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CLASSIFICATION OF EMBEDDED PROJECTIVE MANIFOLDS SWEPT OUT BY RATIONAL HOMOGENEOUS VARIETIES OF CODIMENSION ONE

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We give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one.

1. Introduction

A central problem in the theory of polarized varieties is to classify smooth projective varieties admitting special varieties A as ample divisors. In [Watanabe 2008], we investigated this problem in the case where A is a homogeneous variety. On the other hand, related to the classification problem of polarized varieties, several authors have studied the structure of embedded projective varieties swept out by special varieties [Beltrametti and Ionescu 2008; Muñoz and Solá Conde 2009; Sato 1997; Watanabe 2010]. Inspired by these results, we give a classification of embedded smooth projective varieties swept out by rational homogeneous varieties whose Picard number and codimension are one. Our main result is:

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be a complex smooth projective variety of dimension $n \geq 3$ and A an $(n-1)$-dimensional rational homogeneous variety with* $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathbb{O}_A(1)]$. *Assume that X satisfies either of the following properties.*

(a) *Through a general point $x \in X$, there is a subvariety $Z_x \subset X$ such that $(Z_x, \mathbb{O}_{Z_x}(1))$ is isomorphic to $(A, \mathbb{O}_A(1))$.*

(b) *There is a subvariety $Z \subset X$ such that $(Z, \mathbb{O}_Z(1))$ is isomorphic to $(A, \mathbb{O}_A(1))$ and the normal bundle $N_{Z/X}$ is nef.*

Then we have one of the following:

(i) *X is a projective space $\mathbb{P}^n$*;

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(ii) *X is a quadric hypersurface Q^n;*

(iii) *X is the Grassmannian of lines $G(1, \mathbb{P}^m)$;*

(iv) *X is an E_6 variety $E_6(\omega_1)$, where $E_6(\omega_1) \subset \mathbb{P}^{26}$ is the projectivization of the highest weight vector orbit in the 27-dimensional irreducible representation of a simple algebraic group of Dynkin type E_6; or*

(v) *X admits an extremal contraction of a ray $\varphi : X \to C$ to a smooth curve whose general fibers are projectively equivalent to $(A, \mathbb{O}_A(1))$.*

In cases (i)–(iv), *the corresponding rational homogeneous variety A is one of those in Theorem 2.2.*

We outline the proof of Theorem 1.1. A significant step is to show the existence of a covering family $\mathcal{H}$ of lines on X induced from lines on rational homogeneous varieties of codimension one (Claim 3.2). Then we see that the rationally connected fibration associated to $\mathcal{H}$ is an extremal contraction of the ray $\mathbb{R}_{\geq 0}[\mathcal{H}]$. By applying results from [Watanabe 2008], we obtain our theorem. In this paper, we work over the field of complex numbers.

2. Preliminaries

We denote a simple linear algebraic group of Dynkin type G simply by G, and for a dominant integral weight ω of G, the minimal closed orbit of G in $\mathbb{P}(V_\omega)$ by $G(\omega)$, where V_ω is the irreducible representation space of G with highest weight ω. For example, $E_6(\omega_1)$ is the minimal closed orbit of an algebraic group of type E_6 in $\mathbb{P}(V_{\omega_1})$, where ω_1 is the first fundamental dominant weight in the standard notation of Bourbaki [1968]. For any rational homogeneous variety X of Picard number one, there exists a simple linear algebraic group G and a dominant integral weight ω of G such that X is isomorphic to $G(\omega)$. A rational homogeneous variety A is a Fano variety, that is, the anticanonical divisor of A is ample. If the Picard number of A is one, we have $\mathrm{Pic}(A) \cong \mathbb{Z}[\mathbb{O}_A(1)]$, where $\mathbb{O}_A(1)$ is a very ample line bundle on A. We recall two results on rational homogeneous varieties.

Theorem 2.1 [Hwang and Mok 2005, Main Theorem; 1998, 5.2]. *Let A be a rational homogeneous variety of Picard number one. Let $\rho : X \to Z$ be a smooth proper morphism between two varieties. Suppose for some point y on Z, the fiber X_y is isomorphic to A. Then, for any point z on Z, the fiber X_z is isomorphic to A.*

Theorem 2.2 [Watanabe 2008]. *Let X be a smooth projective variety and A a rational homogeneous variety of Picard number one. If A is an ample divisor on X, (X, A) is isomorphic to $(\mathbb{P}^n, \mathbb{P}^{n-1})$, $(\mathbb{P}^n, Q^{n-1})$, (Q^n, Q^{n-1}), $(G(2, \mathbb{C}^{2l}), C_l(\omega_2))$ or $(E_6(\omega_1), F_4(\omega_4))$.*

For a numerical polynomial $P(t) \in \mathbb{Q}[t]$, write $\mathrm{Hilb}_{P(t)}(X)$ for the Hilbert scheme of X relative to $P(t)$. More generally, for an m-tuple of numerical polynomials $\mathbb{P}(t) := (P_1(t), \dots, P_m(t))$, denote by $\mathrm{FH}_{\mathbb{P}(t)}(X)$ the flag Hilbert scheme of X relative to $\mathbb{P}(t)$ [Sernesi 2006, Section 4.5]. For the Hilbert polynomial of a line $P_1(t)$, an irreducible component of $\mathrm{Hilb}_{P_1(t)}(X)$ is called a *family of lines* on X. Let $\mathrm{Univ}(X)$ be the universal family of $\mathrm{Hilb}(X)$ with the associated morphisms $\pi : \mathrm{Univ}(X) \to \mathrm{Hilb}(X)$ and $\iota : \mathrm{Univ}(X) \to X$. For a subset V of $\mathrm{Hilb}(X)$, $\iota(\pi^{-1}(V))$ is denoted by $\mathrm{Locus}(V) \subset X$. A *covering family of lines* $\mathcal{K}$ means an irreducible component of $F_1(X)$ satisfying $\mathrm{Locus}(\mathcal{K}) = X$. For a covering family of lines, we have the following fibration.

Theorem 2.3 [Campana 1992; Kollár et al. 1992]. *Let $X \subset \mathbb{P}^N$ be a smooth projective variety and $\mathcal{K}$ a covering family of lines. Then there exists an open subset $X^0 \subset X$ and a proper morphism $\varphi : X^0 \to Y^0$ with connected fibers onto a normal variety, such that any two points on the fiber of φ can be joined by a connected chain of finite $\mathcal{K}$-lines.*

We shall call the morphism φ a *rationally connected fibration with respect to* $\mathcal{K}$.

Theorem 2.4 [Bonavero et al. 2007, Theorem 2]. *Under the conditions and notation of Theorem 2.3, assume that $3 \geq \dim Y^0$. Then $\mathbb{R}_{\geq 0}[\mathcal{K}]$ is extremal in the sense of Mori theory and the associated contraction yields a rationally connected fibration with respect to $\mathcal{K}$.*

3. Proof of Theorem 1.1

For a subset $V \subset X$, denote the closure by $\overline{V}$. Let $P_1(t)$, $P_2(t)$ be the Hilbert polynomials of a line $(A, \mathcal{O}_A(1))$ and set $\mathbb{P}(t) := (P_1(t), P_2(t))$. We denote the natural projections by

$$p_i : \mathrm{FH}_{\mathbb{P}(t)}(X) \to \mathrm{Hilb}_{P_i(t)}(X), \text{ where } i = 1, 2. \tag{1}$$

Let $\mathcal{H}$ be the open subscheme of $\mathrm{Hilb}_{P_2(t)}(X)$ parametrizing smooth subvarieties of X with Hilbert polynomial $P_2(t)$. Now we work under the assumption that X satisfies (a) or (b) in Theorem 1.1.

Claim 3.1. *In both cases (a) and (b), there exists a curve $C \subset \mathcal{H}$ that contains a point o corresponding to a subvariety isomorphic to $(A, \mathcal{O}_A(1))$.*

Proof. If the assumption (a) holds, there exists an irreducible component $\mathcal{H}_0$ of $\mathcal{H}$ that contains $o := [Z_x]$ for some $x \in X$ and satisfies $\overline{\mathrm{Locus}(\mathcal{H}_0)} = X$. Then we can take a curve $C \subset \mathcal{H}_0$ that contains o. If the assumption (b) holds, we see that $h^1(N_{Z/X}) = 0$ and $h^0(N_{Z/X}) \geq 1$. Since there is no obstruction in the deformation of Z in X, it turns out that $\mathcal{H}$ is smooth at $o := [Z]$ and $\dim_{[Z]} \mathcal{H} \geq 1$. Then we can also take a curve $C \subset \mathcal{H}_0$ that contains o. □

From now on, we shall not use the assumptions (a) and (b) except through the property proved in Claim 3.1. $\overline{\mathrm{Locus}(C)} = X$. Denote by $\mathcal{H}_0$ an irreducible component of $\mathcal{H}$ that contains C. For the universal family $\pi : \mathcal{U}_0 \to \mathcal{H}_0$ and the normalization $\nu : \tilde{C} \to C \subset \mathcal{H}_0$, we denote $\tilde{C} \times_{\mathcal{H}_0} \mathcal{U}_0$ by $\mathcal{U}_{\tilde{C}}$ and a natural projection by $\tilde{\pi} : \mathcal{U}_{\tilde{C}} \to \tilde{C}$. Let $(\mathcal{U}_{\tilde{C}})_{\mathrm{red}}$ be the reduced scheme associated to $\mathcal{U}_{\tilde{C}}$ and $\Pi : (\mathcal{U}_{\tilde{C}})_{\mathrm{red}} \to \tilde{C}$ the composition of $\tilde{\pi}$ and $(\mathcal{U}_{\tilde{C}})_{\mathrm{red}} \to \mathcal{U}_{\tilde{C}}$. Then we have the following diagram:

$$\begin{array}{ccccc} (\mathcal{U}_{\tilde{C}})_{\mathrm{red}} & \longrightarrow & \mathcal{U}_{\tilde{C}} & \longrightarrow & \mathcal{U}_0 \\ & \overset{\Pi}{\searrow} & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & \tilde{C} & \xrightarrow[\nu]{} & \mathcal{H}_0 \end{array}$$

Now we have an isomorphism between scheme theoretic fibers

$$\tilde{\pi}^{-1}(p) \cong \pi^{-1}(\nu(p))$$

for any closed point $p \in \tilde{C}$. In particular, $\tilde{\pi}^{-1}(p)$ is a smooth projective variety and $\tilde{\pi}^{-1}(\tilde{o}) \cong A$ for a point $\tilde{o} \in \tilde{C}$ corresponding to $o \in C$. Moreover, a natural morphism $\Pi^{-1}(p) \to \tilde{\pi}^{-1}(p)$ is a homeomorphic closed immersion for any closed point $p \in \tilde{C}$. Since $\tilde{\pi}^{-1}(p)$ is reduced, we see that $\Pi^{-1}(p) \cong \tilde{\pi}^{-1}(p)$. Thus we conclude that Π is a proper flat morphism whose fibers on closed points are smooth projective varieties, that is, a proper smooth morphism. Because Π admits a central fiber $\tilde{\Pi}^{-1}(\tilde{o}) \cong A$, it follows that every fiber $\tilde{\Pi}^{-1}(\tilde{p})$ is isomorphic to A from Theorem 2.1. Hence it turns out that every fiber of π over a closed point in C is isomorphic to A. Let consider a constructible subset $p_1(p_2^{-1}(C)) \subset \mathrm{Hilb}_{P_1(t)}(X)$. Since C parametrizes subvarieties isomorphic to $(A, \mathbb{O}_A(1))$ which is covered by lines, we see that

$$\overline{\mathrm{Locus}(p_1(p_2^{-1}(C)))} = X.$$

Claim 3.2. *There exists a covering family of lines $\mathcal{K}$ on X satisfying the following property: Through a general point $x \in X$, there is a subvariety $S_x \subset X$ such that $(S_x, \mathbb{O}_{S_x}(1)) \cong (A, \mathbb{O}_A(1))$ and any line lying in S_x is a member of $\mathcal{K}$.*

Proof. Take an irreducible component $\mathcal{K}^0$ of $p_1(p_2^{-1}(C))$ such that $\overline{\mathrm{Locus}(\mathcal{K}^0)} = X$. Through a general point x on X, there is a line $[l_x]$ in $\mathcal{K}^0$ that is not contained in any irreducible component of $p_1(p_2^{-1}(C))$ except $\mathcal{K}^0$. There is also a subvariety $[S_x]$ in C containing l_x. Because $p_1(p_2^{-1}([S_x]))$ is the Hilbert scheme of lines on S_x, it is irreducible [Landsberg and Manivel 2003, Theorem 4.3; Strickland 2002, Theorem 1]). Therefore $p_1(p_2^{-1}([S_x]))$ is contained in an irreducible component of $p_1(p_2^{-1}(C))$. Since $p_1(p_2^{-1}([S_x]))$ contains $[l_x]$, this implies that $p_1(p_2^{-1}([S_x]))$ is contained in $\mathcal{K}^0$. Thus we put $\mathcal{K}$ as an irreducible component of $\mathrm{Hilb}_{P_1(t)}(X)$ containing $\mathcal{K}^0$. □

Two points on $S_x \cong A$ can be joined by a connected chain of lines in $\mathcal{K}$. This implies that the relative dimension of the rationally connected fibration $\varphi : X \cdots \to Y$ with respect to $\mathcal{K}$ is at least $n-1$. According to Theorem 2.4, $\mathbb{R}_{\geq 0}[\mathcal{K}]$ spans an extremal ray of $NE(X)$ and φ is its extremal contraction. In particular, φ is a morphism that contracts S_x to a point. If $\dim Y = 0$, we see that the Picard number of X is one. This implies that S_x is a very ample divisor on X. From Theorem 2.2, X is $\mathbb{P}^n$, Q^n, $G(1, \mathbb{P}^m)$ or $E_6(\omega_1)$. If $\dim Y = 1$, then Y is a smooth curve C and a general fiber of φ coincides with S_x. Therefore φ is an A-fibration on a smooth curve C. Hence Theorem 1.1 holds.

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