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# SINGULARITIES OF THE PROJECTIVE DUAL VARIETY 

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Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety and let $X^{*} \subset \mathbb{P}^{N^{*}}$ be its projective dual. Let $L \subset \mathbb{P}^{N}$ be a linear space such that $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$ for all $x \in X_{\text {smooth }}$ and such that the lines in $X$ meeting $L$ do not cover $X$. If $x \in X$ is general, we prove that the multiplicity of $X^{*}$ at a general point of $\left\langle L, T_{X, x}\right\rangle^{\perp}$ is strictly greater than the multiplicity of $X^{*}$ at a general point of $L^{\perp}$. This is a strong refinement of Bertini's theorem.

## 1. Introduction

1.1. Multiplicities of the projective dual. Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety over the field of complex numbers. Let $X^{*} \subset \mathbb{P}^{N^{*}}$ be its projective dual, let $L \subset \mathbb{P}^{N}$ be a linear space and $H$ be a general hyperplane containing $L$. Bertini's classical theorem asserts that the tangency locus of $H$ with $X$ is included in $X \cap L$. Very little is known about the hyperplanes whose tangency locus with $X$ lies outside $L \cap X$. It is tempting to think that the multiplicity in $X^{*}$ of such a hyperplane is strictly larger than the multiplicity of a general hyperplane containing $L$. The following example shows that this is not true for every $L$.

Example 1.1.1. Let $X \subset \mathbb{P}^{4}$ be a smooth hyperplane section of $\mathbb{P}^{1} \times \mathbb{P}^{2} \subset \mathbb{P}^{5}$. The variety $X$ is a ruled surface of degree 3 . Its dual is a hypersurface of degree 3 in $\mathbb{P}^{4^{*}}$ which does not contain any points of multiplicity higher than 2 . Let $L$ be the exceptional section of $X$. If $H \subset \mathbb{P}^{4}$ is a general hyperplane which contains $L$, then $H \cap X=L \cup D_{1} \cup D_{2}$, where $D_{1}$ and $D_{2}$ are two distinct lines on $X$ such that $D_{1} . D_{2}=0$ and $L . D_{i}=1$ for $i=1,2$. As a consequence, a general point of $L^{\perp}$ is of multiplicity 2 in $X^{*}$. Now, let $D \subset X$ be a line such that $D . L=1$ and let $x \in D$ such that $x \notin L$. The hyperplane containing $L$ and $T_{X, x}$ is a point of multiplicity exactly 2 in $X^{*}$, that is, the multiplicity of a general point of $L^{\perp}$.

This example shows that, even for general $x \in X$, the multiplicity in $X^{*}$ of a hyperplane containing $L$ and tangent to $X$ at $x$ may well be equal to the multiplicity of a general hyperplane containing $L$. Thus, without extra hypotheses on $L$, it

[^0]seems hopeless to say something about the multiplicity in $X^{*}$ of special points of $L^{\perp}$. For this purpose, we introduce a definition:

Definition 1.1.2. Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety and let $L \subset \mathbb{P}^{N}$ be a linear space. Consider the conormal diagram


Let $F_{1}, \ldots, F_{m}$ be all the irreducible components of $q^{-1}\left(L^{\perp}\right)$ such that the restrictions

$$
\left.q\right|_{F_{i}}: F_{i} \rightarrow L^{\perp}
$$

are surjective. The contact locus of $L$ with $X$, which we denote by $\operatorname{Tan}(L, X)$, is the union of the $p\left(F_{i}\right)$, for $1 \leq i \leq m$.

In the case where $L$ is a hyperplane, the contact locus $\operatorname{Tan}(L, X)$ is called the tangency locus of $L$ with $X$. A tangent hyperplane to $X$ is a hyperplane $H \subset \mathbb{P}^{N}$ such that $\operatorname{Tan}(H, X) \neq \varnothing$.

The contact locus $\operatorname{Tan}(L, X)$ can be thought as the variety covered by the tangency loci of general hyperplanes containing $L$. In case $L^{\perp} \not \subset X^{*}$, this locus is empty. We always have the inclusion

$$
\overline{\left\{x \in X_{\text {smooth }}: T_{X, x} \subset L\right\}} \subset \operatorname{Tan}(L, X),
$$

but if $\operatorname{dim}(L)<N-1$ or if $X$ is not smooth, the former locus can be strictly smaller than the latter. Note also that Bertini's theorem says that $\operatorname{Tan}(L, X) \subset L \cap X$. Finally, the contact locus is well behaved. If for a general hyperplane $H^{\prime}$ containing $L$, we have $\operatorname{dim} \operatorname{Tan}\left(H^{\prime}, X\right)>0$, then

$$
\operatorname{Tan}(H \cap L, H \cap X)=H \cap \operatorname{Tan}(L, X)
$$

for any general hyperplane $H \subset \mathbb{P}^{N}$.
Example 1.1.3. If $X \subset \mathbb{P}^{N}$ is such that $X^{*}$ is a hypersurface and $L=T_{X, x}$, where $x \in X$ is a general point, then $\operatorname{Tan}(L, X)=x$.

- If $X=G(1,7) \subset \mathbb{P}^{27}$ and $L=\left\langle T_{X, y_{1}}, T_{X, y_{2}}\right\rangle$, where $y_{1}, y_{2} \in \mathbb{G}(1,7)$ are two general points, then $\operatorname{Tan}(L, X)=\left\{x \in X: T_{X, x} \subset L\right\}$ is a 4-dimensional quadric, the entry locus of a general point $z \in\left\langle y_{1}, y_{2}\right\rangle$.
- If $X=G(1,4) \subset \mathbb{P}^{9}$ and $L=T_{X, y}$, for any $y \in X$, then $\operatorname{dim} \operatorname{Tan}(L, X)>0$, whereas $\left\{x \in X: T_{X, x} \subset L\right\}=\{y\}$.

Definition 1.1.4. Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety, and let $L \subset \mathbb{P}^{N}$ be a linear subspace. The shadow of $L$ on $X$, which we denote by $\operatorname{Sh}_{X}(L)$, is the closed variety covered by the linear spaces $M \subset X$ such that $\operatorname{dim}(M)=\operatorname{def}(X)+1$ and $\operatorname{dim}(M \cap \operatorname{Tan}(L, X))=\operatorname{def}(X)$.

Here $\operatorname{def}(X)=\operatorname{codim}\left(X^{*}\right)-1$. The shadow is also well behaved. Namely, assume that $\operatorname{def}(X)>0$; then

$$
\operatorname{Sh}_{L}(X)=X \Longleftrightarrow \operatorname{Sh}_{H \cap L}(H \cap X)=H \cap X
$$

for any general hyperplane $H \subset \mathbb{P}^{N}$. Note also that if $x \in X$ is a general point and $L=T_{X, x}$, then $\operatorname{Sh}_{L}(X) \neq X$, unless $X$ is a linear space. Indeed, if $X^{*}$ is a hypersurface, this is obvious since $\operatorname{Tan}\left(T_{X, x}, X\right)=x$ for general $x \in X$. If $X^{*}$ is not a hypersurface, take enough general hyperplane sections of $X$ passing through $x$, so that the corresponding dual is a hypersurface.

Main Theorem 1.1.5. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^{N}$ be a linear space such that $\operatorname{Sh}_{X}(L) \neq X$. Then, for all $x \in$ $X_{\text {smooth }}$ such that $x \notin \operatorname{Sh}_{X}(L)$ and such that $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$, the multiplicity in $X^{*}$ of a general hyperplane containing $\left\langle L, T_{X, x}\right\rangle$ is strictly larger than the multiplicity in $X^{*}$ of a general hyperplane containing $L$.

If $X$ is the ruled cubic surface considered in Example 1.1.1 and $L$ is the directrix of $X$, one notices easily that $\operatorname{Sh}_{X}(L)=X$. This shows that the hypothesis $\operatorname{Sh}_{X}(L) \neq$ $X$ can not be withdrawn. Here is an obvious consequence of Main Theorem 1.1.5:
Corollary 1.1.6. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety. Let $L \subset \mathbb{P}^{N}$ be a linear space such that $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$ for general $x \in X$, and such that the lines in $X$ meeting $L$ do not cover $X$. Then, for general $x \in X$, the multiplicity in $X^{*}$ of a general hyperplane containing $\left\langle L, T_{X, x}\right\rangle$ is strictly larger than the multiplicity in $X^{*}$ of a general hyperplane containing $L$.

### 1.2. Variety of multisecant spaces and duals.

Definition 1.2.1. Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety. Let
$S_{X}^{k^{0}}=\left\{\left(x_{0}, \ldots, x_{k}, u\right) \in X \times \cdots \times X \times \mathbb{P}^{N}: \operatorname{dim}\left\langle x_{0}, \ldots, x_{k}\right\rangle=k, u \in\left\langle x_{0}, \ldots x_{k}\right\rangle\right\}$,
and let $S_{X}^{k}$ be its Zariski closure in $X \times \cdots \times X \times \mathbb{P}^{N}$. Denote by $\phi$ the projection onto $\mathbb{P}^{N}$. The variety $S^{k}(X)=\phi\left(S_{X}^{k}\right)$ is the $k$-th secant variety to $X$.
Theorem 1.2.2 (Terracini's lemma [Zak 1993]). Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety, and let $\left(x_{0}, \ldots, x_{k}\right) \in X \times \cdots \times X$, be general points. If $u$ is general in $\left\langle x_{0}, \ldots, x_{k}\right\rangle$, we have the equality

$$
\left\langle T_{X, x_{0}}, \ldots, T_{X, x_{k}}\right\rangle=T_{S^{k}(X), u}
$$

Definition 1.2.3. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety, and let $k$ be an integer such that $S^{k}(X) \neq \mathbb{P}^{N}$. We say that $X$ is dual $k$-defective if $\operatorname{def}\left(S^{k}(X)\right)>t\left(S^{k}(X)\right)$, where $t\left(S^{k}(X)\right)$ is the dimension of the general fiber of the Gauss map of $S^{k}(X)$.

Note that when $X$ is smooth, then dual 0 -defectiveness is the classical dual defectiveness. I don't know if there exist smooth varieties which are dual $k$-defective for some $k \geq 1$, but which are not dual 0 -defective. I believe it would be interesting to find some examples of such varieties.

Note also that the notion of dual $k$-defectiveness seems to be related to that of $R_{k}$ regularity explored in [Chiantini and Ciliberto 2010].

Here is a consequence of the Main Theorem 1.1.5 and Terracini's lemma:
Proposition 1.2.4. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate, smooth, projective variety. Assume moreover that for all $k$ such that $S^{k}(X) \neq \mathbb{P}^{N}$ the variety $X$ is not dual $k-1$-defective. Then, for any such $k$, we have

$$
S^{k}(X)^{*} \subset X_{k+1}^{*}
$$

where $X_{k+1}^{*}$ is the set of points which have multiplicity at least $k+1$ in $X^{*}$.
Proof. The case $k=0$ is the definition of $S^{0}(X)^{*}=X^{*}$. Let $k \geq 1$ be an integer such that $S^{k}(X) \neq \mathbb{P}^{N}$, let $z \in S^{k-1}(X)$ be a general point and $H$ be a general hyperplane containing $T_{S^{k-1}(X), z}$. Let's prove that

$$
\operatorname{Tan}(H, X)=\left\{x \in X: T_{X, x} \subset T_{S^{k-1}(X), z}\right\}
$$

Let $x_{0}, \ldots, x_{k-1}$ be $k$ general points in $\operatorname{Tan}(H, X)$. Let $z^{\prime}$ be a general point in $\left\langle x_{0}, \ldots, x_{k-1}\right\rangle$, by Terracini's lemma we have

$$
T_{S^{k-1}(X), z^{\prime}}=\left\langle T_{X, x_{0}}, \ldots, T_{X, x_{k-1}}\right\rangle
$$

So $z^{\prime} \in \operatorname{Tan}\left(H, S^{k-1}(X)\right)$. But $\operatorname{def}\left(S^{k-1}(X)\right)=t\left(S^{k-1}(X)\right)$ by hypothesis, and this implies

$$
z^{\prime} \in \overline{\left\{y \in S^{k-1}(X)_{\text {smooth }}: T_{S^{k-1}(X), y}=T_{S^{k-1}(X), z}\right\}}
$$

so that $x_{0}, \ldots, x_{k-1} \in\left\{x \in X: T_{X, x} \subset T_{S^{k-1}(X), z}\right\}$.
We now prove that $\operatorname{Sh}_{X}\left(T_{S^{k-1}(X), z}\right) \neq X$. The argument above shows that

$$
\operatorname{Tan}\left(T_{S^{k-1}(X), z}, X\right)=\left\{x \in X: T_{X, x} \subset T_{S^{k-1}(X), z}\right\}
$$

Assume $\operatorname{Sh}_{X}\left(T_{S^{k-1}(X), z}\right)=X$. For all $x^{\prime \prime} \in X$, there is $x^{\prime} \in\left\{x \in X: T_{X, x} \subset T_{S^{k}(X), z}\right\}$ such that the line $\left\langle x^{\prime \prime}, x^{\prime}\right\rangle$ lies in $X$. But since $X$ is smooth, this line $\left\langle x^{\prime \prime}, x^{\prime}\right\rangle$ lies in $T_{X, x^{\prime}}$. So we have $X \subset T_{S^{k-1}(X), z}$, which contradicts the nondegeneracy.

As a consequence of Main Theorem 1.1.5, we get that for a general $x \in X$, the multiplicity in $X^{*}$ of a general hyperplane containing $\left\langle T_{S^{k-1}(X), z}, T_{X, x}\right\rangle$ is strictly larger than the multiplicity in $X^{*}$ of a general hyperplane containing $T_{S^{k-1}(X), z}$. We apply Terracini's lemma to find that $S^{k}(X)^{*} \subset X_{k+1}^{*}$. This concludes the proof.

A stronger result than Proposition 1.2.4 has been stated for the first time in [Zak 2004], but no proof was given there.

In the second part of this paper we present a proof of Main Theorem 1.1.5, while in the third part we discuss some consequences and open questions.

## 2. Proof of the Main Theorem

When $Z \subset \mathbb{P}^{N}$, we denote by $\mathscr{C}_{z}(Z) \subset \mathbb{P}^{N}$ the embedded tangent cone to $Z$ at $z$ and if $H \subset \mathbb{P}^{N}$ is a hyperplane, then $[h]$ is the corresponding point in $\left(\mathbb{P}^{N}\right)^{*}$.
The proof of Main Theorem 1.1.5 is obvious if $L^{\perp} \not \subset X^{*}$. Thus, we only deal with the case where $L^{\perp} \subset X^{*}$. Moreover, we can restrict to the case where $X^{*}$ is a hypersurface. Indeed, assume that $X^{*}$ has codimension $p \geq 2$. Let $z \in L^{\perp}$ and $z_{x} \in$ $\left\langle L, T_{X, x}\right\rangle^{\perp}$ be general points, let $M \subset \mathbb{P}^{N}$ be a general $\mathbb{P}^{N+1-p}$ passing through $x$, let $X^{\prime}=M \cap X$ and $L^{\prime}=M \cap L$. We have $\operatorname{Sh}_{X^{\prime}}\left(L^{\prime}\right) \neq X^{\prime}$ and $\left\langle T_{X^{\prime}, x}, L^{\prime}\right\rangle \neq \mathbb{P}^{N+1-p}$. Moreover, we have

$$
\left(X^{\prime}\right)^{*}=\pi_{M^{\perp}}\left(X^{*}\right),
$$

where $\pi_{M^{\perp}}$ is the projection from $M^{\perp}$ in $\mathbb{P}^{N^{*}}$. Since $M$ is general, the map $\pi_{M^{\perp}}$ is locally an isomorphism around $z_{x}$. Hence

$$
\operatorname{mult}_{z} X^{*}=\operatorname{mult}_{z_{x}} X^{*} \Longleftrightarrow \operatorname{mult}_{\pi_{M^{\perp}}(z)}\left(X^{\prime}\right)^{*}=\operatorname{mult}_{\pi_{M} \perp\left(z_{x}\right)}\left(X^{\prime}\right)^{*} .
$$

Finally, note that $\pi_{M^{\perp}}(z)$ is a general point of $\left(L^{\prime}\right)^{\perp}$ and that $\pi_{M^{\perp}}\left(z_{x}\right)$ is a general point of $\left\langle L^{\prime}, T_{X^{\prime}, x}\right\rangle^{\perp}$. As a consequence, it is sufficient to prove the theorem for $X^{\prime}$, whose dual is a hypersurface.

Let's start with a plan of the proof. We assume that $X^{*}$ has constant multiplicity along a smooth curve $S \subset L^{\perp}$ passing through $\left\langle L, T_{X, x}\right\rangle^{\perp}$ and through a general point of $L^{\perp}$ and we find a contradiction. More precisely:

- We prove that the equimultiplicity of $X^{*}$ along $S$ implies that the family of the tangent cones to $X^{*}$ at the points of $S$ is flat.
- Then, we show that the flatness of the family of the tangent cones to $X^{*}$ at the points of $S$ leads to the flatness of the family of the conormal spaces of these tangent cones. As a consequence, we have $\left|\mathscr{C}_{s}\left(X^{*}\right)\right|^{*} \subset L$ for all $s \in S$.
- Finally, we relate the tangent cone to $X^{*}$ at $z$ to the set of tangent hyperplanes to $X^{*}$ at $z$ (when $z$ is a smooth point of $X^{*}$; this is the reflexivity theorem [Kleiman 1986]). Using the fact that $\operatorname{Sh}_{L}(X) \neq X$, we deduce that $\left|\mathscr{C}_{S}\left(X^{*}\right)\right|^{*} \not \subset$ $L$ for $s \in\left\langle L, T_{X, x}\right\rangle^{\perp}$ and thus a contradiction.
2.1. Normal flatness and Lagrangian specialization principle. Let $S \subset Z \subset \mathbb{P}^{N}$ be two varieties. We recall some properties of the tangent cones $\mathscr{C}_{s}(Z), s \in S$ when $Z$ is equimultiple along $S$.

Definition 2.1.1. Let $S \subset Z$ be two varieties. We say that $Z$ is equimultiple along $S$ if the multiplicity of the local ring $0_{Z, s}$ is constant for $s \in S$.

Proposition 2.1.2 [Hironaka 1964, Corollary 2, p. 197]. Let $Z \subset \mathbb{P}^{N}$ be a hypersurface and $S$ a connected smooth subvariety (not necessarily closed) of $Z$ such that $Z$ is equimultiple along $S$.

Then, for all $s \in S$, there exists an open neighborhood $U$ of $s$ in $S$ containing $s$ and a closed subscheme $\mathscr{G}(Z) \subset \mathbb{P}^{N} \times U$ such that the natural projection $p$ : $\mathscr{G}(Z) \rightarrow U$ is a flat and surjective morphism whose fiber $\mathscr{G}(Z)_{s^{\prime}}$ over any $s^{\prime} \in U$ is $\mathscr{C}_{s^{\prime}}(Z)$.

We assume that our theorem is not true, that is for general $x \in X$, the multiplicity of $X^{*}$ at a general point of $\left\langle L, T_{X, x}\right\rangle^{\perp}$ is equal to the multiplicity at a general point of $L^{\perp}$.

Let [ $h$ ] be a general point of $\left\langle L, T_{X, x}\right\rangle^{\perp}$ and let $S \subset L^{\perp}$ be a smooth (not necessarily closed) connected curve passing through [ $h$ ] and through a general point of $L^{\perp}$. We apply the proposition to $X^{*}$ and $S$. Then there exists a scheme $\mathscr{G}\left(X^{*}\right) \subset \mathbb{P}^{N^{*}} \times S$ such that the natural projection $p: \mathscr{G}\left(X^{*}\right) \rightarrow S$ is a flat and surjective morphism whose fiber over $s \in S$ is the tangent cone to $X^{*}$ at $s$. Let $\Gamma\left(X^{*}\right)=\left|\mathscr{G}\left(X^{*}\right)\right|$. The induced morphism $\Gamma\left(X^{*}\right) \rightarrow S$ is flat and for general $s \in S$ the fiber $\Gamma\left(X^{*}\right)_{s}$ is exactly $\left|\mathscr{C}_{s}\left(X^{*}\right)\right|$.

Now we study the family of the duals of the reduced tangent cones of $X^{*}$ at points of $S$. Applying the Lagrangian specialization principle [Lê and Teissier 1988; Kleiman 1984] to $\Gamma\left(X^{*}\right)$ and $S$, we find:

Theorem 2.1.3. Let $S \subset X^{*}$ be a smooth curve such that $X^{*}$ is equimultiple along $S$. There esists a variety $I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)$ with the following properties.
(i) For general $s \in S$, the following equality holds in $\mathbb{P}^{N} \times \Gamma\left(X^{*}\right)_{s}$ :

$$
I\left(\left|\mathscr{C}_{s}\left(X^{*}\right)\right| / \mathbb{P}^{N^{*}}\right)=I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}
$$

(ii) The morphism $I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right) \rightarrow S$ is flat and surjective.
(iii) For all $s \in S$, the conormal space $I\left(\left|\mathscr{C}_{s}\left(X^{*}\right)\right| / \mathbb{P}^{N^{*}}\right)$ is a union of irreducible components of the reduced fiber $\left|I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}\right|$.
As a consequence, the image in $\mathbb{P}^{N}$ of the fiber $I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}$, for general $s \in S$, is $\left|\mathscr{C}_{s}\left(X^{*}\right)\right|^{*}$. Moreover, for any $s \in S$, the image of the reduced fiber $\left|I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}\right|$ contains $\left|\mathscr{C}_{s}\left(X^{*}\right)\right|^{*}$.
2.2. Polar varieties and duals of tangent cones. We discuss an extension of the reflexivity theorem proved in [Lê and Teissier 1988]. The main results of this section will be applied to $X^{*}$ when it is a hypersurface, so we restrict our study to that case.

Definition 2.2.1. Let $Z \subset \mathbb{P}^{N}$ be a reduced and irreducible hypersurface and let $D \subset \mathbb{P}^{N}$ be a linear space. The polar variety of $Z$ associated to $D$, which we denote by $P(Z, D)$, is the closure of the set $\left\{z \in Z_{\text {smooth }}: D \subset T_{Z, z}\right\}$.

If $D=\varnothing$ (that is, $D$ has dimension -1 ), then we put $P(Z, D)=Z$.
Remark 2.2.2. If $Z$ is normal, if $u=\left[u_{0}, \ldots, u_{N}\right]$ in an homogeneous system of coordinates on $\mathbb{P}^{N}$ and $f$ is an equation of $Z$ in this system then $P(Z, u)$ is given by the equations $f=0$ and $u_{0} \partial f / \partial x_{0}+\cdots+u_{N} \partial f / \partial x_{N}=0$.

If $Z$ is not normal, then all irreducible components of $Z_{\text {sing }}$ which are of dimension $N-2$ are irreducible components of the scheme defined by $f=0$ and $u_{0} \partial f / \partial x_{0}+\cdots+u_{N} \partial f / \partial x_{N}=0$, but they are not irreducible components of $P(X, u)$.

Proposition 2.2.3. Let $Z \subset \mathbb{P}^{N}$ be a reduced, irreducible hypersurface and let $D \subset \mathbb{P}^{N}$ be a general linear space of dimension $k$. Then $P(Z, D)$ is empty or of codimension $k+1$ in $Z$.

We state a result of Lê and Teissier which relates the duals of the tangent cones at $z$ of some polar varieties of $Z$ with the tangency locus of $z^{\perp}$ with $Z^{*}$. See [Lê and Teissier 1988, Proposition 2.2.1]. For any $z \in Z$, recall that $\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)$ is the tangency locus of $z$ along $Z^{*}$ (see conormal diagram on page 2).
Theorem 2.2.4. Let $Z \subset \mathbb{P}^{N}$ be a reduced and irreducible hypersurface and let $z \in Z$ be a point.
(i) The dual of $\left|G_{z}(Z)\right|$ is a union of reduced spaces underlying (possibly embedded) components of $\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)$.
(ii) Any irreducible component of $\left|\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)\right|$ is dual to an irreducible component of $\left|\mathscr{C}_{z}(P(Z, D))\right|$ for general $D \in \mathbb{G}(k, N)$ and for some integer $k \in$ $\{-1, \ldots, N-2\}$.

Remark 2.2.5. Part (ii) of the theorem has to be explained. Assume that there is an irreducible component (say $T$ ) of $\left|\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)\right|$ which is not dual to an irreducible component of $\left|\mathscr{C}_{z}(Z)\right|$. Then, there is $k \in\{0, \ldots, N-2\}$ such that for general $D \in \mathbb{G}(k, N)$, we have $z \in P(Z, D)$. Moreover, as $D$ varies in a dense open subset of $\mathbb{G}(k, D)$, the cones $\mathscr{C}_{z}(P(D, Z))$ have a fixed irreducible component in common whose reduced locus is $T^{*}$.

Note also that if $z \in Z_{\text {smooth }}$ then for $k \geq 0$ and for $D$ general in $\mathbb{G}(k, N)$, we have $z \notin P(Z, D)$. As a consequence of the (ii) of the above theorem, we find
$\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)=T_{Z, z}^{\perp}$ for $z \in Z_{\text {smooth }}$. This is the way the (obvious corollary of the) reflexivity theorem is often stated.

When $\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)$ is irreducible, one may expect $\left|\mathscr{C}_{z}(Z)\right|^{*}=\left|\operatorname{Tan}\left(z^{\perp}, Z^{*}\right)\right|$. But this is not true:
Example 2.2.6. Let $X \subset \mathbb{P}^{4}$ be the smooth ruled surface of degree 3 considered in example 1.1.1 and let $X^{*}$ its dual. The hypersurface $X^{*}$ has also degree 3 and its singular locus is a $\mathbb{P}^{2}$, the dual of the exceptional section of $X$ (which we denote by $L$ ). Let $C \subset L^{\perp}=X_{\text {sing }}^{*}$ be the conic corresponding to the hyperplanes which are tangent to $X$ along a ruling of $X$ and let $z \in C$.

The tangent cone $\mathscr{C}_{z}\left(X^{*}\right)$ is a doubled $\mathbb{P}^{3}$ so that $\left|\mathscr{C}_{z}\left(X^{*}\right)\right|^{*} \neq \operatorname{Tan}\left(z^{\perp}, X\right)$. We also note that the scheme-theoretic tangency locus of $z^{\perp}$ along $X$ is a line with an embedded point. The embedded point is dual to $\left|\mathscr{C}_{z}\left(X^{*}\right)\right|$ and the line is dual to $\left|\mathscr{C}_{z}\left(P\left(X^{*}, u\right)\right)\right|$, for general $u \in \mathbb{P}^{4^{*}}$.
Notations 2.2.7. Let $f: Y \rightarrow T$ be a quasiprojective morphism between quasiprojective schemes, let $T^{\prime} \subset T$ be a smooth variety and let $s \in T^{\prime}$ be any point. Let $Y_{1}, \ldots, Y_{m}$ be the irreducible components of $f^{-1}\left(T^{\prime}\right)$ such that the restrictions

$$
\left.f\right|_{Y_{i}}: Y_{i} \rightarrow T^{\prime}
$$

are surjective. Define the scheme

$$
\operatorname{limflat}_{\left\{t \rightarrow s, t \in T^{\prime}\right\}} f^{-1}(t):=\left.f\right|_{Y_{1} \cup \ldots \cup Y_{m}} ^{-1}(s)
$$

If $\operatorname{dim}\left(T^{\prime}\right)=1$ and the $Y_{i}$ are all reduced, this is the classical flat limit taken along a smooth curve. If $\left.f\right|_{f^{-1}\left(T^{\prime}\right)}: f^{-1}\left(T^{\prime}\right) \rightarrow T^{\prime}$ is flat, then

$$
\operatorname{limflat}_{\left\{t \rightarrow s, t \in T^{\prime}\right\}} f^{-1}(t)=\left.f\right|_{f^{-1}\left(T^{\prime}\right)} ^{-1}(s)
$$

Proof of Main Theorem 1.1.5. We recall the setting for the convenience of the reader. The projective variety $X \subset \mathbb{P}^{N}$ is irreducible and nondegenerate. The linear space $L \subset \mathbb{P}^{N}$ is such that $\operatorname{Sh}_{X}(L) \neq X$ and $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$ for all $x \in X_{\text {smooth }}$. We want to prove that for all $x \in X_{\text {smooth }}$ such that $x \notin \operatorname{Sh}_{X}(L)$, the multiplicity in $X^{*}$ of a general hyperplane containing $\left\langle L, T_{X, x}\right\rangle$ is strictly greater than that of a general hyperplane containing $L$.

The result is obvious if $L^{\perp} \not \subset X^{*}$ and we have already seen that we can restrict to the case where $X^{*}$ is a hypersurface. So we only consider the case where $L^{\perp} \subset X^{*}$ and $X^{*}$ is a hypersurface and we assume that our result is not true. Let $x \in X_{\text {smooth }}$ with $x \notin \mathrm{Sh}_{X}(L)$ and let [ $h$ ] be a general point in $\left\langle L, T_{X, x}\right\rangle^{\perp}$. By the results of the previous section, there exists a smooth (not necessarily closed) curve $S \subset L^{\perp}$ with $[h] \in S$ and a flat morphism

$$
I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right) \rightarrow S
$$

whose fiber $I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}$ is the conormal space of $\left|\mathscr{C}_{S}\left(X^{*}\right)\right|$, for general $s \in S$. Further, the conormal space of $\left|\mathscr{C}_{s}\left(X^{*}\right)\right|$ is included in $\left|I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right)_{s}\right|$ for all $s \in S$.

Theorem 2.2.4(i) implies that

$$
\left|\mathscr{C}_{s}\left(X^{*}\right)\right|^{*} \subset p\left(\left|q^{-1}(s)\right|\right)
$$

for all $s \in S$, where $p$ and $q$ are as in the conormal diagram of page 2 . The flatness of $I_{S}\left(\Gamma\left(X^{*}\right) / \mathbb{P}^{N^{*}} \times S\right) \rightarrow S$ gives the inclusion

$$
\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*} \subset p\left(\operatorname{limflat}_{\{s \rightarrow[h], s \in S\}}\left|q^{-1}(s)\right|\right)
$$

By Definition 1.1.2, the right-hand side is contained in $\operatorname{Tan}(L, X) \subset L$.
Let $\mathscr{F}$ be an irreducible component of $\operatorname{Tan}(H, X)$ passing through $x$. By Theorem 2.2.4, there is an integer $k \in\{-1, \ldots, N-2\}$ such that $|\mathscr{F}|$ is dual to an irreducible component of $\left|\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right)\right|$, for general $D \in \mathbb{G}(k, N)$. Since $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*} \subset L$, we have $k \geq 0$.

Let $x_{0} \in \mathscr{F}$ be a general point. Duality implies $T_{\left|\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right)\right|, z} \subset x_{0}^{\perp}$ for some general $z$ in the irreducible component of $\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right.$ ) whose reduced locus is $|\mathscr{F}|^{*}$. Note that $\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right) \subset \mathscr{C}_{[h]}\left(X^{*}\right)$. Let $T_{\left[\mathscr{C}_{[h]}\left(X^{*}\right) \mid, z\right.}$ be a limit of tangent spaces to $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|$ at $z$. The point $z$ is general in $\left|\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right)\right|$, so $T_{\left[\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right) \mid, z\right.} \subset T_{\left[\mathscr{C}_{[h]}\left(X^{*}\right) \mid, z\right.}$.

As a consequence of this, we have $T_{\left[\mathscr{C}_{[h]}\left(P\left(X^{*}, D\right)\right) \mid, z\right.} \subset x_{0}^{\perp} \cap T_{\mathscr{C}_{[h]}\left(X^{*}\right), z}$. That is,

$$
\left\langle x_{0}, T_{\left[\mathscr{C}_{[h]}\left(X^{*}\right) \mid, z\right.}^{\perp}\right\rangle \subset \mathscr{F} \subset X
$$

But $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*} \subset \operatorname{Tan}(L, X)$, so $T_{\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|, z}^{\perp} \in \operatorname{Tan}(L, X)$, and the inclusion above says that $x_{0} \in \operatorname{Sh}_{X}(L)$. This is a contradiction.

## 3. Corollaries and open questions

We present here some corollaries of the Main Theorem and related open questions.
3.1. Zak's conjecture on varieties with minimal codegree. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety. We recall, following Zak, that the order of $X$ is ord $X=\min \left\{k, S^{k-1}(X)=\mathbb{P}^{N}\right\}$ and the $k$-th secant-defect is $\delta_{k}=$ $\operatorname{dim} X+\operatorname{dim} S^{k-1}(X)+1-\operatorname{dim} S^{k}(X)$, for all $k \leq \operatorname{ord} X-1$.

Zak [1993] proved an important result related to secant defects.
Theorem 3.1.1 (Zak's superadditivity theorem). Let $X \subset \mathbb{P}^{N}$ an irreducible, nondegenerate projective variety such that $\delta_{1}>0$. For all $k \leq$ ord $X-1$, we have the inequality

$$
\delta_{k} \geq \delta_{k-1}+\delta_{1}
$$

The varieties on the boundary are called Scorza varieties. More precisely:

Definition 3.1.2. An irreducible, smooth, nondegenerate projective variety $X \subset$ $\mathbb{P}^{N}$ is a Scorza variety if the following conditions hold:
(i) $\delta_{1}>0$ and $N>2 n+1-\delta_{1}$,
(ii) $\delta_{k}=\delta_{k-1}+\delta_{1}$ for all $k \leq \operatorname{ord} X-1$,
(iii) ord $X-1=\left[\operatorname{dim} X / \delta_{1}\right]$, where [ ] denotes the integral part.

Theorem 3.1.3 (Classification of Scorza varieties [Zak 1993]). Any Scorza variety $X$ is of one of the following types:
(i) $X=v_{2}\left(\mathbb{P}^{n}\right) \subset \mathbb{P}^{n(n+3) / 2}\left(2^{\text {nd }}\right.$ Veronese $)$ and $\operatorname{deg} X^{*}=n+1$;
(ii) $X=\mathbb{P}^{n} \times \mathbb{P}^{n} \subset \mathbb{P}^{n(n+2)}$ and $\operatorname{deg} X^{*}=n+1$;
(iii) $X=\mathbb{G}(1,2 n+1) \subset \mathbb{P}\left(\Lambda^{2} \mathbb{C}^{2 n+2}\right)$ and $\operatorname{deg}\left(X^{*}\right)=n+1$;
(iv) $X \subset \mathbb{P}^{26}$ is the 16 -dimensional variety corresponding to the orbit of highest weight vector in the lowest nontrivial representation of the group of type $E_{6}$ and $\operatorname{deg} X^{*}=3$.

In [Zak 2004] an important consequence of the assertion $S^{k}(X)^{*} \subset X_{k+1}^{*}$ (where $X_{k}^{*}$ is the set of points of multiplicity at least $k$ in $X^{*}$ ) was discovered. We state that result in the setting where we are able to prove it.

Proposition 3.1.4. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate, smooth, projective variety. Assume that $X$ is not $k$ dual defective for $k<\operatorname{ord} X-1$, then

$$
\operatorname{deg} X^{*} \geq \operatorname{ord} X
$$

Proof. With the assumptions above, Proposition 1.2.4 implies that there is a point of multiplicity ord $X-1$ in $X^{*}$. Since $X$ is nondegenerate, its dual is not a cone and so $\operatorname{deg} X^{*} \geq \operatorname{ord} X$.

If $X$ is a Scorza variety then $\operatorname{deg} X^{*}=$ ord $X$. The converse statement in conjectured in [Zak 2004]. We formulate the conjecture in the setting where we can prove the inequality: $\operatorname{deg} X^{*} \geq \operatorname{ord} X$.

Conjecture 3.1.5 [Zak 2004]. Let $X \subset \mathbb{P}^{N}$ be an irreducible, smooth, nondegenerate, projective variety. Assume that $X$ is not $k$ dual defective for all $k<\operatorname{ord} X$ and that $\operatorname{deg} X^{*}=\operatorname{ord} X+1$, then $X$ is a hyperquadric or a Scorza variety.

It is proved in [Zak 1993], without any hypothesis on the dual defectiveness of $X$, that smooth varieties with $\operatorname{deg}\left(X^{*}\right)=3$ and ord $X=3$ are Severi varieties. In particular, they are Scorza varieties. Note, however, that the smoothness assumption seems to be necessary in his proof. I believe it would be very interesting to have a classification of all varieties whose duals have degree 3 .
3.2. Varieties with unexpected equisingular linear spaces. We come back to our usual setting. Let $L \subset \mathbb{P}^{N}$ be a linear space such that for all $x \in X_{\text {smooth }}$, we have $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$. We have seen in example 1.1.1 that a hyperplane containing the join $\left\langle L, T_{X, x}\right\rangle$ may have the same multiplicity in $X^{*}$ as the general hyperplane containing $L$, even if $x$ is a general point of $X$. The following definition is convenient to describe this situation.

Definition 3.2.1. Let $X \subset \mathbb{P}^{N}$ be an irreducible, nondegenerate projective variety such that $X^{*}$ is a hypersurface. Let $L \subset \mathbb{P}^{N}$ be a linear space such that for all $x \in X_{\text {smooth }}$, we have $\left\langle L, T_{X, x}\right\rangle \neq \mathbb{P}^{N}$. We say that $L^{\perp}$ is an unexpected equisingular linear space in $X^{*}$ if for all $x \in X_{\text {smooth }}$, the general hyperplane containing $\left\langle L, T_{X, x}\right\rangle$ has the same multiplicity in $X^{*}$ as the general hyperplane containing $L$.

The variety in Example 1.1 .1 is rather special since it is a scroll surface (see [Zak 2004] for interesting discussions about this variety). It is not a coincidence that the directrix of this variety is an unexpected equisingular linear space in its dual. Indeed, we have:
Theorem 3.2.2. Let $X \subset \mathbb{P}^{N}$ be an irreducible, smooth, nondegenerate projective variety such that $X^{*}$ is a hypersurface. Let $L \subset X$ be a linear space with $\operatorname{dim}(L)=$ $\operatorname{dim}(X)-1$. Assume that $L^{\perp}$ is an unexpected equisingular linear space in $X^{*}$ such that mult $L_{\perp^{\perp}} X^{*}=2$. Then $X$ is the cubic scroll surface in $\mathbb{P}^{4}$.

Here mult $L_{L^{\perp}} X^{*}$ denotes the multiplicity in $X^{*}$ of a general point of $L^{\perp}$. Before diving into the proof of Theorem 3.2.2, we describe the tangency locus of any point $[h] \subset X^{*}$, such that mult $_{[h]} X^{*}=2$.
Proposition 3.2.3. Let $X \subset \mathbb{P}^{N}$ be a smooth, irreducible, nondegenerate projective variety such that $X^{*}$ is a hypersurface. Let $[h] \in X^{*}$ be such that $\operatorname{mult}_{[h]} X^{*}=2$. The scheme theoretic tangency locus of $H$ with $X$ is either
(i) an irreducible hyperquadric and in this case $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*}=\operatorname{Tan}(H, X)$,
(ii) the union of two (not necessarily distinct) linear spaces, or
(iii) a linear space with at least one embedded component.

We postpone the proof of this result to the Appendix.
Proof of Theorem 3.2.2. Let $H$ be a general hyperplane containing $L$. We have $H \cap X=L \cup D_{H}$, where $D_{H}$ is a divisor such that

$$
D_{H} \cap L=\operatorname{Tan}(H, X)
$$

Let $x \in X$ be a general point and let $H_{x}$ be a general hyperplane containing $\left\langle L, T_{X, x}\right\rangle$. Then $\operatorname{Tan}\left(H_{x}, X\right)$ contains $x$ and

$$
\xi:=p\left(\operatorname{limflat}_{\left\{[h] \rightarrow\left[h_{x}\right],[h] \in L^{\perp}\right\}} q^{-1}([h])\right) .
$$

By hypothesis, we have

$$
\operatorname{mult}_{\left[h_{x}\right]} X^{*}=\operatorname{mult}_{[h]} X^{*}=2,
$$

for all $[h] \in L^{\perp}$. Proposition 3.2.3 hence implies that the irreducible component of $\operatorname{Tan}\left(H_{x}, X\right)$ containing $x$, which we denote by $R_{H_{x}}$, also contains $\xi$. Moreover, $\xi \subset L$, so $\operatorname{dim} R_{H_{x}}>\operatorname{dim} \xi$, for general $[h] \in L^{\perp}$. As a consequence, $\operatorname{dim} R_{H_{x}}=$ $n-1$.

On the other hand, since

$$
\operatorname{mult}_{\left[h_{x}\right]} X^{*}=\operatorname{mult}_{[h]} X^{*}=2,
$$

for all $[h] \in L^{\perp}$, we have $\left|\mathscr{C}_{\left[h_{x}\right]}\left(X^{*}\right)\right|^{*} \neq\left|R_{H_{x}}\right|$. We apply again Proposition 3.2.3 and we find that $\left|R_{H_{x}}\right|$ is necessarily a linear space of dimension $n-1$. Thus,

$$
\operatorname{dim}\left\langle L, T_{X, x}\right\rangle=n+1
$$

Note that Bertini's theorem implies that

$$
R_{H_{x}} \subset\left\langle L, T_{X, x}\right\rangle \cap X,
$$

for general $H_{x}$ containing $\left\langle L, T_{X, x}\right\rangle$. As a consequence $R_{H_{x}}$ is an irreducible component of $\left\langle L, T_{X, x}\right\rangle \cap X$, for general $H_{x}$. Thus $R_{H_{x}}$ does not depend on $H_{x}$, for general $H_{x}$ containing $\left\langle L, T_{X, x}\right\rangle$. We deduce that $\left\langle L, T_{X, x}\right\rangle$ is tangent to $X$ along a linear space of dimension $n-1$. By the theorem on tangencies, we have $n-1 \leq 1$, that is $n=2$ (obviously, $X$ is not a curve). So $X \subset \mathbb{P}^{N}$ is a nondegenerate surface containing a distinguished line $L$, such that for general $x \in X$, there is a $\mathbb{P}^{3}$ tangent to $X$ along a line passing through $x$ and meeting $L$. This means that $X$ is the projection of a scroll of type $S_{1, d-1}$. By hypothesis, we have mult $L_{\perp^{\perp}} X^{*}=2$, hence of [Ciliberto et al. 2008, Proposition 1.6] implies that $X=S_{1,2} \subset \mathbb{P}^{4}$.

## Appendix: Tangency loci of points of multiplicity 2 in the dual

The goal of this appendix is to prove the following proposition.
Proposition 3.2.3. Let $X \subset \mathbb{P}^{N}$ be a smooth, irreducible, nondegenerate projective variety such that $X^{*}$ is a hypersurface. Let $[h] \in X^{*}$ be such that $\operatorname{mult}_{[h]} X^{*}=2$. The scheme theoretic tangency locus of $H$ with $X$ is either
(i) an irreducible hyperquadric and in this case $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*}=\operatorname{Tan}(H, X)$,
(ii) the union of two (not necessarily distinct) linear spaces, or
(iii) a linear space with at least one embedded component.

Example A.1. All three cases are encountered in nature:
(i) If $X=v_{2}\left(\mathbb{P}^{2}\right) \subset \mathbb{P}^{5}$, then for all $[h] \in v_{2}\left(\mathbb{P}^{2 *}\right) \subset X^{*}$, we have mult ${ }_{[h]} X^{*}=2$ and $\operatorname{Tan}(H, X)$ is a smooth conic.
(ii) If $X$ is a complete intersection of large multidegree and large codimension, then there are points $\left[h_{1}\right],\left[h_{2}\right] \in X^{*}$ such that $\operatorname{mult}_{\left[h_{i}\right]} X^{*}=2$ and $\operatorname{Tan}\left(H_{1}, X\right)$ is exactly two distinct points, whereas $\operatorname{Tan}\left(H_{2}, X\right)$ is a single double point.
(iii) If $X$ is the cubic scroll of Example 1.1.1, then there is a conic $C \subset X^{*}$, such that for all $[h] \in C$, we have $\operatorname{mult}_{[h]} X^{*}=2$ and $\operatorname{Tan}(H, X)$ is a line with an embedded point.

A doubled linear space will be considered as the union of two (not distinct) linear spaces. By Theorem 2.2.4, we know that the irreducible components of $\operatorname{Tan}(H, X)$ are dual to irreducible components of the reduced spaces underlying some $\mathscr{C}_{[h]}\left(P\left(X^{*}, D_{k}\right)\right)$ for general $D_{k} \in \mathbb{G}(k, N)$. When mult ${ }_{[h]} X^{*}=2$, the cones $\mathscr{C}_{[h]}\left(P\left(X^{*}, D_{k}\right)\right)$ are rather easy to describe. Let's start with some notation.
Notations A.2. Let $Z \subset \mathbb{P}^{N}$ be a reduced and irreducible hypersurface. Let $D \in$ $\mathfrak{G}(k, N)$ and let $f_{Z}$ be an equation for $Z$ in some coordinate system of $\mathbb{P}^{N}$. We denote by $P\left(f_{Z}, D\right)$ the subscheme of $\mathbb{P}^{N}$ whose ideal is generated by the equations

$$
u_{0} \frac{\partial f_{Z}}{\partial t_{0}}+\cdots+u_{N} \frac{\partial f_{Z}}{\partial t_{N}}
$$

for $u=\left[u_{0}, \ldots, u_{N}\right]$ varying in $D$.
Let $D \in \mathbb{G}(k, N)$ be a general $k$-plane. Note that if $\operatorname{dim}\left(Z_{\text {sing }}\right)<\operatorname{dim} P(Z, D)$ (that is $\operatorname{dim} Z_{\text {sing }} \leq N-k-3$ ), then $P(Z, D)=P\left(f_{Z}, D\right) \cap Z$. In the other case, the irreducible components of maximal dimension of $Z_{\text {sing }}$ are irreducible components of $P\left(f_{Z}, D\right) \cap Z$.

Lemma A.3. Let $Z \subset \mathbb{P}^{N}$ be an irreducible and reduced hypersurface. Let $z \in Z$ and let $k \in\{-1, \ldots, N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have
(1) $z \notin P(Z, D)$, or
(2) $\operatorname{mult}_{z} P(Z, D)=\operatorname{mult}_{z}(Z) . \operatorname{mult}_{z} P\left(f_{Z}, D\right)$, if $\operatorname{dim}\left(Z_{\text {sing }}^{(z)}\right)<\operatorname{dim} P(Z, D)$, where $Z_{\text {sing }}^{(z)}$ is an irreducible component of $Z_{\text {sing }}$ of maximal dimension passing through $z$, or
(3) $\operatorname{mult}_{z} P(Z, D)<\operatorname{mult}_{z}(Z) . \operatorname{mult}_{z} P\left(f_{Z}, D\right)$, if $\operatorname{dim}\left(Z_{\text {sing }}^{(z)}\right) \geq \operatorname{dim} P(Z, D)$, where $Z_{\mathrm{sing}}^{(z)}$ is an irreducible component of $Z_{\text {sing }}$ of maximal dimension passing through $z$.
Proof. If $z \in P(Z, D)$ for general $D \in \mathbb{G}(k, N)$, we will prove the lemma only in the case $P\left(f_{Z}, D\right)$ is smooth at $z$, for two reasons. The general case is obtained by the same methods, this is only more technical, and we will use the result only in the case $P\left(f_{Z}, D\right)$ is smooth at $z$.

Moreover if $z \in P(Z, D)$ for general $D$, we will only concentrate on the case $\operatorname{dim}\left(Z_{\text {sing }}^{(s)}\right)<\operatorname{dim} P(Z, D)$. In this case, we have locally around $z$ the equality
$P(Z, D)=P\left(f_{Z}, D\right) \cap Z$ for general $D \in \mathbb{G}(k, N)$. The situation where an irreducible component $Z_{\text {sing }}$ containing $z$ is an irreducible component of $P\left(f_{Z}, D\right) \cap$ $Z$ - this is case (3) of the lemma - is dealt with exactly in the same way.

Now, we work locally around $z$, so that $P\left(f_{Z}, D\right) \cap Z=P(Z, D) \subset \mathbb{A}^{N}$, for general $D \in \mathbb{G}(k, N)$. Let $\left(Z_{i}\right)_{i \in I}$ be a stratification of $Z$ such that $Z_{i}$ is smooth and $Z$ is normally flat along $Z_{i}$, for all $i \in I$. Such a stratification exists, due to the open nature of normal flatness (see [Hironaka 1964, Chapter II]). Consider the Gauss map $G: Z \rightarrow \mathbb{P}^{N^{*}}$. It restricts to a map $G_{i}: Z_{i} \rightarrow \mathbb{P}^{N^{*}}$. We have

$$
P\left(f_{Z}, D\right) \cap Z=P(Z, D)=G^{-1}\left(D^{\perp}\right)
$$

so that $P\left(f_{Z}, D\right) \cap Z_{i}=G_{i}^{-1}\left(D^{\perp}\right)$, for all $i$.
Now, we apply Kleiman's transversality theorem to find that for all $i$ and for general $D \in \mathbb{G}(k, N)$, the inverse images $G_{i}^{-1}\left(D^{\perp}\right)$ are either empty or smooth of the expected dimension.

Let $i$ such that $z$ is in $Z_{i}$. If $z \notin G_{i}^{-1}\left(D^{\perp}\right)$ for general $D \in \mathbb{G}(k, N)$, then $z \notin P(Z, D)$ and we are in the case 1 of the lemma. Otherwise, $z$ is a smooth point of $G_{i}^{-1}\left(D^{\perp}\right)$, so $T_{P\left(f_{Z}, D\right), z}$ and $T_{Z_{i}, z}$ are transverse.

Assume that $\operatorname{mult}_{z} P(Z, D)>\operatorname{mult}_{z} Z . \operatorname{mult}_{z} P\left(f_{Z}, D\right)$. Since $P\left(f_{Z}, D\right)$ is smooth at $z$, this implies that $T_{P\left(f_{Z}, D\right), z}$ and $\mathscr{C}_{z}(Z)$ are not transverse. In particular, the linear spaces $T_{P\left(f_{Z}, D\right), z}$ and $\operatorname{Vert}\left(\mathscr{C}_{z}(Z)\right)$ are not transverse (here $\operatorname{Vert}\left(\mathscr{C}_{z}(Z)\right)$ is the vertex of the cone $\left.\mathscr{C}_{z}(Z)\right)$. But $Z$ is normally flat along $Z_{i}$, so we have $T_{Z_{i}, z} \subset \operatorname{Vert}\left(\mathscr{C}_{z}(Z)\right)$ (see [Hironaka 1964, Theorem 2, p. 195]). This is a contradiction.

Corollary A.4. Let $Z \subset \mathbb{P}^{N}$ be a reduced, irreducible hypersurface. Let $z \in Z$ such that mult $_{z} Z=2$ and let $k \in\{-1, \ldots, N-2\}$. Then, for general $D \in \mathbb{G}(k, N)$, we have

$$
\operatorname{mult}_{z} P(Z, D) \leq 2
$$

Proof. The result is obvious for $k=-1$, since in this case $P(Z, D)=Z$. Assume that $k \geq 0$ and let $D \in \mathbb{G}(k, N)$ be a general $k$-plane. Let $u \in D$ be a general point ans let $\pi_{u}$ be the projection from $u$. Then, the projections

$$
\left.\pi_{u}\right|_{P(Z, u)}: P(Z, u) \rightarrow \pi_{u}(P(Z, u))
$$

and

$$
\left.\pi_{u}\right|_{P(Z, D)}: P(Z, D) \rightarrow \pi_{u}(P(Z, D))
$$

are locally isomorphisms around $z$. Moreover, we have the following equality (see [Teissier 1982]):

$$
\pi_{u}(P(Z, D))=P\left(\pi_{u}(P(Z, u)), \pi_{u}(D)\right)
$$

As a consequence, it is sufficient to prove the result for $k=0$. But in this case, this is an obvious application of the lemma above. Indeed, for general $u \in \mathbb{P}^{N}$,

$$
\operatorname{mult}_{z} P\left(f_{Z}, u\right)=\operatorname{mult}_{z} Z-1=1
$$

We also need the following result.
Proposition A.5. Let $X \subset \mathbb{P}^{N}$ be an irreducible projective variety such that $X^{*}$ is a hypersurface. Let $[h] \in X^{*}$ be such that $\operatorname{Tan}(H, X)$ has $m$ components (some of which may be embedded components), then there exists $k \in\{-1, \ldots, N-2\}$, such that for general $D \in G(k, N)$, we have

$$
\operatorname{mult}_{[h]} P\left(X^{*}, D\right) \geq m .
$$

Proof. We only prove the result when $\operatorname{Tan}(H, X)$ is reduced and pure dimensional. The general case is done using the same ideas; it's just more technical.

Assume that

$$
\operatorname{Tan}(H, X)=Y_{1} \cup \cdots \cup Y_{m}
$$

where the $Y_{i}$ have the same codimension, say $c$. Let $D \subset \mathbb{P}^{N^{*}}$ be a general $\mathbb{P}^{N-1-c}$. Then

$$
\pi_{D}\left(P\left(X^{*}, D\right)\right)=\left(D^{\perp} \cap X\right)^{*}
$$

where $\pi_{D}$ is the projection from $D$. Moreover, we have $[h] \in P\left(X^{*}, D\right)$ and

$$
\operatorname{Tan}\left(D^{\perp} \cap H, D^{\perp} \cap X\right)=D^{\perp} \cap \operatorname{Tan}(H, X)
$$

As a consequence, $\operatorname{Tan}\left(D^{\perp} \cap H, D^{\perp} \cap X\right)$ is a 0 -dimensional scheme of degree at least $m$. In this case, it is clear that

$$
\operatorname{mult}_{\pi_{D}([h])} \pi_{D}\left(P\left(X^{*}, D\right)\right) \geq m
$$

On the other hand, since $D$ is general, the morphism

$$
\pi_{D}: P\left(X^{*}, D\right) \rightarrow \pi_{D}\left(P\left(X^{*}, D\right)\right)
$$

is locally an isomorphism around [ $h$ ], so that

$$
\operatorname{mult}_{[h]} P\left(X^{*}, D\right) \geq m
$$

Proof of Proposition 3.2.3. Let $T_{1} \cup \cdots \cup T_{m}$ be the decomposition of $\operatorname{Tan}(H, X)$ into irreducible components. If $m \geq 3$, then Proposition A. 5 implies that $\operatorname{mult}_{[h]}\left(X^{*}\right) \geq$ 3 , this is impossible, so that $m \leq 2$.

Assume that $m=2$. The proof of Proposition A. 5 shows that these two irreducible components are scheme-theoretically linear spaces.

Assume that $m=1$ and let $k \in\{-1, \ldots, N-2\}$ such that $T_{1}$ is dual to some irreducible components of the reduced space underlying $\mathscr{C}_{[h]} P\left(X^{*}, D\right)$, for general $D \in \mathbb{G}(k, N)$. By Corollary A.4, the cone $\mathscr{C}_{[h]} P\left(X^{*}, D\right)$ is either a hyperquadric
or a linear space. Assume that it is an irreducible hyperquadric. If $k \geq 0$, we know by Theorem 2.2.4 that $\left|\mathscr{G}_{[h]}\left(X^{*}\right)\right|^{*}$ is the reduced space underlying some embedded component of $\operatorname{Tan}(H, X)$. Taking $q=\operatorname{dim} \operatorname{Tan}(H, X)$ general hyperplane sections of $\operatorname{Tan}(H, X)$ passing through $\left|\mathscr{C}_{[h]}\left(X^{*}\right)\right|^{*}$, we see as in the proof of Proposition A. 5 that for general $D^{\prime} \in \mathbb{G}(q-1, N)$, we have

$$
\operatorname{mult}_{[h]} P\left(X^{*}, D^{\prime}\right) \geq 3
$$

This is impossible by Corollary A.4. Thus, if $\mathscr{C}_{[h]} P\left(X^{*}, D\right)$ is an irreducible hyperquadric, then $k=-1$, and we are in the case 1 of the proposition.

Finally, if $\mathscr{C}_{[h]} P\left(X^{*}, D\right)$ is a the union of two linear spaces or a unique linear space, then we are in case 2 or 3 of the proposition. This concludes the proof of Proposition 3.2.3.

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## References

[Chiantini and Ciliberto 2010] L. Chiantini and C. Ciliberto, "On the dimension of secant varieties", J. Eur. Math. Soc. 12:5 (2010), 1267-1291. MR 2011m:14088 Zbl 1201.14038
[Ciliberto et al. 2008] C. Ciliberto, F. Russo, and A. Simis, "Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian", Adv. Math. 218:6 (2008), 1759-1805. MR 2009j:14056 Zbl 1144.14009
[Hironaka 1964] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero, I", Ann. of Math. (2) 79 (1964), 109-203. MR 33 \#7333
[Kleiman 1984] S. L. Kleiman, "About the conormal scheme", pp. 161-197 in Complete intersections (Acireale, 1983), edited by S. Greco and R. Strano, Lecture Notes in Math. 1092, Springer, Berlin, 1984. MR 87g:14060 Zbl 0547.14031
[Kleiman 1986] S. L. Kleiman, "Tangency and duality", pp. 163-225 in Proceedings of the 1984 Vancouver conference in algebraic geometry, edited by J. Carrell et al., CMS Conf. Proc. 6, Amer. Math. Soc., Providence, RI, 1986. MR 87i:14046 Zbl 0601.14046
[Lê and Teissier 1988] Lê D. T. and B. Teissier, "Limites d'espaces tangents en géométrie analytique", Comment. Math. Helv. 63:4 (1988), 540-578. MR 89m:32025 Zbl 0658.32010
[Teissier 1982] B. Teissier, "Variétés polaires, II: multiplicités polaires, sections planes, et conditions de Whitney", pp. 314-491 in Algebraic geometry (La Rábida, 1981), edited by J. M. Aroca et al., Lecture Notes in Math. 961, Springer, Berlin, 1982. MR 85i:32019 Zbl 0585.14008
[Zak 1993] F. L. Zak, Tangents and secants of algebraic varieties, Transl. Math. Monographs 127, American Mathematical Society, Providence, RI, 1993. MR 94i:14053 Zbl 0795.14018
[Zak 2004] F. L. Zak, "Determinants of projective varieties and their degrees", pp. 207-238 in Algebraic transformation groups and algebraic varieties, edited by V. L. Popov, Encyclopaedia Math. Sci. 132, Springer, Berlin, 2004. MR 2005h:14128 Zbl 1063.14058

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