathcal{H}(\mathfrak{Q}) < \infty$; see [García and García Iglesias $\geq$ 2011, Proposition 4.2]). In particular, this holds for $n = 3, 4, 5$.

7A. $\widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$-*comodule algebras.* We will construct families of comodule algebras over quadratic approximations of Nichols algebras. These families are large enough to classify module categories in all of our examples.

Definition 7.1. Let $F < G$ be a subgroup and $\psi \in Z^2(F, \Bbbk^\times)$. If $Y \subseteq X$ is a subset such that $F \cdot Y \subseteq Y$, that is, $F < \operatorname{Stab} \mathcal{K}_Y$, then we will say that a family of scalars $\xi = \{\xi_C\}_{C \in \mathcal{R}}$ with $\xi_C \in \Bbbk$ is *compatible* with the triple (Y, F, ψ) if, for any

$f \in \operatorname{Stab} \mathcal{K}_Y$, we have

$$\xi_{f\cdot C}\, \chi_i(f)\chi_j(f) = \xi_C\, \psi(f, g_i g_j)\, \psi(f g_i g_j, f^{-1}) \quad \text{if } C \in \mathcal{R}_1^Y,\ (i,j) \in C;$$
$$\xi_{f\cdot C}\, \chi_i^2(f)\chi_j(f) = \xi_C\, \psi(f, g_i g_j g_i)\, \psi(f g_i g_j g_i, f^{-1}) \quad \text{if } C \in \mathcal{R}_2^Y,\ (i,j) \in C;$$
$$\xi_{C_i} = \xi_{C_j} = 0 \quad \text{if } C \in \mathcal{R}_2^Y,\ (i,j) \in C.$$

We will assume that the family ξ is normalized by $\xi_C = 0$ if either $C \in \mathcal{R}_1^Y$, $(i, j) \in C$, and $g_i g_j \notin F$, or if $C \in \mathcal{R}_2^Y$, $(i, j) \in C$, and $g_i g_j g_i \notin F$.

We now introduce the comodule algebras we will work with.

Definition 7.2. Let $F < G$ be a subgroup, $\psi \in Z^2(F, \Bbbk^\times)$, and $Y \subseteq X$ a subset such that $F \cdot Y \subseteq Y$. Let $\xi = \{\xi_C\}_{C \in \mathcal{R}'}$ be compatible with (Y, F, ψ). Define $\mathcal{A}(Y, F, \psi, \xi)$ to be the algebra generated by $\{y_l, e_f : l \in Y, f \in F\}$ and relations

$$e_1 = 1 \quad \text{and} \quad e_r e_s = \psi(r,s)\, e_{rs} \quad \text{for } r, s \in F, \tag{7-1}$$

$$e_f\, y_l = \chi_l(f)\, y_{f\cdot l}\, e_f \quad \text{for } f \in F,\ l \in Y, \tag{7-2}$$

$$\vartheta_{C,Y}(\{y_l\}_{l\in X}) = \begin{cases} \xi_C\, e_C & \text{if } e_C \in F \\ 0 & \text{if } e_C \notin F \end{cases} \qquad \text{for } C \in \mathcal{R}. \tag{7-3}$$

Here, $\vartheta_{C,Y}$ was defined in (6-4) while the element e_C is defined by

$$e_C = \begin{cases} e_{g_i g_j} & \text{if } C \in \mathcal{R}_1^Y \text{ and } (i,j) \in C, \\ e_{g_i g_j g_i} & \text{if } C \in \mathcal{R}_2^Y \text{ and } (i,j) \in C \cap Y \times Y, \\ 0, & \text{if } C \in \mathcal{R}_3^Y. \end{cases} \tag{7-4}$$

If $Z \subseteq X$ is a subset invariant under the action of F, we define $\mathcal{B}(Z, F, \psi, \xi)$ as the subalgebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by the elements $\{y_l, e_f : l \in Z, f \in F\}$.

Remark 7.3. (a) Applying $\operatorname{ad}(f)$, with $f \in \operatorname{Stab} \mathcal{K}_Y$, to Equation (7-3) and using (5-3) one can deduce the equations in Definition 7.1.

(b) It may happen that $\mathcal{B}(Z, F, \psi, \xi) \neq \mathcal{A}(Z, F, \psi, \xi)$.

Let $\lambda : \mathcal{A}(Y, F, \psi, \xi) \to \mathcal{H} \otimes \mathcal{A}(Y, F, \psi, \xi)$ be the map defined by

$$\lambda(e_f) = f \otimes e_f, \qquad \lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l, \tag{7-5}$$

for all $f \in F$, $l \in Y$.

Lemma 7.4. *$\mathcal{A}(Y, F, \psi, \xi)$ is a left $\mathcal{H}$-comodule algebra with coaction λ as in* (7-5) *and $\mathcal{B}(Z, F, \psi, \xi)$ is a subcomodule algebra of $\mathcal{A}(X, F, \psi, \xi)$.*

Proof. We first prove that the map λ is well defined. It is easy to see that $\lambda(e_f\, y_l) = \chi_l(f)\, \lambda(y_{f\cdot l}\, e_g)$ for any $f \in F$, $l \in X$.

Let $C \in \mathcal{R}_1^Y$ and $(i, j) \in C$. In this case, $\vartheta_C = \phi_C$. We will prove that $\lambda(\phi_C(\{y_l\}_{l\in X})) = \lambda(\xi_C\, e_{g_i g_j})$. Using the definition of the polynomial ϕ_C, we obtain that

$$\begin{aligned}\lambda(\phi_C(\{y_l\}_{l\in X})) &= \sum_{h=1}^{n(C)} \eta_h(C)\, x_{i_{h+1}} x_{i_h} \otimes 1 + x_{i_{h+1}} g_{i_h} \otimes y_{i_h} \\ &\qquad\qquad + g_{i_{h+1}} x_{i_h} \otimes y_{i_{h+1}} + g_{i_{h+1}} g_{i_h} \otimes y_{i_{h+1}} y_{i_h} \\ &= \phi_C(\{x_l\}_{l\in X}) \otimes 1 + g_i g_j \otimes \phi_C(\{y_l\}_{l\in X}) \\ &= \xi_C\; g_i g_j \otimes e_{g_i g_j} = \lambda(\xi_C\, e_{g_i g_j}).\end{aligned}$$

The second equality follows since $i_{n(C)+1} = i_1$,

$$g_{i_{h+1}} x_{i_h} = q_{i_{h+1} i_h}\, x_{i_{h+2}} g_{i_{h+1}} \quad \text{and} \quad \eta_h(C) q_{i_{h+1} i_h} = -\eta_{h+1}(C).$$

Now, let $C \in \mathcal{R}_2^Y$, $(i, j) \in C$ and $i \triangleright j \notin Y$. In this case relation (7-3) is

$$y_i y_j y_i + q_{i\triangleright j\, i}\, y_j y_i y_j = \xi_C\, e_{g_i g_j g_i}.$$

Note that assumption $\xi_{C_i} = \xi_{C_j} = 0$ implies that $y_i^2 = 0 = y_j^2$. The proof that $\lambda(y_i y_j y_i + q_{i\triangleright j\, i}\, y_j y_i y_j) = \xi_C \lambda(e_{g_i g_j g_i})$ is a straightforward computation. □

Theorem 7.5. *Let $Y \subseteq X$ be an F-invariant subset. If $\mathcal{A}(X, F, \psi, \xi) \neq 0$, then the following statements hold*:

1. *The algebras $\mathcal{A}(X, G, \psi, \xi)$ are left $\mathcal{H}$-Galois extensions.*
2. *If ξ satisfies*

$$\text{(7-6)} \qquad \xi_C = \begin{cases} -\lambda_C & \text{if } \lambda_C \neq 0, \\ 0 & \text{if } \lambda_C = 0 \text{ and } g_j g_i \neq 1, \\ \text{arbitrary} & \text{if } \lambda_C = 0 \text{ and } g_j g_i = 1, \end{cases}$$

   then $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$-biGalois object.

3. *$\mathcal{B}(Y, F, \psi, \xi)_0 = \Bbbk_\psi F$, and thus $\mathcal{B}(Y, F, \psi, \xi)$ is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra.*
4. *There is an isomorphism of comodule algebras* $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{H}_Y \# \Bbbk_\psi F$.
5. *There is an isomorphism $\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{B}(Y', F', \psi', \xi')$ of comodule algebras if and only if $Y = Y'$, $F = F'$, $\psi = \psi'$ and $\xi = \xi'$.*

Proof. Step 1: To prove that $\mathcal{A}(X, G, \psi, \xi)$ is a Galois extension, observe that the canonical map

$$\begin{aligned}&\operatorname{can}: \mathcal{A}(X, G, \psi, \xi) \otimes \mathcal{A}(X, G, \psi, \xi) \to \mathcal{H} \otimes \mathcal{A}(X, G, \psi, \xi), \\ &\operatorname{can}(x \otimes y) = x_{(-1)} \otimes x_{(0)} y,\end{aligned}$$

is surjective. Indeed, for any $f \in G$, $l \in X$, we have $\operatorname{can}(e_f \otimes e_{f^{-1}}) = f \otimes 1$ and

$$\operatorname{can}(y_l \otimes 1 - e_{g_l} \otimes e_{g_l^{-1}} y_l) = x_l \otimes 1.$$

Step 2: Define the map $\rho : \mathcal{A}(X, G, 1, \xi) \to \mathcal{A}(X, G, 1, \xi) \otimes \mathcal{H}(\mathcal{Q})$ by

$$\rho(e_f) = e_f \otimes H_f \quad \text{and} \quad \rho(y_l) = y_l \otimes 1 + e_{g_l} \otimes a_l \quad \text{for } l \in X,\ f \in G.$$

The map ρ is well defined. Indeed, if $C \in \mathcal{R}$ and $(i, j) \in C$, then

$$\begin{aligned}\rho(\phi_C(\{y_l\}_{l\in X})) &= \phi_C(\{y_l\}_{l\in X}) \otimes 1 + e_{g_i g_j} \otimes \phi_C(\{a_l\}_{l\in X})\\ &= \xi_C\, e_{g_i g_j} \otimes 1 + \lambda_C\, e_{g_i g_j} \otimes (1 - H_{g_i g_j}).\end{aligned}$$

Clearly, if ξ satisfies (7-6), then $\rho(\phi_C(\{y_l\}_{l\in X})) = \xi_C\, \rho(e_{g_i g_j})$. The proof that $\mathcal{A}(X, G, 1, \xi)$ is a $(\mathcal{H}, \mathcal{H}(\mathcal{Q}))$-bicomodule and a right $\mathcal{H}(\mathcal{Q})$-Galois object is done by a straightforward computation.

Step 3: If $\mathcal{A}(X, F, \psi, \xi) \neq 0$, then there is a group $\overline{F}$ with a projection $F \twoheadrightarrow \overline{F}$ such that $\mathcal{A}(Y, F, \psi, \xi)_0 = \Bbbk_{\overline{\psi}} \overline{F}$. The map $\mathcal{A}(Y, F, \psi, \xi)_0 \otimes \mathcal{A}(Y, F, \psi, \xi)_0 \to \Bbbk F \otimes \mathcal{A}(Y, F, \psi, \xi)_0$, defined by $e_f \otimes e_g \mapsto f \otimes \psi(f, g)\, e_{fg}$, is surjective. Hence, $F = \overline{F}$. This implies that $\mathcal{B}(Z, F, \psi, \xi)_0 = \Bbbk_\psi F$ and, by [Mombelli 2010, Prop. 4.4], it follows that $\mathcal{B}(Z, F, \psi, \xi)$ is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra.

Step 4: It follows from Theorem 3.2(3) that $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \simeq \mathcal{K} \# \Bbbk_\psi F$ for some homogeneous left coideal subalgebra $\mathcal{K} \subseteq \widehat{\mathfrak{B}}_2(X, q)$. Recall that $\mathcal{K}$ is identified with the subalgebra of $\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi)$ given by

$$\big\{a \in \operatorname{gr}\mathcal{A}(Y, F, \psi, \xi) : (\mathrm{id} \otimes \pi)\lambda(a) \in \mathcal{H} \otimes 1\big\};$$

see [Mombelli 2010, Proposition 7.3 (3)]. There, it is also proved that the composition

$$\operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \xrightarrow{(\vartheta\otimes\pi)\lambda} \mathcal{K} \# \Bbbk_\psi F \xrightarrow{\mu} \operatorname{gr}\mathcal{B}(Y, F, \psi, \xi),$$

is the identity map, where $\vartheta : \mathcal{H} \to \widehat{\mathfrak{B}}_2(X, q)$ and $\pi : \operatorname{gr}\mathcal{B}(Y, F, \psi, \xi) \to \Bbbk_\psi F$ are the canonical projections, and μ is the multiplication map. Both maps are bijections and, since for any $l \in Y$ we have$(\vartheta \otimes \pi)\lambda(y_l) = x_l$, it follows that $\mathcal{K} = \mathcal{K}_Y$.

Step 5: Let $\beta : \mathcal{B}(Y, F, \psi, \xi) \to \mathcal{B}(Y', F', \psi', \xi')$ be a comodule algebra isomorphism. The restriction of β to $\mathcal{B}(Y, F, \psi, \xi)_0$ induces an isomorphism between $\Bbbk_\psi F$ and $\Bbbk_{\psi'} F'$, and thus $F = F'$ and $\psi = \psi'$. Since β is a comodule morphism, it is clear that $Y = Y'$ and $\xi_C = \xi'_C$ for any $C \in \mathcal{R}$. □

Corollary 7.6. *If $\mathcal{A}(X, G, 1, \xi) \neq 0$ for some ξ satisfying* (7-6), *then*

1. *the Hopf algebras $\mathcal{H} = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$ and $\mathcal{H}(\mathcal{Q})$ are cocycle deformations of each other*;
2. *there is a bijective correspondence between equivalence classes of exact module categories over* $\operatorname{Rep}(\mathcal{H})$ *and* $\operatorname{Rep}(\mathcal{H}(\mathcal{Q}))$.

Remark 7.7. Under the assumptions in Corollary 7.8, we obtain in particular that $\operatorname{gr}\mathcal{H}(\mathcal{Q}) = \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk G$, since the latter is a quotient of the first.

The following corollary uses Propositions A.14 and A.18, where certain algebras are shown to be not null. These propositions will be proven in the Appendix, and their proofs are independent of the other results in the article.

Corollary 7.8. *If H is a nontrivial pointed Hopf algebra over $\mathbb{S}_3$ or $\mathbb{S}_4$, then H is a cocycle deformation of* $\operatorname{gr} H$.

Proof. Finite-dimensional Nichols algebras over $\mathbb{S}_3$ and $\mathbb{S}_4$ coincide with their quadratic approximations. That is, if H is a finite-dimensional pointed Hopf algebra over $\mathbb{S}_n$ with $n = 3, 4$, then $\operatorname{gr} H \cong \widehat{\mathfrak{B}}_2(X, q) \# \Bbbk\mathbb{S}_n$. By Main Theorem of [García and García Iglesias ≥ 2011] we know that $H \cong \mathcal{H}(\mathfrak{Q})$. Therefore, the theorem follows from Corollary 7.6, since in Propositions A.14 and A.18 we show the existence of nonzero $(\operatorname{gr}\mathcal{H}(\mathfrak{Q}), \mathcal{H}(\mathfrak{Q}))$-biGalois objects in these cases.

When dealing with either $\mathfrak{Q}_4^{-1}[t]$ or $\mathfrak{D}[t]$, notice that the condition $\xi_2 = 2\xi_1$ in Proposition A.18 does not interfere with the proof, since, by (7-6), ξ_1 and respectively ξ_2 can be chosen arbitrarily. □

Remark 7.9. In [2008, Theorem A1], Masuoka proved that the Hopf algebras $u(\mathfrak{D}, \lambda, \mu)$ associated to a datum of finite Cartan type $\mathfrak{D}$ appearing in the classification of [Andruskiewitsch and Schneider 2010] are cocycle deformations of the associated graded Hopf algebras $u(\mathfrak{D}, 0, 0)$.

Corollaries 7.6(1) and 7.8 provide a similar result for some families of Hopf algebras constructed from Nichols algebras not of diagonal type. It would be interesting to generalize this kind of result for larger classes of Nichols algebras.

7B. ***Module categories over*** **$\operatorname{Rep}(\mathcal{H}(\mathfrak{Q}))$.** Let A be a $\mathcal{H}$-comodule algebra with $\operatorname{gr} A = \mathcal{K}_Y \# \Bbbk_\psi F$ for $F \le \operatorname{Stab}\mathcal{K}_Y$ and $\psi \in Z^2(F, \Bbbk^*)$. Let Z be such that, as sets, $X = Y \sqcup Z$. Notice that $F \le \operatorname{Stab}\mathcal{K}_Z$.

Lemma 7.10. *Under these assumptions, there exists a family of scalars ξ compatible with (X, F, ψ) such that $A \simeq \mathcal{B}(Y, F, \psi, \xi)$ as comodule algebras.*

Proof. The canonical projection $\pi : A_1 \to A_1/A_0 \simeq \mathcal{K}_Y(1) = \Bbbk Y$ is a morphism of $\mathcal{A}_0$-bimodules. Let $\iota : \Bbbk Y \to A_1$ be a section of $\mathcal{A}_0$-bimodules of π. Since the elements $\{x_l : l \in Y\}$ are in the image of π, we can choose elements $\{y_l : l \in Y\}$ in A_1 such that $\iota(x_l) = y_l$ for any $l \in Y$. It is straightforward to verify that $\lambda(y_l) = x_l \otimes 1 + g_l \otimes y_l$ and $e_f\, y_l = \chi_l(f)\, y_{f\cdot l}\, e_f$ for $f \in F$, $l \in Y$. Since $\operatorname{gr} A$ is generated by the elements $\{x_l, e_f : l \in Y, f \in F\}$, it follows that A is generated as an algebra by the elements $\{y_l, e_f : l \in Y, f \in F\}$.

Now, let $B = A \otimes \mathcal{K}_Z$. Then, B has an comodule algebra structure for which the canonical inclusion $A \hookrightarrow A \otimes 1 \subset B$ is a homomorphism. The algebra structure is given as follows:

For $i \in Y$, $j \in Z$, $f \in F$, set

$$
\begin{aligned}
&(e_f \otimes 1)(1 \otimes y_j) = e_f \otimes y_j,\\
&(1 \otimes y_j)(e_f \otimes 1) = \chi_j^{-1}(f) e_f \otimes y_{f^{-1}.j},\\
&(y_i \otimes 1)(1 \otimes y_j) = (y_i \otimes y_j),\\
&(1 \otimes y_j)(y_i \otimes 1) =\\
&\qquad \begin{cases} q_{ji} y_i \otimes y_j + \xi_C e_C \otimes 1 & \text{if } i \triangleright j = j,\\ q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C e_C \otimes 1 & \text{if } i \triangleright j \neq j,\ i \triangleright j \in Y,\\ q_{ji} 1 \otimes y_{j\triangleright i} y_j - q_{ji} q_{j\triangleright i\, j} y_i \otimes y_{j\triangleright i} + \xi_C e_C \otimes 1 & \text{if } i \triangleright j \neq j,\ i \triangleright j \notin Y. \end{cases}
\end{aligned}
$$

Here, C stands for the class $C \in \mathcal{R}'$ such that $(j, i) \in C$. Recall that, by definition, $\xi_C = 0$ if $g_C \notin F$. Then, the map

$$m : B \to \mathcal{A}(X, F, \psi, \xi), \quad a \otimes x \mapsto ax, \tag{7-7}$$

is an algebra epimorphism. Now, if

$$A \ni a \longmapsto a_{(-1)} \otimes a_{(0)} \in \mathcal{H} \otimes A \quad \text{and} \quad \mathcal{H}_Z \ni x \longmapsto x_{(-1)} \otimes x_{(0)} \in \mathcal{H} \otimes \mathcal{H}_Z$$

denote the corresponding coactions, define $\lambda : B \to \mathcal{H} \otimes B$ by $\lambda(a \otimes x) = a_{(-1)} x_{(-1)} \otimes a_{(0)} \otimes x_{(0)}$. It is straightforward to check that λ is well defined. We check this case by case in the above definition of the multiplication of B. For instance, if $i \triangleright j \neq j$ and $i \triangleright j \in Y$, then we have

$$
\begin{aligned}
&\lambda(1 \otimes y_j)\lambda(y_i \otimes 1)\\
&\quad = (g_j \otimes (1 \otimes y_j) + x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1) + x_i \otimes (1 \otimes 1))\\
&\quad = (g_j \otimes (1 \otimes y_j))(g_i \otimes (y_i \otimes 1)) + (x_j \otimes (1 \otimes 1))(g_i \otimes (y_i \otimes 1))\\
&\qquad\qquad + (g_j \otimes (1 \otimes y_j))(x_i \otimes (1 \otimes 1)) + (x_j \otimes (1 \otimes 1))(x_i \otimes (1 \otimes 1))\\
&\quad = g_j g_i \otimes (1 \otimes y_j)(y_i \otimes 1) + x_j g_i \otimes (y_i \otimes 1)\\
&\qquad\qquad + q_{ji} x_{j\triangleright i} g_j \otimes (1 \otimes y_j) + x_j x_i \otimes (1 \otimes 1)\\
&\quad = g_j g_i \otimes (q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C g_C \otimes 1)\\
&\qquad\qquad + x_j g_i \otimes (y_i \otimes 1) + q_{ji} x_{j\triangleright i} g_j \otimes (1 \otimes y_j)\\
&\qquad\qquad + (q_{ji} x_{j\triangleright i} x_j - q_{ji} q_{j\triangleright i\, j} x_i x_{j\triangleright i} \otimes 1) \otimes (1 \otimes 1),
\end{aligned}
$$

which coincides with $\lambda(q_{ji} y_{j\triangleright i} \otimes y_j - q_{ji} q_{j\triangleright i\, j} y_i y_{j\triangleright i} \otimes 1 + \xi_C g_C \otimes 1)$.

Thus, B is an $\mathcal{H}$-comodule algebra, with

$$\dim B = \dim A \dim \mathcal{H}_Z = \dim \mathcal{H}_Y \dim \mathcal{H}_Z |F| = \dim \mathcal{A}(X, F, \psi, \xi)$$

by Remark 6.11. Then, the map m from (7-7) is an isomorphism. □

We can now formulate the main result of the paper. For any $h \in G$, we write $\xi_C^h = \xi_{h^{-1}\cdot C}$. Recall that we denote by $\mathcal{B}(Y, F, \psi, \xi)$ the sub-comodule algebra of $\mathcal{A}(X, F, \psi, \xi)$ generated by $\{y_i\}_{i \in Y}$.

Theorem 7.11. (1) *Let $\mathcal{M}$ be an exact indecomposable module category over* $\mathrm{Rep}(\mathcal{H}(\mathcal{Q}))$. *There exist*

(i) *a subgroup $F < G$ and a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$,*
(ii) *a subset $Y \subset X$ with $F \cdot Y \subset Y$, and*
(iii) *a family of scalars $\{\xi_C\}_{C \in \mathcal{R}'}$ compatible with (X, F, ψ),*

such that there is a module equivalence $\mathcal{M} \simeq {}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$.

(2) *Let (Y, F, ψ, ξ) and (Y', F', ψ', ξ') be two families as before. There is an equivalence of module categories ${}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M} \simeq {}_{\mathcal{B}(Y',F',\psi',\xi')}\mathcal{M}$ if and only if there exists an element $h \in G$ such that $F' = hFh^{-1}$, $\psi' = \psi^h$, $Y' = h \cdot Y$ and $\xi' = \xi^h$.*

Proof. Step 1: By Corollary 7.8, we can assume that $\mathcal{M}$ is an exact indecomposable module category over $\mathrm{gr}\,\mathcal{H}(\mathcal{Q}) = \mathcal{H}$. It follows from [Andruskiewitsch and Mombelli 2007, Theorem 3.3] that there is a right $\mathcal{H}$-simple left $\mathcal{H}$-comodule algebra $\mathcal{A}$ such that $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$. Theorem 3.2 implies that there is a subgroup $F < G$, a 2-cocycle $\psi \in Z^2(F, \Bbbk^\times)$ and a subset $Y \subset X$ with $F \cdot Y \subset Y$, such that $\mathrm{gr}\,\mathcal{A} = \mathcal{K}_Y \# \Bbbk_\psi F$. Here, $\mathcal{A}_0 = \Bbbk_\psi F$. The result then follows from Lemma 7.10.

Step 2: If the module categories ${}_{\mathcal{B}(Y,F,\psi,\xi)}\mathcal{M}$ and ${}_{\mathcal{B}(Y',F',\psi',\xi')}\mathcal{M}$ are equivalent, then Theorem 4.2 implies that there exists an element $h \in G$ such that $\mathcal{B}(Y', F', \psi', \xi') \simeq h\mathcal{B}(Y, F, \psi, \xi)h^{-1}$ as H-comodule algebras.

The algebra map $\alpha : h\mathcal{B}(Y, F, \psi, \xi)h^{-1} \to \mathcal{B}(h\cdot Y, hFh^{-1}, \psi^h, \xi^h)$, defined by $\alpha(he_f h^{-1}) = e_{hfh^{-1}}$ and $\alpha(hy_l h^{-1}) = \chi_l(h)\, y_{h\cdot l}$ for all $f \in F$ and $l \in Y$, is a well-defined comodule algebra isomorphism. It follows that $\mathcal{B}(Y', F', \psi', \xi') \simeq \mathcal{B}(h \cdot Y, hFh^{-1}, \psi^h, \xi^h)$ and, by using Theorem 7.5(3), we get the result. □

As a consequence of Theorem 7.11 we have:

Corollary 7.12. *Any $\mathcal{H}$-Galois object is of the form $\mathcal{A}(X, G, \psi, \xi)$.*

Proof. Let A be a $\mathcal{H}$-Galois object. Then, ${}_A\mathcal{M}$ is an exact module category over $\mathrm{Rep}\,\mathcal{H}$. Moreover, ${}_A\mathcal{M}$ is indecomposable; otherwise, by [Andruskiewitsch and Mombelli 2007, Proposition 1.18], there would exist a proper bilateral ideal $J \subset A$ $\mathcal{H}$-stable. Thus, $\mathrm{can}(A \otimes J) = \mathrm{can}(J \otimes A)$, which contradicts the bijectivity of can. By Theorem 7.11, there exists (X, G, ψ, ξ) such that $A \cong \mathcal{A}(X, G, \psi, \xi)$. □

7C. ***Modules categories over*** $\mathfrak{B}(\mathbb{O}_2^3, -1) \# \Bbbk\mathbb{S}_3$**.** We apply Theorem 7.11 to exhibit explicitly all module categories in this particular case. In this case the rack is

$$\mathbb{O}_2^3 = \{(12), (13), (23)\}.$$

For each $i \in \mathcal{O}_2^3$, we denote by g_i the element i when thought of as an element of the group $\mathbb{S}_3$. We will show in the Appendix that the algebras in the following result are not null; then, the next corollary will follow from Theorem 7.11.

Corollary 7.13. *Let $\mathcal{M}$ be an indecomposable exact module category over*

$$\operatorname{Rep}(\mathfrak{B}(\mathcal{O}_2^3, -1)\,\#\,\Bbbk\mathbb{S}_3).$$

There is a module equivalence $\mathcal{M} \simeq {}_{\mathcal{A}}\mathcal{M}$ where $\mathcal{A}$ is one (and only one) of the comodule algebras in following list, where $i, j, k \in \mathcal{O}_2^3$ and $\xi, \mu, \eta \in \Bbbk$.

(1) *For any subgroup $F \subseteq \mathbb{S}_3$, $\psi \in Z^2(F, \Bbbk^\times)$, the twisted group algebra $\Bbbk_\psi F$.*

(2) *The algebra $\mathcal{A}(\{i\}, \xi, 1) = \langle y_i : y_i^2 = \xi 1\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$.*

(3) *The algebra $\mathcal{A}(\{i\}, \xi, \mathbb{Z}_2) = \langle y_i, h : y_i^2 = \xi 1,\ h^2 = 1,\ hy_i = -y_i h\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$ and $\lambda(h) = g_i \otimes h$.*

(4) *The algebra $\mathcal{A}(\{i, j\}, 1) = \langle y_i, y_j : y_i^2 = y_j^2 = 0,\ y_i y_j y_i = y_j y_i y_j\rangle$ with coaction $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$ and $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$.*

(5) *The algebra*

$$\mathcal{A}(\{i, j\}, \mathbb{Z}_2) = \langle y_i, y_j, h : y_i^2 = y_j^2 = 0,\ h^2 = 1,\ hy_i = -y_j h,\ y_i y_j y_i = y_j y_i y_j\rangle$$

with coaction determined by $\lambda(y_i) = x_i \otimes 1 + g_i \otimes y_i$, $\lambda(y_j) = x_j \otimes 1 + g_j \otimes y_j$ and $\lambda(h) = g_k \otimes h$, where $k \neq i, j$.

(6) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, 1)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = 0,$$
$$y_{(13)} y_{(12)} + y_{(23)} y_{(13)} + y_{(12)} y_{(23)} = 0.$$

The coaction is determined by $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

(7) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mathbb{Z}_2)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^2 = 1, \quad hy_{(12)} = -y_{(12)} h, \quad hy_{(13)} = -y_{(23)} h,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = 0.$$

The coaction is determined by $\lambda(h) = g_{(12)} \otimes h$ and $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ for any $s \in \mathcal{O}_2^3$.

(8) *The algebra $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \eta, \mathbb{Z}_3)$ generated by $\{y_{(12)}, y_{(13)}, y_{(23)}, h\}$ with relations*

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad h^3 = 1,$$
$$hy_{(12)} = y_{(13)} h, \quad hy_{(13)} = y_{(23)} h, \quad hy_{(23)} = y_{(12)} h,$$
$$y_{(12)} y_{(13)} + y_{(13)} y_{(23)} + y_{(23)} y_{(12)} = \mu\, h,$$

$$y_{(13)}y_{(12)} + y_{(23)}y_{(13)} + y_{(12)}y_{(23)} = \eta\, h^2.$$

The coaction is determined by $\lambda(h) = g_{(132)} \otimes h$ *and* $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ *for any* $s \in \mathcal{O}_2^3$.

(9) *For each* $\psi \in Z^2(\mathbb{S}_3, \Bbbk^\times)$, *the algebra* $\mathcal{A}(\mathcal{O}_2^3, \xi, \mu, \mathbb{S}_3, \psi)$ *generated by* $\{y_{(12)}, y_{(13)}, y_{(23)}, e_h : h \in \mathbb{S}_3\}$ *with relations*

$$e_h e_t = \psi(h, t)\, e_{ht} \quad \text{and} \quad e_h y_s = -y_{h\cdot s} e_h \quad \text{for } h, t \in \mathbb{S}_3, s \in \mathcal{O}_2^3,$$

$$y_{(12)}^2 = y_{(13)}^2 = y_{(23)}^2 = \xi 1, \quad y_{(12)}y_{(13)} + y_{(13)}y_{(23)} + y_{(23)}y_{(12)} = \mu\, e_{(123)}.$$

The coaction is determined by $\lambda(e_h) = h \otimes e_h$ *and* $\lambda(y_s) = x_s \otimes 1 + g_s \otimes y_s$ *for any* $s \in \mathcal{O}_2^3$.

Appendix: $\mathcal{A}(Y, F, \psi, \xi) \neq 0$

We will complete the proofs of Corollaries 7.8 and 7.13, by showing that the algebras in their statements are not null.

Proposition A.14. *If* $\mathcal{A}(Y, F, \psi, \xi)$ *is one of the algebras in Corollary 7.13, then* $\mathcal{A}(Y, F, \psi, \xi) \neq 0$.

Proof. The case $Y \neq \mathcal{O}_2^3$ is clear. Set $Y = \mathcal{O}_2^3$. Note that each one of these algebras is naturally a right $\Bbbk F$-module via $a \leftharpoonup t = ae_t$ for $a \in \mathcal{A}(Y, F, \psi, \xi)$, $t \in F$. Thus, we can consider the induced representation $W = \mathcal{A}(Y, F, \psi, \xi) \otimes_{\Bbbk F} W_\varepsilon$, where $W_\varepsilon = \Bbbk\{z\}$ is the trivial $\Bbbk F$-module. Let

$$B = \{1,\ y_{(12)},\ y_{(13)},\ y_{(23)},\ y_{(13)}y_{(12)},\ y_{(12)}y_{(13)},\ y_{(12)}y_{(23)},\ y_{(13)}y_{(23)},\\ y_{(12)}y_{(13)}y_{(23)},\ y_{(13)}y_{(12)}y_{(23)},\ y_{(12)}y_{(13)}y_{(12)},\ y_{(12)}y_{(13)}y_{(12)}y_{(23)}\}$$

and consider the linear subspace V of W generated by $B \otimes z$. We show that this is a nontrivial submodule in the four cases left, namely $F = 1,\ \mathbb{Z}_2,\ \mathbb{Z}_3$ or $\mathbb{S}_3$. In all these cases, the action of $y_{(12)}$ is determined by the matrix

$$y_{(12)} = \begin{bmatrix} 0 & \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now, take $F = \mathbb{S}_3$, $\psi \equiv 1$. The action of $e_{(12)}$ and $e_{(13)}$ is determined, respectively, by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \mu & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & -\mu & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \mu & \xi & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \xi & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \mu & \mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \mu & 0 & \xi & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & -\mu & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & \xi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -\mu \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The action of $e_{(23)}$ is given by $e_{(12)}e_{(13)}e_{(12)}$. Finally, we use Mathematica to check that these matrices satisfy the relations defining the algebra in each case.

We deal now with a generic 2-cocycle $\psi \in Z^2(\mathbb{S}_3, \Bbbk^\times)$. Fix $\mathcal{A} = \mathcal{A}(Y, F, 1, \xi)$ and $\mathcal{A}' = \mathcal{A}(Y, F, \psi, \xi)$. Also, set $U = \mathcal{H}_Y \# \Bbbk F$ and $U' = \mathcal{H}_Y \# \Bbbk_\psi F$. If $\overline{\psi} \in Z^2(U)$ is the 2-cocycle such that $\overline{\psi}_{F\times F} = \psi$ (see Lemma 4.1), it follows that $U' = U^{\overline{\psi}}$. Now, as $\mathcal{A}$ is an U-comodule algebra which is isomorphic to U as U-comodules, it follows that there exists a 2-cocycle $\gamma \in Z^2(U)$ such that $\mathcal{A} \cong {}_\gamma U$ (see [Montgomery 1993, Sec. 7 & 8]). It is easy to check then that $\mathcal{A}' = {}_\gamma U'$ by computing the multiplication on the generators. Thus, $\mathcal{A}' \neq 0$. □

To finish the proof of Corollary 7.8, we present three families of nontrivial algebras $\mathcal{A}(X, G, 1, \xi)$ for $X = \mathcal{O}_2^4$, $G = \mathbb{S}_4$, and certain collections of scalars $\{\xi_C\}_{C\in\mathcal{R}'}$ satisfying (7-6). In Proposition A.18, we will show that $\mathcal{A}(X, G, 1, \xi) \neq 0$.

Definition A.15. Let $\psi \in Z^2(\mathbb{S}_4, \Bbbk^\times)$ and $\alpha, \beta \in \Bbbk$.

(1) $\mathscr{A}_\psi^{-1}(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathbb{O}_2^4, g \in \mathbb{S}_4\}$ with relations

$$e_1 = 1, \quad e_r e_s = \psi(r, s)\, e_{rs} \ \text{ for } r, s \in \mathbb{S}_4,$$
$$e_g y_l = \operatorname{sgn}(g)\, y_{g\cdot l}\, e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathbb{O}_2^4,$$
$$y_{(12)}^2 = \alpha\, 1, \quad y_{(12)}y_{(34)} + y_{(34)}y_{(12)} = 2\alpha\, e_{(12)(34)},$$
$$y_{(12)}y_{(23)} + y_{(23)}y_{(13)} + y_{(13)}y_{(12)} = \beta\, e_{(132)}.$$

(2) $\mathscr{A}_\psi^{4}(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathbb{O}_4^4, g \in \mathbb{S}_4\}$ with relations

$$e_1 = 1, \quad e_r e_s = \psi(r, s)\, e_{rs} \ \text{ for } r, s \in \mathbb{S}_4,$$
$$e_g y_l = \operatorname{sgn}(g)\, y_{g\cdot l} e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathbb{O}_4^4,$$
$$y_{(1234)}^2 = \alpha\, e_{(13)(24)}, \quad y_{(1234)}y_{(1432)} + y_{(1432)}y_{(1234)} = 2\alpha 1,$$
$$y_{(1234)}y_{(1243)} + y_{(1243)}y_{(1423)} + y_{(1423)}y_{(1234)} = \beta\, e_{(132)}.$$

(3) $\mathscr{A}_\psi^{\chi}(\alpha, \beta)$ is the algebra generated by $\{y_i, e_g : i \in \mathbb{O}_2^4, g \in \mathbb{S}_4\}$ with relations

$$e_1 = 1, \quad e_r e_s = \psi(r, s)\, e_{rs} \ \text{ for } r, s \in \mathbb{S}_4,$$
$$e_g\, y_l = \chi_l(g)\, y_{g\cdot l}\, e_g \ \text{ for } g \in \mathbb{S}_4,\ l \in \mathbb{O}_2^4,$$
$$y_{(12)}^2 = \alpha\, 1, \quad y_{(12)}y_{(34)} - y_{(34)}y_{(12)} = 0,$$
$$y_{(12)}y_{(23)} - y_{(23)}y_{(13)} - y_{(13)}y_{(12)} = \beta\, e_{(132)}.$$

Remark A.16. Let $\mathfrak{Q} = \mathfrak{Q}^{-1}[t]$. It is clear $\mathscr{A}_\psi^{-1}(\alpha, \beta) \cong \mathscr{A}(\mathbb{O}_2^4, \mathbb{S}_4, \psi, \xi)$ for the family $\xi = \{\xi_C\}_{C\in\mathscr{R}}$ where, for $i = 1, 2, 3$, $\xi_C = \xi_i$ is constant in the classes C with the same cardinality $|C| = i$ and where, in this case, $\xi_1 = \alpha$, $\xi_2 = 2\alpha$, $\xi_3 = \beta$.

Analogously, if $\mathfrak{Q} = \mathfrak{Q}^\chi[t]$, then $\mathscr{A}_\psi^{\chi}(\alpha, \beta)$ is the algebra $\mathscr{A}(\mathbb{O}_2^4, \mathbb{S}_4, \psi, \xi)$ for a certain family ξ subject to similar conditions as in the previous paragraph. The same holds for $\mathfrak{Q} = \mathfrak{D}[t]$, $\mathscr{A}_\psi^{4}(\alpha, \beta)$ and $\mathscr{A}(\mathbb{O}_4^4, \mathbb{S}_4, \psi, \xi)$.

Recall that there is a group epimorphism $\pi : \mathbb{S}_4 \to \mathbb{S}_3$ with kernel $H = \langle(12)(34), (13)(24), (23)(14)\rangle$. Moreover, $\pi(\mathbb{O}_2^4) = \mathbb{O}_2^3$. Let $\mathfrak{Q}$ be one of the ql-data from Section 5D for $n = 4$.

Lemma A.17. *Let $\mathfrak{Q}$ be as above, and take $\gamma = 0$ if $\mathfrak{Q} = \mathfrak{Q}_4^{-1}$. There is an epimorphism of algebras $\mathscr{H}(\mathfrak{Q}) \twoheadrightarrow \mathscr{H}(\mathfrak{Q}_3^{-1}[\lambda])$.*

Proof. Consider the ideal I in $\mathscr{H}(\mathfrak{Q})$ generated by the element $H_{(12)}H_{(34)} - 1$, and let $\mathscr{L} = \mathscr{H}(\mathfrak{Q})/I$. We have

$$H_{(14)}H_{(23)} = \operatorname{ad}(H_{(24)})(H_{(12)}H_{(34)}), \quad \text{so } H_{(14)}H_{(23)} = 1 \quad \text{in } \mathscr{L},$$
$$a_{(34)} = \operatorname{ad}(H_{(14)}H_{(23)})(a_{12}), \qquad \text{so } a_{(34)} = a_{(12)} \qquad \text{in } \mathscr{L}.$$

Analogously, $H_{(13)} = H_{(24)}$, $a_{(14)} = a_{(23)}$ and $a_{(24)} = a_{(13)}$ in $\mathscr{L}$. Since, for this ql-data, the action $\cdot : \mathbb{S}_4 \times X \to X$ is given by conjugation, and $g : X \to \mathbb{S}_4$ is the inclusion, the relations (5-6) and (5-7) in the definition of $\mathscr{H}(\mathfrak{Q})$ are satisfied in the quotient. It is now easy to check that the quadratic relations (5-8) defining $\mathscr{H}(\mathfrak{Q})$ become in the quotient the corresponding ones defining the algebra $\mathscr{H}(\mathfrak{Q}_3^{-1}[\lambda])$. □

Proposition A.18. *Assume that* (Y, F, ψ, ξ) *satisfies* $\xi_{C_i} = \xi_{C_j}$ *for all* $i, j \in Y$. *If* $\mathfrak{Q} \neq \mathfrak{Q}_4^{\chi}(\lambda)$, *assume in addition that*

$$i, j \in Y,\ i \triangleright j = j\ (i, j) \in C \ \Rightarrow\ \xi_C = 2\xi_i.$$

Then, the algebra $\mathscr{A}(Y, F, \psi, \xi)$ *is not null.*

Proof. Assume first that $\psi \equiv 1$. Now, given a datum (Y, F, ψ, ξ), we have $\pi(F) < \mathbb{S}_3$ and it is easy to see that $\pi(Y)$ is a subrack of $\mathcal{O}_2^3$. Moreover, it follows that ξ is compatible with the triple $(\pi(Y), \pi(F), \psi)$. Then, we have the algebra $\mathscr{A}(\pi(Y), \pi(F), \psi, \xi)$. As in Lemma A.17, it is easy to see that, if we quotient out by the ideal generated by $\langle e_f e_g : fg^{-1} \in N\rangle$, then we have an algebra epimorphism $\mathscr{A}(Y, F, \psi, \xi) \twoheadrightarrow \mathscr{A}(\pi(Y), \pi(F), \psi, \xi)$. As these algebras are nonzero by Proposition A.14, so is $\mathscr{A}(Y, F, \psi, \xi)$.

Notice that, in the case in which (Y, F, ψ, ξ) is associated with the ql-datum $\mathfrak{Q}_4^{\chi}(\lambda)$, assumption (ii) is not needed, since the first equation in Definition 7.1 implies that, if $i, j \in Y$ are such that $i \triangleright j = i$ and $C \in \mathscr{R}'$ is the corresponding class, then $\xi_C = 0$ and this relation is contained in the ideal by which we quotient.

The case $\psi \neq 1$ follows now as in the proof of Proposition A.14. □

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