Pacific Journal of Mathematics

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Volume 252 No. 2 August 2011

PACIFIC JOURNAL OF MATHEMATICS
Vol. 252, No. 2, 2011

ON INTRINSICALLY KNOTTED OR COMPLETELY 3-LINKED GRAPHS

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We say that a graph is intrinsically knotted or completely 3-linked if every embedding of the graph into the 3-sphere contains a nontrivial knot or a 3-component link each of whose 2-component sublinks is nonsplittable. We show that a graph obtained from the complete graph on seven vertices by a finite sequence of ΔY-exchanges and YΔ-exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let f be an embedding of a finite graph G into the 3-sphere. Then f is called a *spatial embedding* of G and $f(G)$ is called a *spatial graph*. We denote the set of all spatial embeddings of G by $\mathrm{SE}(G)$. We call a subgraph γ of G that is homeomorphic to the circle a *cycle* of G. For a positive integer n, let $\Gamma^{(n)}(G)$ denote the set of all cycles of G if $n = 1$ and the set of all unions of n mutually disjoint cycles of G if $n \geq 2$. For simplicity, we also write $\Gamma(G)$ for $\Gamma^{(1)}(G)$. For an element λ in $\Gamma^{(n)}(G)$ and a spatial embedding f of G, $f(\lambda)$ is a knot if $n = 1$ and an n-component link if $n \geq 2$.

A graph G is said to be *intrinsically linked* (IL) if for every spatial embedding f of G, $f(G)$ contains a nonsplittable 2-component link. Conway and Gordon [1983] and Sachs [1984] showed that K_6 is IL, where K_m denotes the *complete graph* on m vertices. Also, IL graphs have been completely characterized as follows. For a graph G and an edge e of G, we denote the subgraph $G \setminus \operatorname{int} e$ by $G - e$. Let $e = \overline{uv}$ be an edge of G that is not a loop. We call the graph obtained from $G - e$ by identifying the end vertices u and v the *edge contraction of G along e*, and denote it by G/e. A graph H is called a *minor* of a graph G if there exists a subgraph G' of G and edges $e_1, e_2, \ldots, e_m$ of G' such that H is obtained from G' by a

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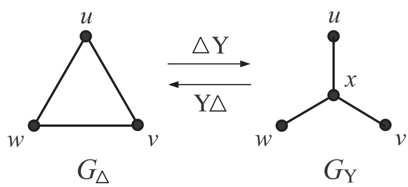
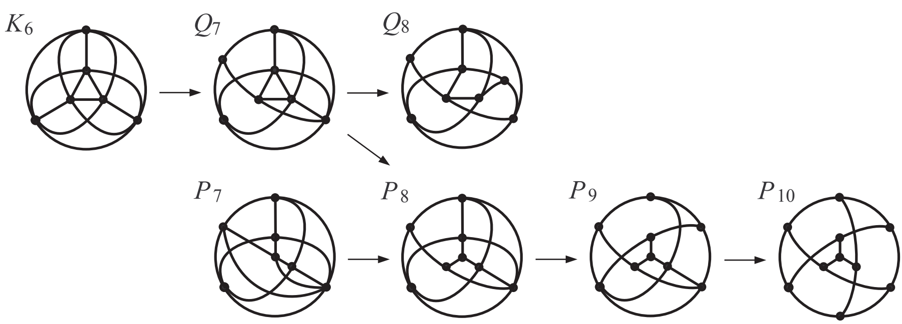
Nikkuni was partially supported by Grant-in-Aid for Young Scientists (B) (No. 21740046), Japan Society for the Promotion of Science. Taniyama was partially supported by Grant-in-Aid for Scientific Research (C) (No. 21540099), Japan Society for the Promotion of Science.
MSC2000: primary 57M15; secondary 57M25.
Keywords: spatial graph, intrinsic knottedness, △Y-exchange, Y△-exchange.

sequence of edge contractions along $e_1, e_2, \ldots, e_m$. A minor H of G is called a *proper minor* if H does not equal G. Let $\mathcal{P}$ be a property for graphs that is *closed* under minor reductions; that is, for any graph G that does not have $\mathcal{P}$, all minors of G also do not have $\mathcal{P}$. A graph G is said to be *minor-minimal* with respect to $\mathcal{P}$ if G has $\mathcal{P}$ but all proper minors of G do not have $\mathcal{P}$. Note that G has $\mathcal{P}$ if and only if G has a minor-minimal graph with respect to $\mathcal{P}$ as a minor. By the famous theorem of Robertson and Seymour [2004], there are finitely many minor-minimal graphs with respect to $\mathcal{P}$. Nešetřil and Thomas [1985] showed that IL is closed under minor reductions, and Robertson, Seymour and Thomas [Robertson et al. 1995] showed that the set of all minor-minimal graphs with respect to IL equals the *Petersen family*, which is the set of all graphs obtained from K_6 by a finite sequence of $\triangle$Y-*exchanges* and Y$\triangle$-*exchanges*. A $\triangle$Y-exchange is the left-to-right operation shown here:



That is, a graph $G_\triangle$ containing a three-edge cycle $\triangle$ is changed into a new graph G_Y by removing the edges of the cycle and adding a new vertex x connected to each of the vertices of the deleted cycle, thus forming a Y. A Y$\triangle$-exchange is the reverse of this operation. $\triangle$Y- and Y$\triangle$-exchanges preserve IL: if $G_\triangle$ is IL, so is G_Y [Motwani et al. 1988], and if G_Y is IL, so is $G_\triangle$ [Robertson et al. 1995].

The Petersen family contains seven graphs, including the *Petersen graph* P_{10}:



(An arrow between two graphs indicates the application of a single $\triangle$Y-exchange.)

A graph G is said to be *intrinsically knotted* (IK) if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot. Conway and Gordon [1983] showed that K_7 is IK. Fellows and Langston [1988] showed that IK is closed under minor reductions. Motwani, Raghunathan, and Saran [Motwani et al. 1988] showed that K_7 is a minor-minimal IK graph, and additional minor-minimal IK graphs were given in [Kohara and Suzuki 1992] and [Foisy 2002; 2003].

IK graphs have not been completely characterized yet. If $G_\triangle$ is IK then G_Y is also IK [Motwani et al. 1988], but if G_Y is IK then $G_\triangle$ may not always be IK. That is, the Y△-exchange does not preserve IK in general. Flapan and Naimi [2008] showed that there exists a graph G_{FN} obtained from K_7 by five △Y-exchanges and two Y△-exchanges that is not IK. We call the set of all graphs obtained from K_7 by a finite sequence of △Y and Y△-exchanges the *Heawood family*.[1] This family contains exactly twenty graphs, as illustrated in Figure 1; of these, C_{14} is the *Heawood graph* (Remark 4.7).

Kohara and Suzuki [1992] showed that a graph G in the Heawood family is a minor-minimal IK graph if G is obtained from K_7 by a finite sequence of △Y-exchanges, that is, if G is one of fourteen graphs K_7, H_8, $H_9, \dots, H_{12}$, F_9, F_{10}, E_{10}, E_{11} and $C_{11}, C_{12}, \dots, C_{14}$.[2] On the other hand, N'_{10} is isomorphic to G_{FN}, that is, N'_{10} is not IK. Our first purpose in this paper is to determine completely when a graph in the Heawood family is IK.

Theorem 1.1. *For a graph G in the Heawood family, the following are equivalent:*

(1) *G is IK.*

(2) *G is obtained from K_7 by a finite sequence of △Y-exchanges.*

(3) *$\Gamma^{(3)}(G)$ is the empty set.*

Hence the members N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} of the Heawood family are not IK, and only they contain a union of three mutually disjoint cycles.

Our second purpose is to show that any of the graphs in the Heawood family is a minor-minimal graph with respect to a certain kind of intrinsic nontriviality even if it is not IK. We say that a graph G is *intrinsically knotted or completely 3-linked* — I(K or C3L) for short — if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot or a 3-component link all of whose 2-component sublinks are nonsplittable. An IK graph is I(K or C3L). As we show in Proposition 2.2, I(K or C3L) is closed under minor reductions.

Theorem 1.2. *All graphs in the Heawood family are minor-minimal I(K or C3L) graphs.*

As we have seen, N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} are not IK, but they are but I(K or C3L) and are minor-minimal with respect to I(K or C3L).

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[1] Van der Holst [2006] calls the set of all graphs obtained from K_7 or $K_{3,3,1,1}$ by a finite sequence of △Y-exchanges and Y△-exchanges the Heawood family, where $K_{3,3,1,1}$ is the complete 4-partite graph on $3+3+1+1$ vertices.

[2] One edge of F_{10} in [Kohara and Suzuki 1992, Figure 5] is wanting.

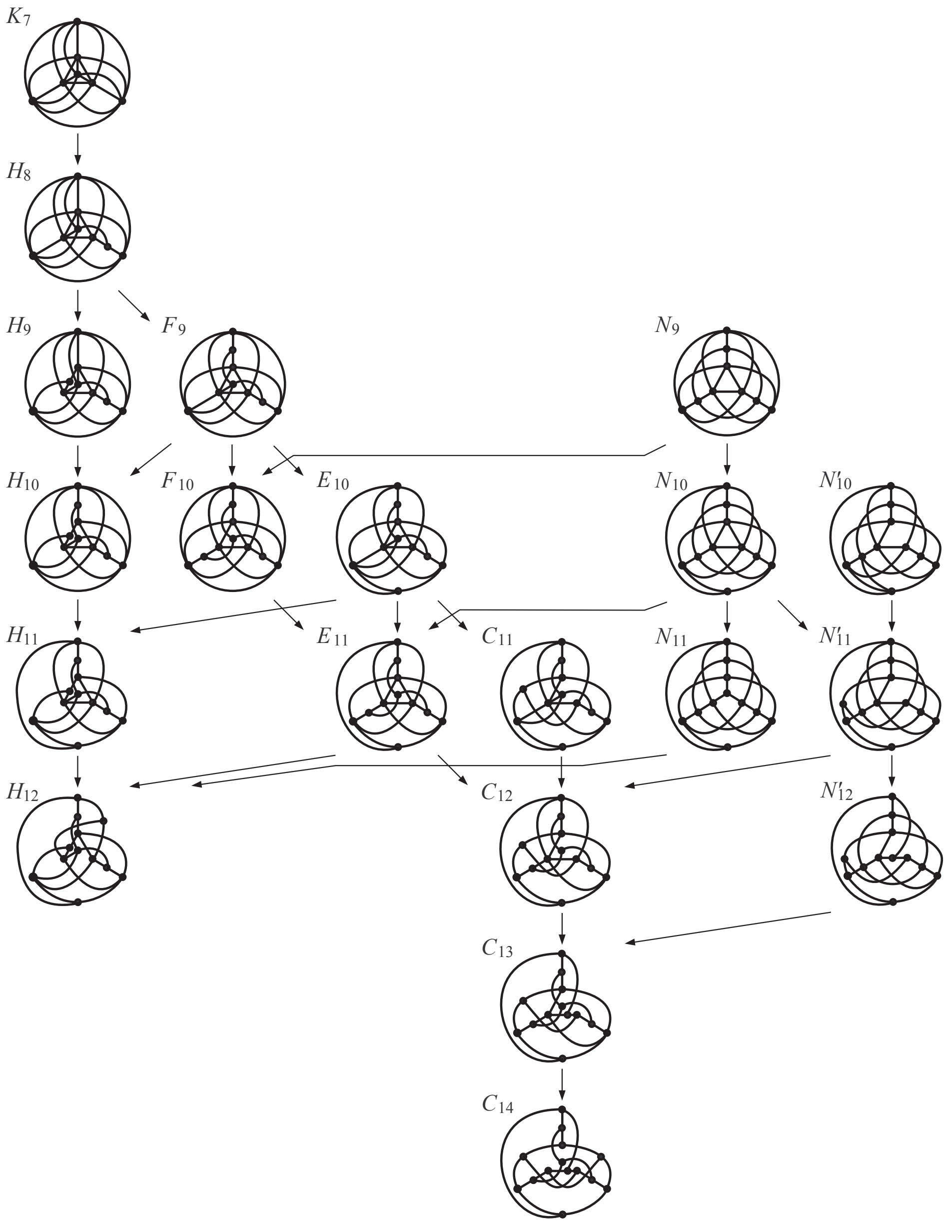


Figure 1. The Heawood family. An arrow between two graphs indicates the application of a single $\triangle$Y-exchange.

Remark 1.3. A graph G is said to be *intrinsically n-linked* (InL) if for every spatial embedding f of G, $f(G)$ contains a nonsplittable n-component link [Flapan et al. 2001a; 2001b]. I2L coincides with IL. Let G be a graph in the Heawood family that is not IK. Then we show in Example 4.6 that there exists a spatial embedding f of G such that $f(G)$ does not contain a nonsplittable 3-component link. That is, G is neither IK nor I3L.

Remark 1.4. A graph G is called *intrinsically knotted or* 3*-linked* — I(K or 3L) for short — if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot or a nonsplittable 3-component link. Clearly I(K or C3L) implies I(K or 3L), but the converse is not true: [Foisy 2006] exhibits an I(K or 3L) graph G and a spatial embedding f of G such that $f(G)$ contains no nontrivial knot and all nonsplittable 3-component links contained in $f(G)$ have split 2-component sublinks.

The rest of this paper is organized as follows. Section 2 contains general results about graph minors, $\triangle$Y-exchanges and spatial graphs. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

2. Graph minors, ΔY-exchanges and spatial graphs

Let H be a minor of a graph G. Then there exists a natural injection

$$\Psi^{(n)} = \Psi^{(n)}_{H,G} : \Gamma^{(n)}(H) \longrightarrow \Gamma^{(n)}(G)$$

for any positive integer n. We write Ψ for $\Psi^{(1)}$. Let f be a spatial embedding of G and e an edge of G that is not a loop. Then by contracting $f(e)$ into one point, we obtain a spatial embedding $\psi(f)$ of G/e. Similarly, we can also obtain a spatial embedding $\psi(f)$ of H from f. Thus we obtain a map

$$\psi = \psi_{G,H} : \mathrm{SE}(G) \longrightarrow \mathrm{SE}(H).$$

Then we immediately have:

Proposition 2.1. *For a spatial embedding f of G and an element λ in $\Gamma^{(n)}(H)$, $\psi(f)(\lambda)$ is ambient isotopic to $f(\Psi^{(n)}(\lambda))$.* □

Proposition 2.2. *I(K or C3L) is closed under minor reductions.*

Proof. Let G be a graph that is not I(K or C3L), and H be a minor of G. Let f be a spatial embedding of G that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Then by Proposition 2.1, $\psi(f)$ has the same property. This implies that H is not I(K or C3L). □

Remark 2.3. Proposition 2.1 also implies that IK, InL and I(K or 3L) are closed under minor reductions.

Let $G_\triangle$ and G_Y be two graphs such that G_Y is obtained from $G_\triangle$ by a single $\triangle$Y-exchange, as in the previous section. Let λ be an element in $\Gamma^{(n)}(G_\triangle)$ that does not contain $\triangle$. Then there exists an element $\Phi^{(n)}(\lambda)$ in $\Gamma^{(n)}(G_Y)$ such that

$\lambda \setminus \triangle = \Phi^{(n)}(\lambda) \setminus \mathrm{Y}$. Thus we obtain a map

$$\Phi^{(n)} = \Phi^{(n)}_{G_\triangle, G_\mathrm{Y}} : \{\lambda \in \Gamma^{(n)}(G_\triangle) \mid \lambda \not\supset \triangle\} \longrightarrow \Gamma^{(n)}(G_\mathrm{Y}),$$

for any positive integer n. We denote $\Phi^{(1)}$ by Φ. Note that $\Phi^{(n)}$ is surjective and the inverse image of λ by $\Phi^{(n)}$ contains at most two elements in $\Gamma^{(n)}(G_\triangle)$ for any element λ in $\Gamma^{(n)}(G_\mathrm{Y})$. The surjectivity of $\Phi^{(n)}$ implies Proposition 2.4.

Proposition 2.4. *For $n \geq 2$, if $\Gamma^{(n)}(G_\triangle) = \varnothing$, then $\Gamma^{(n)}(G_\mathrm{Y}) = \varnothing$.* □

Let f be a spatial embedding of G_Y, and let D be a 2-disk in the 3-sphere such that $D \cap f(G_\mathrm{Y}) = f(\mathrm{Y})$ and $\partial D \cap f(G_\mathrm{Y}) = \{f(u), f(v), f(w)\}$. (Throughout the paper we use u, v, w, x for the vertices of the Y of interest, as in the first figure on page 408), Let $\varphi(f)$ be a spatial embedding of $G_\triangle$ such that $\varphi(f)(x) = f(x)$ for $x \in G_\mathrm{Y} \setminus \mathrm{Y}$ and $\varphi(f)(G_\triangle) = (f(G_\mathrm{Y}) \setminus f(\mathrm{Y})) \cup \partial D$. Then we obtain a map

$$\varphi = \varphi_{G_\mathrm{Y}, G_\triangle} : \mathrm{SE}(G_\mathrm{Y}) \longrightarrow \mathrm{SE}(G_\triangle),$$

and we immediately have Proposition 2.5.

Proposition 2.5. *For a spatial embedding f of G_Y and an element λ in $\Gamma^{(n)}(G_\mathrm{Y})$, $f(\lambda)$ is ambient isotopic to $\varphi(f)(\lambda')$ for each element λ' in the inverse image of λ by $\Phi^{(n)}$.* □

Lemma 2.6. *If $G_\triangle$ is I(K or C3L), then G_Y is also I(K or C3L).*

Proof. Assume that G_Y is not I(K or C3L), that is, that there exists a spatial embedding f of G_Y that contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. We show that $\varphi(f)(G_\triangle)$ also has the same property.

Let γ be an element in $\Gamma(G_\triangle)$. If γ is not $\triangle$, then $\varphi(f)(\gamma)$ is ambient isotopic to $f(\Phi(\gamma))$ by Proposition 2.5, and $f(\Phi(\gamma))$ is a trivial knot by the assumption. Since $\varphi(f)(\triangle)$ is also a trivial knot, it follows that $\varphi(f)(G_\triangle)$ does not contain a nontrivial knot. Let λ be an element in $\Gamma^{(3)}(G_\triangle)$. If λ does not contain $\triangle$, then $\varphi(f)(\lambda)$ is ambient isotopic to $f(\Phi^{(3)}(\lambda))$ by Proposition 2.5, and $f(\Phi^{(3)}(\lambda))$ is a 3-component link that contains a split 2-component sublink by the assumption. If λ contains $\triangle$, then $\varphi(f)(\lambda)$ is a split 3-component link. Thus we see that $\varphi(f)(G_\triangle)$ does not contain a 3-component link with a nonsplittable 2-component sublink. □

Lemma 2.7. *If G_Y is minor-minimal for I(K or C3L), then $G_\triangle$ is also minor-minimal for I(K or C3L).*

Proof. (This lemma has already been proven in more general form [Ozawa and Tsutsumi 2007, Lemma 3.1, Exercise 3.2], but we prove it here for convenience.) We show that for any edge e of $G_\triangle$ that is not a loop, there exist a spatial embedding f of $G_\triangle - e$ and a spatial embedding g of $G_\triangle / e$ such that each of $f(G_\triangle - e)$ and

$g(G_\triangle/e)$ contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublink are nonsplittable. If e is not one of the edges $\overline{uv}$, $\overline{vw}$ or $\overline{wu}$ of the $\triangle$ then there exist a spatial embedding f' of $G_Y - e$ and a spatial embedding g' of G_Y/e such that both $f'(G_Y - e)$ and $g'(G_Y/e)$ contain neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. The graph $G_Y - e$ is obtained from $G_\triangle - e$, and likewise G_Y/e from $G_\triangle/e$, by a single $\triangle$Y-exchange at the same $\triangle$. Then we see that each of $\varphi(f')(G_\triangle - e)$ and $\varphi(g')(G_\triangle/e)$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks, in a way similar to the proof of Lemma 2.6. If e is one of $\overline{uv}$, $\overline{vw}$ and $\overline{wu}$, we may assume that $e = \overline{uv}$ without loss of generality. Now there exists a spatial embedding f' of $G_Y/\overline{xw}$ such that $f'(G_Y/\overline{xw})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublinks. Then we can see that $G_\triangle - \overline{uv} = G_Y/\overline{xw}$. On the other hand, there exists a spatial embedding g' of $G_Y/\overline{xv}/\overline{xu}$ such that $g'(G_Y/\overline{xv}/\overline{xu})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. Take a 2-disk D' in the 3-sphere such that $D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = g'(\overline{uw})$ and $\partial D' \cap g'(G_Y/\overline{xv}/\overline{xu}) = \{g'(u), g'(w)\}$. Then $(g'(G_Y/\overline{xv}/\overline{xu}) \setminus \operatorname{int} g'(\overline{uw})) \cup \partial D'$ may be regarded as the image of a spatial embedding of $G_\triangle/\overline{uv}$, denoted by g. Clearly $g(G_\triangle/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link having only nonsplittable 2-component sublink. □

3. Proof of Theorem 1.1

Lemma 3.1. *Each of the graphs N_9, N_{10}, N_{11}, N'_{10}, N'_{11} and N'_{12} in the Heawood family is not IK.*

Proof. For N'_{10}, see [Flapan and Naimi 2008]. We show that N_9, N_{10}, N_{11}, N'_{11} and N'_{12} are not IK. Let f_9 be the spatial embedding of N_9 illustrated in Figure 2. It can be checked directly that $f_9(N_9)$ does not contain a nontrivial knot. Thus N_9 is

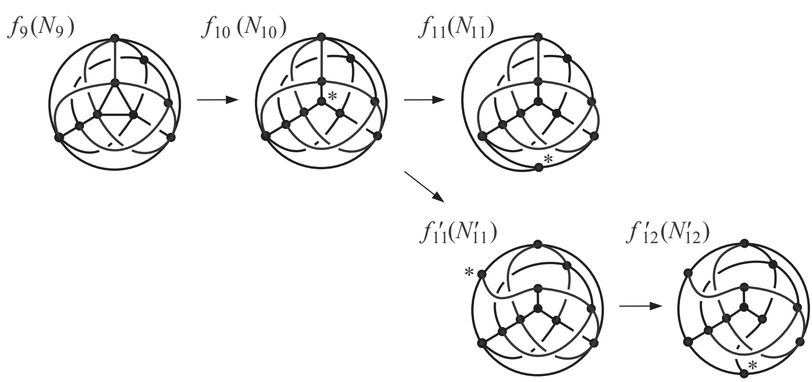
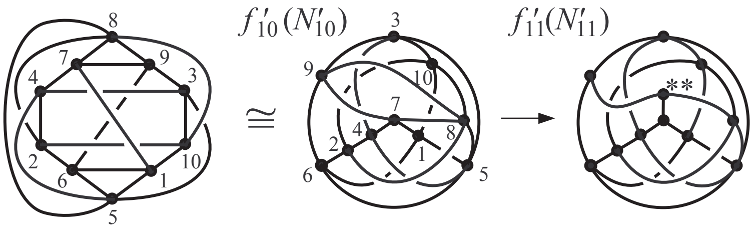


Figure 2

not IK. Let f_{10} be the spatial embedding of N_{10} illustrated in Figure 2. Let φ_{N_{10},N_9} be the map from $\mathrm{SE}(N_{10})$ to $\mathrm{SE}(N_9)$ induced by the Y△-exchange from N_{10} to N_9 at the Y-fork marked $*$ in Figure 2. Then clearly $\varphi(f_{10}) = f_9$. Since $f_9(N_9)$ does not contain a nontrivial knot, by Proposition 2.5 it follows that $f_{10}(N_{10})$ also does not contain a nontrivial knot. Thus, N_{10} is not IK. By repeating this argument, we can see that each of the graphs N_{11}, N'_{11} and N'_{12} is also not IK; see Figure 2. □

Proof of Theorem 1.1. First we show that (1) and (2) are equivalent. Since we already know that (2) implies (1), we show that (1) implies (2). If G is IK, then by Lemma 3.1 we see that G is not one of N_9, N_{10}, N_{11}, N'_{10}, N'_{11} or N'_{12}. Thus G is obtained from K_7 by a finite sequence of △Y-exchanges. Next we show that (2) and (3) are equivalent. Assume that G is obtained from K_7 by a finite sequence of △Y-exchanges. $\Gamma^{(3)}(K_7)$ is the empty set. Thus, by Proposition 2.4, we see that $\Gamma^{(3)}(G)$ is the empty set. Conversely, if G is one of N_9, N_{10}, N_{11}, N'_{10}, N'_{11}, and N'_{12}, then $\Gamma^{(3)}(G)$ is not the empty set. This completes the proof. □

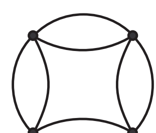
Remark 3.2. Let f'_{11} be the spatial embedding of N'_{11} illustrated in Figure 2, and let f'_{10} be the spatial embedding of N'_{10} illustrated in the figure below. Let $\varphi_{N'_{11},N'_{10}}$ be the map from $\mathrm{SE}(N'_{11})$ to $\mathrm{SE}(N'_{10})$ induced by the Y△-exchange from N'_{11} to N'_{10} at the Y-fork marked $**$. Then clearly $\varphi(f'_{11}) = f'_{10}$. Also, we can see that f'_{10} coincides with Flapan and Naimi's example [2008] of a spatial embedding of N'_{10} whose image does not contain a nontrivial knot, as illustrated in the leftmost diagram:



4. Proof of Theorem 1.2

Lemma 4.1 [Conway and Gordon 1983; Taniyama and Yasuhara 2001]. *Let G be a graph in the Petersen family and f a spatial embedding of G. Then there exists an element λ in $\Gamma^{(2)}(G)$ such that $\mathrm{lk}(f(\lambda))$ is odd, where* lk *denotes the linking number in the* 3*-sphere.*

Let D_4 be the graph illustrated on the right. We denote the set of all cycles of D_4 with exactly four edges by $\Gamma_4(D_4)$. For a spatial embedding f of D_4, we define



D_4

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\gamma)) \pmod 2,$$

where a_2 denotes the second coefficient of the *Conway polynomial*. Note that $a_2(K)$ of a knot K is congruent to the *Arf invariant* modulo 2 [Kauffman 1983].

Lemma 4.2 [Taniyama and Yasuhara 2001]. *Let f be a spatial embedding of D_4 and λ, λ' all elements in $\Gamma^{(2)}(D_4)$. If both $\mathrm{lk}(f(\lambda))$ and $\mathrm{lk}(f(\lambda'))$ are odd, then $\alpha(f) = 1$.*

Let G be a graph that contains D_4 as a minor and f a spatial embedding of G. Then we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2\big(f(\Psi_{D_4,G}(\gamma))\big) \pmod 2.$$

Lemma 4.3. *Let G be a graph that contains D_4 as a minor and let f be a spatial embedding of G. For two elements μ and μ' in $\Psi^{(2)}_{D_4,G}(\Gamma^{(2)}(D_4))$, if both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'))$ are odd, then $\alpha(f) = 1$.*

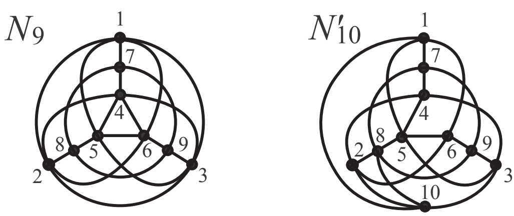
Proof. For two elements λ and λ' in $\Gamma^{(2)}(D_4)$, we see that both $\mathrm{lk}\big(f(\Psi^{(2)}_{D_4,G}(\lambda))\big)$ and $\mathrm{lk}\big(f(\Psi^{(2)}_{D_4,G}(\lambda'))\big)$ are odd by the assumption. Then by Proposition 2.1, it follows that $\mathrm{lk}(\psi_{G,D_4}(f)(\lambda))$ and $\mathrm{lk}(\psi_{G,D_4}(f)(\lambda'))$ are also odd. Therefore, by Lemma 4.2, we have that

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2\big(f(\Psi_{D_4,G}(\gamma))\big) = \sum_{\gamma \in \Gamma_4(D_4)} a_2(\psi_{G,D_4}(f)(\gamma)) \equiv 1 \pmod 2. \quad \square$$

The next theorem is the most important part of the proof of Theorem 1.2.

Theorem 4.4. *Let G be N_9 or N'_{10}. For every spatial embedding f of G, there exists an element γ in $\Gamma(G)$ such that $a_2(f(\gamma))$ is odd, or there exists an element λ in $\Gamma^{(3)}(G)$ such that each 2-component sublink of $f(\lambda)$ has an odd linking number.*

Proof. We will denote by $[i_1\, i_2\, \ldots\, i_k]$ the cycle $\overline{i_1 i_2} \cup \overline{i_2 i_3} \cup \cdots \cup \overline{i_{k-1} i_k} \cup \overline{i_k i_1}$ of G. We label each vertex of G as follows:



First we show the case of $G = N_9$. Let f be a spatial embedding of N_9. Note that N_9 contains K_6 as the proper minor

$$(((N_9 - \overline{78}) - \overline{89}) - \overline{97})/\overline{47}/\overline{58}/\overline{69}.$$

By Lemma 4.1, there is thus an element ν in $\Gamma^{(2)}(K_6)$ such that $\mathrm{lk}(\psi_{N_9,K_6}(f)(\nu))$ is odd. Hence, by Proposition 2.1, there exists an element μ in $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ such that $\mathrm{lk}(f(\mu))$ is odd. $\Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6))$ consists of ten elements, and by the symmetry of N_9, we may assume that $\mu = [1\,7\,4\,3] \cup [2\,6\,5\,8]$ or $[1\,2\,3] \cup [4\,5\,6]$ without loss of generality.

Case 1. Let $\mu = [1\,7\,4\,3] \cup [2\,6\,5\,8]$. Note that N_9 contains P_7 as the proper minor

$$(((((N_9 - \overline{6\,1}) - \overline{6\,2}) - \overline{6\,4}) - \overline{6\,5}) - \overline{6\,9})/\overline{3\,9}.$$

Thus, by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N_9,P_7}(f)(\nu'))$ is odd. Hence, by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7,N_9}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7,N_9}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\mu'_1 = [3\,4\,5] \cup [1\,2\,8\,7], \quad \mu'_2 = [1\,5\,4\,7] \cup [2\,3\,9\,8], \quad \mu'_3 = [2\,8\,5\,4] \cup [3\,1\,7\,9],$$
$$\mu'_4 = [1\,2\,4\,7] \cup [3\,5\,8\,9], \quad \mu'_5 = [1\,2\,3] \cup [4\,7\,8\,5], \quad \mu'_6 = [1\,2\,8\,5] \cup [3\,4\,7\,9],$$
$$\mu'_7 = [2\,3\,4] \cup [1\,5\,8\,7], \quad \mu'_8 = [7\,8\,9] \cup [1\,2\,4\,5], \quad \mu'_9 = [1\,5\,3] \cup [2\,8\,7\,4].$$

For $i = 1, 2, \dots, 9$, let J^i be the subgraph of N_9 that is $\mu \cup \mu'_i \cup \overline{6\,9}$ if $i = 3, 6$ and $\mu \cup \mu'_i$ if $i \neq 3, 6$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some $i \neq 8$. Then it can be easily seen that J^i contains a graph D^i as a minor, such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu'_8))$ is odd. We denote two elements $[7\,8\,9] \cup [1\,2\,6\,5]$ and $[7\,8\,9] \cup [4\,2\,6\,5]$ in $\Gamma^{(2)}(J^8)$ by $\mu'_{8,1}$ and $\mu'_{8,2}$, respectively. We denote the subgraph $\mu \cup \mu'_{8,j}$ of J^8 by $J^{8,j}$ $(j = 1, 2)$. Then it can be easily seen that $J^{8,j}$ contains a graph $D^{8,j}$ as a minor, such that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu'_{8,j}\} = \Psi^{(2)}_{D^{8,j},J^{8,j}}(\Gamma^{(2)}(D^{8,j}))$ $(j = 1, 2)$. Note that

$$[1\,2\,4\,5] = [1\,2\,6\,5] + [4\,2\,6\,5]$$

in $H_1(J^8; \mathbb{Z}_2)$, where $H_*(\cdot\,; \mathbb{Z}_2)$ denotes the homology group with $\mathbb{Z}_2$-coefficients. Then, by the homological property of the linking number, we have that

$$1 \equiv \mathrm{lk}(f(\mu'_8)) \equiv \mathrm{lk}(f(\mu'_{8,1})) + \mathrm{lk}(f(\mu'_{8,2})) \pmod 2.$$

Thus we see that $\mathrm{lk}(f(\mu'_{8,1}))$ is odd or $\mathrm{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd.

Case 2. Let $\mu = [1\,2\,3] \cup [4\,5\,6]$. Note that N_9 contains P_9 as the proper minor

$$(((((N_9 - \overline{1\,2}) - \overline{2\,3}) - \overline{3\,1}) - \overline{4\,5}) - \overline{5\,6}) - \overline{6\,4}.$$

Thus, by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N_9,P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N_9}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N_9}(\Gamma^{(2)}(P_9))$ consists of seven elements, and by the symmetry of N_9, we may assume, without loss of generality, that $\mu' = [1\,5\,8\,7] \cup [2\,6\,9\,3\,4]$ or $[7\,8\,9] \cup [1\,5\,3\,4\,2\,6]$. Denote by J the subgraph $\mu \cup \mu'$ of N_9. Assume that $\mu' = [1\,5\,8\,7] \cup [2\,6\,9\,3\,4]$. We denote the two elements $[1\,5\,8\,7] \cup [4\,3\,2]$ and $[1\,5\,8\,7] \cup [6\,9\,3\,2]$ in $\Gamma^{(2)}(J)$ by μ'_1 and μ'_2, respectively. We denote the subgraph $\mu \cup \mu'_i$ of J by J^i $(i = 1, 2)$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and

$$\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i)) \quad (i = 1, 2).$$

Since $[2\,6\,9\,3\,4] = [4\,3\,2] + [6\,9\,3\,2]$ in $H_1(J; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu')) \equiv \mathrm{lk}(f(\mu'_1)) + \mathrm{lk}(f(\mu'_2)) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu'_1))$ is odd or $\mathrm{lk}(f(\mu'_2))$ is odd. In both cases, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mu' = [7\,8\,9] \cup [1\,5\,3\,4\,2\,6]$. We denote four elements $[7\,8\,9] \cup [3\,4\,5]$, $[7\,8\,9] \cup [4\,5\,6]$, $[7\,8\,9] \cup [1\,5\,6]$ and $[7\,8\,9] \cup [2\,4\,6]$ in $\Gamma^{(2)}(J)$ by μ'_1, μ'_2, μ'_3 and μ'_4, respectively. Since $[1\,5\,3\,4\,2\,6] = [3\,4\,5] + [4\,5\,6] + [1\,5\,6] + [2\,4\,6]$ in $H_1(J; \mathbb{Z}_2)$, we get

$$1 \equiv \mathrm{lk}(\mu') \equiv \mathrm{lk}(\mu'_1) + \mathrm{lk}(\mu'_2) + \mathrm{lk}(\mu'_3) + \mathrm{lk}(\mu'_4) \pmod 2.$$

This implies that $\mathrm{lk}(\mu'_i)$ is odd for some $i = 1, 2, 3$ or 4. Moreover, by the symmetry of J, we may assume that $\mathrm{lk}(\mu'_1)$ is odd or $\mathrm{lk}(\mu'_2)$ is odd without loss of generality. Assume that $\mathrm{lk}(\mu'_1)$ is odd. We denote the subgraph $\mu \cup \mu'_1 \cup \overline{1\,7} \cup \overline{6\,9}$ of N_9 by J^1. Then J^1 contains a graph D^1 as a minor such that D^1 is isomorphic to D_4 and $\{\mu, \mu'_1\} = \Psi^{(2)}_{D^1, J^1}(\Gamma^{(2)}(D^1))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_1))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^1)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(\mu'_2)$ is odd. We denote four elements $[7\,8\,9] \cup [1\,2\,6]$, $[7\,8\,9] \cup [1\,2\,3]$, $[7\,8\,9] \cup [2\,3\,4]$ and $[7\,8\,9] \cup [1\,3\,5]$ in $\Gamma^{(2)}(J)$ by μ'_5, μ'_6, μ'_7 and μ'_8, respectively. Since $[1\,5\,3\,4\,2\,6] = [1\,2\,6] + [1\,2\,3] + [2\,3\,4] + [1\,3\,5]$ in $H_1(J; \mathbb{Z}_2)$, we have

$$1 \equiv \mathrm{lk}(\mu') \equiv \mathrm{lk}(\mu'_5) + \mathrm{lk}(\mu'_6) + \mathrm{lk}(\mu'_7) + \mathrm{lk}(\mu'_8) \pmod 2.$$

Thus we see that $\mathrm{lk}(\mu'_i)$ is odd for some $i = 5, 6, 7$ or 8. Moreover, by the symmetry of J, we may assume that $\mathrm{lk}(\mu'_5)$ is odd or $\mathrm{lk}(\mu'_6)$ is odd without loss of generality. Assume that $\mathrm{lk}(\mu'_5)$ is odd. We denote the subgraph $\mu \cup \mu'_5 \cup \overline{4\,7} \cup \overline{3\,9}$ of N_9 by J^5. Then J^5 contains a graph D^5 as a minor such that D^5 is isomorphic to D_4 and $\{\mu, \mu'_5\} = \Psi^{(2)}_{D^5, J^5}(\Gamma^{(2)}(D^5))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_5))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^5)$ such that $a_2(f(\gamma))$ is odd. Finally, assume that $\mathrm{lk}(\mu'_6)$ is odd. Let us consider the 3-component link $L = f([1\,2\,3] \cup [4\,5\,6] \cup [7\,8\,9])$. Since all 2-component sublinks of L are $f(\mu)$, $f(\mu'_2)$ and $f(\mu'_6)$, each of the 2-component sublinks of L has an odd linking number.

Now we show the case of $G = N'_{10}$. Let f be a spatial embedding of N'_{10}. Note that N'_{10} contains P_7 as the proper minor

$$(((N'_{10} - \overline{7\,8}) - \overline{8\,9}) - \overline{9\,7})/\overline{4\,7}/\overline{5\,8}/\overline{6\,9}.$$

Thus by Lemma 4.1, there is an element ν in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N'_{10},P_7}(f)(\nu))$ is odd. Hence by Proposition 2.1, there exists an element μ in $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu))$ is odd. $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ consists of nine elements, and by the symmetry of N'_{10}, we may assume that $\mu = [1\,7\,4\,5] \cup [2\,10\,3\,9\,6]$, $[2\,4\,5\,8] \cup [1\,10\,3\,9\,6]$, $[3\,10\,8\,5] \cup [1\,6\,2\,4\,7]$, $[3\,4\,5] \cup [1\,10\,2\,6]$ or $[2\,8\,10] \cup [1\,6\,9\,3\,4\,7]$ without loss of generality.

Case 1. Let $\mu = [1\,7\,4\,5] \cup [2\,10\,3\,9\,6]$. Note that N'_{10} contains P_9 as the proper minor

$$(((((N'_{10} - \overline{5\,1}) - \overline{5\,3}) - \overline{5\,4}) - \overline{5\,6}) - \overline{5\,8}) - \overline{7\,9}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of seven elements

$$\mu'_1 = [3\,10\,8\,9] \cup [1\,6\,2\,4\,7], \quad \mu'_2 = [1\,7\,8\,10] \cup [2\,4\,3\,9\,6],$$
$$\mu'_3 = [1\,10\,2\,6] \cup [3\,4\,7\,8\,9], \quad \mu'_4 = [2\,4\,3\,10] \cup [1\,7\,8\,9\,6],$$
$$\mu'_5 = [2\,4\,7\,8] \cup [1\,10\,3\,9\,6], \quad \mu'_6 = [2\,8\,9\,6] \cup [1\,10\,3\,4\,7],$$
$$\mu'_7 = [2\,8\,10] \cup [1\,6\,9\,3\,4\,7].$$

For $i = 1, 2, \ldots, 7$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_i \cup \overline{5\,8}$ if $i = 1, 6, 7$ and $\mu \cup \mu'_i$ if $i = 2, 3, 4, 5$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu'_i\} = \Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Because both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 2. Let $\mu = [2\,4\,5\,8] \cup [1\,10\,3\,9\,6]$. Note that N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10} - \overline{8\,2}) - \overline{8\,5}) - \overline{8\,7}) - \overline{8\,9}) - \overline{8\,10}) - \overline{3\,4}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\mu'_1 = [1\,6\,9\,7] \cup [2\,4\,5\,3\,10], \quad \mu'_2 = [1\,7\,4\,5] \cup [2\,10\,3\,9\,6],$$
$$\mu'_3 = [3\,5\,6\,9] \cup [1\,10\,2\,4\,7], \quad \mu'_4 = [1\,5\,3\,10] \cup [2\,4\,7\,9\,6],$$
$$\mu'_5 = [1\,10\,2\,6] \cup [3\,9\,7\,4\,5], \quad \mu'_6 = [1\,5\,6] \cup [2\,4\,7\,9\,3\,10],$$
$$\mu'_7 = [2\,4\,5\,6] \cup [1\,10\,3\,9\,7].$$

For $i = 1, 2, \ldots, 7$, let J^i be the subgraph of N'_{10} that is $\mu \cup \mu'_i \cup \overline{7\,8}$ if $i = 1, 7$ and $\mu \cup \mu'_i$ if $i \neq 1, 7$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and

$$\{\mu, \mu_i'\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i)).$$

Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu_i'))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 3. Let $\mu = [3\,10\,8\,5] \cup [1\,6\,2\,4\,7]$. Let P_9 be the proper minor of N_{10}' and μ_i' the element in

$$\Psi^{(2)}_{P_9, N_{10}'}(\Gamma^{(2)}(P_9)) \quad (i = 1, 2, \ldots, 7)$$

as in Case 2. For $i = 1, 2, \ldots, 7$, let J^i be the subgraph of N_{10}' that is $\mu \cup \mu_i' \cup \overline{8\,9}$ if $i = 1, 4$ and $\mu \cup \mu_i'$ if $i \neq 1, 4$. Assume that $\mathrm{lk}(f(\mu_i'))$ is odd for some i. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu_i'\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i))$. Because both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu_i'))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

Case 4. Let $\mu = [3\,4\,5] \cup [1\,10\,2\,6]$. Note that N_{10}' contains another P_7 as the proper minor

$$(((N_{10}' - \overline{3\,4}) - \overline{4\,5}) - \overline{5\,3}) / \overline{3\,9} / \overline{4\,7} / \overline{5\,8}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N_{10}', P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7, N_{10}'}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7, N_{10}'}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\begin{aligned}
&\mu_1' = [5\,6\,9\,8] \cup [1\,10\,2\,4\,7], && \mu_2' = [3\,10\,8\,9] \cup [1\,6\,2\,4\,7],\\
&\mu_3' = [1\,5\,8\,10] \cup [2\,4\,7\,9\,6], && \mu_4' = [7\,8\,9] \cup [1\,10\,2\,6],\\
&\mu_5' = [2\,8\,10] \cup [1\,6\,9\,7], && \mu_6' = [2\,8\,5\,6] \cup [1\,10\,3\,9\,7],\\
&\mu_7' = [1\,7\,8\,5] \cup [2\,10\,3\,9\,6], && \mu_8' = [1\,5\,6] \cup [2\,4\,7\,9\,3\,10],\\
&\mu_9' = [2\,4\,7\,8] \cup [1\,10\,3\,9\,6].
\end{aligned}$$

For $i = 1, 2, \ldots, 9$, let J^i be the subgraph of N_{10}' that is $\mu \cup \mu_5' \cup \overline{4\,7} \cup \overline{5\,8}$ if $i = 5$ and $\mu \cup \mu_i'$ if $i \neq 5$. Assume that $\mathrm{lk}(f(\mu_i'))$ is odd for some $i \neq 4, 8$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu, \mu_i'\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i))$. Since both $\mathrm{lk}(f(\mu))$ and $\mathrm{lk}(f(\mu_i'))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu_8'))$ is odd. We denote two elements $[1\,5\,6] \cup [2\,4\,3\,10]$ and $[1\,5\,6] \cup [3\,4\,7\,9]$ in $\Gamma^{(2)}(J^8)$ by $\mu_{8,1}'$ and $\mu_{8,2}'$, respectively. We denote the subgraph $\mu \cup \mu_{8,1}'$ of J^8 by $J^{8,1}$ and the subgraph $\mu \cup \mu_{8,2}' \cup \overline{8\,9} \cup \overline{8\,10}$ of N_{10}' by $J^{8,2}$. Then $J^{8,j}$ contains a graph $D^{8,j}$ as a minor such that $D^{8,j}$ is isomorphic to D_4 and $\{\mu, \mu_{8,j}'\} = \Psi^{(2)}_{D^{8,j}, J^{8,j}}(\Gamma^{(2)}(D^{8,j}))$ $(j = 1, 2)$. Since $[2\,4\,7\,9\,3\,10] = [2\,4\,3\,10] + [3\,4\,7\,9]$ in $H_1(J^8; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu_8')) \equiv \mathrm{lk}(f(\mu_{8,1}')) + \mathrm{lk}(f(\mu_{8,2}')) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu'_{8,1}))$ is odd or $\mathrm{lk}(f(\mu'_{8,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{8,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\mathrm{lk}(f(\mu'_4))$ is odd. Note that N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10} - \overline{2\,4}) - \overline{2\,6}) - \overline{2\,8}) - \overline{2\,10}) - \overline{5\,1}) - \overline{5\,3}.$$

Thus by Lemma 4.1, there is an element ν'' in $\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists an element μ'' in $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu''))$ is odd. $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\mu''_1 = [5\,6\,9\,8] \cup [1\,10\,3\,4\,7], \quad \mu''_2 = [4\,5\,8\,7] \cup [1\,10\,3\,9\,6],$$
$$\mu''_3 = [1\,7\,8\,10] \cup [3\,4\,5\,6\,9], \quad \mu''_4 = [3\,10\,8\,9] \cup [1\,7\,4\,5\,6],$$
$$\mu''_5 = [1\,6\,9\,7] \cup [3\,4\,5\,8\,10], \quad \mu''_6 = [3\,9\,7\,4] \cup [1\,10\,8\,5\,6],$$
$$\mu''_7 = [7\,8\,9] \cup [1\,10\,3\,4\,5\,6].$$

For $j = 1, 2, \ldots, 7$, let $J^{4,j}$ be the subgraph of N'_{10} which is $\mu'_4 \cup \mu''_j \cup \overline{2\,4}$ if $j = 2, 6$ and $\mu'_4 \cup \mu''_j$ if $j \neq 2, 6$. Assume that $\mathrm{lk}(f(\mu''_j))$ is odd for some $j \neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor such that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4, \mu''_i\} = \Psi^{(2)}_{D^{4,j},J^{4,j}}(\Gamma^{(2)}(D^{4,j}))$. Since both $\mathrm{lk}(f(\mu'_4))$ and $\mathrm{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu''_7))$ is odd. We denote three elements $[7\,8\,9] \cup [1\,5\,3\,10]$, $[7\,8\,9] \cup [1\,5\,6]$ and $[7\,8\,9] \cup [3\,4\,5]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$, $\mu''_{7,2}$ and $\mu''_{7,3}$. We denote the subgraph $\mu \cup \mu''_{7,k} \cup \overline{4\,7} \cup \overline{2\,8}$ of N'_{10} by $J^{4,7,k}$ $(k = 1, 2)$. Then $J^{4,7,k}$ contains a graph $D^{4,7,k}$ as a minor such that $D^{4,7,k}$ is isomorphic to D_4 and $\{\mu, \mu''_{7,k}\} = \Psi^{(2)}_{D^{4,7,k},J^{4,7,k}}(\Gamma^{(2)}(D^{4,7,k}))$ $(k = 1, 2)$. Since $[1\,10\,3\,4\,5\,6] = [1\,5\,3\,10] + [1\,5\,6] + [3\,4\,5]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu'_7)) \equiv \mathrm{lk}(f(\mu''_{7,1})) + \mathrm{lk}(f(\mu''_{7,2})) + \mathrm{lk}(f(\mu''_{7,3})) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu''_{7,k}))$ is odd for some k. If $\mathrm{lk}(f(\mu''_{7,1}))$ is odd or $\mathrm{lk}(f(\mu''_{7,2}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7,k})$ such that $a_2(f(\gamma))$ is odd. If $\mathrm{lk}(f(\mu''_{7,3}))$ is odd, let us consider the 3-component link

$$L = f([3\,4\,5] \cup [7\,8\,9] \cup [1\,10\,2\,6]).$$

Since all 2-component sublinks of L are $f(\mu)$, $f(\mu'_4)$ and $f(\mu''_{7,3})$, each of the 2-component sublinks of L has an odd linking number.

Case 5. Let $\mu = [2\,8\,10] \cup [1\,6\,9\,3\,4\,7]$. We denote two elements $[2\,8\,10] \cup [1\,6\,9\,7]$ and $[2\,8\,10] \cup [3\,9\,7\,4]$ in $\Gamma^{(2)}(N'_{10})$ by μ_1 and μ_2, respectively. Since $[1\,6\,9\,3\,4\,7] = [1\,6\,9\,7] + [3\,9\,7\,4]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu)) \equiv \mathrm{lk}(f(\mu_1)) + \mathrm{lk}(f(\mu_2)) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu_1))$ is odd or $\mathrm{lk}(f(\mu_2))$ is odd. By the symmetry of N'_{10}, we may assume that $\mathrm{lk}(f(\mu_1))$ is odd. Note that N'_{10} contains another P_7 as the proper minor

$$(((N'_{10}-\overline{2\,8})-\overline{8\,10})-\overline{10\,2})/\overline{2\,6}/\overline{3\,10}/\overline{5\,8}.$$

Thus by Lemma 4.1, there is an element ν' in $\Gamma^{(2)}(P_7)$ such that $\mathrm{lk}(\psi_{N'_{10},P_7}(f)(\nu'))$ is odd. Hence by Proposition 2.1, there exists an element μ' in $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ such that $\mathrm{lk}(f(\mu'))$ is odd. $\Psi^{(2)}_{P_7,N'_{10}}(\Gamma^{(2)}(P_7))$ consists of the nine elements

$$\begin{aligned}
&\mu'_1=[3\,5\,8\,9]\cup[1\,6\,2\,4\,7], && \mu'_2=[1\,7\,8\,5]\cup[2\,4\,3\,9\,6],\\
&\mu'_3=[1\,5\,6]\cup[3\,9\,7\,4], && \mu'_4=[3\,4\,5]\cup[1\,6\,9\,7],\\
&\mu'_5=[5\,6\,9\,8]\cup[1\,10\,3\,4\,7], && \mu'_6=[4\,5\,8\,7]\cup[1\,10\,3\,9\,6],\\
&\mu'_7=[1\,5\,3\,10]\cup[2\,4\,7\,9\,6], && \mu'_8=[2\,4\,5\,6]\cup[1\,10\,3\,9\,7],\\
&\mu'_9=[7\,8\,9]\cup[1\,10\,3\,4\,2\,6].
\end{aligned}$$

For $i=1,2,\ldots,9$, let J^i be the subgraph of N'_{10} that is $\mu_1\cup\mu'_3\cup\overline{3\,10}\cup\overline{5\,8}$ if $i=3$ and $\mu_1\cup\mu'_i$ if $i\neq 3$. Assume that $\mathrm{lk}(f(\mu'_i))$ is odd for some $i\neq 4,9$. Then J^i contains a graph D^i as a minor such that D^i is isomorphic to D_4 and $\{\mu_1,\mu'_i\}=\Psi^{(2)}_{D^i,J^i}(\Gamma^{(2)}(D^i))$. Since both $\mathrm{lk}(f(\mu_1))$ and $\mathrm{lk}(f(\mu'_i))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu'_9))$ is odd. We denote two elements $[7\,8\,9]\cup[1\,6\,2\,10]$ and $[7\,8\,9]\cup[2\,4\,3\,10]$ in $\Gamma^{(2)}(J^9)$ by $\mu'_{9,1}$ and $\mu'_{9,2}$, respectively. We denote the subgraph $\mu_1\cup\mu'_{8,1}$ of J^9 by $J^{9,1}$ and the subgraph $\mu_1\cup\mu'_{9,2}\cup\overline{5\,3}\cup\overline{5\,1}$ of N'_{10} by $J^{9,2}$. Then $J^{9,j}$ contains a graph $D^{9,j}$ as a minor such that $D^{9,j}$ is isomorphic to D_4 and

$$\{\mu_1,\mu'_{9,j}\}=\Psi^{(2)}_{D^{9,j},J^{9,j}}(\Gamma^{(2)}(D^{9,j}))\quad(j=1,2).$$

Since $[1\,10\,3\,4\,2\,6]=[1\,6\,2\,10]+[2\,4\,3\,10]$ in $H_1(J^9;\mathbb{Z}_2)$, it follows that

$$1\equiv\mathrm{lk}(f(\mu'_9))\equiv\mathrm{lk}(f(\mu'_{9,1}))+\mathrm{lk}(f(\mu'_{9,2}))\pmod 2.$$

This implies that $\mathrm{lk}(f(\mu'_{9,1}))$ is odd or $\mathrm{lk}(f(\mu'_{9,2}))$ is odd. In either case, by Lemma 4.3 there exists an element γ in $\Gamma(J^{9,j})$ such that $a_2(f(\gamma))$ is odd. Finally assume that $\mathrm{lk}(f(\mu'_4))$ is odd. N'_{10} contains another P_9 as the proper minor

$$(((((N'_{10}-\overline{6\,1})-\overline{6\,2})-\overline{6\,5})-\overline{6\,9})-\overline{8\,7})-\overline{8\,10}.$$

Thus, by Lemma 4.1, there is $\nu''\in\Gamma^{(2)}(P_9)$ such that $\mathrm{lk}(\psi_{N'_{10},P_9}(f)(\nu''))$ is odd. Hence by Proposition 2.1, there exists $\mu''\in\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ such that $\mathrm{lk}(f(\mu''))$

is odd. The set $\Psi^{(2)}_{P_9,N'_{10}}(\Gamma^{(2)}(P_9))$ consists of the seven elements

$$\mu''_1 = [3\,5\,8\,9] \cup [1\,10\,2\,4\,7], \quad \mu''_2 = [3\,9\,7\,4] \cup [1\,5\,8\,2\,10],$$
$$\mu''_3 = [1\,7\,4\,5] \cup [2\,8\,9\,3\,10], \quad \mu''_4 = [2\,4\,5\,8] \cup [1\,10\,3\,9\,7],$$
$$\mu''_5 = [2\,4\,3\,10] \cup [1\,5\,8\,9\,7], \quad \mu''_6 = [1\,5\,3\,10] \cup [2\,4\,7\,9\,8],$$
$$\mu''_7 = [3\,4\,5] \cup [1\,10\,2\,8\,9\,7].$$

For $j = 1, 2, \ldots, 7$, let $J^{4,j}$ be the subgraph of N'_{10} that is $\mu'_4 \cup \mu''_j \cup \overline{2\,6}$ if $j = 4, 5$ and $\mu'_4 \cup \mu''_j$ if $j \neq 4, 5$. Assume that $\mathrm{lk}(f(\mu''_j))$ is odd for some $j \neq 7$. Then $J^{4,j}$ contains a graph $D^{4,j}$ as a minor such that $D^{4,j}$ is isomorphic to D_4 and $\{\mu'_4, \mu''_i\} = \Psi^{(2)}_{D^{4,j},J^{4,j}}(\Gamma^{(2)}(D^{4,j}))$. Since both $\mathrm{lk}(f(\mu'_4))$ and $\mathrm{lk}(f(\mu''_j))$ are odd, by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,j})$ such that $a_2(f(\gamma))$ is odd. Next assume that $\mathrm{lk}(f(\mu''_7))$ is odd. We denote two elements $[3\,4\,5] \cup [1\,10\,8\,9\,7]$ and $[3\,4\,5] \cup [2\,8\,10]$ in $\Gamma^{(2)}(N'_{10})$ by $\mu''_{7,1}$ and $\mu''_{7,2}$, respectively. We denote the subgraph $\mu_1 \cup \mu''_{7,1} \cup \overline{2\,4} \cup \overline{5\,6}$ of N'_{10} by $J^{4,7}$. Then $J^{4,7}$ contains a graph $D^{4,7}$ as a minor such that $D^{4,7}$ is isomorphic to D_4 and

$$\{\mu_1, \mu''_{7,1}\} = \Psi^{(2)}_{D^{4,7},J^{4,7}}(\Gamma^{(2)}(D^{4,7})).$$

Since $[1\,10\,2\,8\,9\,7] = [1\,10\,8\,9\,7] + [2\,8\,10]$ in $H_1(N'_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \mathrm{lk}(f(\mu'_7)) \equiv \mathrm{lk}(f(\mu''_{7,1})) + \mathrm{lk}(f(\mu''_{7,2})) \pmod 2.$$

This implies that $\mathrm{lk}(f(\mu''_{7,1}))$ is odd or $\mathrm{lk}(f(\mu''_{7,2}))$ is odd. If $\mathrm{lk}(f(\mu''_{7,1}))$ is odd, then by Lemma 4.3 there exists an element γ in $\Gamma(J^{4,7})$ such that $a_2(f(\gamma))$ is odd. If $\mathrm{lk}(f(\mu''_{7,2}))$ is odd, let us consider the 3-component link

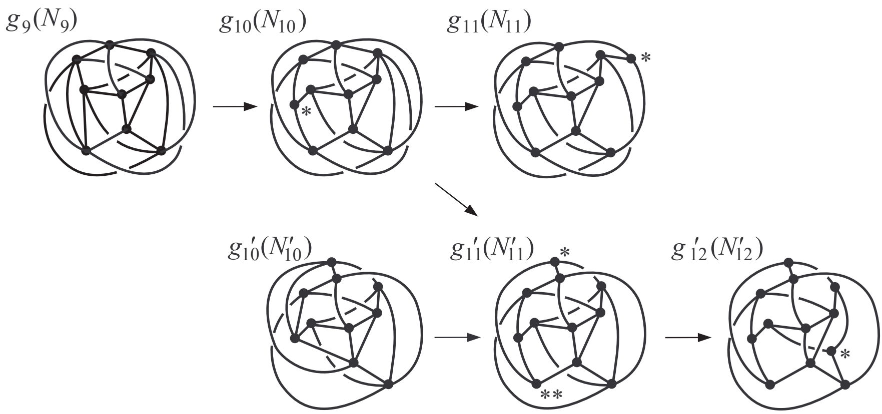
$$L = f([3\,4\,5] \cup [2\,8\,10] \cup [1\,6\,9\,7]).$$

Since all 2-component sublinks of L are $f(\mu_1)$, $f(\mu'_4)$ and $f(\mu''_{7,2})$, each of the 2-component sublinks of L has an odd linking number. This completes the proof. □

Proof of Theorem 1.2. A graph in the Heawood family is obtained from one of K_7, N_9 and N'_{10} by a finite sequence of $\triangle$Y-exchanges. Thus by Lemma 2.6, Theorem 4.4, and the fact that K_7 is IK — and thus I(K or C3L) — it follows that every graph in the Heawood family is I(K or C3L). On the other hand, a graph in the Heawood family is obtained from one of H_{12} and C_{14} by a finite sequence of Y$\triangle$-exchanges. Since each of H_{12} and C_{14} is a minor-minimal IK graph and $\Gamma^{(3)}(H_{12})$ and $\Gamma^{(3)}(C_{14})$ are the empty sets, it follows that H_{12} and C_{14} are minor-minimal I(K or C3L) graphs. By Lemma 2.7, we have the desired conclusion. □

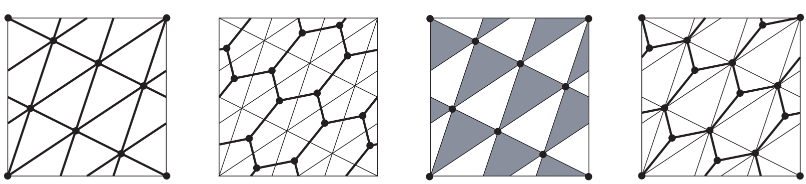
Remark 4.5. A graph is said to be 2-*apex* if it can be embedded in the 2-sphere after the deletion of at most two vertices and all of their incidental edges. It is not hard to see that any 2-apex graph may have a spatial embedding whose image contains neither a nontrivial knot nor a 3-component link all of whose 2-component sublinks are nonsplittable. Thus any 2-apex graph is not I(K or C3L). It is known that every graph of at most twenty edges is 2-apex [Mattman 2011] (see also [Johnson et al. 2010]). Since the number of all edges of every graph in the Heawood family is twenty-one, we see that any proper minor of a graph in the Heawood family is 2-apex, and thus not I(K or C3L). This also implies that any graph in the Heawood family is minor-minimal for I(K or C3L).

Example 4.6. Let g_9 be the spatial embedding of N_9 and g'_{10} the spatial embedding of N'_{10} illustrated here:



Then it can be checked directly that both $g_9(N_9)$ and $g'_{10}(N'_{10})$ do not contain a nonsplittable 3-component link. Thus neither N_9 nor N'_{10} is I3L. Also, we can see that N_{10}, N_{11}, N'_{11} and N'_{12} are not I3L in a similar way as the proof of Lemma 3.1 (see figure above).

Remark 4.7. The Heawood graph is IK. The Heawood graph H is the dual graph of K_7, which is embedded in a torus. It is known that there exists a unique graph C_{14} obtained from K_7 by seven applications of $\triangle$Y-exchanges [Kohara and Suzuki 1992]. The seven triangles correspond to the black triangles of a black-and-white coloring of the torus by K_7. Then C_{14} and H are mapped to each other by a translation of the torus:



Thus they are isomorphic. Since C_{14} is IK, we have the result.

Remark 4.8. It is known that all twenty-six graphs obtained from the complete four-partite graph $K_{3,3,1,1}$ by a finite sequence of △Y-exchanges are minor-minimal IK graphs [Kohara and Suzuki 1992; Foisy 2002]. There exist thirty-two graphs that are obtained from $K_{3,3,1,1}$ by a finite sequence of △Y-exchanges and Y△-exchanges but that cannot be obtained from $K_{3,3,1,1}$ by a finite sequence of △Y-exchanges. Recently, Goldberg, Mattman, and Naimi [2011] announced that these thirty-two graphs are also minor-minimal IK graphs.

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Received August 10, 2010. Revised January 20, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 252 No. 2 August 2011
