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SINGULARITIES OF THE PROJECTIVE DUAL VARIETY

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Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety and let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual. Let $L \subset \mathbb{P}^N$ be a linear space such that $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$ for all $x \in X_{\text{smooth}}$ and such that the lines in X meeting L do not cover X. If $x \in X$ is general, we prove that the multiplicity of X^* at a general point of $\langle L, T_{X,x} \rangle^{\perp}$ is strictly greater than the multiplicity of X^* at a general point of L^{\perp} . This is a strong refinement of Bertini's theorem.

1. Introduction

1.1. *Multiplicities of the projective dual.* Let $X \subset \mathbb{P}^N$ be an irreducible projective variety over the field of complex numbers. Let $X^* \subset \mathbb{P}^{N^*}$ be its projective dual, let $L \subset \mathbb{P}^N$ be a linear space and H be a general hyperplane containing L. Bertini's classical theorem asserts that the tangency locus of H with X is included in $X \cap L$. Very little is known about the hyperplanes whose tangency locus with X lies outside $L \cap X$. It is tempting to think that the multiplicity in X^* of such a hyperplane is strictly larger than the multiplicity of a general hyperplane containing L. The following example shows that this is not true for every L.

Example 1.1.1. Let $X \subset \mathbb{P}^4$ be a smooth hyperplane section of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The variety *X* is a ruled surface of degree 3. Its dual is a hypersurface of degree 3 in \mathbb{P}^{4^*} which does not contain any points of multiplicity higher than 2. Let *L* be the exceptional section of *X*. If $H \subset \mathbb{P}^4$ is a general hyperplane which contains *L*, then $H \cap X = L \cup D_1 \cup D_2$, where D_1 and D_2 are two distinct lines on *X* such that $D_1.D_2 = 0$ and $L.D_i = 1$ for i = 1, 2. As a consequence, a general point of L^{\perp} is of multiplicity 2 in X^* . Now, let $D \subset X$ be a line such that D.L = 1 and let $x \in D$ such that $x \notin L$. The hyperplane containing *L* and $T_{X,x}$ is a point of multiplicity exactly 2 in X^* , that is, the multiplicity of a general point of L^{\perp} .

This example shows that, even for general $x \in X$, the multiplicity in X^* of a hyperplane containing *L* and tangent to *X* at *x* may well be equal to the multiplicity of a general hyperplane containing *L*. Thus, without extra hypotheses on *L*, it

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seems hopeless to say something about the multiplicity in X^* of special points of L^{\perp} . For this purpose, we introduce a definition:

Definition 1.1.2. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety and let $L \subset \mathbb{P}^N$ be a linear space. Consider the conormal diagram

$$I(X/\mathbb{P}^{N}) := \overline{\{(H, x) \in \mathbb{P}^{N^{*}} \times X_{\text{smooth}} : T_{X, x} \subset H\}}$$

$$\downarrow^{q}$$

$$X^{*} \subset \mathbb{P}^{N^{*}}$$

$$X \subset \mathbb{P}^{N}$$

Let F_1, \ldots, F_m be all the irreducible components of $q^{-1}(L^{\perp})$ such that the restrictions

$$q|_{F_i}: F_i \to L^{\perp}$$

are surjective. The *contact locus* of *L* with *X*, which we denote by Tan(L, X), is the union of the $p(F_i)$, for $1 \le i \le m$.

In the case where *L* is a hyperplane, the contact locus Tan(L, X) is called the *tangency locus* of *L* with *X*. A *tangent hyperplane* to *X* is a hyperplane $H \subset \mathbb{P}^N$ such that $Tan(H, X) \neq \emptyset$.

The contact locus $\operatorname{Tan}(L, X)$ can be thought as the variety covered by the tangency loci of general hyperplanes containing L. In case $L^{\perp} \not\subset X^*$, this locus is empty. We always have the inclusion

$$\{x \in X_{\text{smooth}} : T_{X,x} \subset L\} \subset \text{Tan}(L, X),$$

but if dim(L) < N - 1 or if X is not smooth, the former locus can be strictly smaller than the latter. Note also that Bertini's theorem says that Tan $(L, X) \subset L \cap X$. Finally, the contact locus is well behaved. If for a general hyperplane H' containing L, we have dim Tan(H', X) > 0, then

$$\operatorname{Tan}(H \cap L, H \cap X) = H \cap \operatorname{Tan}(L, X),$$

for any general hyperplane $H \subset \mathbb{P}^N$.

Example 1.1.3. If $X \subset \mathbb{P}^N$ is such that X^* is a hypersurface and $L = T_{X,x}$, where $x \in X$ is a general point, then Tan(L, X) = x.

- If $X = G(1,7) \subset \mathbb{P}^{27}$ and $L = \langle T_{X,y_1}, T_{X,y_2} \rangle$, where $y_1, y_2 \in \mathbb{G}(1,7)$ are two general points, then $\operatorname{Tan}(L, X) = \{x \in X : T_{X,x} \subset L\}$ is a 4-dimensional quadric, the entry locus of a general point $z \in \langle y_1, y_2 \rangle$.
- If $X = G(1, 4) \subset \mathbb{P}^9$ and $L = T_{X,y}$, for any $y \in X$, then dim Tan(L, X) > 0, whereas $\{x \in X : T_{X,x} \subset L\} = \{y\}$.

Definition 1.1.4. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety, and let $L \subset \mathbb{P}^N$ be a linear subspace. The *shadow* of *L* on *X*, which we denote by $\text{Sh}_X(L)$, is the closed variety covered by the linear spaces $M \subset X$ such that $\dim(M) = \operatorname{def}(X) + 1$ and $\dim(M \cap \operatorname{Tan}(L, X)) = \operatorname{def}(X)$.

Here $def(X) = codim(X^*) - 1$. The shadow is also well behaved. Namely, assume that def(X) > 0; then

$$\operatorname{Sh}_L(X) = X \iff \operatorname{Sh}_{H \cap L}(H \cap X) = H \cap X,$$

for any general hyperplane $H \subset \mathbb{P}^N$. Note also that if $x \in X$ is a general point and $L = T_{X,x}$, then $\operatorname{Sh}_L(X) \neq X$, unless X is a linear space. Indeed, if X^* is a hypersurface, this is obvious since $\operatorname{Tan}(T_{X,x}, X) = x$ for general $x \in X$. If X^* is not a hypersurface, take enough general hyperplane sections of X passing through x, so that the corresponding dual is a hypersurface.

Main Theorem 1.1.5. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^N$ be a linear space such that $\operatorname{Sh}_X(L) \neq X$. Then, for all $x \in X_{\text{smooth}}$ such that $x \notin \operatorname{Sh}_X(L)$ and such that $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x} \rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing L.

If X is the ruled cubic surface considered in Example 1.1.1 and L is the directrix of X, one notices easily that $\operatorname{Sh}_X(L) = X$. This shows that the hypothesis $\operatorname{Sh}_X(L) \neq X$ can not be withdrawn. Here is an obvious consequence of Main Theorem 1.1.5:

Corollary 1.1.6. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^N$ be a linear space such that $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$ for general $x \in X$, and such that the lines in X meeting L do not cover X. Then, for general $x \in X$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x} \rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing L.

1.2. Variety of multisecant spaces and duals.

Definition 1.2.1. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. Let

 $S_X^{k^0} = \{(x_0, \ldots, x_k, u) \in X \times \cdots \times X \times \mathbb{P}^N : \dim\langle x_0, \ldots, x_k \rangle = k, u \in \langle x_0, \ldots, x_k \rangle \},\$

and let S_X^k be its Zariski closure in $X \times \cdots \times X \times \mathbb{P}^N$. Denote by ϕ the projection onto \mathbb{P}^N . The variety $S^k(X) = \phi(S_X^k)$ is the *k*-th secant variety to X.

Theorem 1.2.2 (Terracini's lemma [Zak 1993]). Let $X \subset \mathbb{P}^N$ be an irreducible projective variety, and let $(x_0, \ldots, x_k) \in X \times \cdots \times X$, be general points. If u is general in $\langle x_0, \ldots, x_k \rangle$, we have the equality

$$\langle T_{X,x_0},\ldots,T_{X,x_k}\rangle = T_{S^k(X),u}.$$

Definition 1.2.3. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety, and let *k* be an integer such that $S^k(X) \neq \mathbb{P}^N$. We say that *X* is *dual k-defective* if def $(S^k(X)) > t(S^k(X))$, where $t(S^k(X))$ is the dimension of the general fiber of the Gauss map of $S^k(X)$.

Note that when X is smooth, then dual 0-defectiveness is the classical dual defectiveness. I don't know if there exist smooth varieties which are dual k-defective for some $k \ge 1$, but which are not dual 0-defective. I believe it would be interesting to find some examples of such varieties.

Note also that the notion of dual k-defectiveness seems to be related to that of R_k regularity explored in [Chiantini and Ciliberto 2010].

Here is a consequence of the Main Theorem 1.1.5 and Terracini's lemma:

Proposition 1.2.4. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume moreover that for all k such that $S^k(X) \neq \mathbb{P}^N$ the variety X is not dual k - 1-defective. Then, for any such k, we have

$$S^k(X)^* \subset X^*_{k+1},$$

where X_{k+1}^* is the set of points which have multiplicity at least k + 1 in X^* .

Proof. The case k = 0 is the definition of $S^0(X)^* = X^*$. Let $k \ge 1$ be an integer such that $S^k(X) \ne \mathbb{P}^N$, let $z \in S^{k-1}(X)$ be a general point and H be a general hyperplane containing $T_{S^{k-1}(X),z}$. Let's prove that

$$Tan(H, X) = \{ x \in X : T_{X,x} \subset T_{S^{k-1}(X),z} \}.$$

Let x_0, \ldots, x_{k-1} be k general points in Tan(H, X). Let z' be a general point in $\langle x_0, \ldots, x_{k-1} \rangle$, by Terracini's lemma we have

$$T_{S^{k-1}(X),z'} = \langle T_{X,x_0}, \ldots, T_{X,x_{k-1}} \rangle$$

So $z' \in \text{Tan}(H, S^{k-1}(X))$. But $\text{def}(S^{k-1}(X)) = t(S^{k-1}(X))$ by hypothesis, and this implies

$$z' \in \overline{\{y \in S^{k-1}(X)_{\text{smooth}} : T_{S^{k-1}(X), y} = T_{S^{k-1}(X), z}\}},$$

so that $x_0, \ldots, x_{k-1} \in \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$

We now prove that $\text{Sh}_X(T_{S^{k-1}(X),z}) \neq X$. The argument above shows that

$$\operatorname{Tan}(T_{S^{k-1}(X),z}, X) = \{x \in X : T_{X,x} \subset T_{S^{k-1}(X),z}\}.$$

Assume $\text{Sh}_X(T_{S^{k-1}(X),z}) = X$. For all $x'' \in X$, there is $x' \in \{x \in X : T_{X,x} \subset T_{S^k(X),z}\}$ such that the line $\langle x'', x' \rangle$ lies in *X*. But since *X* is smooth, this line $\langle x'', x' \rangle$ lies in $T_{X,x'}$. So we have $X \subset T_{S^{k-1}(X),z}$, which contradicts the nondegeneracy. As a consequence of Main Theorem 1.1.5, we get that for a general $x \in X$, the multiplicity in X^* of a general hyperplane containing $\langle T_{S^{k-1}(X),z}, T_{X,x} \rangle$ is strictly larger than the multiplicity in X^* of a general hyperplane containing $T_{S^{k-1}(X),z}$. We apply Terracini's lemma to find that $S^k(X)^* \subset X^*_{k+1}$. This concludes the proof. \Box

A stronger result than Proposition 1.2.4 has been stated for the first time in [Zak 2004], but no proof was given there.

In the second part of this paper we present a proof of Main Theorem 1.1.5, while in the third part we discuss some consequences and open questions.

2. Proof of the Main Theorem

When $Z \subset \mathbb{P}^N$, we denote by $\mathscr{C}_z(Z) \subset \mathbb{P}^N$ the embedded tangent cone to Z at z and if $H \subset \mathbb{P}^N$ is a hyperplane, then [*h*] is the corresponding point in $(\mathbb{P}^N)^*$.

The proof of Main Theorem 1.1.5 is obvious if $L^{\perp} \not\subset X^*$. Thus, we only deal with the case where $L^{\perp} \subset X^*$. Moreover, we can restrict to the case where X^* is a hypersurface. Indeed, assume that X^* has codimension $p \ge 2$. Let $z \in L^{\perp}$ and $z_x \in$ $\langle L, T_{X,x} \rangle^{\perp}$ be general points, let $M \subset \mathbb{P}^N$ be a general \mathbb{P}^{N+1-p} passing through x, let $X' = M \cap X$ and $L' = M \cap L$. We have $\operatorname{Sh}_{X'}(L') \neq X'$ and $\langle T_{X',x}, L' \rangle \neq \mathbb{P}^{N+1-p}$. Moreover, we have

$$(X')^* = \pi_{M^\perp}(X^*),$$

where $\pi_{M^{\perp}}$ is the projection from M^{\perp} in \mathbb{P}^{N^*} . Since *M* is general, the map $\pi_{M^{\perp}}$ is locally an isomorphism around z_x . Hence

 $\operatorname{mult}_{z} X^* = \operatorname{mult}_{z_X} X^* \iff \operatorname{mult}_{\pi_{M^{\perp}}(z)}(X')^* = \operatorname{mult}_{\pi_{M^{\perp}}(z_X)}(X')^*.$

Finally, note that $\pi_{M^{\perp}}(z)$ is a general point of $(L')^{\perp}$ and that $\pi_{M^{\perp}}(z_x)$ is a general point of $\langle L', T_{X',x} \rangle^{\perp}$. As a consequence, it is sufficient to prove the theorem for X', whose dual is a hypersurface.

Let's start with a plan of the proof. We assume that X^* has constant multiplicity along a smooth curve $S \subset L^{\perp}$ passing through $\langle L, T_{X,x} \rangle^{\perp}$ and through a general point of L^{\perp} and we find a contradiction. More precisely:

- We prove that the equimultiplicity of X^* along S implies that the family of the tangent cones to X^* at the points of S is flat.
- Then, we show that the flatness of the family of the tangent cones to X^* at the points of *S* leads to the flatness of the family of the conormal spaces of these tangent cones. As a consequence, we have $|\mathscr{C}_s(X^*)|^* \subset L$ for all $s \in S$.
- Finally, we relate the tangent cone to X^* at z to the set of tangent hyperplanes to X^* at z (when z is a smooth point of X^* ; this is the reflexivity theorem [Kleiman 1986]). Using the fact that $\operatorname{Sh}_L(X) \neq X$, we deduce that $|\mathscr{C}_s(X^*)|^* \not\subset$ L for $s \in \langle L, T_{X,x} \rangle^{\perp}$ and thus a contradiction.

2.1. *Normal flatness and Lagrangian specialization principle.* Let $S \subset Z \subset \mathbb{P}^N$ be two varieties. We recall some properties of the tangent cones $\mathscr{C}_s(Z)$, $s \in S$ when *Z* is equimultiple along *S*.

Definition 2.1.1. Let $S \subset Z$ be two varieties. We say that Z is *equimultiple* along S if the multiplicity of the local ring $\mathbb{O}_{Z,s}$ is constant for $s \in S$.

Proposition 2.1.2 [Hironaka 1964, Corollary 2, p. 197]. Let $Z \subset \mathbb{P}^N$ be a hypersurface and S a connected smooth subvariety (not necessarily closed) of Z such that Z is equimultiple along S.

Then, for all $s \in S$, there exists an open neighborhood U of s in S containing s and a closed subscheme $\mathfrak{G}(Z) \subset \mathbb{P}^N \times U$ such that the natural projection p: $\mathfrak{G}(Z) \to U$ is a flat and surjective morphism whose fiber $\mathfrak{G}(Z)_{s'}$ over any $s' \in U$ is $\mathfrak{C}_{s'}(Z)$.

We assume that our theorem is not true, that is for general $x \in X$, the multiplicity of X^* at a general point of $\langle L, T_{X,x} \rangle^{\perp}$ is equal to the multiplicity at a general point of L^{\perp} .

Let [h] be a general point of $\langle L, T_{X,x} \rangle^{\perp}$ and let $S \subset L^{\perp}$ be a smooth (not necessarily closed) connected curve passing through [h] and through a general point of L^{\perp} . We apply the proposition to X^* and S. Then there exists a scheme $\mathscr{G}(X^*) \subset \mathbb{P}^{N^*} \times S$ such that the natural projection $p : \mathscr{G}(X^*) \to S$ is a flat and surjective morphism whose fiber over $s \in S$ is the tangent cone to X^* at s. Let $\Gamma(X^*) = |\mathscr{G}(X^*)|$. The induced morphism $\Gamma(X^*) \to S$ is flat and for general $s \in S$ the fiber $\Gamma(X^*)_s$ is exactly $|\mathscr{C}_s(X^*)|$.

Now we study the family of the duals of the reduced tangent cones of X^* at points of *S*. Applying the Lagrangian specialization principle [Lê and Teissier 1988; Kleiman 1984] to $\Gamma(X^*)$ and *S*, we find:

Theorem 2.1.3. Let $S \subset X^*$ be a smooth curve such that X^* is equimultiple along *S*. There esists a variety $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)$ with the following properties.

(i) For general $s \in S$, the following equality holds in $\mathbb{P}^N \times \Gamma(X^*)_s$:

$$I(|\mathscr{C}_{s}(X^{*})|/\mathbb{P}^{N^{*}}) = I_{S}(\Gamma(X^{*})/\mathbb{P}^{N^{*}} \times S)_{s}.$$

- (ii) The morphism $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S) \to S$ is flat and surjective.
- (iii) For all $s \in S$, the conormal space $I(|\mathscr{C}_s(X^*)|/\mathbb{P}^{N^*})$ is a union of irreducible components of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$.

As a consequence, the image in \mathbb{P}^N of the fiber $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s$, for general $s \in S$, is $|\mathscr{C}_s(X^*)|^*$. Moreover, for any $s \in S$, the image of the reduced fiber $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$ contains $|\mathscr{C}_s(X^*)|^*$.

2.2. *Polar varieties and duals of tangent cones.* We discuss an extension of the reflexivity theorem proved in [Lê and Teissier 1988]. The main results of this section will be applied to X^* when it is a hypersurface, so we restrict our study to that case.

Definition 2.2.1. Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a linear space. The *polar variety* of *Z* associated to *D*, which we denote by P(Z, D), is the closure of the set $\{z \in Z_{smooth} : D \subset T_{Z,Z}\}$.

If $D = \emptyset$ (that is, D has dimension -1), then we put P(Z, D) = Z.

Remark 2.2.2. If *Z* is normal, if $u = [u_0, ..., u_N]$ in an homogeneous system of coordinates on \mathbb{P}^N and *f* is an equation of *Z* in this system then P(Z, u) is given by the equations f = 0 and $u_0 \partial f / \partial x_0 + \cdots + u_N \partial f / \partial x_N = 0$.

If Z is not normal, then all irreducible components of Z_{sing} which are of dimension N - 2 are irreducible components of the scheme defined by f = 0 and $u_0 \partial f / \partial x_0 + \cdots + u_N \partial f / \partial x_N = 0$, but they are not irreducible components of P(X, u).

Proposition 2.2.3. Let $Z \subset \mathbb{P}^N$ be a reduced, irreducible hypersurface and let $D \subset \mathbb{P}^N$ be a general linear space of dimension k. Then P(Z, D) is empty or of codimension k + 1 in Z.

We state a result of Lê and Teissier which relates the duals of the tangent cones at z of some polar varieties of Z with the tangency locus of z^{\perp} with Z*. See [Lê and Teissier 1988, Proposition 2.2.1]. For any $z \in Z$, recall that $\text{Tan}(z^{\perp}, Z^*)$ is the tangency locus of z along Z* (see conormal diagram on page 2).

Theorem 2.2.4. Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface and let $z \in Z$ be a point.

- (i) The dual of |𝔅_z(Z)| is a union of reduced spaces underlying (possibly embedded) components of Tan(z[⊥], Z*).
- (ii) Any irreducible component of $|\operatorname{Tan}(z^{\perp}, Z^*)|$ is dual to an irreducible component of $|\mathscr{C}_z(P(Z, D))|$ for general $D \in \mathbb{G}(k, N)$ and for some integer $k \in \{-1, \ldots, N-2\}$.

Remark 2.2.5. Part (ii) of the theorem has to be explained. Assume that there is an irreducible component (say *T*) of $|\text{Tan}(z^{\perp}, Z^*)|$ which is not dual to an irreducible component of $|\mathscr{C}_z(Z)|$. Then, there is $k \in \{0, \ldots, N-2\}$ such that for general $D \in \mathbb{G}(k, N)$, we have $z \in P(Z, D)$. Moreover, as *D* varies in a dense open subset of $\mathbb{G}(k, D)$, the cones $\mathscr{C}_z(P(D, Z))$ have a fixed irreducible component in common whose reduced locus is T^* .

Note also that if $z \in Z_{\text{smooth}}$ then for $k \ge 0$ and for D general in $\mathbb{G}(k, N)$, we have $z \notin P(Z, D)$. As a consequence of the (ii) of the above theorem, we find

Tan $(z^{\perp}, Z^*) = T_{Z,z}^{\perp}$ for $z \in Z_{\text{smooth}}$. This is the way the (obvious corollary of the) reflexivity theorem is often stated.

When $\operatorname{Tan}(z^{\perp}, Z^*)$ is irreducible, one may expect $|\mathscr{C}_z(Z)|^* = |\operatorname{Tan}(z^{\perp}, Z^*)|$. But this is not true:

Example 2.2.6. Let $X \subset \mathbb{P}^4$ be the smooth ruled surface of degree 3 considered in example 1.1.1 and let X^* its dual. The hypersurface X^* has also degree 3 and its singular locus is a \mathbb{P}^2 , the dual of the exceptional section of X (which we denote by L). Let $C \subset L^{\perp} = X^*_{\text{sing}}$ be the conic corresponding to the hyperplanes which are tangent to X along a ruling of X and let $z \in C$.

The tangent cone $\mathscr{C}_z(X^*)$ is a doubled \mathbb{P}^3 so that $|\mathscr{C}_z(X^*)|^* \neq \operatorname{Tan}(z^{\perp}, X)$. We also note that the scheme-theoretic tangency locus of z^{\perp} along X is a line with an embedded point. The embedded point is dual to $|\mathscr{C}_z(X^*)|$ and the line is dual to $|\mathscr{C}_z(P(X^*, u))|$, for general $u \in \mathbb{P}^{4^*}$.

Notations 2.2.7. Let $f: Y \to T$ be a quasiprojective morphism between quasiprojective schemes, let $T' \subset T$ be a smooth variety and let $s \in T'$ be any point. Let Y_1, \ldots, Y_m be the irreducible components of $f^{-1}(T')$ such that the restrictions

$$f|_{Y_i}: Y_i \to T',$$

are surjective. Define the scheme

$$\operatorname{limflat}_{\{t \to s, t \in T'\}} f^{-1}(t) := f|_{Y_1 \cup \dots \cup Y_m}^{-1}(s)$$

If dim(T') = 1 and the Y_i are all reduced, this is the classical flat limit taken along a smooth curve. If $f|_{f^{-1}(T')} : f^{-1}(T') \to T'$ is flat, then

$$\operatorname{limflat}_{\{t \to s, t \in T'\}} f^{-1}(t) = f|_{f^{-1}(T')}^{-1}(s).$$

Proof of Main Theorem 1.1.5. We recall the setting for the convenience of the reader. The projective variety $X \subset \mathbb{P}^N$ is irreducible and nondegenerate. The linear space $L \subset \mathbb{P}^N$ is such that $\operatorname{Sh}_X(L) \neq X$ and $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$ for all $x \in X_{\text{smooth}}$. We want to prove that for all $x \in X_{\text{smooth}}$ such that $x \notin \operatorname{Sh}_X(L)$, the multiplicity in X^* of a general hyperplane containing $\langle L, T_{X,x} \rangle$ is strictly greater than that of a general hyperplane containing L.

The result is obvious if $L^{\perp} \not\subset X^*$ and we have already seen that we can restrict to the case where X^* is a hypersurface. So we only consider the case where $L^{\perp} \subset X^*$ and X^* is a hypersurface and we assume that our result is not true. Let $x \in X_{\text{smooth}}$ with $x \notin \text{Sh}_X(L)$ and let [h] be a general point in $\langle L, T_{X,x} \rangle^{\perp}$. By the results of the previous section, there exists a smooth (not necessarily closed) curve $S \subset L^{\perp}$ with $[h] \in S$ and a flat morphism

$$I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S) \to S,$$

whose fiber $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s$ is the conormal space of $|\mathscr{C}_s(X^*)|$, for general $s \in S$. Further, the conormal space of $|\mathscr{C}_s(X^*)|$ is included in $|I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S)_s|$ for all $s \in S$.

Theorem 2.2.4(i) implies that

$$|\mathscr{C}_s(X^*)|^* \subset p(|q^{-1}(s)|),$$

for all $s \in S$, where *p* and *q* are as in the conormal diagram of page 2. The flatness of $I_S(\Gamma(X^*)/\mathbb{P}^{N^*} \times S) \to S$ gives the inclusion

$$|\mathscr{C}_{[h]}(X^*)|^* \subset p(\text{limflat}_{\{s \to [h], s \in S\}} |q^{-1}(s)|).$$

By Definition 1.1.2, the right-hand side is contained in $Tan(L, X) \subset L$.

Let \mathcal{F} be an irreducible component of $\operatorname{Tan}(H, X)$ passing through x. By Theorem 2.2.4, there is an integer $k \in \{-1, \ldots, N-2\}$ such that $|\mathcal{F}|$ is dual to an irreducible component of $|\mathscr{C}_{[h]}(P(X^*, D))|$, for general $D \in \mathbb{G}(k, N)$. Since $|\mathscr{C}_{[h]}(X^*)|^* \subset L$, we have $k \ge 0$.

Let $x_0 \in \mathcal{F}$ be a general point. Duality implies $T_{[\mathscr{C}_{[h]}(P(X^*,D))|,z} \subset x_0^{\perp}$ for some general z in the irreducible component of $\mathscr{C}_{[h]}(P(X^*, D))$ whose reduced locus is $|\mathscr{F}|^*$. Note that $\mathscr{C}_{[h]}(P(X^*, D)) \subset \mathscr{C}_{[h]}(X^*)$. Let $T_{[\mathscr{C}_{[h]}(X^*)|,z}$ be a limit of tangent spaces to $|\mathscr{C}_{[h]}(X^*)|$ at z. The point z is general in $|\mathscr{C}_{[h]}(P(X^*, D))|$, so $T_{[\mathscr{C}_{[h]}(P(X^*,D))|,z} \subset T_{[\mathscr{C}_{[h]}(X^*)|,z}$.

As a consequence of this, we have $T_{[\mathscr{C}_{[h]}(P(X^*,D))],z} \subset x_0^{\perp} \cap T_{\mathscr{C}_{[h]}(X^*),z}$. That is,

$$\langle x_0, T_{|\mathscr{C}_{[h]}(X^*)|, z}^{\perp} \rangle \subset \mathscr{F} \subset X$$

But $|\mathscr{C}_{[h]}(X^*)|^* \subset \operatorname{Tan}(L, X)$, so $T_{|\mathscr{C}_{[h]}(X^*)|, z}^{\perp} \in \operatorname{Tan}(L, X)$, and the inclusion above says that $x_0 \in \operatorname{Sh}_X(L)$. This is a contradiction.

3. Corollaries and open questions

We present here some corollaries of the Main Theorem and related open questions.

3.1. *Zak's conjecture on varieties with minimal codegree.* Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety. We recall, following Zak, that the order of X is ord $X = \min\{k, S^{k-1}(X) = \mathbb{P}^N\}$ and the *k*-th secant-defect is $\delta_k = \dim X + \dim S^{k-1}(X) + 1 - \dim S^k(X)$, for all $k \leq \operatorname{ord} X - 1$.

Zak [1993] proved an important result related to secant defects.

Theorem 3.1.1 (Zak's superadditivity theorem). Let $X \subset \mathbb{P}^N$ an irreducible, nondegenerate projective variety such that $\delta_1 > 0$. For all $k \leq \text{ord } X - 1$, we have the inequality

$$\delta_k \ge \delta_{k-1} + \delta_1.$$

The varieties on the boundary are called Scorza varieties. More precisely:

Definition 3.1.2. An irreducible, smooth, nondegenerate projective variety $X \subset \mathbb{P}^N$ is a Scorza variety if the following conditions hold:

- (i) $\delta_1 > 0$ and $N > 2n + 1 \delta_1$,
- (ii) $\delta_k = \delta_{k-1} + \delta_1$ for all $k \leq \text{ord } X 1$,
- (iii) ord $X 1 = [\dim X/\delta_1]$, where [] denotes the integral part.

Theorem 3.1.3 (Classification of Scorza varieties [Zak 1993]). *Any Scorza variety X is of one of the following types*:

- (i) $X = v_2(\mathbb{P}^n) \subset \mathbb{P}^{n(n+3)/2}$ (2nd Veronese) and deg $X^* = n + 1$;
- (ii) $X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{n(n+2)}$ and deg $X^* = n + 1$;
- (iii) $X = \mathbb{G}(1, 2n+1) \subset \mathbb{P}(\Lambda^2 \mathbb{C}^{2n+2})$ and $deg(X^*) = n+1$;
- (iv) $X \subset \mathbb{P}^{26}$ is the 16-dimensional variety corresponding to the orbit of highest weight vector in the lowest nontrivial representation of the group of type E_6 and deg $X^* = 3$.

In [Zak 2004] an important consequence of the assertion $S^k(X)^* \subset X_{k+1}^*$ (where X_k^* is the set of points of multiplicity at least k in X^*) was discovered. We state that result in the setting where we are able to prove it.

Proposition 3.1.4. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate, smooth, projective variety. Assume that X is not k dual defective for $k < \operatorname{ord} X - 1$, then

$$\deg X^* \ge \operatorname{ord} X.$$

Proof. With the assumptions above, Proposition 1.2.4 implies that there is a point of multiplicity ord X - 1 in X^* . Since X is nondegenerate, its dual is not a cone and so deg $X^* \ge$ ord X.

If X is a Scorza variety then deg $X^* = \operatorname{ord} X$. The converse statement in conjectured in [Zak 2004]. We formulate the conjecture in the setting where we can prove the inequality: deg $X^* \ge \operatorname{ord} X$.

Conjecture 3.1.5 [Zak 2004]. Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate, projective variety. Assume that X is not k dual defective for all $k < \operatorname{ord} X$ and that deg $X^* = \operatorname{ord} X + 1$, then X is a hyperquadric or a Scorza variety.

It is proved in [Zak 1993], without any hypothesis on the dual defectiveness of X, that smooth varieties with deg(X^*) = 3 and ord X = 3 are Severi varieties. In particular, they are Scorza varieties. Note, however, that the smoothness assumption seems to be necessary in his proof. I believe it would be very interesting to have a classification of all varieties whose duals have degree 3.

3.2. Varieties with unexpected equisingular linear spaces. We come back to our usual setting. Let $L \subset \mathbb{P}^N$ be a linear space such that for all $x \in X_{\text{smooth}}$, we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We have seen in example 1.1.1 that a hyperplane containing the join $\langle L, T_{X,x} \rangle$ may have the same multiplicity in X^* as the general hyperplane containing L, even if x is a general point of X. The following definition is convenient to describe this situation.

Definition 3.2.1. Let $X \subset \mathbb{P}^N$ be an irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset \mathbb{P}^N$ be a linear space such that for all $x \in X_{\text{smooth}}$, we have $\langle L, T_{X,x} \rangle \neq \mathbb{P}^N$. We say that L^{\perp} is an *unexpected equisingular linear space* in X^* if for all $x \in X_{\text{smooth}}$, the general hyperplane containing $\langle L, T_{X,x} \rangle$ has the same multiplicity in X^* as the general hyperplane containing L.

The variety in Example 1.1.1 is rather special since it is a scroll surface (see [Zak 2004] for interesting discussions about this variety). It is not a coincidence that the directrix of this variety is an unexpected equisingular linear space in its dual. Indeed, we have:

Theorem 3.2.2. Let $X \subset \mathbb{P}^N$ be an irreducible, smooth, nondegenerate projective variety such that X^* is a hypersurface. Let $L \subset X$ be a linear space with dim $(L) = \dim(X) - 1$. Assume that L^{\perp} is an unexpected equisingular linear space in X^* such that $\operatorname{mult}_{L^{\perp}} X^* = 2$. Then X is the cubic scroll surface in \mathbb{P}^4 .

Here mult_{*L*[⊥]} X^* denotes the multiplicity in X^* of a general point of L^{\perp} . Before diving into the proof of Theorem 3.2.2, we describe the tangency locus of any point $[h] \subset X^*$, such that mult_[h] $X^* = 2$.

Proposition 3.2.3. Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that $\operatorname{mult}_{[h]} X^* = 2$. The scheme theoretic tangency locus of H with X is either

- (i) an irreducible hyperquadric and in this case $|\mathscr{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$,
- (ii) the union of two (not necessarily distinct) linear spaces, or
- (iii) a linear space with at least one embedded component.

We postpone the proof of this result to the Appendix.

Proof of Theorem 3.2.2. Let *H* be a general hyperplane containing *L*. We have $H \cap X = L \cup D_H$, where D_H is a divisor such that

$$D_H \cap L = \operatorname{Tan}(H, X).$$

Let $x \in X$ be a general point and let H_x be a general hyperplane containing $\langle L, T_{X,x} \rangle$. Then Tan (H_x, X) contains x and

$$\xi := p \big(\operatorname{limflat}_{\{[h] \to [h_x], [h] \in L^{\perp}\}} q^{-1}([h]) \big).$$

By hypothesis, we have

$$\operatorname{mult}_{[h_x]} X^* = \operatorname{mult}_{[h]} X^* = 2,$$

for all $[h] \in L^{\perp}$. Proposition 3.2.3 hence implies that the irreducible component of Tan(H_x , X) containing x, which we denote by R_{H_x} , also contains ξ . Moreover, $\xi \subset L$, so dim $R_{H_x} > \dim \xi$, for general $[h] \in L^{\perp}$. As a consequence, dim $R_{H_x} = n-1$.

On the other hand, since

$$\operatorname{mult}_{[h_x]} X^* = \operatorname{mult}_{[h]} X^* = 2,$$

for all $[h] \in L^{\perp}$, we have $|\mathscr{C}_{[h_x]}(X^*)|^* \neq |R_{H_x}|$. We apply again Proposition 3.2.3 and we find that $|R_{H_x}|$ is necessarily a linear space of dimension n-1. Thus,

$$\dim \langle L, T_{X,x} \rangle = n+1.$$

Note that Bertini's theorem implies that

$$R_{H_x} \subset \langle L, T_{X,x} \rangle \cap X,$$

for general H_x containing $\langle L, T_{X,x} \rangle$. As a consequence R_{H_x} is an irreducible component of $\langle L, T_{X,x} \rangle \cap X$, for general H_x . Thus R_{H_x} does not depend on H_x , for general H_x containing $\langle L, T_{X,x} \rangle$. We deduce that $\langle L, T_{X,x} \rangle$ is tangent to X along a linear space of dimension n-1. By the theorem on tangencies, we have $n-1 \leq 1$, that is n = 2 (obviously, X is not a curve). So $X \subset \mathbb{P}^N$ is a nondegenerate surface containing a distinguished line L, such that for general $x \in X$, there is a \mathbb{P}^3 tangent to X along a line passing through x and meeting L. This means that X is the projection of a scroll of type $S_{1,d-1}$. By hypothesis, we have $\operatorname{mult}_{L^{\perp}} X^* = 2$, hence of [Ciliberto et al. 2008, Proposition 1.6] implies that $X = S_{1,2} \subset \mathbb{P}^4$. \Box

Appendix: Tangency loci of points of multiplicity 2 in the dual

The goal of this appendix is to prove the following proposition.

Proposition 3.2.3. Let $X \subset \mathbb{P}^N$ be a smooth, irreducible, nondegenerate projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that $\operatorname{mult}_{[h]} X^* = 2$. The scheme theoretic tangency locus of H with X is either

- (i) an irreducible hyperquadric and in this case $|\mathscr{C}_{[h]}(X^*)|^* = \operatorname{Tan}(H, X)$,
- (ii) the union of two (not necessarily distinct) linear spaces, or
- (iii) a linear space with at least one embedded component.

Example A.1. All three cases are encountered in nature:

(i) If $X = v_2(\mathbb{P}^2) \subset \mathbb{P}^5$, then for all $[h] \in v_2(\mathbb{P}^{2^*}) \subset X^*$, we have $\operatorname{mult}_{[h]} X^* = 2$ and $\operatorname{Tan}(H, X)$ is a smooth conic.

(ii) If X is a complete intersection of large multidegree and large codimension, then there are points $[h_1], [h_2] \in X^*$ such that $\operatorname{mult}_{[h_i]} X^* = 2$ and $\operatorname{Tan}(H_1, X)$ is exactly two distinct points, whereas $\operatorname{Tan}(H_2, X)$ is a single double point.

(iii) If X is the cubic scroll of Example 1.1.1, then there is a conic $C \subset X^*$, such that for all $[h] \in C$, we have $\operatorname{mult}_{[h]} X^* = 2$ and $\operatorname{Tan}(H, X)$ is a line with an embedded point.

A doubled linear space will be considered as the union of two (not distinct) linear spaces. By Theorem 2.2.4, we know that the irreducible components of Tan(H, X) are dual to irreducible components of the reduced spaces underlying some $\mathscr{C}_{[h]}(P(X^*, D_k))$ for general $D_k \in \mathbb{G}(k, N)$. When $\operatorname{mult}_{[h]} X^* = 2$, the cones $\mathscr{C}_{[h]}(P(X^*, D_k))$ are rather easy to describe. Let's start with some notation.

Notations A.2. Let $Z \subset \mathbb{P}^N$ be a reduced and irreducible hypersurface. Let $D \in \mathbb{G}(k, N)$ and let f_Z be an equation for Z in some coordinate system of \mathbb{P}^N . We denote by $P(f_Z, D)$ the subscheme of \mathbb{P}^N whose ideal is generated by the equations

$$u_0\frac{\partial f_Z}{\partial t_0}+\cdots+u_N\frac{\partial f_Z}{\partial t_N},$$

for $u = [u_0, \ldots, u_N]$ varying in D.

Let $D \in \mathbb{G}(k, N)$ be a general *k*-plane. Note that if $\dim(Z_{\text{sing}}) < \dim P(Z, D)$ (that is $\dim Z_{\text{sing}} \le N - k - 3$), then $P(Z, D) = P(f_Z, D) \cap Z$. In the other case, the irreducible components of maximal dimension of Z_{sing} are irreducible components of $P(f_Z, D) \cap Z$.

Lemma A.3. Let $Z \subset \mathbb{P}^N$ be an irreducible and reduced hypersurface. Let $z \in Z$ and let $k \in \{-1, ..., N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have

- (1) $z \notin P(Z, D)$, or
- (2) $\operatorname{mult}_{z} P(Z, D) = \operatorname{mult}_{z}(Z)$. $\operatorname{mult}_{z} P(f_{Z}, D)$, if $\operatorname{dim}(Z_{\operatorname{sing}}^{(z)}) < \operatorname{dim} P(Z, D)$, where $Z_{\operatorname{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through *z*, or
- (3) $\operatorname{mult}_{z} P(Z, D) < \operatorname{mult}_{z}(Z)$. $\operatorname{mult}_{z} P(f_{Z}, D)$, if $\dim(Z_{\operatorname{sing}}^{(z)}) \ge \dim P(Z, D)$, where $Z_{\operatorname{sing}}^{(z)}$ is an irreducible component of Z_{sing} of maximal dimension passing through z.

Proof. If $z \in P(Z, D)$ for general $D \in \mathbb{G}(k, N)$, we will prove the lemma only in the case $P(f_Z, D)$ is smooth at z, for two reasons. The general case is obtained by the same methods, this is only more technical, and we will use the result only in the case $P(f_Z, D)$ is smooth at z.

Moreover if $z \in P(Z, D)$ for general D, we will only concentrate on the case $\dim(Z_{\text{sing}}^{(s)}) < \dim P(Z, D)$. In this case, we have locally around z the equality

 $P(Z, D) = P(f_Z, D) \cap Z$ for general $D \in \mathbb{G}(k, N)$. The situation where an irreducible component Z_{sing} containing *z* is an irreducible component of $P(f_Z, D) \cap Z$ —this is case (3) of the lemma—is dealt with exactly in the same way.

Now, we work locally around z, so that $P(f_Z, D) \cap Z = P(Z, D) \subset \mathbb{A}^N$, for general $D \in \mathbb{G}(k, N)$. Let $(Z_i)_{i \in I}$ be a stratification of Z such that Z_i is smooth and Z is normally flat along Z_i , for all $i \in I$. Such a stratification exists, due to the open nature of normal flatness (see [Hironaka 1964, Chapter II]). Consider the Gauss map $G : Z \to \mathbb{P}^{N^*}$. It restricts to a map $G_i : Z_i \to \mathbb{P}^{N^*}$. We have

$$P(f_Z, D) \cap Z = P(Z, D) = G^{-1}(D^{\perp}),$$

so that $P(f_Z, D) \cap Z_i = G_i^{-1}(D^{\perp})$, for all *i*.

Now, we apply Kleiman's transversality theorem to find that for all *i* and for general $D \in \mathbb{G}(k, N)$, the inverse images $G_i^{-1}(D^{\perp})$ are either empty or smooth of the expected dimension.

Let *i* such that *z* is in Z_i . If $z \notin G_i^{-1}(D^{\perp})$ for general $D \in \mathbb{G}(k, N)$, then $z \notin P(Z, D)$ and we are in the case 1 of the lemma. Otherwise, *z* is a smooth point of $G_i^{-1}(D^{\perp})$, so $T_{P(f_Z,D),z}$ and $T_{Z_i,z}$ are transverse.

Assume that $\operatorname{mult}_z P(Z, D) > \operatorname{mult}_z Z$. $\operatorname{mult}_z P(f_Z, D)$. Since $P(f_Z, D)$ is smooth at *z*, this implies that $T_{P(f_Z,D),z}$ and $\mathscr{C}_z(Z)$ are not transverse. In particular, the linear spaces $T_{P(f_Z,D),z}$ and $\operatorname{Vert}(\mathscr{C}_z(Z))$ are not transverse (here $\operatorname{Vert}(\mathscr{C}_z(Z))$ is the vertex of the cone $\mathscr{C}_z(Z)$). But *Z* is normally flat along Z_i , so we have $T_{Z_i,z} \subset \operatorname{Vert}(\mathscr{C}_z(Z))$ (see [Hironaka 1964, Theorem 2, p. 195]). This is a contradiction.

Corollary A.4. Let $Z \subset \mathbb{P}^N$ be a reduced, irreducible hypersurface. Let $z \in Z$ such that $\text{mult}_z Z = 2$ and let $k \in \{-1, ..., N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have

$$\operatorname{mult}_{z} P(Z, D) \leq 2.$$

Proof. The result is obvious for k = -1, since in this case P(Z, D) = Z. Assume that $k \ge 0$ and let $D \in \mathbb{G}(k, N)$ be a general k-plane. Let $u \in D$ be a general point ans let π_u be the projection from u. Then, the projections

$$\pi_u|_{P(Z,u)}: P(Z,u) \to \pi_u(P(Z,u))$$

and

$$\pi_u|_{P(Z,D)}: P(Z,D) \to \pi_u(P(Z,D))$$

are locally isomorphisms around *z*. Moreover, we have the following equality (see [Teissier 1982]):

$$\pi_u(P(Z, D)) = P(\pi_u(P(Z, u)), \pi_u(D)).$$

As a consequence, it is sufficient to prove the result for k = 0. But in this case, this is an obvious application of the lemma above. Indeed, for general $u \in \mathbb{P}^N$,

$$\operatorname{mult}_{z} P(f_{Z}, u) = \operatorname{mult}_{z} Z - 1 = 1.$$

We also need the following result.

Proposition A.5. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety such that X^* is a hypersurface. Let $[h] \in X^*$ be such that $\operatorname{Tan}(H, X)$ has m components (some of which may be embedded components), then there exists $k \in \{-1, \ldots, N-2\}$, such that for general $D \in G(k, N)$, we have

$$\operatorname{mult}_{[h]} P(X^*, D) \ge m.$$

Proof. We only prove the result when Tan(H, X) is reduced and pure dimensional. The general case is done using the same ideas; it's just more technical.

Assume that

$$\operatorname{Tan}(H, X) = Y_1 \cup \cdots \cup Y_m,$$

where the Y_i have the same codimension, say c. Let $D \subset \mathbb{P}^{N^*}$ be a general \mathbb{P}^{N-1-c} . Then

$$\pi_D(P(X^*, D)) = (D^{\perp} \cap X)^*$$

where π_D is the projection from *D*. Moreover, we have $[h] \in P(X^*, D)$ and

$$\operatorname{Tan}(D^{\perp} \cap H, D^{\perp} \cap X) = D^{\perp} \cap \operatorname{Tan}(H, X).$$

As a consequence, $\operatorname{Tan}(D^{\perp} \cap H, D^{\perp} \cap X)$ is a 0-dimensional scheme of degree at least *m*. In this case, it is clear that

$$\operatorname{mult}_{\pi_D([h])} \pi_D(P(X^*, D)) \ge m.$$

On the other hand, since D is general, the morphism

$$\pi_D: P(X^*, D) \to \pi_D(P(X^*, D))$$

is locally an isomorphism around [h], so that

$$\operatorname{mult}_{[h]} P(X^*, D) \ge m.$$

Proof of Proposition 3.2.3. Let $T_1 \cup \cdots \cup T_m$ be the decomposition of Tan(H, X) into irreducible components. If $m \ge 3$, then Proposition A.5 implies that $mult_{[h]}(X^*) \ge 3$, this is impossible, so that $m \le 2$.

Assume that m = 2. The proof of Proposition A.5 shows that these two irreducible components are scheme-theoretically linear spaces.

Assume that m = 1 and let $k \in \{-1, ..., N - 2\}$ such that T_1 is dual to some irreducible components of the reduced space underlying $\mathscr{C}_{[h]}P(X^*, D)$, for general $D \in \mathbb{G}(k, N)$. By Corollary A.4, the cone $\mathscr{C}_{[h]}P(X^*, D)$ is either a hyperquadric

or a linear space. Assume that it is an irreducible hyperquadric. If $k \ge 0$, we know by Theorem 2.2.4 that $|\mathscr{C}_{[h]}(X^*)|^*$ is the reduced space underlying some embedded component of $\operatorname{Tan}(H, X)$. Taking $q = \dim \operatorname{Tan}(H, X)$ general hyperplane sections of $\operatorname{Tan}(H, X)$ passing through $|\mathscr{C}_{[h]}(X^*)|^*$, we see as in the proof of Proposition A.5 that for general $D' \in \mathbb{G}(q-1, N)$, we have

$$\operatorname{mult}_{[h]} P(X^*, D') \ge 3.$$

This is impossible by Corollary A.4. Thus, if $\mathscr{C}_{[h]}P(X^*, D)$ is an irreducible hyperquadric, then k = -1, and we are in the case 1 of the proposition.

Finally, if $\mathscr{C}_{[h]}P(X^*, D)$ is a the union of two linear spaces or a unique linear space, then we are in case 2 or 3 of the proposition. This concludes the proof of Proposition 3.2.3.

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Volume 253 No. 1 September 2011

Singularities of the projective dual variety ROLAND ABUAF	1
Eigenvalue estimates for hypersurfaces in $\mathbb{H}^m \times \mathbb{R}$ and applications PIERRE BÉRARD, PHILIPPE CASTILLON and MARCOS CAVALCANTE	19
Conformal Invariants associated to a measure: Conformally covariant operators	37
SUN-YUNG A. CHANG, MATTHEW J. GURSKY and PAUL YANG	
Compact symmetric spaces, triangular factorization, and Cayley coordinates DEREK HABERMAS	57
Automorphisms of the three-torus preserving a genus-three Heegaard splitting JESSE JOHNSON	75
The rationality problem for purely monomial group actions HIDETAKA KITAYAMA	95
On a Neumann problem with <i>p</i> -Laplacian and noncoercive resonant nonlinearity	103
SALVATORE A. MARANO and NIKOLAOS S. PAPAGEORGIOU	
Minimal ramification in nilpotent extensions NADYA MARKIN and STEPHEN V. ULLOM	125
Regularity of weakly harmonic maps from a Finsler surface into an <i>n</i> -sphere XIAOHUAN MO and LIANG ZHAO	145
On the sum of powered distances to certain sets of points on the circle NIKOLAI NIKOLOV and RAFAEL RAFAILOV	157
Formal geometric quantization II PAUL-ÉMILE PARADAN	169
Embedded constant-curvature curves on convex surfaces HAROLD ROSENBERG and MATTHIAS SCHNEIDER	213
A topological construction for all two-row Springer varieties HEATHER M. RUSSELL	221

