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SALVATORE A. MARANO AND NIKOLAOS S. PAPAGEORGIOU

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ON A NEUMANN PROBLEM WITH p-LAPLACIAN AND NONCOERCIVE RESONANT NONLINEARITY

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Using variational techniques and Morse theory, we establish three existence results for a Neumann boundary-value problem with p-Laplacian and Carathéodory reaction term, which can be $(p-1)$-asymptotically linear or sublinear at infinity. The hypotheses taken on permit resonance and make the corresponding energy functional noncoercive.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, having a smooth boundary $\partial\Omega$ and let $1 < p < +\infty$. This paper treats the existence of weak solutions $\hat{u} \in W^{1,p}(\Omega)$ to the boundary value problem

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = j(x, u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the reaction term $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, and $\partial u/\partial n_p := |\nabla u|^{p-2}\nabla u \cdot n$, with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.

Let $\{\lambda_n\}$ be the sequence of eigenvalues of $(-\Delta_p, W^{1,p}(\Omega))$. It is known that $0 = \lambda_1 < \lambda_2$. Three existence results are established here; see Theorems 2.1– 2.3 below. The first of them allows resonance with respect to λ_1 and requires that $t \mapsto j(x, t)$ be $(p-1)$-asymptotically super-linear at zero. In Theorem 2.2 the function $t \mapsto j(x, t)$ is $(p-1)$-asymptotically linear both at zero and at infinity, but resonance cannot occur. Finally, the third result examines the case $p = 2$, where the reaction term behaves — roughly speaking — as in Theorem 2.2, and resonance with respect to λ_2 is allowed.

From a technical point of view, the approach adopted combines variational methods of min-max type with Morse theory. Standard regularity arguments then provide $\hat{u} \in C^1(\overline{\Omega})$.

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Noncoercive, linear or sublinear Neumann problems have been widely investigated in the framework of semilinear equations (i.e., for $p = 2$) under sign conditions, monotonicity assumptions, and hypotheses of Landesman–Lazer type. We refer the reader to [Tang 2001] and the bibliography therein.

The p-Laplacian operator Δ_p arises from a variety of physical phenomena. For instance, it is employed in the mathematical modeling of non-Newtonian fluids, some reaction-diffusion problems, as well as flows through porous media. Nevertheless, no much attention has been payed to Neumann problems with p-Laplacian until few years ago. Previous results on this topic can be found in [Marano and Papageorgiou 2006; Motreanu et al. 2009] and the references mentioned there.

1. Preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\overline{V}$ for the closure of V and ∂V for the boundary of V. Given $\rho > 0$, the symbol B_ρ indicates the open ball of radius ρ centered at the origin of X. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X and X^*. Let $\Phi : X \to \mathbb{R}$. The function Φ is called locally Lipschitz continuous when to every $x \in X$ there corresponds a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall\, z, w \in V_x.$$

If $\lim_{\|x\|\to+\infty} \Phi(x) = +\infty$ then we say that Φ is coercive. Define

$$\Phi^c := \{x \in X : \Phi(x) \leq c\}, \quad c \in \mathbb{R}.$$

Now, let $\Phi \in C^1(X)$. The classical Palais–Smale condition for Φ reads as follows.

$(\mathrm{PS})_\Phi$ *Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and that*

$$\lim_{n\to+\infty} \|\Phi'(x_n)\|_{X^*} = 0$$

has a convergent subsequence.

We shall employ also the next compactness hypothesis, which includes $(\mathrm{PS})_\Phi$.

$(\mathrm{C})_\Phi$ *Every sequence $\{x_n\} \subseteq X$ such that $\{\Phi(x_n)\}$ is bounded and that*

$$\lim_{n\to+\infty} (1 + \|x_n\|)\|\Phi'(x_n)\|_{X^*} = 0$$

has a convergent subsequence.

Finally, $K(\Phi)$ indicates the critical set of Φ while

$$K_c(\Phi) := \{x \in K(\Phi) : \Phi(x) = c\}.$$

The critical point result below is a very special case of [Bonanno and Marano 2010, Theorem 2.2]; see also [Livrea and Marano 2009, Theorem 3.1].

Let Q be a compact topological manifold in X having a nonempty boundary Q_0. Set

$$\Gamma := \left\{\gamma \in C^0(Q, X) : \gamma|_{Q_0} = \mathrm{id}\,|_{Q_0}\right\}, \quad c := \inf_{\gamma\in\Gamma} \sup_{x\in Q} \Phi(\gamma(x)).$$

Theorem 1.1. *Suppose Φ satisfies condition* $(\mathrm{C})_\Phi$ *and there exists a nonempty closed subset F of X such that*

$$(\gamma(Q)\cap F)\setminus Q_0 \neq \varnothing \quad \forall\,\gamma\in\Gamma \quad \textit{and} \quad \sup_{x\in Q_0} \Phi(x) \le \inf_{x\in F} \Phi(x).$$

Then $K_c(\Phi) \neq \varnothing$. Moreover, $K_c(\Phi)\cap F \neq \varnothing$ as soon as $\inf_{x\in F} \Phi(x) = c$.

Let (A, B) be a topological pair fulfilling $B \subset A \subseteq X$. The symbol $H_k(A, B)$, $k \in \mathbb{N}_0$, indicates the k-th relative singular homology group of (A, B) with integer coefficients. If $x_0 \in K_c(\Phi)$ is an isolated point of $K(\Phi)$ then

$$C_k(\Phi, x_0) := H_k(\Phi^c \cap U, \Phi^c \cap U \setminus \{x_0\}), \quad k \in \mathbb{N}_0,$$

are the critical groups of Φ at x_0. Here, U stands for any neighborhood of x_0 such that $K(\Phi)\cap\Phi^c\cap U = \{x_0\}$. By excision, critical groups turn out to be independent of U. When $\Phi|_{K(\Phi)}$ is bounded below and $c < \inf_{x\in K(\Phi)} \Phi(x)$ we define

$$C_k(\Phi, \infty) := H_k(X, \Phi^c), \quad k \in \mathbb{N}_0.$$

For general references on this subject, see [Ambrosetti and Malchiodi 2007; Chang 1993; Granas and Dugundji 2003].

Throughout the paper, Ω denotes a bounded domain of real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \ge 3$, with a smooth boundary $\partial\Omega$, $p \in (1, +\infty)$, $p' := p/(p-1)$, $\|\cdot\|_p$ is the usual norm of $L^p(\Omega)$, $X := W^{1,p}(\Omega)$, and

$$\|u\| := \left(\|\nabla u\|_p^p + \|u\|_p^p\right)^{1/p}, \quad u \in X,$$

where

$$\|\nabla u\|_p := \left(\int_\Omega |\nabla u(x)|^p\, dx\right)^{1/p}.$$

Write p^* for the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \subseteq L^q(\Omega)$. Recall that $p^* = N/(N-p)$ if $p < N$, $p^* = +\infty$ otherwise, and the embedding is compact whenever $1 \le q < p^*$. The symbol $m(E)$ indicates the Lebesgue measure of E. If $m(E) > 0$, then we say that E is nonnegligible. Set, for any $w : \Omega \to \mathbb{R}$, $w^- := \max\{-w, 0\}$ and $w^+ := \max\{w, 0\}$.

Let $A : X \to X^*$ be the nonlinear operator defined by

$$\langle A(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)\, dx \quad \forall\, u, v \in X.$$

A standard argument [Chabrowski 1997, p. 3] yields this auxiliary result:

Proposition 1.1. *Assume $u_n \rightharpoonup u$ in X and $\limsup_{n\to+\infty}\langle A(u_n), u_n - u\rangle \le 0$. Then $u_n \to u$ in X.*

We shall employ some facts on the spectrum $\sigma(-\Delta_p)$ of the operator $-\Delta_p$ with homogeneous Neumann boundary conditions, i.e., $(-\Delta_p, X)$. The situation looks very nice when $p = 2$ (linear case), whereas it is more involved if $p \neq 2$. In fact, consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-1}$$

Lyusternik–Schnirelman theory still provides a strictly increasing sequence $\{\lambda_n\} \subseteq \mathbb{R}_0^+$ of eigenvalues for (1-1). However, we do not know whether they are all the eigenvalues of the operator $(-\Delta_p, X)$. When $p = 2$, denote by $E(\lambda_n)$ the eigenspace corresponding to λ_n, $n \in \mathbb{N}$. If $p \neq 2$ then we can characterize $E(\lambda_1)$ only. Proposition 3 in [Motreanu and Papageorgiou 2007] ensures that:

(p$_1$) $\lambda_1 = \inf\left\{\dfrac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in X,\ u \neq 0\right\} = 0.$

Further, λ_1 is isolated, simple, and $E(\lambda_1) = \mathbb{R}$.

(p$_2$) *The functions $\pm\hat{u}_0$ given by*

$$\hat{u}_0(x) := m(\Omega)^{-1/p} \quad \forall\, x \in \overline{\Omega}, \tag{1-2}$$

are the only constant-sign L^p-normalized eigenfunctions of $(-\Delta_p, X)$ corresponding to λ_1.

From [Motreanu and Papageorgiou 2007, Proposition 4] we next obtain:

(p$_3$) *Define*

$$C(p) =: \left\{u \in X : \int_\Omega |u(x)|^{p-2}u(x)\,dx = 0\right\}. \tag{1-3}$$

Then

$$\lambda_2 = \inf\left\{\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in C(p),\ u \neq 0\right\} = \inf\{\lambda \in \sigma(-\Delta_p) : \lambda > 0\}.$$

A different characterization of λ_2 will be used in Section 2. For the proof we refer the reader to [Aizicovici et al. 2009, Proposition 2].

(p$_4$) *Write*

$$\begin{aligned} &S := \{u \in X : \|u\|_p = 1\}, \\ &\Gamma_0 := \{\gamma_0 \in C^0([-1, 1], S) : \gamma_0(-1) = -\hat{u}_0,\ \gamma_0(1) = \hat{u}_0\}. \end{aligned} \tag{1-4}$$

Then

$$\lambda_2 = \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} \|\nabla \gamma(t)\|_p^p.$$

Finally, let $m \in L^\infty(\Omega) \setminus \{0\}$ satisfy $m \geq 0$ in Ω. Consider the weighted nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \hat{\lambda} m(x) |u|^{p-2} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-5}$$

As before, the Lyusternik–Schnirelman theory gives a strictly increasing sequence $\{\hat{\lambda}_n(m)\}$ of eigenvalues for (1-5). Moreover, one has [Aizicovici et al. 2009, Section 3]:

($\hat{\mathrm{p}}_1$) $\hat{\lambda}_1(m) = 0$ *and* $E(\hat{\lambda}_1(m)) = \mathbb{R}$.

($\hat{\mathrm{p}}_2$) *If* $m', m'' \in L^\infty(\Omega) \setminus \{0\}$ *and* $0 \leq m' < m''$ *in* Ω *then* $\hat{\lambda}_2(m'') < \hat{\lambda}_2(m')$.

($\hat{\mathrm{p}}_3$) *If* $m', m'' \in L^\infty(\Omega) \setminus \{0\}$, $0 \leq m' \leq m''$ *in* Ω, $m' < m''$ *on a nonnegligible subset of* Ω, *and* $p = 2$ *then* $\hat{\lambda}_n(m'') < \hat{\lambda}_n(m')$ *for all* $n \in \mathbb{N}$.

2. Existence results

The following hypotheses on the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ will be used in the sequel. To avoid unnecessary technicalities, "for every $x \in \Omega$" takes the place of "for almost every $x \in \Omega$".

(j_1) $x \mapsto j(x,t)$ *is measurable for all* $t \in \mathbb{R}$.

(j_2) $t \mapsto j(x,t)$ *is continuous and* $j(x,0) = 0$ *for every* $x \in \Omega$.

(j_3) *There exists a constant* $a_1 > 0$ *such that*

$$|j(x,t)| \leq a_1 \left(1 + |t|^{p-1}\right) \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

For $(x, \xi) \in \Omega \times \mathbb{R}$, define

$$J(x,\xi) := \int_0^\xi j(x,t)\,dt.$$

(j_4) *There are constants* $a_2 \in [0, \lambda_2)$, $r \in [1, p]$ *such that*

$$0 \leq \liminf_{|\xi| \to +\infty} \frac{pJ(x,\xi)}{|\xi|^p} \leq \limsup_{|\xi| \to +\infty} \frac{pJ(x,\xi)}{|\xi|^p} \leq a_2$$

and

$$\liminf_{|\xi| \to +\infty} \frac{pJ(x,\xi) - j(x,\xi)\xi}{|\xi|^r} > 0$$

uniformly in $x \in \Omega$.

(j_5) *There exist* $\delta > 0$, $\mu \in [1, p)$, $q \in (p, p^*)$, *and* $a_3, a_4 > 0$ *such that*

$$j(x,t)t > 0 \quad \textit{if} \quad x \in \Omega,\ 0 < |t| \le \delta$$

and

$$\mu J(x,\xi) - j(x,\xi)\xi \ge a_3|\xi|^p - a_4|\xi|^q \quad \forall\,(x,\xi) \in \Omega \times \mathbb{R}.$$

Example 2.1. A simple verification shows that the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ given by setting, for all $(x,t) \in \Omega \times \mathbb{R}$,

$$j(x,t) := \begin{cases} |t|^{\mu-2}t - |t|^{p-2}t + b|t|^{q-2}t & \text{if } |t| \le 1, \\ a_2|t|^{s-2}t + (b-a_2)/t & \text{otherwise,} \end{cases}$$

where $1 < \mu < p < q$, $s < p$, and $0 < a_2 \le b$, fulfills (j_1)–(j_5).

Now, define

$$\Phi(u) := \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega J(x,u(x))\,dx \quad \forall\,u \in X.$$

Due to (j_1)–(j_3) one clearly has $\Phi \in C^1(X)$.

Proposition 2.1. *If hypotheses* (j_1)–(j_4) *hold true*, Φ *satisfies condition* $(\mathrm{C})_\Phi$.

Proof. Pick a sequence $\{u_n\} \subseteq X$ such that $\{\Phi(u_n)\}$ is bounded and

$$\lim_{n\to+\infty}(1+\|u_n\|)\|\Phi'(u_n)\|_{X^*} = 0.$$

This implies

$$\left|\langle A(u_n), v\rangle - \int_\Omega j(x,u_n(x))v(x)\,dx\right| \le \frac{\varepsilon_n}{1+\|u_n\|}\|v\| \quad \forall n \in \mathbb{N},\ v \in X, \tag{2-1}$$

where $\varepsilon_n \to 0^+$. Setting $v := u_n$ yields

$$\|\nabla u_n\|_p^p - \int_\Omega j(x,u_n(x))u_n(x)\,dx \le \varepsilon_n. \tag{2-2}$$

Since $\{\Phi(u_n)\}$ is bounded, there exists $c_1 > 0$ fulfilling

$$-\|\nabla u_n\|_p^p + \int_\Omega pJ(x,u_n(x))\,dx \le c_1 \quad \forall\,n \in \mathbb{N}.$$

Therefore,

$$\int_\Omega [pJ(x,u_n(x)) - j(x,u_n(x))u_n(x)]\,dx \le c_2, \quad n \in \mathbb{N}, \tag{2-3}$$

where $c_2 > 0$. Combining (j_3) with (j_4) produces constants $c_3, c_4 > 0$ such that

$$c_3|\xi|^r - c_4 \le pJ(x,\xi) - j(x,\xi)\xi \quad \forall\,(x,\xi) \in \Omega \times \mathbb{R}.$$

So, on account of (2-3), the sequence $\{u_n\}$ turns out to be bounded in $L^r(\Omega)$. Since $r \le p < p^*$ we can find $\tau \in [0, 1)$ satisfying

$$\frac{1}{p} = \frac{1-\tau}{r} + \frac{\tau}{p^*}.$$

The interpolation inequality gives

$$\|u_n\|_p \le \|u_n\|_r^{1-\tau} \|u_n\|_{p^*}^{\tau},$$

which easily leads to

$$\|u_n\|_p^p \le c_5 \|u_n\|^{\tau p} \quad \forall n \in \mathbb{N}, \tag{2-4}$$

where $c_5 > 0$. By (2-2), (j_3), and (2-4), it follows that

$$\begin{aligned} \|\nabla u_n\|_p^p &\le \varepsilon_n + \int_\Omega j(x, u_n(x)) u_n(x)\, dx \le \varepsilon_n + \int_\Omega a_1(|u_n(x)| + |u_n(x)|^p)\, dx \\ &\le \varepsilon_n + c_6 m(\Omega)^{1-1/r} + a_1 c_5 \|u_n\|^{\tau p}, \quad n \in \mathbb{N}, \end{aligned}$$

for some $c_6 > 0$. Using (2-4) in this inequality one has

$$\|u_n\|^p \le \varepsilon_n + c_6 m(\Omega)^{1-1/r} + c_5(1 + a_1) \|u_n\|^{\tau p} \quad \forall n \in \mathbb{N},$$

namely, the sequence $\{u_n\}$ turns out to be bounded in X because $\tau < 1$. We may thus assume that $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$, where a subsequence is considered when necessary. Hypothesis (j_3) yields

$$\lim_{n\to+\infty} \int_\Omega j(x, u_n(x))(u_n(x) - u(x))\, dx = 0.$$

Hence, from (2-1) written for $v := u_n - u$ it follows

$$\lim_{n\to+\infty} \langle A(u_n), u_n - u \rangle = 0,$$

which, on account of Proposition 1.1, leads to the conclusion. □

From now on, F will denote the closed symmetric cone

$$F := \{u \in X : \|\nabla u\|_p^p \ge \lambda_2 \|u\|_p^p\}. \tag{2-5}$$

Proposition 2.2. *Let* (j_1)–(j_4) *be satisfied. Then the function* $\Phi|_F$ *is coercive. Moreover*, $\inf_{u\in F} \Phi(u) > -\infty$.

Proof. Hypotheses (j_3)–(j_4) provide constants $c_7 \in (0, \lambda_2)$, $c_8 > 0$ such that

$$J(x, \xi) \le \frac{c_7}{p} |\xi|^p + c_8 \quad \forall (x, \xi) \in \Omega \times \mathbb{R}.$$

Consequently, if $u \in F$ then

$$\begin{aligned}\Phi(u) &\geq \frac{1}{p}\|\nabla u\|_p^p - \frac{c_7}{p}\|u\|_p^p - c_8 m(\Omega)\\ &\geq \frac{1}{p}\left(1-\frac{c_7}{\lambda_2}\right)\|\nabla u\|_p^p - c_8 m(\Omega) \geq \frac{\lambda_2 - c_7}{p(\lambda_2+1)}\|u\|^p - c_8 m(\Omega).\end{aligned}$$

Since $c_7 < \lambda_2$, we evidently have

$$\lim_{\|u\|\to+\infty} \Phi|_F(u) = +\infty \quad \text{as well as} \quad \inf_{u\in F} \Phi(u) \geq -c_8 m(\Omega) > -\infty.$$

This completes the proof. □

Proposition 2.3. *If* (j_1)–(j_4) *hold then* $\lim_{\xi\to\pm\infty} \Phi|_{\mathbb{R}}(\xi) = -\infty$.

Proof. Condition (j_4) yields $c_9, c_{10} > 0$ such that

$$\frac{d}{dt}\left(\frac{J(x,t)}{t^p}\right) = \frac{j(x,t)t - pJ(x,t)}{t^{p+1}} \leq -c_9 \frac{1}{t^{p-r+1}}$$

for any $x \in \Omega$, $t \geq c_{10}$. Without loss of generality we can assume $r < p$. So,

$$\frac{J(x,z)}{z^p} - \frac{J(x,\xi)}{\xi^p} \leq \frac{c_9}{p-r}\left(\frac{1}{z^{p-r}} - \frac{1}{\xi^{p-r}}\right)$$

provided $z \geq \xi \geq c_{10}$. By (j_4) this forces, as $z \to +\infty$,

$$J(x,\xi) \geq \frac{c_9}{p-r}\xi^r, \quad \xi \geq c_{10}.$$

Hence,

$$\lim_{\xi\to+\infty} J(x,\xi) = +\infty \quad \text{uniformly in } x \in \Omega,$$

which evidently leads to $\lim_{\xi\to+\infty} \Phi|_{\mathbb{R}}(\xi) = -\infty$. A similar reasoning then gives $\lim_{\xi\to-\infty} \Phi|_{\mathbb{R}}(\xi) = -\infty$. □

Through Propositions 2.2 and 2.3 we obtain $\xi_0 > 0$ such that

$$\Phi(\pm\xi_0) < \inf_{u\in F} \Phi(u). \tag{2-6}$$

Define

$$Q_0 := \{\pm\xi_0\}, \quad Q := [-\xi_0, \xi_0] \subseteq \mathbb{R}, \quad \Gamma := \{\gamma \in C^0(Q, X) : \gamma|_{Q_0} = \mathrm{id}\,|_{Q_0}\}. \tag{2-7}$$

Proposition 2.4. *Let* F *be as in* (2-5) *and let* Q, Q_0, Γ *be as in* (2-7). *Then*

$$Q_0 \cap F = \varnothing \quad \textit{and} \quad \gamma(Q) \cap F \neq \varnothing \quad \forall\, \gamma \in \Gamma.$$

Proof. The first assertion immediately follows from (2-6). Let us next verify that $-\xi_0$ and ξ_0 belong to different path components of $X \setminus F$. Indeed, if the conclusion was false then there would exist a continuous function $\hat{\gamma} : [-1, 1] \to X$ fulfilling

$$\hat{\gamma}(-1) = -\xi_0, \quad \hat{\gamma}(1) = \xi_0, \quad \hat{\gamma}([-1, 1]) \subseteq X \setminus F.$$

Therefore,

$$\frac{\|\nabla \hat{\gamma}(t)\|_p^p}{\|\hat{\gamma}(t)\|_p^p} < \lambda_2$$

for all $t \in [-1, 1]$. However, this contradicts (p_4). Now, pick any $\gamma \in \Gamma$ and define $\hat{\gamma}(t) := \gamma(t\xi_0)$, $t \in [-1, 1]$. Since $\hat{\gamma}([-1, 1]) \cap \partial(X \setminus F) \neq \varnothing$ while $\partial(X \setminus F) = \partial F \subseteq F$, we actually have $\gamma(Q) \cap F \neq \varnothing$, as desired. □

Theorem 2.1. *If hypotheses* (j_1)–(j_5) *are satisfied*, (P) *possesses a nontrivial solution* $\hat{u} \in C^1(\overline{\Omega})$.

Proof. Propositions 2.1 and 2.4, besides (2-6), ensure that Φ, Q, Q_0, F comply with all the assumptions of Theorem 1.1. Thus, there is $\hat{u} \in X$ such that $\Phi(\hat{u}) = c$, $\Phi'(\hat{u}) = 0$. Reasoning exactly as in [Marano and Papageorgiou 2006, pp. 1310–1311] then provides

$$-\Delta_p \hat{u}(x) = j(x, \hat{u}(x)) \quad \text{a.e. in } \Omega, \qquad \frac{\partial \hat{u}}{\partial n_p} = 0 \quad \text{on } \partial\Omega, \tag{2-8}$$

i.e., the function $\hat{u}$ turns out to be a weak solution of (P). By (j_1)–(j_3), (2-8), and standard results from nonlinear regularity theory one has $\hat{u} \in C^1(\overline{\Omega})$; see for instance [Kristály and Papageorgiou 2010, p. 8]. So, it remains to verify that $\hat{u} \neq 0$. Proposition 3.2 in [Kristály and Papageorgiou 2010], which requires (j_5), yields $C_n(\Phi, 0) = 0$ for all $n \in \mathbb{N}_0$. Without loss of generality, suppose $K_c(\Phi)$ isolated. Thanks to Theorem 1.5 on p. 89 of [Chang 1993] we thus obtain $C_1(\Phi, \hat{u}) \neq 0$. Consequently, $\hat{u} \neq 0$, and the conclusion follows. □

Because of (j_5) the function $\xi \mapsto J(x, \xi)$ grows as $|\xi|^\mu$ near zero. Thus,

$$\lim_{\xi \to 0} \frac{J(x, \xi)}{|\xi|^p} = +\infty \quad \text{for any } x \in \Omega.$$

The next result treats the case when this limit is finite, namely $j(x, \cdot)$ turns out to be $(p-1)$-asymptotically linear at zero.

We shall also assume that:

(j_4') *There are constants* $a_5, a_6 \in (0, \lambda_2)$ *such that*

$$a_5 \le \liminf_{|t| \to +\infty} \frac{j(x, t)}{|t|^{p-2} t} \le \limsup_{|t| \to +\infty} \frac{j(x, t)}{|t|^{p-2} t} \le a_6$$

uniformly in Ω.

(j_5') *For some* $\lambda \in (\lambda_2, +\infty) \setminus \sigma(-\Delta_p)$ *one has*

$$\lim_{t\to 0} \frac{j(x,t)}{|t|^{p-2}t} = \lambda$$

uniformly with respect to $x \in \Omega$.

Example 2.2. A simple verification shows that the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ given by setting, for all $(x,t) \in \Omega \times \mathbb{R}$,

$$j(x,t) := \begin{cases} \lambda |t|^{p-2}t & if\, |t| \le 1, \\ a_6|t|^{p-2}t + (\lambda - a_6)|t|^{s-2}t & \text{otherwise,} \end{cases}$$

where $0 < a_6 < \lambda_2 < \lambda$, $\lambda \notin \sigma(-\Delta_p)$, while $1 < s < p$, fulfills (j_4') and (j_5') besides (j_1)–(j_3).

Proposition 2.5. *If* (j_1)–(j_3) *and* (j_4') *hold true*, Φ *satisfies condition* $(\mathrm{PS})_\Phi$.

Proof. Pick a sequence $\{u_n\} \subseteq X$ such that $\{\Phi(u_n)\}$ is bounded and

$$\lim_{n\to+\infty} \|\Phi'(u_n)\|_{X^*} = 0. \tag{2-9}$$

We claim that $\{u_n\}$ turns out to be bounded. Indeed, if the assertion was false then, passing to a subsequence when necessary,

$$\lim_{n\to+\infty} \|u_n\| = +\infty. \tag{2-10}$$

Define

$$w_n := \frac{u_n}{\|u_n\|}, \quad n \in \mathbb{N}.$$

Obviously, we may suppose

$$w_n \rightharpoonup w \quad \text{in } X \quad \text{and} \quad w_n \to w \quad \text{in } L^p(\Omega) \tag{2-11}$$

because $\{w_n\} \subseteq X$ is bounded. From (2-9) it follows that

$$\left| \langle A(w_n), v\rangle - \frac{1}{\|u_n\|^{p-1}} \int_\Omega j(x, u_n(x)) v(x)\, dx \right| \le \frac{\varepsilon_n}{\|u_n\|^{p-1}} \|v\| \quad \forall\, v \in X, \tag{2-12}$$

where $\varepsilon_n \to 0^+$. Since, on account of (j_3) and (2-11),

$$\lim_{n\to+\infty} \frac{1}{\|u_n\|^{p-1}} \int_\Omega j(x, u_n(x))(w_n(x) - w(x))\, dx = 0,$$

inequality (2-12) written for $v := w_n - w$ provides

$$\lim_{n\to+\infty} \langle A(w_n), w_n - w\rangle = 0.$$

Hence, thanks to Proposition 1.1,

$$\lim_{n\to+\infty} w_n = w \quad \text{in } X, \tag{2-13}$$

which evidently forces

$$\|w\| = 1. \tag{2-14}$$

By (j_3) again the sequence $\{\|u_n\|^{-p+1} j(\cdot, u_n(\cdot))\} \subseteq L^{p'}(\Omega)$ is bounded. Through the same arguments exploited in [Motreanu et al. 2007, Proposition 5] we thus obtain a function $\alpha \in L^\infty(\Omega)$ such that $a_5 \le \alpha \le a_6$ in Ω and

$$\frac{1}{\|u_n\|^{p-1}} j(\cdot, u_n(\cdot)) \rightharpoonup \alpha|w|^{p-2}w \quad \text{in } L^{p'}(\Omega).$$

Because of (2-12) and (2-13) this implies

$$\langle A(w), v\rangle = \int_\Omega \alpha(x)|w(x)|^{p-2}w(x)v(x)\,dx \quad \forall\, v \in X,$$

namely the function w turns out to be a weak solution of the problem

$$-\Delta_p u = \alpha(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n_p} = 0 \quad \text{on } \partial\Omega.$$

Now, recalling that $a_6 < \lambda_2$, property (p_2) yields

$$1 = \hat\lambda_2(\lambda_2) < \hat\lambda_2(\alpha),$$

namely

$$0 = \hat\lambda_1(\alpha) < 1 < \hat\lambda_2(\alpha).$$

Consequently $w = 0$, which contradicts (2-14). The boundedness of $\{u_n\}$ leads to

$$u_n \rightharpoonup u \quad \text{in } X, \qquad u_n \to u \quad \text{in } L^p(\Omega), \tag{2-15}$$

where a subsequence is considered when necessary. As we already did for $\{w_n\}$, through (2-12) and (2-15) we finally achieve $u_n \to u$ in X. □

Next, let $\lambda \in \mathbb{R}$ and let $\Psi(\lambda) : X \to \mathbb{R}$ be defined by

$$\Psi(\lambda)(u) := \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{p}\|u\|_p^p \quad \forall\, u \in X.$$

Proposition 2.6. $C_0(\Psi(\lambda), 0) = C_1(\Psi(\lambda), 0) = 0$ *for all* $\lambda \in (\lambda_2, +\infty) \setminus \sigma(-\Delta_p)$.

Proof. Pick $\lambda \in (\lambda_2, +\infty) \setminus \sigma(-\Delta_p)$ and write $G := \{u \in X : \|\nabla u\|_p^p < \lambda\|u\|_p^p\}$. Obviously, $\hat u_0 \in G$, with $\hat u_0$ being as in (1-2). We first claim that the set G turns out to be path-wise connected. Indeed, let $u \in G$ and let G_u the path component of G containing u. If

$$m_u := \inf_{w \in G_u} \frac{\|\nabla w\|_p^p}{\|w\|_p^p}$$

then there exists $\{w_n\} \subseteq G_u$ fulfilling

$$\|w_n\|_p = 1, \quad \|\nabla w_n\|_p^p < m_u + \frac{1}{n^2} \quad \forall n \in \mathbb{N}. \tag{2-16}$$

Along a subsequence when necessary, this gives

$$w_n \rightharpoonup w_0 \quad \text{in } X, \quad w_n \to w_0 \quad \text{in } L^p(\Omega). \tag{2-17}$$

Since $\Psi(\lambda)$ is p-homogeneous, we may restrict ourselves to the C^1 Banach manifold S defined in (1-4). Set $\xi(u) := \|\nabla u\|_p^p$, $u \in X$. By Ekeland's variational principle, there exists a sequence $\{v_n\} \subseteq \overline{G_u \cap S}$ such that

$$\xi(v_n) \le \xi(w_n) < m_u + \frac{1}{n^2}, \quad \|v_n - w_n\| \le \frac{1}{n}, \quad n \in \mathbb{N}, \tag{i}$$

and

$$\xi(v_n) \le \xi(v) + \frac{1}{n}\|v - v_n\| \quad \forall n \in \mathbb{N},\ v \in \overline{G_u \cap S}. \tag{ii}$$

If $v_n \in \partial(\overline{G_u \cap S})$ for infinitely many n then Lemma 3.5 of [Cuesta et al. 1999] and (i) force

$$\lambda = \xi(v_n) \le \xi(w_n) < m_u + \frac{1}{n^2} < \lambda,$$

which is impossible. So, $v_n \in G_u \cap S$ for all n large enough. Thus, exploiting (ii) yields

$$\lim_{n\to+\infty} \|(\xi|_S)'(v_n)\|_{X^*} = 0.$$

Arguing as in the proof of Proposition 2.1 we see that $\xi|_S$ satisfies condition $(\mathrm{C})_{\xi|_S}$. Therefore, up to subsequences, $v_n \to w_0$ in X and, a fortiori, $w_0 \in \overline{G_u \cap S}$. Now, observe that $G \cap S$ is open in S while $G_u \cap S$ turns out to be a component of $G \cap S$. So, if $w_0 \in \partial(G_u \cap S)$ then, thanks to [Cuesta et al. 1999, Lemma 3.5], $w_0 \notin G \cap S$. On the other hand, by (2-16)–(2-17) one has

$$\|w_0\|_p = 1, \quad \|\nabla w_0\|_p^p \le m_u < \lambda,$$

i.e., $w_0 \in G \cap S$, a contradiction. Hence, $w_0 \in G_u \cap S$, and the assertion follows once we show that $\hat{u}_0$ can be joined with w_0 through a path contained in G. This is an immediate consequence of (p_4) as soon as $w_0 \le 0$, because in such a case (p_2) yields $w_0 = -\hat{u}_0$. Suppose thus $w_0^+ \ne 0$ and define

$$w(t) := \frac{w_0^+ - (1-t)w_0^-}{\|w_0^+ - (1-t)w_0^-\|_p}, \quad t \in [0, 1].$$

Since

$$\langle A(w_0), v\rangle = m_u \int_\Omega |w_0(x)|^{p-2} w_0(x) v(x)\, dx \quad \forall v \in X,$$

choosing $v := w_0^+$ and $v := -w_0^-$ provides, respectively,

$$\|\nabla w_0^+\|_p^p = m_u \|w_0^+\|_p^p, \quad \|\nabla w_0^-\|_p^p = m_u \|w_0^-\|_p^p,$$

which evidently forces

$$\|\nabla w(t)\|_p^p = m_u \|w(t)\|_p^p = m_u, \quad t \in [0, 1].$$

Hence, $w(t) \in G$ for all $t \in [0, 1]$, $w(0) = w_0$, and

$$w(1) = \frac{w_0^+}{\|w_0^+\|_p} = \hat{u}_0$$

on account of (p_2) again. The function $t \mapsto w(t)$, $t \in [0, 1]$, represents the desired arc. From the path-wise connectedness of G it follows

$$H_0(G, *) = 0, \quad * \in G. \tag{2-18}$$

Let $* \in G$. The set $\Psi(\lambda)^0$ is contractible, because $\Psi(\lambda)$ is p-homogeneous. So, thanks to [Granas and Dugundji 2003, Section 14, Proposition 4.9], we get

$$H_k(\Psi(\lambda)^0, *) = 0 \quad \forall\, k \in \mathbb{N}_0. \tag{2-19}$$

Now, Theorem 5.1.33 of [Gasiński and Papageorgiou 2006] ensures that $\Psi(\lambda)^0 \setminus \{0\}$ and $\Psi(\lambda)^{-\varepsilon}$ are homotopically equivalent. Since the same holds for $G = \operatorname{int}(\Psi(\lambda)^0)$ and $\Psi(\lambda)^{-\varepsilon}$ whenever $\varepsilon > 0$ is suitably small (see [Granas and Dugundji 2003, p. 407]), the sets $\Psi(\lambda)^0 \setminus \{0\}$ and G turn out to be homotopically equivalent too. This implies

$$H_k(\Psi(\lambda)^0 \setminus \{0\}, *) = H_k(G, *), \quad k \in \mathbb{N}_0. \tag{2-20}$$

Gathering (2-18) and (2-20) together we obtain

$$H_0(\Psi(\lambda)^0 \setminus \{0\}, *) = 0. \tag{2-21}$$

On account of Theorem 4.8 in [Granas and Dugundji 2003, Section 14] the reduced homology sequence

$$\begin{aligned} \ldots H_k(\Psi(\lambda)^0 \setminus \{0\}, *) \to H_k(\Psi(\lambda)^0, *) \xrightarrow{i_*} \\ H_k(\Psi(\lambda)^0, \Psi(\lambda)^0 \setminus \{0\}) \xrightarrow{\partial_*} H_{k-1}(\Psi(\lambda)^0 \setminus \{0\}, *) \ldots \to 0, \end{aligned} \tag{2-22}$$

where i_* denotes the group homomorphism arising from the inclusion map while ∂_* stands for the boundary homomorphism, is exact. Therefore, by (2-19),

$$\operatorname{Ker} \partial_* = \operatorname{Im} i_* = \{0\}\ .$$

This means that ∂_* is an isomorphism between $H_k(\Psi(\lambda)^0, \Psi(\lambda)^0 \setminus \{0\})$ and a subgroup of $H_{k-1}(\Psi(\lambda)^0 \setminus \{0\}, *)$. Using (2-21), this results in

$$C_1(\Psi(\lambda), 0) = H_1(\Psi(\lambda)^0, \Psi(\lambda)^0 \setminus \{0\}) = 0.$$

Finally, due to (2-22), one directly has

$$C_0(\Psi(\lambda), 0) = H_0(\Psi(\lambda)^0, \Psi(\lambda)^0 \setminus \{0\}) = 0,$$

which completes the proof. □

Write, as usual,

$$\delta_{k,h} Z = \begin{cases} Z & \text{when } k = h, \\ \{0\} & \text{otherwise.} \end{cases}$$

Proposition 2.7. (i) *If* $\lambda < \lambda_1$ *then* $C_k(\Psi(\lambda), 0) = \delta_{k,0} Z$ *for all* $k \in \mathbb{N}_0$.

(ii) *If* $\lambda \in (\lambda_1, \lambda_2)$ *then* $C_k(\Psi(\lambda), 0) = \delta_{k,1} Z$ *for every* $k \in \mathbb{N}_0$.

Proof. Pick $\lambda < \lambda_1 = 0$. The functional $\Psi(\lambda)$ is bounded from below and satisfies condition $(\mathrm{PS})_c$, $c \in \mathbb{R}$. Thus, choosing $c < \inf_{u \in X} \Psi(\lambda)(u)$ yields

$$C_k(\Psi(\lambda), \infty) := H_k(X, \Psi(\lambda)^c) = \delta_{k,0} Z\,, \quad k \in \mathbb{N}_0. \tag{2-23}$$

From $\lambda \notin \sigma(-\Delta_p)$ it easily follows $K(\Psi(\lambda)) = \{0\}$. Hence, by [Bartsch and Li 1997, Proposition 3.6] we get

$$C_k(\Psi(\lambda), 0) = C_k(\Psi(\lambda), \infty). \tag{2-24}$$

Now, assertion (i) is an immediate consequence of (2-23)–(2-24).

Let us next verify (ii). Fix $\lambda \in (\lambda_1, \lambda_2)$. It is evident that

$$\Psi(\lambda)|_{\mathbb{R}} \le 0, \quad \Psi(\lambda)|_{C(p)\setminus\{0\}} > 0,$$

where $C(p)$ is as in (1-3). If $U := X$, $Q := [-\hat{u}_0, \hat{u}_0]$, $Q_0 := \{\pm\hat{u}_0\}$, and $F := C(p)$, while $i_{1*} : H_0(Q_0) \to H_0(U \setminus F)$ and $i_{2*} : H_0(Q_0) \to H_0(Q)$ denote the group homomorphisms induced by the corresponding inclusion maps, then

$$\operatorname{rank}(i_{1*}) - \operatorname{rank}(i_{2*}) = 2 - 1 = 1.$$

Therefore, on account of [Perera 1998, Theorem 3.1], one has

$$\operatorname{rank}\, C_1(\Psi(\lambda), 0) \ge 1. \tag{2-25}$$

Through the long exact homology sequence

$$\dots H_k(\Psi(\lambda)^\varepsilon, \Psi(\lambda)^{-\varepsilon}) \xrightarrow{i_*} H_k(X, \Psi(\lambda)^{-\varepsilon}) \xrightarrow{j_*}$$
$$H_k(X, \Psi(\lambda)^\varepsilon) \xrightarrow{\partial_*} H_{k-1}(\Psi(\lambda)^\varepsilon, \Psi(\lambda)^{-\varepsilon}) \dots$$

for the topological pair $(\Psi(\lambda)^{\varepsilon}, \Psi(\lambda)^{-\varepsilon})$, where $\varepsilon > 0$ is suitably small, we obtain

$$\operatorname{rank} H_k(X, \Psi(\lambda)^{-\varepsilon}) = \operatorname{rank} \operatorname{Ker} j_* + \operatorname{rank} \operatorname{Im} j_* = \operatorname{rank} \operatorname{Ker} j_*$$

because $\operatorname{rank} H_k(X, \Psi(\lambda)^{\varepsilon}) = 0$. Thus, by (2-25),

$$\operatorname{rank} H_k(X, \Psi(\lambda)^{-\varepsilon}) = \operatorname{rank} \operatorname{Im} i_* \leq 1,$$

which implies assertion (ii). □

Proposition 2.8. *Let hypotheses* (j_1)–(j_3) *and* (j_4') *be satisfied. If, moreover,* $p \geq 2$, *then* $C_k(\Phi, \infty) = \delta_{k,1} Z$ *for all* $k \in \mathbb{N}_0$.

Proof. Fix $\mu \in (0, \lambda_2)$ and define, provided $(t, u) \in [0, 1] \times X$,

$$h_1(t, u) := (1-t)\Phi(u) + t\Psi(\mu)(u), \quad h_2(t, u) := t\Phi(u) + (1-t)\Psi(\mu)(u).$$

We claim that for some $R > 0$ one has

$$\inf\left\{\|h_1(t, \cdot)'(u)\|_{X^*} : t \in [0, 1], \ \|u\| > R\right\} > 0. \tag{2-26}$$

Indeed, if (2-26) were false then there would exist $\{t_n\} \subseteq [0, 1]$, $t \in [0, 1]$, and $\{u_n\} \subseteq X$ fulfilling

$$\lim_{n\to+\infty} t_n = t, \quad \lim_{n\to+\infty} \|u_n\| = +\infty, \quad h_1(t_n, \cdot)'(u_n) = 0 \quad \forall\, n \in \mathbb{N}.$$

Write

$$w_n := \frac{u_n}{\|u_n\|}, \quad n \in \mathbb{N}.$$

The same arguments exploited in the proof of Proposition 2.5 yield a weak solution $w \in X$ to the problem

$$-\Delta_p u = [(1-t)\alpha(x) + t\mu]|u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n_p} = 0 \quad \text{on } \partial\Omega$$

that satisfies (2-14). Since

$$(1-t)\alpha(x) + t\mu \leq (1-t)a_6 + t\mu < \lambda_2,$$

property $(\hat{\mathrm{p}}_2)$ yields

$$1 = \hat{\lambda}_2(\lambda_2) < \hat{\lambda}_2((1-t)\alpha + t\mu),$$

namely, on account of $(\hat{\mathrm{p}}_1)$,

$$0 = \hat{\lambda}_1((1-t)\alpha + t\mu) < 1 < \hat{\lambda}_2((1-t)\alpha + t\mu).$$

Consequently, $w = 0$, which contradicts (2-14).

A similar argument ensures that

$$\inf\left\{\|h_2(t, \cdot)'(u)\|_{X^*} : t \in [0, 1], \ \|u\| > R\right\} > 0 \tag{2-27}$$

for any sufficiently large $R > 0$.

Now, bearing in mind (2-26), Theorem 5.1.19 of [Gasiński and Papageorgiou 2006] can be applied, and there exists a pseudogradient vector field

$$\hat{v} := (v_0, v) : [0, 1] \times (X \setminus \overline{B}_R) \to [0, 1] \times X$$

such that $v_0(t, u) = h_1(\,\cdot\,, u)'(t)$ and, moreover, $v(t, \cdot)$ is a locally Lipschitz continuous pseudogradient vector field of $h_1(t, \cdot)$ for every $t \in [0, 1]$. Observe that $A : X \to X^*$ turns out to be locally Lipschitz continuous too, because $p \geq 2$. So, setting

$$w(t, u) := -\frac{|h_1(\,\cdot\,, u)'(t)|}{\|h_1(t, \cdot)'(u)\|_{X^*}^2} v(t, u), \quad u \in X \setminus \overline{B}_R,$$

we evidently obtain a locally Lipschitz continuous function. If

$$b < \inf\{h_i(t, u) : \ (t, u) \in [0, 1] \times \overline{B}_R\}, \quad i = 1, 2, \tag{2-28}$$

then, due to (2-26)–(2-27), the constant b is not a critical value of $h_i(t, \cdot)$, $t \in [0, 1]$. By (j_4') the functional Φ turns out to be unbounded below. Thus, there exists $u_0 \in X$ such that $\Phi(u_0) \leq b$. Using Theorem 5.1.21 of the same reference provides a local flow $x(t)$ of the Cauchy problem

$$x' = w(t, x), \quad x(0) = u_0.$$

Hence, for every $t \geq 0$ sufficiently small we have $\dfrac{dh_1(t, x(t))}{dt} \leq 0$, which clearly forces

$$h_1(t, x(t)) \leq h_1(0, x(0)) = h_1(0, u_0) = \Phi(u_0) \leq b.$$

Bearing in mind (2-28) this implies $\|x(t)\| > R$. Thanks to (2-26) we thus get $h_1(t, \cdot)'(x(t)) \neq 0$ for any $t \geq 0$ small enough. Therefore, the flow $x(t)$ turns out to be global on [0, 1]. Consequently,

$$\Phi^b = h_1(0, \cdot)^b \quad \text{is homeomorphic to a subset of} \quad \Psi(\mu)^b = h_1(1, \cdot)^b. \tag{2-29}$$

Replacing h_1 with h_2 then yields

$$\Psi(\mu)^b = h_2(0, \cdot)^b \quad \text{is homeomorphic to a subset of} \quad \Phi^b = h_2(1, \cdot)^b. \tag{2-30}$$

From (2-29)–(2-30) it evidently follows that Φ^b and $\Psi(\mu)^b$ are of the same homotopy type. So,

$$C_k(\Phi, \infty) = H_k(X, \Phi^b) = H_k(X, \Psi(\mu)^b) = C_k(\Psi(\mu), \infty) \quad \forall\, k \in \mathbb{N}_0. \tag{2-31}$$

Since $\mu \in (\lambda_1, \lambda_2)$, the functional $\Psi(\mu)$ possesses only one critical point, i.e., $u \equiv 0$. By [Bartsch and Li 1997, Proposition 3.6] we have

$$C_k(\Psi(\mu), \infty) = C_k(\Psi(\mu), 0), \quad k \in \mathbb{N}_0. \tag{2-32}$$

At this point the conclusion is a direct consequence of (2-31), (2-32), and assertion (ii) in Proposition 2.7. □

Theorem 2.2. *If* $p \geq 2$ *and* (j_1)–(j_3), (j_4'), *and* (j_5') *hold true*, (P) *has a nontrivial solution* $\hat{u} \in C^1(\overline{\Omega})$.

Proof. Thanks to Proposition 2.5 the functional Φ satisfies condition $(\mathrm{PS})_\Phi$. Thus, in view of [Perera 2003, Lemma 4.1], there exist $\hat{\Phi} \in C^1(X)$, $r > 0$ such that

$$\hat{\Phi}(u) = \Psi(\lambda)(u) \quad \forall\, u \in \overline{B}_r\,, \quad \hat{\Phi}(u) = \Phi(u) \quad \forall\, u \in X \setminus \overline{B}_{2r} \tag{2-33}$$

as well as

$$K(\Phi) \cap \overline{B}_{2r} = K(\hat{\Phi}) \cap \overline{B}_{2r} = \{0\}. \tag{2-34}$$

Through (2-26) we easily obtain $K(\Phi)$, $K(\hat{\Phi}) \subseteq \overline{B}_R$ for some $R > 2r$. So, if

$$c < \min\Big\{ \inf_{u \in \overline{B}_R} \Phi(u),\ \inf_{u \in \overline{B}_R} \hat{\Phi}(u) \Big\},$$

then, by (2-33),

$$H_k(X, \Phi^c) = H_k(X, \hat{\Phi}^c), \quad k \in \mathbb{N}_0.$$

Bearing in mind Proposition 2.8, this implies

$$C_k(\hat{\Phi}, \infty) = C_k(\Phi, \infty) = \delta_{k,1} Z \quad \forall\, k \in \mathbb{N}_0. \tag{2-35}$$

On the other hand, due to Proposition 2.6 one has

$$C_i(\hat{\Phi}, 0) = C_i(\Psi(\lambda), 0) = 0, \quad i = 0, 1. \tag{2-36}$$

Now, gathering (2-35)–(2-36) together and using [Bartsch and Li 1997, Proposition 3.6], we obtain a point $\hat{u} \in K(\hat{\Phi}) \setminus \{0\}$. By (2-34) one must have $\|\hat{u}\| > 2r$. Therefore, on account of (2-33), it follows that $\hat{u} \in K(\Phi) \setminus \{0\}$. The same argument of [Marano and Papageorgiou 2006, pp. 1310–1311] ensures that the function $\hat{u}$ is a nontrivial weak solution to (P), namely (2-8) holds true. Finally, by (j_1)–(j_3), (2-8), and standard results of nonlinear regularity theory, we get $\hat{u} \in C^1(\overline{\Omega})$; see for instance [Kristály and Papageorgiou 2010, p. 8]. □

There are two interesting questions arising from Theorem 2.2.

(q_1) *Is it possible to remove the restriction* $p \geq 2$ *and consider differential operators* $\Delta_p u$ *which are singular on the set* $\{x \in \Omega :\ \nabla u(x) = 0\}$?

(q_2) *Can the case of resonance at infinity with respect to* λ_2 *be treated*?

Both problems remain open in their full generality. However, concerning (q_2), a positive answer can be given when $p = 2$. Indeed, in this case, the eigenfunctions of $-\Delta$ with homogeneous Neumann boundary conditions, i.e., $(-\Delta, H^1(\Omega))$, exhibit the unique continuation property [Gasiński and Papageorgiou 2006, Section 6.6].

So, the monotonicity of weighted eigenvalues holds true once weights differ only on a nonnegligible set; cf. (p_3).

From now on, fix $X := H^1(\Omega)$ and let $\{\lambda_n\}$ be the sequence of eigenvalues of $(-\Delta, X)$. The following assumptions will be used in the sequel.

(j_4') *There are* $\beta, \eta \in L^\infty(\Omega) \setminus \{0\}$ *such that* $0 \le \eta \le \lambda_2$ *in* Ω, $\eta < \lambda_2$ *on a nonnegligible subset of* Ω, *as well as*

$$0 \le \beta(x) \le \liminf_{|t|\to+\infty} \frac{j(x,t)}{t} \le \limsup_{|t|\to+\infty} \frac{j(x,t)}{t} \le \lambda_2, \quad \limsup_{|\xi|\to+\infty} \frac{2J(x,\xi)}{\xi^2} \le \eta(x)$$

uniformly in Ω.

(j_5') *For some* $\theta \in L^\infty(\Omega)$, $k \ge 2$ *one has* $\lambda_k \le \theta \le \lambda_{k+1}$ *in* Ω, $\lambda_k < \theta < \lambda_{k+1}$ *on a nonnegligible subset of* Ω, *and*

$$\lim_{t\to 0} \frac{j(x,t)}{t} = \theta(x)$$

uniformly in Ω.

Example 2.3. A simple verification shows that the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ given by setting, for all $(x,t) \in \Omega \times \mathbb{R}$,

$$j(x,t) := \begin{cases} a_7 t & \text{if } |t| \le \sqrt{\pi/2}, \\ a_8 t + (\lambda_2 - a_8)t\cos t^2 + a_9 & \text{otherwise}, \end{cases}$$

where $\lambda_k < a_7 < \lambda_{k+1}$ for some $k \ge 2$, $\lambda_2/2 \le a_8 < \lambda_2$, while $a_9 := (a_7 - a_8)\sqrt{\pi/2}$, complies with ($j_4'$) and ($j_5'$), besides ($j_1$)–($j_3$).

Proposition 2.9. *If* $p \ge 2$ *and* (j_1)–(j_3) *and* (j_4') *hold true*, Φ *satisfies condition* $(\mathrm{PS})_\Phi$.

Proof. Reasoning exactly as in the proof of Proposition 2.5, with the same notation, we obtain a weak solution $w \in X$ to the problem

$$-\Delta u = \alpha(x)u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n_2} = 0 \quad \text{on } \partial\Omega,$$

where $\alpha \in L^\infty(\Omega)$ and $\beta \le \alpha \le \lambda_2$ in Ω, which fulfills (2-13) and (2-14). If $\alpha(x) < \lambda_2$ on a nonnegligible subset of Ω then by ($\hat{p}_3$) one has $1 = \hat{\lambda}_2(\lambda_2) < \hat{\lambda}_2(\alpha)$, which leads to

$$0 = \hat{\lambda}_1(\alpha) < 1 < \hat{\lambda}_2(\alpha).$$

Consequently $w = 0$, against (2-14). Otherwise,

$$w \in E(\lambda_2) \tag{2-37}$$

and thus $w \neq 0$. Since $\{\Phi(u_n)\}$ is bounded, there exists $c_{11} > 0$ fulfilling

$$\|\nabla w_n\|_2^2 - \int_\Omega \frac{2J(x, u_n(x))}{\|u_n\|^2}\, dx \le \frac{c_{11}}{\|u_n\|^2} \quad \forall n \in \mathbb{N}. \tag{2-38}$$

Through (j_3) we immediately see that the sequence

$$\left\{ \frac{2J(\cdot\,, u_n(\cdot))}{\|u_n\|^2} \right\} \subseteq L^1(\Omega)$$

is bounded too. Hence, on account of (j_4'), the same argument exploited in [Motreanu et al. 2007, Proposition 5] provides a function $\hat{\alpha} \in L^\infty(\Omega)$ such that $\hat{\alpha} \le \eta$ in Ω and

$$\frac{2J(\cdot\,, u_n(\cdot))}{\|u_n\|^2} \rightharpoonup \hat{\alpha} w^2 \quad \text{in } L^1(\Omega). \tag{2-39}$$

Combining (2-38) with (2-39) results in

$$\|\nabla w\|_2^2 \le \int_\Omega \hat{\alpha}(x) w(x)^2\, dx \le \int_\Omega \eta(x) w(x)^2\, dx < \lambda_2 \|w\|_2^2.$$

However, this contradicts (2-37). Therefore, the sequence $\{u_n\}$ turns out to be bounded. The rest of the proof is as that of Proposition 2.5. □

Proposition 2.10. *Let* $p = 2$ *and let* (j_1)–(j_3) *and* (j_4') *be satisfied. Then*

$$C_k(\Phi, \infty) = \delta_{k,1} Z \quad \forall k \in \mathbb{N}_0.$$

Proof. Keep the same notation introduced in the proof of Proposition 2.8. We claim that for suitable $c \in \mathbb{R}$, $R > 0$ one has

$$\inf\{\|h_1(t, \cdot)'(u)\|_{X^*} : (t, u) \in h_1^c\} \ge R. \tag{2-40}$$

Indeed, if (2-40) were false then there would exist $\{t_n\} \subseteq [0, 1]$, $t \in [0, 1]$, and $\{u_n\} \subseteq X$ fulfilling

$$\lim_{n\to+\infty} t_n = t, \qquad \lim_{n\to+\infty} \|u_n\| = +\infty, \qquad \lim_{n\to+\infty} h_1(t_n, u_n) = -\infty$$

as well as

$$\lim_{n\to+\infty} \|h_1(t, \cdot)'(u_n)\|_{X^*} = 0. \tag{2-41}$$

Write $w_n := \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Obviously, we may suppose

$$w_n \rightharpoonup w \quad \text{in } X \quad \text{and} \quad w_n \to w \quad \text{in } L^2(\Omega)$$

because $\{w_n\} \subseteq X$ is bounded. From (2-41) it follows that

$$\left| \langle A(w_n), v\rangle - \frac{1 - t_n}{\|u_n\|} \int_\Omega j(x, u_n(x)) v(x)\, dx - t_n \mu \int_\Omega w_n(x) v(x)\, dx \right| \le \varepsilon_n \|v\|$$

for all $v \in X$, where $\varepsilon_n \to 0^+$. Arguing exactly as in the proof of Proposition 2.5, one then obtains a weak solution $w \in X$ to the problem

$$\begin{cases} -\Delta u = \alpha(x)u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_2} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in L^\infty(\Omega)$ and $\beta \leq \alpha \leq \lambda_2$ in Ω, which fulfills (2-13)–(2-14). However, this is impossible; see the proof of Proposition 2.9. Hence, (2-40) holds. Through [Li et al. 2001, Theorem 3.1] we thus achieve

$$\begin{aligned} C_k(\Phi,\infty) &= C_k(h_1(0,\cdot),\infty) = C_k(h_1(1,\cdot),\infty) \\ &= C_k(\Psi(\mu),\infty) \quad \forall\, k \in \mathbb{N}_0. \end{aligned} \tag{2-42}$$

At this point, the same reasoning exploited to get Proposition 2.8, but with (2-31) replaced by (2-42), yields the conclusion. □

The next existence result can be established via Propositions 2.9 and 2.10. The proof is analogous to that of Theorem 2.2. So, we omit it.

Theorem 2.3. *If $p = 2$ and hypotheses* (j_1)–(j_3)*,* (j_4')*, and* (j_5') *are satisfied*, (P) *possesses a nontrivial solution $\hat{u} \in C^1(\overline{\Omega})$.*

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SALVATORE A. MARANO
DIPARTIMENTO DI MATEMATICA E INFORMATICA
UNIVERSITÀ DEGLI STUDI DI CATANIA
VIALE A. DORIA 6
95125 CATANIA
ITALY
marano@dmi.unict.it

NIKOLAOS S. PAPAGEORGIOU
DEPARTMENT OF MATHEMATICS
NATIONAL TECHNICAL UNIVERSITY
ZOGRAFOU CAMPUS
ATHENS 15780
GREECE
npapg@math.ntua.gr

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