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## CLASSIFICATION OF EMBEDDED PROJECTIVE MANIFOLDS SWEPT OUT BY RATIONAL HOMOGENEOUS VARIETIES OF CODIMENSION ONE


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# CLASSIFICATION OF EMBEDDED PROJECTIVE MANIFOLDS SWEPT OUT BY RATIONAL HOMOGENEOUS VARIETIES OF CODIMENSION ONE

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**We give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one.**

## 1. Introduction

A central problem in the theory of polarized varieties is to classify smooth projective varieties admitting special varieties $A$ as ample divisors. In [Watanabe 2008], we investigated this problem in the case where $A$ is a homogeneous variety. On the other hand, related to the classification problem of polarized varieties, several authors have studied the structure of embedded projective varieties swept out by special varieties [Beltrametti and Ionescu 2008; Muñoz and Solá Conde 2009; Sato 1997; Watanabe 2010]. Inspired by these results, we give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one. Our main result is:

**Theorem 1.1.** *Let $X \subset \mathbb{P}^N$ be a complex smooth projective variety of dimension $n \geq 3$ and $A$ an $(n-1)$-dimensional rational homogeneous variety with* $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathbb{O}_A(1)]$. *Assume that $X$ satisfies either of the following properties.*

(a) *Through a general point $x \in X$, there is a subvariety $Z_x \subset X$ such that $(Z_x, \mathbb{O}_{Z_x}(1))$ is isomorphic to $(A, \mathbb{O}_A(1))$.*

(b) *There is a subvariety $Z \subset X$ such that $(Z, \mathbb{O}_Z(1))$ is isomorphic to $(A, \mathbb{O}_A(1))$ and the normal bundle $N_{Z/X}$ is nef.*

*Then we have one of the following*:

(i) *$X$ is a projective space $\mathbb{P}^n$*;

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(ii) *X is a quadric hypersurface* $Q^n$;

(iii) *X is the Grassmannian of lines* $G(1, \mathbb{P}^m)$;

(iv) *X is an* $E_6$ *variety* $E_6(\omega_1)$, *where* $E_6(\omega_1) \subset \mathbb{P}^{26}$ *is the projectivization of the highest weight vector orbit in the 27-dimensional irreducible representation of a simple algebraic group of Dynkin type* $E_6$; *or*

(v) *X admits an extremal contraction of a ray* $\varphi : X \to C$ *to a smooth curve whose general fibers are projectively equivalent to* $(A, \mathcal{O}_A(1))$.

*In cases* (i)–(iv), *the corresponding rational homogeneous variety A is one of those in Theorem 2.2.*

We outline the proof of Theorem 1.1. A significant step is to show the existence of a covering family $\mathcal{K}$ of lines on $X$ induced from lines on rational homogeneous varieties of codimension one (Claim 3.2). Then we see that the rationally connected fibration associated to $\mathcal{K}$ is an extremal contraction of the ray $\mathbb{R}_{\geq 0}[\mathcal{K}]$. By applying results from [Watanabe 2008], we obtain our theorem. In this paper, we work over the field of complex numbers.

## 2. Preliminaries

We denote a simple linear algebraic group of Dynkin type $G$ simply by $G$, and for a dominant integral weight $\omega$ of $G$, the minimal closed orbit of $G$ in $\mathbb{P}(V_\omega)$ by $G(\omega)$, where $V_\omega$ is the irreducible representation space of $G$ with highest weight $\omega$. For example, $E_6(\omega_1)$ is the minimal closed orbit of an algebraic group of type $E_6$ in $\mathbb{P}(V_{\omega_1})$, where $\omega_1$ is the first fundamental dominant weight in the standard notation of Bourbaki [1968]. For any rational homogeneous variety $X$ of Picard number one, there exists a simple linear algebraic group $G$ and a dominant integral weight $\omega$ of $G$ such that $X$ is isomorphic to $G(\omega)$. A rational homogeneous variety $A$ is a Fano variety, that is, the anticanonical divisor of $A$ is ample. If the Picard number of $A$ is one, we have $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathcal{O}_A(1)]$, where $\mathcal{O}_A(1)$ is a very ample line bundle on $A$. We recall two results on rational homogeneous varieties.

**Theorem 2.1** [Hwang and Mok 2005, Main Theorem; 1998, 5.2]. *Let A be a rational homogeneous variety of Picard number one. Let* $\rho : X \to Z$ *be a smooth proper morphism between two varieties. Suppose for some point y on Z, the fiber* $X_y$ *is isomorphic to A. Then, for any point z on Z, the fiber* $X_z$ *is isomorphic to A.*

**Theorem 2.2** [Watanabe 2008]. *Let X be a smooth projective variety and A a rational homogeneous variety of Picard number one. If A is an ample divisor on X,* $(X, A)$ *is isomorphic to* $(\mathbb{P}^n, \mathbb{P}^{n-1})$, $(\mathbb{P}^n, Q^{n-1})$, $(Q^n, Q^{n-1})$, $(G(2, \mathbb{C}^{2l}), C_l(\omega_2))$ *or* $(E_6(\omega_1), F_4(\omega_4))$.

For a numerical polynomial $P(t) \in \mathbb{Q}[t]$, write $\mathrm{Hilb}_{P(t)}(X)$ for the Hilbert scheme of $X$ relative to $P(t)$. More generally, for an $m$-tuple of numerical polynomials $\mathbb{P}(t) := (P_1(t), \dots, P_m(t))$, denote by $\mathrm{FH}_{\mathbb{P}(t)}(X)$ the flag Hilbert scheme of $X$ relative to $\mathbb{P}(t)$ [Sernesi 2006, Section 4.5]. For the Hilbert polynomial of a line $P_1(t)$, an irreducible component of $\mathrm{Hilb}_{P_1(t)}(X)$ is called a *family of lines* on $X$. Let $\mathrm{Univ}(X)$ be the universal family of $\mathrm{Hilb}(X)$ with the associated morphisms $\pi : \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ and $\iota : \mathrm{Univ}(X) \to X$. For a subset $V$ of $\mathrm{Hilb}(X)$, $\iota(\pi^{-1}(V))$ is denoted by $\mathrm{Locus}(V) \subset X$. A *covering family of lines* $\mathcal{K}$ means an irreducible component of $F_1(X)$ satisfying $\mathrm{Locus}(\mathcal{K}) = X$. For a covering family of lines, we have the following fibration.

**Theorem 2.3** [Campana 1992; Kollár et al. 1992]. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $\mathcal{K}$ a covering family of lines. Then there exists an open subset $X^0 \subset X$ and a proper morphism $\varphi : X^0 \to Y^0$ with connected fibers onto a normal variety, such that any two points on the fiber of $\varphi$ can be joined by a connected chain of finite $\mathcal{K}$-lines.*

We shall call the morphism $\varphi$ a *rationally connected fibration with respect to $\mathcal{K}$.*

**Theorem 2.4** [Bonavero et al. 2007, Theorem 2]. *Under the conditions and notation of Theorem 2.3, assume that $3 \geq \dim Y^0$. Then $\mathbb{R}_{\geq 0}[\mathcal{K}]$ is extremal in the sense of Mori theory and the associated contraction yields a rationally connected fibration with respect to $\mathcal{K}$.*

## 3. Proof of Theorem 1.1

For a subset $V \subset X$, denote the closure by $\overline{V}$. Let $P_1(t)$, $P_2(t)$ be the Hilbert polynomials of a line $(A, \mathcal{O}_A(1))$ and set $\mathbb{P}(t) := (P_1(t), P_2(t))$. We denote the natural projections by

$$p_i : \mathrm{FH}_{\mathbb{P}(t)}(X) \to \mathrm{Hilb}_{P_i(t)}(X), \text{ where } i = 1, 2. \tag{1}$$

Let $\mathcal{H}$ be the open subscheme of $\mathrm{Hilb}_{P_2(t)}(X)$ parametrizing smooth subvarieties of $X$ with Hilbert polynomial $P_2(t)$. Now we work under the assumption that $X$ satisfies (a) or (b) in Theorem 1.1.

**Claim 3.1.** *In both cases* (a) *and* (b), *there exists a curve $C \subset \mathcal{H}$ that contains a point $o$ corresponding to a subvariety isomorphic to $(A, \mathcal{O}_A(1))$.*

*Proof.* If the assumption (a) holds, there exists an irreducible component $\mathcal{H}_0$ of $\mathcal{H}$ that contains $o := [Z_x]$ for some $x \in X$ and satisfies $\overline{\mathrm{Locus}(\mathcal{H}_0)} = X$. Then we can take a curve $C \subset \mathcal{H}_0$ that contains $o$. If the assumption (b) holds, we see that $h^1(N_{Z/X}) = 0$ and $h^0(N_{Z/X}) \geq 1$. Since there is no obstruction in the deformation of $Z$ in $X$, it turns out that $\mathcal{H}$ is smooth at $o := [Z]$ and $\dim_{[Z]} \mathcal{H} \geq 1$. Then we can also take a curve $C \subset \mathcal{H}_0$ that contains $o$. □

From now on, we shall not use the assumptions (a) and (b) except through the property proved in Claim 3.1. $\overline{\text{Locus}(C)} = X$. Denote by $\mathcal{H}_0$ an irreducible component of $\mathcal{H}$ that contains $C$. For the universal family $\pi : \mathcal{U}_0 \to \mathcal{H}_0$ and the normalization $\nu : \tilde{C} \to C \subset \mathcal{H}_0$, we denote $\tilde{C} \times_{\mathcal{H}_0} \mathcal{U}_0$ by $\mathcal{U}_{\tilde{C}}$ and a natural projection by $\tilde{\pi} : \mathcal{U}_{\tilde{C}} \to \tilde{C}$. Let $(\mathcal{U}_{\tilde{C}})_{\text{red}}$ be the reduced scheme associated to $\mathcal{U}_{\tilde{C}}$ and $\Pi : (\mathcal{U}_{\tilde{C}})_{\text{red}} \to \tilde{C}$ the composition of $\tilde{\pi}$ and $(\mathcal{U}_{\tilde{C}})_{\text{red}} \to \mathcal{U}_{\tilde{C}}$. Then we have the following diagram:

$$\begin{array}{ccccc} (\mathcal{U}_{\tilde{C}})_{\text{red}} & \longrightarrow & \mathcal{U}_{\tilde{C}} & \longrightarrow & \mathcal{U}_0 \\ & \overset{\Pi}{\searrow} & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & \tilde{C} & \underset{\nu}{\longrightarrow} & \mathcal{H}_0 \end{array}$$

Now we have an isomorphism between scheme theoretic fibers

$$\tilde{\pi}^{-1}(p) \cong \pi^{-1}(\nu(p))$$

for any closed point $p \in \tilde{C}$. In particular, $\tilde{\pi}^{-1}(p)$ is a smooth projective variety and $\tilde{\pi}^{-1}(\tilde{o}) \cong A$ for a point $\tilde{o} \in \tilde{C}$ corresponding to $o \in C$. Moreover, a natural morphism $\Pi^{-1}(p) \to \tilde{\pi}^{-1}(p)$ is a homeomorphic closed immersion for any closed point $p \in \tilde{C}$. Since $\tilde{\pi}^{-1}(p)$ is reduced, we see that $\Pi^{-1}(p) \cong \tilde{\pi}^{-1}(p)$. Thus we conclude that $\Pi$ is a proper flat morphism whose fibers on closed points are smooth projective varieties, that is, a proper smooth morphism. Because $\Pi$ admits a central fiber $\tilde{\Pi}^{-1}(\tilde{o}) \cong A$, it follows that every fiber $\tilde{\Pi}^{-1}(\tilde{p})$ is isomorphic to $A$ from Theorem 2.1. Hence it turns out that every fiber of $\pi$ over a closed point in $C$ is isomorphic to $A$. Let consider a constructible subset $p_1(p_2^{-1}(C)) \subset \text{Hilb}_{P_1(t)}(X)$. Since $C$ parametrizes subvarieties isomorphic to $(A, \mathcal{O}_A(1))$ which is covered by lines, we see that

$$\overline{\text{Locus}(p_1(p_2^{-1}(C)))} = X.$$

**Claim 3.2.** *There exists a covering family of lines $\mathcal{K}$ on $X$ satisfying the following property: Through a general point $x \in X$, there is a subvariety $S_x \subset X$ such that $(S_x, \mathcal{O}_{S_x}(1)) \cong (A, \mathcal{O}_A(1))$ and any line lying in $S_x$ is a member of $\mathcal{K}$.*

*Proof.* Take an irreducible component $\mathcal{K}^0$ of $p_1(p_2^{-1}(C))$ such that $\overline{\text{Locus}(\mathcal{K}^0)} = X$. Through a general point $x$ on $X$, there is a line $[l_x]$ in $\mathcal{K}^0$ that is not contained in any irreducible component of $p_1(p_2^{-1}(C))$ except $\mathcal{K}^0$. There is also a subvariety $[S_x]$ in $C$ containing $l_x$. Because $p_1(p_2^{-1}([S_x]))$ is the Hilbert scheme of lines on $S_x$, it is irreducible [Landsberg and Manivel 2003, Theorem 4.3; Strickland 2002, Theorem 1]). Therefore $p_1(p_2^{-1}([S_x]))$ is contained in an irreducible component of $p_1(p_2^{-1}(C))$. Since $p_1(p_2^{-1}([S_x]))$ contains $[l_x]$, this implies that $p_1(p_2^{-1}([S_x]))$ is contained in $\mathcal{K}^0$. Thus we put $\mathcal{K}$ as an irreducible component of $\text{Hilb}_{P_1(t)}(X)$ containing $\mathcal{K}^0$. $\square$

Two points on $S_x \cong A$ can be joined by a connected chain of lines in $\mathscr{K}$. This implies that the relative dimension of the rationally connected fibration $\varphi : X \cdots \to Y$ with respect to $\mathscr{K}$ is at least $n-1$. According to Theorem 2.4, $\mathbb{R}_{\geq 0}[\mathscr{K}]$ spans an extremal ray of $NE(X)$ and $\varphi$ is its extremal contraction. In particular, $\varphi$ is a morphism that contracts $S_x$ to a point. If $\dim Y = 0$, we see that the Picard number of $X$ is one. This implies that $S_x$ is a very ample divisor on $X$. From Theorem 2.2, $X$ is $\mathbb{P}^n$, $Q^n$, $G(1, \mathbb{P}^m)$ or $E_6(\omega_1)$. If $\dim Y = 1$, then $Y$ is a smooth curve $C$ and a general fiber of $\varphi$ coincides with $S_x$. Therefore $\varphi$ is an $A$-fibration on a smooth curve $C$. Hence Theorem 1.1 holds.

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