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TYPE II ALMOST-HOMOGENEOUS MANIFOLDS OF COHOMOGENEITY ONE

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This paper continues our investigation on the existence of extremal metrics of the general affine and type II almost-homogeneous manifolds of cohomogeneity one. It deals with the general type II cases with hypersurface ends: more precisely, with manifolds having certain $\mathbb{C}P^n \times (\mathbb{C}P^n)^*$- or $\mathbb{C}P^2$-bundle structures. In particular, we study the existence of Kähler–Einstein metrics on these manifolds and obtain new Kähler–Einstein manifolds as well as Fano manifolds without Kähler–Einstein metrics.

1. Introduction

The theory of simply connected compact Kähler homogeneous manifolds has applications in many branches of mathematics and physics. These complex manifolds possess significant properties: for example, they are projective, Fano, Kähler–Einstein, and rational.

One more general class of Kähler manifolds of potential use is that of almost-homogeneous compact Kähler manifolds with two orbits, especially those having cohomogeneity one. If we assume simple connectedness, such manifolds are automatically projective. It is interesting to ask when they are Fano, Kähler–Einstein, and so on [Guan 2009].

This paper is one in a series in which we answer the questions above, completing the study of the existence of Calabi extremal metrics in any Kähler class on any compact almost-homogeneous manifold of cohomogeneity one. That is, we have dealt with all the compact Kähler manifolds on which we could use ordinary differential equations instead of partial differential equations for these geometric analysis problems.

There are three types of manifolds of this kind (see [Guan 2002] for details). Type III compact complex almost-homogeneous manifolds of real cohomogeneity

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one were dealt with in [Guan 1995]. Not much stability is found there (but see [Guan 2003] for the stability of related constructions). The type I case was dealt with in [Guan 2011a; 2011b; ≥ 2011b], while the type II case is the subject of this paper and [Guan 2009].* This is the first class of manifolds for which a criterion for the existence of Calabi extremal metrics has been completely elucidated; it is equivalent to geodesic stability.

Specifically, in this paper we conclude the task of proving that there is always a Kähler metric of constant scalar curvature on a type II almost-homogeneous manifold of cohomogeneity one whose generalized Futaki invariant is positive; see Theorems 15, 15′, 23, and 24. We prove the converse in [Guan ≥ 2011a]. In [Guan 2002; 2003; 2006; Guan and Chen 2000] we dealt with some examples, and in [Guan 2009] we dealt with the two most conceptually difficult series of manifolds.

We should mention that our concept of the generalized Futaki invariant is not the same as the one in [Ding and Tian 1992], although it looks similar in our case. The generalized Futaki invariant in this paper comes from a kind of combination of the generalized Futaki invariants along the maximal geodesic rays in the moduli space of Kähler metrics, but does not necessarily come directly from any one of them, as discussed in [Guan 2003; 2006].

In this paper, we first treat manifolds that are fiber bundles with typical fibers of the first and fifth cases in [Akhiezer 1983, p. 73] as one situation. Let G be a complex Lie subgroup of the automorphism group of our manifold M, and assume G has an open orbit O on M. Then M is a fiber bundle over a compact homogeneous space Q. We have $Q = G/P$ with P a parabolic subgroup of G, and $P = SS_1R$ with R the radical of P and S, S_1 semisimple factors of G. The group S_1R acts on the fiber F trivially. In our case $S = A_n$ acts on the central fiber. The fiber is just $\mathbb{C}P^n \times (\mathbb{C}P^n)^*$, which is isotropic and is the first manifold in the list of [Akhiezer 1983, p. 67]. It is also of affine type and therefore of type II. Therefore, to finish the affine case and the type II case, we have to deal with this case. Individually, these manifolds seem easier to deal with than those in [Guan 2009]. However, there are more of them, and it turns out that as a group and analytically, they are technically more involved.

We stress the difference between the open orbits of the manifolds with $S = A_n$ actions and those of the manifolds treated in [Guan 2002; 2003]. For example, the isotropic group U in the A_n action case is $\mathrm{GL}(n, \mathbb{C})$, corresponding to the first manifold in [Akhiezer 1983, Table 2, p. 67], while the isotropic groups of the manifolds in [Guan 2002; 2003] are not reductive at all. Also, the manifolds in [Akhiezer 1983, p. 67] are all homogeneous, which is not true for the examples

*Originally, the two paper and [Guan 2009] were one. Because of its length, it was split.

in [Guan 2002; 2003]. We shall come to some generalizations of these latter examples in Theorems 18 and 22. Similar calculations appear in Sections 3 and 4. However, the manifolds we considered in [Guan 2002; 2003] are manifolds with $S = A_1$ actions on the fiber and are special cases of those treated in this paper. It is interesting that the first examples we treated in [Guan and Chen 2000; Guan 2002; 2003] are both type II and isotropic (having a similar complex structure to the type I case); they served as sample cases for both type I and type II manifolds, which led us to breakthroughs for both cases.

Outline of the paper. We follow the method introduced in [Guan 2002; 2003]. In Section 2 we look back, from a Lie group point of view, at what we did in those papers and in [Guan and Chen 2000]. From that viewpoint, the method can be regarded as a nilpotent path method: we consider a path starting from the singular real orbit, and generated by the action of a one-parameter subgroup generated by a nilpotent element. (One could also consider the path as generated by a semisimple element H_α, where α is the root that generates the $\mathfrak{sl}(2)$ Lie algebra $\mathscr{A}$.) Then we apply the same argument given in Section 3 of [Guan 2006; 2009] to the affine A_n-action case. We find that the same method works for the complex structure of both the affine and the type II cases. At the end of Section 2, we work as in [Guan 2002] to give a comparison of two different methods for the homogeneous case.

In Section 3, we find that the same argument works for the Kähler structure. We deal there with many different possibilities of the pairs of groups (A_n, G). This shows that the affine and type II classes are very big and are not extraordinary at all (see also the proof of Lemma 13 for many examples of this kind of manifold). A new ingredient is that, in contrast with [Guan 2009], our B here can be either positive or negative.

Section 4 is a central part of the paper. To calculate the Ricci curvature, we apply a trick inspired by [Koszul 1955, p. 567–570], as we did in [Guan 2006; 2009]. The formula we used from [Dorfmeister and Guan 1991, 4.11] is due to Professor Dorfmeister.

We calculate the scalar curvature in Section 5 and set up equations in Section 6. The pattern of these equations makes it possible to reduce a fourth-order ODE to a second-order ODE as in [Guan 2006; 2009].

We finally prove our existence theorem (Theorem 15) in Section 7.

We treat the type II case in Section 8 and the Kähler–Einstein case in Section 9, generalizing results from [Guan 2002; 2003]. At the end of Section 9, we give a very uniform description for the generalized Futaki invariant (Theorems 23 and 24). These results confirm our calculation in [Guan 2006].

In all our calculations we need to deal carefully with the change of the invariant inner products when we restrict our calculation to a typical subgroup S in G.

2. The complex structures of isotropic affine almost-homogeneous manifolds

Let G be a semisimple complex Lie group and U_G its 1-subgroup. Recall that there is a parabolic subgroup

$$P = SS_1R \tag{1}$$

with S, S_1 semisimple and R solvable such that

$$U_G = US_1R, \tag{2}$$

where U is a 1-subgroup of S. The manifold is a fibration over G/P with the completion of

$$\frac{P}{U_G} = \frac{S}{U} \tag{3}$$

as the affine almost-homogeneous fiber F. In this case, the root system of S is a subsystem of the root system of G.

Let $\mathscr{H}$ be the corresponding Cartan subalgebra of G. The Lie algebra $\mathscr{G}$ of G has a decomposition

$$\mathscr{H} + \sum_{\alpha\in\Delta} \mathbb{C}E_\alpha$$

with a Chevalley lattice generated by h_α, E_α; see [Humphreys 1978, p. 147]. Assume that a maximal compact Lie subalgebra is generated by

$$F_\alpha = E_\alpha - E_{-\alpha}, \quad G_\alpha = i(E_\alpha + E_\alpha), \quad H_\alpha = i[E_\alpha,\ E_{-\alpha}] = ih_\alpha. \tag{4}$$

We have

$$[H_\alpha,\ E_\alpha] = 2iE_\alpha. \tag{5}$$

Let $\mathscr{A} = \mathfrak{su}(2)$ be the commutator of a generic compact isotropic subgroup and let p_t be a curve generated by a nilpotent element in the complexification of $\mathscr{A}$. In the Lie algebra of G, we have F_α, G_α for those roots of G not in S. The tangent space of G/U_G along p_t is decomposed into irreducible $\mathscr{A}$ representations. F_α, G_α are in the complement representation of $\mathscr{S}$. But $JF_\alpha = -G_\alpha \pmod{\mathscr{S}}$, since it lies in the tangent space of G/P. Therefore, we have $JF_\alpha = -G_\alpha$ for any α not in the root system of S. This discussion corresponds to the one in the last paragraph of [Guan 2006, Section 2].

As stated in [Kobayashi and Nomizu 1981, p. 38], we can always identify the Lie algebra as the left invariant vector fields on the Lie group. For example, if G is $\mathrm{GL}_n(\mathbb{C})$ and $B(t)$ a curve on G with tangent vector X_0 at $B(0) = I$, then $AB(t)$ is a curve starting at A, and AX_0 with $A \in G$ is a left-invariant vector field on G. That is, the left-invariant vector fields can be described as AX_0 for some X_0.

Let $X_0 = (b_{ij})$ and $Y_0 = (c_{ij})$. The Lie bracket of two left-invariant vector fields AX_0, AY_0 is

$$[AX_0, AY_0] = \left[a_{ij}b_{jl}\frac{\partial}{\partial a_{il}},\ a_{ks}c_{st}\frac{\partial}{\partial a_{kt}}\right] = a_{ij}b_{jl}c_{lt}\frac{\partial}{\partial a_{it}} - a_{ks}c_{st}b_{tl}\frac{\partial}{\partial a_{kl}}$$
$$= a_{ij}\left(b_{jl}c_{lt}\frac{\partial}{\partial a_{it}} - c_{jt}b_{tl}\frac{\partial}{\partial a_{il}}\right) = a_{ij}(b_{jl}c_{lt} - c_{jl}b_{lt})\frac{\partial}{\partial a_{it}} = A[X_0, Y_0],$$

which is comparable with the Lie bracket of the Lie algebra $gl_n(\mathbb{C})$.

In our case we have the $S = A_n = \mathrm{SL}(n+1, \mathbb{C})$ action of [Akhiezer 1983, p. 73], which includes both the first case and the fifth case there.

Let us look at the case for $n = 1$ first. The action is

$$A\begin{bmatrix}1\\0\end{bmatrix} \times [1, 0]A^{-1}, \tag{6}$$

where $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $[1, 0]$ represent points in $\mathbb{C}P^1$. We have

$$E_{\alpha_1} = \begin{bmatrix}0&1\\0&0\end{bmatrix},\quad E_{-\alpha_1} = E_{\alpha_1}^T = \begin{bmatrix}0&0\\1&0\end{bmatrix},\quad H = H_{\alpha_1} = \begin{bmatrix}i&0\\0&-i\end{bmatrix}. \tag{7}$$

Thus

$$\exp(tE_{\alpha_1})\begin{bmatrix}1\\0\end{bmatrix} \times [1.0]\exp(-tE_{\alpha_1}) = \begin{bmatrix}1&-t\\0&0\end{bmatrix} = p_t, \tag{8}$$

$$p_\infty = \begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \times [0, 1]. \tag{9}$$

We let

$$F = F_{\alpha_1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix},\quad G = G_{\alpha_1} = \begin{bmatrix}0&i\\i&0\end{bmatrix}. \tag{10}$$

Using the coordinates $[1, z]^T \times [1, w]$, we can check that along p_t, H acts as vector $(z, w) = (0, -2it)$. The tangent vector T of p_t is $(0, -1)$. F acts as $(-1, -1-t^2)$ and G acts as $(i, -i(1-t^2))$ along p_t. Moreover, $F + (1+t^2)T$ is $(-1, 0)$. Hence

$$JF = i(-1, -1-t^2) = -\big(i, -i(1-t^2) + 2i\big) = -G + \frac{H}{t} \tag{11}$$

and

$$JH = i(0, -2it) = -2tT. \tag{12}$$

In general, if $S = \mathrm{SL}(n+1, \mathbb{C}) = A_n$, then S has simple roots $\alpha_i = e_i - e_{i+1}$. The affine fiber $\mathbb{C}^n$ is generated by the root vectors with the roots $e_1 - e_j$, $1 < j \le n+1$.

The action is

$$A[1,0,\dots,0]^T\times[1,0,\dots,0]A^{-1}. \tag{13}$$

We can choose

$$E_{e_i-e_j}=E_{ij} \tag{14}$$

as a square matrix $(a_{kl})_{(n+1)\times(n+1)}$; that is, all the elements a_{kl} are zero except $a_{ij}=1$. We also let $H_{e_i-e_j}=iE_{ii}-iE_{jj}$. We have $[E_{ij},\ E_{kl}]=0$ if $j\neq k$, $i\neq l$ and

$$[E_{ij},\ E_{jk}]=E_{ij}E_{jk}-E_{jk}E_{ij}=E_{ik}-0=E_{ik}$$

if $i\neq k$. As above, $F=F_{\alpha_1}$, $G=G_{\alpha_1}$ and $H=H_{\alpha_1}$. We write

$$p_t=\exp(tE_{\alpha_1})[1,0,\dots,0]^T\times[1,0,\dots,0]\exp(-tE_{\alpha_1}) \tag{15}$$

$$=[1,0,0,\dots,0]^T\times[1,-t,0.\dots,0], \tag{16}$$

$$JF=-G+\frac{H}{t},\quad JH=-2tT, \tag{17}$$

$$JF_{e_2-e_j}=G_{e_2-e_j},\quad JF_{e_1-e_j}=-G_{e_1-e_j}-\frac{2G_{e_2-e_j}}{t}\quad\text{for } 2<j, \tag{18}$$

$$F_{e_k-e_j}=G_{e_k-e_j}=0\quad\text{for } 2<k<j. \tag{19}$$

We also have

$$J\left(F_{e_1-e_j}+\frac{F_{e_2-e_j}}{t}\right)=-\left(G_{e_1-e_j}+\frac{G_{e_2-e_j}}{t}\right). \tag{20}$$

Actually, if we let $[1,z_1,\dots,z_n]\times[1,w_1,\dots,w_n]$ be the coordinate, then F_{1j} is the same as $z_k=w_k=0$ for $k\neq j$, and $z_j=w_j=-1$. For F_{2j} we have $z_k=w_l=0$ for $l\neq j$, and $w_j=t$. Therefore, $F_{1j}+t^{-1}F_{2j}$ has $z_k=w_j=0$ for $k\neq j$, and $z_j=-1$. At p_∞,

$$JF_{e_1-e_k}=-G_{e_1-e_k},\quad JF_{e_2-e_k}=G_{e_2-e_k}, \tag{21}$$

$$F_{e_i-e_k}=G_{e_i-e_k}=0\quad\text{for } 2<i<k. \tag{22}$$

Let

$$F_{ij}=E_{ij}-E_{ji},\quad G_{ij}=i(E_{ij}+E_{ji}). \tag{23}$$

We have

$$[F_{ij},\ G_{jk}]=G_{ik}\quad\text{for } i\neq k. \tag{24}$$

In our case of $S=A_n$, the bigger complex Lie group G can be any complex semisimple Lie group. That is quite different from that in [Guan 2009]. This makes our argument more involved in this paper starting from the next section.

We can also use a similar method from [Guan and Chen 2000; Guan 2002; 2003] to understand the complex structure. Let the complex bilinear form be given by

$$(z, w) = z_0 w_0 + z_1 w_1 + \cdots + z_n w_n, \tag{25}$$

where

$$[z, w] = \big([z_0, z_1, \ldots, z_n]; [w_0, w_1, \ldots, w_n]\big) \in \mathbb{C}P^n \times (\mathbb{C}P^n)^*.$$

(This is different from the form in [Guan and Chen 2000; Guan 2002; 2003], where (z, w) represents the inner product.) Then the hypersurface end is just $(z, w) = 0$, and the singular $\mathrm{SU}(n+1)$ orbit is $w = \bar{z}$, or if we let

$$\gamma = \frac{|(z, w)|^2}{|z|^2 |w|^2}, \tag{26}$$

the singular orbit is just $\gamma = 1$. Note that this γ is different from θ in [Guan and Chen 2000; Guan 2002; 2003], which corresponds to $1 - \gamma$. (That θ is like the square of the cosine, while and γ is like the square of the sine; compare with [Guan 2006, Section 3].)

3. The Kähler structures

Now we calculate the Kähler form by different methods. First, if $G = S = A_n$, let

$$\omega = a\omega_1 + b\omega_2 + i\partial\bar{\partial} F,$$

with $\omega_1 = \partial\bar{\partial}\log|z|^2$, $\omega_2 = \partial\bar{\partial}\log|w|^2$, and F an $\mathrm{SU}(n+1)$-invariant smooth function. We see that $F = F(\gamma)$.

Let $f = \gamma F'$, the derivative being with respect to γ. At p_t we have $\gamma = 1/(1+t^2)$, and we can write

$$\partial\bar{\partial}\log\gamma = -\partial\bar{\partial}\big(\log|z|^2 + \log|w|^2\big),$$

$$\partial\log\gamma = \partial\big(\log(z, w) - \log|z|^2 - \log|w|^2\big) = -t\left(dz_1 - \frac{dw_1}{|w|^2}\right),$$

$$\begin{aligned}\omega &= a\omega_1 + b\omega_2 + \gamma f'\partial\log\gamma \wedge \bar{\partial}\log\gamma + f\partial\bar{\partial}\log\gamma \\ &= (a - f)\, dz \wedge d\bar{z} + (b - f)\left(\frac{dw_1 \wedge d\bar{w}_1}{|w|^4} + |w|^{-2}\sum_{j>1} dw_j \wedge d\bar{w}_j\right) \\ &\qquad + \gamma f' |w_1|^2 (dz_1 - |w|^{-2} dw_1) \wedge (d\bar{z}_1 - |w|^{-2} d\bar{w}_1).\end{aligned}$$

The difference between this formula and the one in [Guan 2003] is that here we do not have the second term on the right, since (z, w) here is holomorphic. We notice that the subspaces $W = \{\partial/\partial z_1, \partial/\partial w_1\}$ and $\mathbb{C}\partial/\partial z_j$, $\mathbb{C}\partial/\partial w_j$ for $j > 1$ are orthogonal to each other. Let us calculate the determinant τ of W.

We have

$$\tau=\begin{vmatrix} a-f+(1-\gamma)f' & -(1-\gamma)f'|w|^{-2} \\ -(1-\gamma)f'|w|^{-2} & (b-f+(1-\gamma)f')|w|^{-4} \end{vmatrix} = \frac{1}{|w|^4}\big((a-f)(b-f)+(1-\gamma)(a+b-2f)f'\big).$$

Likewise, for the standard metric we have $a=b=n+1,\ f=0$, and $\tau_0=\dfrac{(n+1)^2}{|z|^4|w|^4}$. Therefore,

$$\tau=-\frac{1}{|z|^4|w|^4}D', \tag{27}$$

with

$$D=(a-f)(b-f)(1-\gamma). \tag{28}$$

The determinant of $\mathbb{C}\partial/\partial z^i$ for $i>1$ is $|z|^{-2}(a-f)$. The determinant of $\mathbb{C}\partial/\partial w^i$ for $i>1$ is $|w|^{-2}(b-f)$. Therefore, the volume form is

$$V=\frac{-D^{n-1}D'}{(|z||w|)^{2n+2}(1-\gamma)^{n-1}}\,dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n \wedge dw^1\wedge d\bar{w}^1\wedge\cdots\wedge dw^n\wedge d\bar{w}^n. \tag{29}$$

Next, by regarding the open A_n orbit as a homogeneous space, a vector field corresponding to the Lie algebra is the pushdown of a right-invariant vector field on the Lie group A_n. As in [Guan 2006], we study the corresponding left-invariant vector fields on the Lie group. To make the things simpler, we still use our original notation for left-invariant vector fields. Since the Kähler form is (left-)invariant under the action of the maximal compact Lie subalgebra $\mathcal{K}$ of the complex Lie algebra $\mathcal{A}_n$, the pullback of this Kähler form is a left-$\mathcal{K}$-invariant form on A_n. We also extend t to be $\mathcal{K}$-invariant; hence so is T, since it is the derivative of t. Therefore, we have (by [Kobayashi and Nomizu 1981, p. 36] and [Huybrechts 2005, p. 283]; we use the convention in the latter reference)

$$\begin{aligned} 0&=d\omega(T,X,Y)\\ &=T(\omega(X,Y))-X(\omega(T,Y))+Y(\omega(T,X))\\ &\qquad -\omega([T,X],Y)+\omega([T,Y],X)-\omega([X,Y],T)\\ &=T(\omega(X,Y))-\omega([X,Y],T). \end{aligned}$$

Thus $T(\omega(X,Y))=-\omega(T,[X,Y])$ for any left-invariant $X,Y\in\mathcal{K}$.

Now,

$$\begin{aligned} T(\omega(G,H))&=-2\omega(T,F)=-2\omega(JT,JF)\\ &=-\omega\left(\frac{H}{t},\ -G+\frac{H}{t}\right)=-t^{-1}\omega(G,H); \end{aligned}$$

that is, $\omega(G,\ H) = Ct^{-1}$ for a constant C. Then $C = 0$; otherwise $\omega(G,\ H)$ is infinite at p_0. It follows that $\omega(G,\ H) = \omega(T,\ F) = 0$.

Similarly,

$$tT(\omega(H, F)) - T(\omega(F, G)) = 2\omega\left(tT, -G + \frac{H}{t}\right) = 2\omega(tJT, J^2F) = -\omega(H, F);$$

that is, $T\big(t\omega(H, F) - \omega(F, G)\big) = 0$. We have

$$\omega(F, G) = t\omega(H, F) + A.$$

Let $(\ ,\ )_A$ be an invariant metric on $\mathcal{K}$ such that $(H, H)_A = 1$. If there is no confusion, we write $(\ ,\ ) = (\ ,\ )_A$. Then H, G, F is a unitary basis of the Lie algebra $\mathcal{A}$. Therefore

$$[X, Y] = ([X, Y], H)H + ([X, Y], F)F + ([X, Y], G)G + [X, Y]_l + [X, Y]_{(\mathcal{A}+l)^\perp},$$

which implies Therefore,

$$\omega(T, [X, Y]) = ([X, Y], H)\omega(T, H) + ([X, Y], G)\omega(T, G) + \omega(T, [X, Y]_{(\mathcal{A}+l)^\perp}).$$

But

$$\omega(T, [X, Y]_{(\mathcal{A}+l)^\perp}) = \omega\big((2t)^{-1}H, J([X, Y]_{(\mathcal{A}+l)^\perp})\big) = 0,$$

since $JX \in (\mathcal{A}+l)^\perp$ if $X \in (\mathcal{A}+l)^\perp$. We also have

$$\omega(X, Y) = (g_1H + g_2F + g_3G + I, [X, Y])$$

with I in the center of l. We conclude that

$$\omega(G, H) = (g_1H + g_2F + g_3G + I, [G, H]) = 2(g_2F, F) = g_2 = 0,$$

that is, $g_2 = 0$. Hence, for left-invariant X, Y, we have

$$\begin{aligned} T(\omega(X, Y)) &= (\dot{g}_1H + \dot{g}_3G + \dot{I}, [X, Y]) = -\omega(T, [X, Y]) \\ &= -\big([X, Y], \omega(T, H)H + \omega(T, G)G\big), \end{aligned}$$

where the dot denotes the derivative with respect to t. It follows that $\dot{I} = 0$ and $\dot{g}_1 = -\omega(T, H)$, $\dot{g}_3 = -\omega(T, G)$. (The last two equalities are already known to us.) We have obtained

$$\begin{aligned} \omega\left(T, -G + \frac{H}{t}\right) = \dot{c} - \frac{\dot{a}}{t} &= \omega(JT, J^2F) = -\omega\left(\frac{H}{2t}, F\right) \\ &= -t^{-1}(g_1H + g_3G, G) = -g_3t^{-1}; \end{aligned}$$

that is, $t\dot{g}_3 + g_3 = \dot{g}_1$. Therefore, $g_1 = tg_3 + C$; that is,

$$\omega(F, G) = 2g_1 = 2tg_3 + 2C = t\omega(H, F) + 2C.$$

(We already have this equality with $A = 2C$.) We also see that $g_3(0) = 0$ since $H(0) = 0$. The first equality, $I' = 0$, means that I does not depend on t. In other words, if we let

$$I_0 = \frac{n-1}{n+1}(e_1 + e_2) - \frac{2}{n+1}\sum_{i=3}^{n+1} e_i,$$

then $I = BiI_0$ for some constant B.

Set $g = g_3$. Then $g_1 = tg + C$ and the Kähler form is

$$\omega(X, Y) = \big((tg(t) + C)H + g(t)G + BiI_0, [X, Y]\big) = (H(t), [X, Y])$$

for left-invariant X, Y, where $H(t) = g_1 H + gG + I = (tg + C) + gH + H$.

As an observation, we see that if

$$V_1 = \operatorname{span}(T, F_\alpha) \quad \text{and} \quad V_2 = \operatorname{span}(H, G_\alpha),$$

then

$$JV_1 = V_2 \quad \text{and} \quad V_1^{\perp} = V_2$$

with respect to ω. Moreover,

$$[V_1, V_1], [V_2, V_2] \subset V_1 \quad \text{and} \quad [V_1, V_2] \subset V_2.$$

The Kähler metric is a direct sum of its restriction on the subspaces

(30) $W = \operatorname{span}(T, H, F, G)$,

(31) $$W_1 = \operatorname{span}\big(E_\alpha | \alpha = e_i - e_j, i \neq j, \{i, j\} \cap \{1, 2\} \neq 0\big).$$

On W the metric is

$$\begin{bmatrix} \omega(T, JT) & \omega(T, JF) \\ \omega(F, JT) & \omega(F, JF) \end{bmatrix} = \begin{bmatrix} \omega\Big(T, \dfrac{H}{2t}\Big) & \omega(JT, -F) \\ \omega\Big(F, \dfrac{H}{2t}\Big) & \omega\Big(F, -G + \dfrac{H}{t}\Big) \end{bmatrix}$$
$$= \begin{bmatrix} -(t\dot{g} + g)/2t & -g/t \\ -g/t & -2(1 + t^2)g/t - 2C \end{bmatrix}.$$

The determinant is equal to

$$(2t)^{-1} \det \begin{bmatrix} \omega(T, H) & \omega(T, -G) \\ \omega(F, H) & \omega(F, -G) \end{bmatrix} = (2t)^{-1} \det \begin{bmatrix} -\dot{g}_1 & \dot{g} \\ -2g & -2g_1 \end{bmatrix}$$
$$= t^{-1}(g_1\dot{g}_1 + g\dot{g}) = \frac{\dot{U}}{2t},$$

where

(32) $$U = g_1^2 + g^2.$$

We notice that U is the square norm $(H(t), H(t))$ up to a constant; in other words, the energy of $H(t)$ up to a constant.

We also see that U is increasing. We also see that $g(0) = 0$, $-(\dot{tg}) > 0$ when $t > 0$, therefore, $-g > 0$ when $t > 0$ and $-tg$ is increasing. We also notice that $g(-t)/-t = g(t)/t$, that is, $g(t)$ is an odd function.

We first consider $n = 2$. Then $W_1 = \text{span}(E_\alpha|_{\alpha=\pm\alpha_2,\pm(\alpha_1+\alpha_2)})$. On W_1 we have

$$\begin{bmatrix} \omega(F_{\alpha_2}, JF_{\alpha_2}) & \omega(F_{\alpha_2}, JF_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_2}) & \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1+\alpha_2}) \end{bmatrix} = \begin{bmatrix} -g_1 + B & g \\ g & -g_1 - B - 2g/t \end{bmatrix}.$$

The determinant is equal to $U - B^2$. Since $F_{\alpha_2}(0) = 0$, we have $g_1(0) = C = B$ and $U(0) = B^2$. Since U is increasing, $U - B^2 > 0$.

When $n > 2$, we have 2-strings $e_2 - e_j$, $e_1 - e_j$ of α_1. The calculation is exactly the same and the determinant is $U - B^2$. Therefore, the volume form is

$$\dot{U}(2t)^{-1}(U - B^2)^{n-1}. \tag{33}$$

This fits well with our earlier volume formula (29).

Moreover, along p_t, we have

$$\omega(F_{23},\ JF_{23}) = 2t^2 \frac{b - f}{|w|^2} = \frac{2t^2(b - f)}{1 + t^2}, \tag{34}$$

$$\omega\left(F_{13} + \frac{F_{23}}{t},\ J\left(F_{13} + \frac{F_{23}}{t}\right)\right) = 2(a - f). \tag{35}$$

Then

$$-g = \frac{2t(b - f)}{1 + t^2}, \tag{36}$$

$$-t^{-1}(1 + t^2)g - 2B = 2(a - f). \tag{37}$$

Therefore, $2(b - f) + 2B = (a - f)$, that is,

$$B = b - a. \tag{38}$$

We also have $-t^{-1}g = 2\gamma(b - f)$ and

$$-tg = \frac{2t^2}{1 + t^2}(b - f). \tag{39}$$

Therefore, when $t \to 0$, we get $-\dot{g}(0) = 2(b - f(1))$ and $\lim_{t\to+\infty} tg = -2b$. That is, $-tg$ is nonnegative and increasing with a limit $2b$. In particular, both B and $l = \lim_{t\to+\infty} tg = -2b$ are topological invariants of the given Kähler class.

Moreover,

$$(40)\qquad D=(1-\gamma)(a-f)(b-f)=4^{-1}(1-\gamma)(t\gamma)^{-2}g(g-2Bt\gamma)$$
$$=4^{-1}g\big((1+t^2)g+2tB\big)=4^{-1}(U-B^2).$$

When $n=1$, we have

$$\omega(T,JT)=(2t)^{-1}\omega(T,H)=-(2t)^{-1}\dot g_1=2\frac{b-f+\theta f'}{(1+t^2)^2},$$
$$\omega\big(T,J(F-(1+t^2)T)\big)=\omega\Big(T,-G+\frac{H}{t}-\frac{1+t^2}{2t}H\Big)$$
$$=\dot g+\frac{t^2-1}{2t}\dot g_1=-2\theta f'(1+t^2)^{-2}.$$

Therefore,

$$-(2t)^{-1}\dot g_1=2\frac{b-f}{(1+t^2)^2}-\frac{2t\dot g+(t^2-1)\dot g_1}{2t(1+t^2)}.$$

We have

$$2\frac{b-f}{1+t^2}=\dot g-t^{-1}\dot g_1=-t^{-1}g$$

as above.

To get the formula for B, we similarly have

$$2(a-f+\theta f')=\omega\big(F-(1+t^2)T,\ J(F-(1+t^2)T)\big)$$
$$=-2g_1+\frac{t^2-1}{t}g-(1+t^2)\dot g-\frac{t^4-1}{2t}\dot g_1$$
$$=-2g_1+\frac{t^2}{t}g+2\theta f'=-\frac{t^2-2B+1}{t}g+2\theta f'.$$

That is,

$$2(a-f)=-\frac{t^2+1}{t}g-2B=2(b-f)-2B$$

as before. Hence, again $B=b-a$.

As in [Guan 2009], all the I and therefore the coefficients B depend on the chosen inner product $(\ ,\ )$. In general, G might be bigger than $S=A_n$. And, we can write the volume formula as

$$M\dot U t^{-1}(U-B^2)^{k-1}\prod(a_i^2-U).$$

For each string, by changing the sign of the eigenvalues we can exchange the eigenvectors. This induces a *mirror symmetry* of the eigenvectors. Formally, we can let $c=0$ (and assume $a\neq 0$); then $(aH+I,\beta_i)=k_{\beta_i}(a_i\pm a)$ for each eigenvector β_i. Therefore, we can choose $a_i=-|(I,\beta_i)/(H,\beta_i)|$ if $(H,\beta_i)\neq 0$. If β_{i_1},β_{i_2} are mirror symmetric to each other, we have the same a_i. We call a *mirror symmetry* *class* the set $[i]$ of two different roots that are mirror symmetric to each other, and we define $a_{[i]} = a_i$ for $i \in [i]$. By $\mathcal{I}$ we denote the set of all mirror symmetry classes.

Similar to what is in [Guan 2006; 2009], we have:

Theorem 1. *For the affine isotropic case, that is, when $S = A_n$, the volume is*

$$V = \frac{M\dot{U}}{t}(U - B^2)^{n-1} \prod_{[i]\in\mathcal{I}} (a_i^2 - U) \tag{41}$$

for some positive numbers M and a_i^2 with

$$a_i = -\left|\frac{(I_G, \beta_i)}{(H, \beta_i)}\right|.$$

Moreover, $U(0) = B^2$ and $B^2 \leq U < a_i^2$. In particular, if $G = S$, we have that $V = Mt^{-1}\dot{U}(U - B^2)^{n-1}$.

Proof. We need to take care of the case $S = A_n,\ G \neq S$.

If $G = A_{m+n+k}$ and $S = A_n$ is generated by simple roots $e_{m+1} - e_{m+2}, \ldots, e_{m+n} - e_{m+n+1}$, then α_{m+1} has other 2-strings with determinants $a_j^2 - U$ for some constants a_j.

As we saw in the last section, in the general case of $S = A_n$, the group G can be any semisimple Lie group. To see that Theorem 1 still holds, we have to deal with pairs of roots. There is a classification in [Humphreys 1978, p. 44–45]. We have the following three lemmas:

Lemma 2. *If α has a 1-string, then the 1-string and α generate an $A_1 \times A_1$ type of complex Lie subalgebra. In this case, the determinant is a positive constant.*

Proof. The Lie algebra is a rank 2 algebra. Since the action of α_1 is trivial on the 1-string β, the minimal Lie algebra including both triples must be $A_1 \times A_1$. The restricted ω is $(aH + cG + M\beta, [X, Y])$ for a constant M. The positivity comes from the positivity of the metric. □

Lemma 3. *If α has a 3-string generated by β, then β has twice the length of α, and α, β generate a B_2 type of complex Lie subalgebra, which has an induced cohomogeneity one action. The determinant is $-8M(M^2 - U)$ for a real negative number M.*

Proof. The Lie algebra has rank 2. Since the representation of $\mathcal{A}$ has length 3, it cannot be $A_1 \times A_1$, A_2, or G_2. It must be B_2. The calculation of the volume follows from a similar argument for 3-strings in [Guan 2009]. □

Before we go further, we check that the other possible strings are 4-strings and 2-strings. While the 4-strings can only occur in G_2, the 2-strings are more complicated than the cases above, which only involved Lie subalgebras of type A_2.

We have basically dealt with the G_2 case in [Guan 2006]. The only possible case for a 4-string is $G = G_2$, and $S = A_1$ is generated by the short root $\alpha = \alpha_1$. In this case, the 4-string is α_2, $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$. The restricted metric ω is $\big(aH + cG + B_1 i(3\alpha_1 + 2\alpha_2), [X, Y]\big)$. The determinant is equal to

$$\det(\omega(F_{\alpha_i}, -G_{\alpha_j})) = (B_1^2 - U)(9B_1^2 - U);$$

see [Guan 2006]. We let $a_1 = B_1$ and $a_2 = 3B_1$.

If a simple root α has a 2-string generated by β and the length of β is the same as that of α, then they generate an A_2. This case includes all the cases for $G = A_n$, D_n, E_k.

If a simple root α has a 2-string generated by β and the length of β is half that of α, then they generate a B_2 type of complex Lie subalgebra. Assuming that $\alpha = e_1 - e_2$ and $\beta = e_2$, we see that the 2-string is e_2, e_1. Then the restricted metric ω is

$$\big(aH + cG + B_1 i(e_1 + e_2), [X, Y]\big).$$

The determinant is $B_1^2 - U$. This includes the long simple roots in B_n, C_n, F_4. Together with the paragraph above, we have dealt with all the possibilities except the case in which $G = G_2$.

If a simple root α has a 2-string generated by β and the length of β is a third of that of α, then α and β generate a G_2 type of complex Lie algebra. This only occurs in G_2. Then $\alpha = \alpha_2$ is the long simple root, and β can be either α_1 or $3\alpha_1 + \alpha_2$. But this latter case cannot occur, since $3\alpha_1 + \alpha_2$ has the same length as α_2, and they generate an A_2 type of complex Lie subalgebra. Therefore, $\beta = \alpha_1$. We have $H = \frac{1}{3} H_{\alpha_2}$ and, since $(H, H)_A = 1$, we have $(H_{\alpha_2}, H_{\alpha_2})_A = 9$ and

$$\omega(X, Y) = \big(g_1 H + gG + B_1 i(2\alpha_1 + \alpha_2), [X, Y]\big).$$

The restricted metric is

$$\begin{bmatrix} \omega(F_{\alpha_1}, JF_{\alpha_1}) & \omega(F_{\alpha_1}, JF_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1}) & \omega(F_{\alpha_1+\alpha_2}, JF_{\alpha_1+\alpha_2}) \end{bmatrix} = \begin{bmatrix} \omega(F_{\alpha_1}, -G_{\alpha_1}) & \omega(F_{\alpha_1}, -G_{\alpha_1+\alpha_2}) \\ \omega(F_{\alpha_1+\alpha_2}, -G_{\alpha_1}) & \omega(F_{\alpha_1+\alpha_2}, -G_{\alpha_1+\alpha_2}) \end{bmatrix} = \begin{bmatrix} 3g_1 - 3B_1 & -3g \\ -3g & -3g_1 - 3B_1 \end{bmatrix}.$$

Therefore, the determinant is $9(B_1^2 - U)$.

Lemma 4. *If α has a 2-string, the determinant is $M(d - U)$ for some numbers M and $d > 0$. If α has a 4-string, the determinant is $(d - U)(9d - U)$ for a positive number d.*

From these three lemmas, we obtain our Theorem 1. □

4. Calculating the Ricci curvature

Let α_1 be the root that generates $\mathscr{A}$ and $h = \log V$. Following [Koszul 1955, p. 567], we have

$$\rho(X, JY) = \frac{L_{J[X_r, JY_r]}(\omega^n)(T, JT, F, JF, F_\alpha, JF_\alpha)}{2\omega^n(T, JT, F, JF, F_\alpha, JF_\alpha)}, \tag{42}$$

where X_r, Y_r are the corresponding right-invariant vector fields, and where we use F_α, JF_α to represent

$$F_{\alpha_2}, JF_{\alpha_2}, \ldots, F_{\alpha_l}, JF_{\alpha_l},$$

the array of F_α with its conjugate for positive roots α other than α_1 that have nonzero F_α and G_α.

We now use a method similar to that of [Guan and Chen 2000; Guan 2002; 2003] to calculate the Ricci curvature for the case $S = G = A_n$; later we shall compare the conclusion to Koszul's method. By the volume formula (29) — or by (33) and (41) — we have

$$a_\rho = n + 1 = b_\rho, \tag{43}$$

$$F_\rho = -(n-1)(\log D - \log(1-\gamma)) + \log(-D'). \tag{44}$$

Therefore,

$$\begin{aligned} f_\rho = \gamma F'_\rho &= -\gamma\big[(n-1)(D'D^{-1} + (1-\gamma)^{-1}) + D''(D')^{-1}\big] \\ &= -\frac{n-1}{t^2} + 2 + \frac{1+t^2}{2t}\dot{h}, \end{aligned} \tag{45}$$

and by (36), (38) we have

$$g_\rho = -\frac{2t}{1+t^2}(b_\rho - f_\rho) = \dot{h} - \frac{2(n-1)}{t} \quad \text{for } B_\rho = 0. \tag{46}$$

To use Koszul's method, we need to consider X, Y taking the values H, $G - H/t$, and then F, F. We have

$$\Big[H, J\Big(G - \frac{H}{t}\Big)\Big] = [H, F] = 2G,$$

$$J\Big[H_r, J\Big(G - \frac{H}{t}\Big)_r\Big] = -2JG = -2J\Big(G - \frac{H}{t} + \frac{H}{t}\Big) = 2(2T - F),$$

$$[F, JF] = \Big[F, -G + \frac{H}{t}\Big] = -2H - \frac{2G}{t},$$

$$J[F_r, JF_r] = J\Big(2H + \frac{2G}{t}\Big) = 2J\Big(-2tT + \frac{F - 2T}{t}\Big) = 2\frac{F - 2(1+t^2)T}{t}.$$

Again as in [Koszul 1955, p. 567–570], usually it is not clear how to find JX for a right-invariant vector field X along p_t and to deal with the left-invariant form with right-invariant vector fields. Therefore, the argument in [Spiro 2003] does not work for our situation. We need something similar to Koszul's trick [1955, p. 567–570]. It turns out that all the arguments there still go through for our situation once both X, JY are in the maximal compact Lie algebra $\mathcal{K}$. Therefore, we have

$$\begin{aligned}
&\rho\Big(H, J\Big(G-\frac{H}{t}\Big)\Big)\\
&\quad= 2\dot{h}+\frac{1}{2\omega^n(T, JT, F, JF, F_\alpha, JF_\alpha)}\\
&\qquad\quad\times\Big(\omega^n\big([2(F-2T), T]-J[2G, T], JT, F, JF, F_\alpha, JF_\alpha\big)\\
&\qquad\qquad\quad+\omega^n\big(T, [2(F-2T), JT]-J[2G, JT], F, JF, F_\alpha, JF_\alpha\big)\\
&\qquad\qquad\quad+\omega^n\big(T, JT, [2(F-2T), F]-J[2G, F], JF, F_\alpha, JF_\alpha\big)\\
&\qquad\qquad\quad+\omega^n\big(T, JT, F, [2(F-2T), JF]-J[G, JF], F_\alpha, JF_\alpha\big)\\
&\qquad\qquad\quad+\omega^n\big(T, JT, F, JF, [2(F-2T), F_\alpha]-J[2G, F_\alpha], JF_\alpha\big)\\
&\qquad\qquad\quad+\omega^5\big(T, JT, F, JF, F_\alpha, [2(F-2T), JF_\alpha]-J[2G, JF_\alpha]\big)\Big)\\
&\quad= 2\dot{h}-\frac{4(n-1)}{t}.
\end{aligned}$$

Here we have used the notation

$$\omega^n\big(\dots, [A, F_\alpha]-J[B, F_\alpha], JF_\alpha\big)$$

to represent

$$\begin{aligned}
&\omega^n\big(\dots, [A, F_{\alpha_2}]-J[B, F_{\alpha_2}], JF_{\alpha_2}, \dots, F_{\alpha_l}, JF_{\alpha_l}\big)\\
&+\cdots\\
&+\omega^n\big(\dots, F_{\alpha_2}, JF_{\alpha_2}, \dots, [A, F_{\alpha_l}]-J[B, F_{\alpha_l}], JF_{\alpha_l}\big),
\end{aligned}$$

which is the sum of

$$\omega^n\big(\dots, F_{\alpha_2}, JF_{\alpha_2}, \dots, [A, F_\alpha]-J[B, F_\alpha], JF_\alpha, \dots, F_{\alpha_l}, JF_{\alpha_l}\big)$$

for all the positive roots α other than α_1, and we have used the notation

$$\omega^n\big(\dots, F_\alpha, [A, JF_\alpha]-J[B, JF_\alpha]\big)$$

to represent a similar sum.

Another way to understand the calculation is by regarding the volume tensor formally as a product of the two determinant tensors. When $n=2$, these determinants are τ, τ_1 of the subspaces W, W_i. We have the formula

$$\rho(X, JY)=\tfrac{1}{2}J[X_r, JY_r](h)+\frac{A_{X,Y}(\tau)}{2\tau}+\frac{A_{X,Y}(\tau_1)}{2\tau_1}, \tag{47}$$

where

$$A_{X,Y}(\tau) = \sum_i \tau\big(\dots, [J[X, JY], X_i] - J[[X, JY], X_i], \dots\big). \tag{48}$$

Applying this formula, we have the components that come from the determinants τ and τ_1:

$$\frac{A_{H,G-H/t}(\tau)}{2\tau} = 0,$$

since

$$\begin{gathered}
[F-2T, T] = -J[G, T] = 0,\\
[F-2T, JT] = \big[F-2T, H/(2t)\big] = -G/t + H/t^2 = t^{-1}JF,\\
-J[G, JT] = -J\big[G, H/(2t)\big] = -t^{-1}JF,\\
[F-2T, F] = 0, \quad -J[G, F] = -2JH = 4tT,\\
[F-2T, JF] = [F-2T, -G+t^{-1}H] = -2H - 2t^{-1}G + 2t^{-2}H = 2t^{-1}JF - 2H,\\
-J[G, JF] = -t^{-1}J[G, H] = -2t^{-1}JF;
\end{gathered}$$

and

$$\frac{A_{H,G-H/t}(\tau_1)}{2\tau_1} = -\frac{4}{t},$$

since

$$\begin{gathered}
[F-2T, F_{23}] = F_{13},\\
-J[G, F_{23}] = -JG_{13} = -J(G_{13} + 2t^{-1}G_{23} - 2t^{-1}G_{23}) = -2t^{-1}F_{23} - F_{13},\\
[F-2T, JF_{23}] = [F-2T, G_{23}] = G_{13} = -JF_{13} - 2t^{-1}JF_{23},\\
-J[G, JF_{23}] = -J[G, G_{23}] = JF_{13},\\
[F-2T, F_{13}] = -F_{23}, \quad -J[G, F_{13}] = -JG_{23} = F_{23},\\
\begin{aligned}[F-2T, JF_{13}] = [F-2T, -G_{13} - 2t^{-1}G_{23}] &= G_{23} - 2t^{-1}G_{13} - 4t^{-2}G_{23}\\
&= JF_{23} + 2t^{-1}JF_{13},\end{aligned}\\
-J[G, JF_{13}] = -J[G, -G_{13} - 2t^{-1}G_{23}] = -JF_{23} - 2t^{-1}JF_{13}.
\end{gathered}$$

Similarly, we have:

Theorem 5. *If the fiber with $S = A_n$ action is affine and isotropic, then $g_\rho = \dot{h} - 2(n-1)/t$. Moreover, $B_\rho = 0$. Other coefficients, that is, other parts of I_ρ, come from the Ricci curvature of G/P, which is $-(q_{G/P}, [X, Y])_0$ with $q_{G/P} = \sum_{\alpha\in\Delta^+-\Delta_P} H_\alpha$ with the standard inner product.*

Proof. As above, we consider X, Y taking the values H, $G-H/t$ and F, F. First,

$$\Big[H, J\Big(G-\frac{H}{t}\Big)\Big]=2G, \quad J\Big[H_r, J\Big(G-\frac{H}{t}\Big)_r\Big]=2(2T-F).$$

As above, the contribution of T, JT, F, JF is zero. The contribution of e_2-e_j, e_1-e_j is $-4/t$. When $G\neq S$, the contribution from the roots outside S is zero. Therefore,

$$2g_\rho=\rho(H,F)=\rho(H,-J^2F)=\rho\Big(H, J\Big(G-\frac{H}{t}\Big)\Big)=2\Big(\dot{h}-\frac{2(n-1)}{t}\Big).$$

That is, $g_\rho=\dot{h}-2(n-1)/t$.

Next,

$$[F, JF]=-2H-\frac{2G}{t}, \quad J[F_r, JF_r]=\frac{2}{t}\big(F-2(1+t^2)T\big).$$

The contribution of T, JT, F, JF is zero. The contribution of e_2-e_j, e_1-e_j is $4(t+1/t)$. When $G\neq S$, the contribution from the roots outside S is zero. Therefore,

$$\rho(F, JF)=-2\Big(t+\frac{1}{t}\Big)\Big(\dot{h}-\frac{2(n-1)}{t}\Big),$$

and $B_\rho=0$.

Other coefficients come from the $q_{G/P}$ as above. □

5. Calculating the scalar curvature

To calculate the scalar curvature, we separate our subspaces into five kinds of spaces. The first W is generated by T, JT, F, JF. The second, third, fourth and fifth are the subspaces of 1-, 2-, 3-, and 4-strings. The Ricci curvature is a sum of its restrictions to each subspace:

(49) $$\rho=\sum_i \rho_i.$$

Similarly,

(50) $$\omega=\sum_i \omega_i.$$

Then, by Theorem 1, we have

(51) $$V=\frac{M\dot{U}Q(U)}{t}=\frac{M\dot{U}}{t}(U-B^2)^{k-1}Q_1(U)$$

and

(52) $$\rho\wedge\omega^{n-1}=\sum_i \Omega_i,$$

where

$$\Omega_i = \rho_i \wedge \omega^{n-1}. \tag{53}$$

Let

$$U_\rho = (g_1 H + gG, t g_\rho H + g_\rho G); \tag{54}$$

then

$$U_\rho(0) = 0. \tag{55}$$

Furthermore,

$$\Omega_W = (n-1)!\, K \dot{U}_\rho Q(U)/t$$

if the determinant of W is $K\dot{U}/t$. For 1-strings,

$$\Omega_i = K_i \dot{U} Q(U)/t.$$

For 2-strings,

$$\Omega_i = -2(n-1)!(U_\rho - a_i a_{\rho,i})V/q_i,$$

where $q_i = a_i^2 - U$ is the linear factor of Q introduced by the given 2-string. Similarly, we can see, by a direct calculation, that for a 3-string

$$\Omega_i = -\Big(2U_\rho - 2a_i a_{\rho,i} + \frac{a_{\rho,i}}{a_i}(U - a_i^2)\Big)\frac{(n-1)!V}{q_i}.$$

The case of 4-strings only occurs when $G = G_2$ and H corresponds to the short root. In this case,

$$\begin{aligned}
\Omega_1 &= \rho_1 \wedge \omega^{n-1} \\
&= -4\big(U_\rho(5B_1^2 - U) + B_1 B_{\rho,1}(5U - 9B_1^2)\big)\frac{(n-1)!V}{(B_1^2 - U)(9B_1^2 - U)} \\
&= -2\big[U_\rho\big((B_1^2 - U) + (9B_1^2 - U)\big) - B_1 B_{\rho,1}\big(9(B_1^2 - U) + (9B_1^2 - U)\big)\big] \\
&\qquad \times \frac{(n-1)!V}{(B_1^2 - U)(9B_1^2 - U)} \\
&= -2(U_\rho - 9B_1 B_{\rho,1})\frac{(n-1)!V}{9B_1^2 - U} - 2(U_\rho - B_1 B_{\rho,1})\frac{(n-1)!V}{B_1^2 - U}.
\end{aligned}$$

Therefore,

$$\rho \wedge \omega^{n-1} = (n-1)!\, M \frac{(U_\rho \dot{Q}(U)) + p_0(U)\dot{U}}{t}. \tag{56}$$

Theorem 6. *The scalar curvature is*

$$\frac{2(U_\rho \dot{Q}) + p\dot{U}}{\dot{U} Q},$$

where p is a polynomial in U of the form $p(U) = (U - B^2)^{n-1} P_1(U)$; furthermore the polynomial $P_1(U)$ is a positive linear sum of

(1) *Q_1 and*

(2) *the products of* $\deg Q_1 - 1$ *linear factors of Q_1.*

Only 1-strings and 3-strings contribute to (1); the contribution of each 1-string and 3-string is $c_{\rho,l}/c_l$ for the Q_1 term, where $c_i = \omega(F_{\alpha_i}, JF_{\alpha_i})$ for 1-strings and $c_i = a_i$ for 3-strings. Only 2-strings, 3-strings and 4-strings contribute to (2); the contribution of each 2-string and 4-string related to the products of $\deg Q_1 - 1$ *linear factors of Q_1 is $2(a_{\rho,i} a_i Q_1)/q_i$. In particular, if $G = S$, we have $p(U) = 0$.*

6. Setting up the equations

Now, we set up the equations for the metrics with constant scalar curvature. Before we do that, we shall understand more about the metrics.

Theorem 7. *If $S = A_n$, ω is a metric on the open orbit if and only if $B < -\frac{1}{2}\dot{g}(0)$ and g is an odd function with $\dot{g}(0) < 0$, $t^{-1}\dot{U} > 0$, and $U < a_i^2$.*

Proof. From the metric formula for the metrics, we need

$$\lim_{t\to 0} \frac{t\dot{g} + g}{t} = 2\dot{g}(0) < 0,$$

$$\lim_{t\to 0} \left(\frac{(1+t^2)g}{t} + B \right) = \dot{g}(0) + B < 0,$$

$$\lim_{t\to 0} (tg + 2B + 2t^{-1}g) = 2B + 2\dot{g}(0) < 0,$$

$$\lim_{t\to 0} t^{-1}g = \dot{g}(0) < 0,$$

$$\lim_{t\to 0} t^{-1}\dot{U} = 2\dot{g}(0)B + (\dot{g}(0))^2 > 0$$

and

$$t^{-1}U > 0. \qquad \square$$

This result is somehow quite different from those in [Guan 2006; 2009]. Therefore, with the property also that $B_\rho = 0$ in Theorem 5, we prefer to call the manifolds in the case $S = A_n$ *type IV manifolds*.

To understand the metrics near the hypersurface orbit, we can let $\theta = t^2/(1+t^2)$, and we see that

$$\dot{\theta} = \frac{2t}{1+t^2} - \frac{2t^3}{(1+t^2)^2} = \frac{2t}{(1+t^2)^2}.$$

We can also see that $U_\theta(1) = \lim_{t\to+\infty} (1+t^2)^2 \dot{U}/2t > 0$ exists. In particular, U is bounded, and so is tg. This was done in Section 3. Let $l = \lim_{t\to+\infty} tg$.

The closure D of the orbit Ω of the complex Lie group SL(2, $\mathbb{C}$) generated by α_1 is a cohomogeneity one fiber bundle with a $\mathbb{C}P^1$ as the base and another $\mathbb{C}P^1$ as the fiber. Since Ω is a $\mathbb{C}$ bundle over $\mathbb{C}P^1$, D is affine compact almost-homogeneous manifold with the SL(2, $\mathbb{C}$) action. That is, D is exactly the $S = A_1$ action manifold and is $\mathbb{C}P^1 \times \mathbb{C}P^1$. A calculation in Section 3 for the $S = A_1$ action also gives the bounded property of U and l. The restriction of the metric to D also gives us the same topological invariants B and l.

Theorem 8. *The metric ω of Section 3 extends to a Kähler metric over the exceptional divisor if and only if* $\lim_{t\to+\infty} tg = l > a_i - B$ *and* $U_\theta(1) > 0$.

Now, for any given pair B, l with $0 > l > a_i - B$, we can check that $g(t) = lt/(1+t^2)$ satisfies Theorems 7 and 8. We shall see later on that this actually gives us the solutions of our equations for the homogeneous cases, that is, when $G = S$. Thus:

Theorem 9. *The Kähler classes are in one-to-one correspondence with the elements in the set*

$$\Gamma = \big\{(B, l) \mid 0 > l > a_i - B \text{ and } B < -l/2\big\}.$$

To calculate the total volume, we notice that

$$T \wedge JT \wedge F \wedge JF \textstyle\bigwedge_{\alpha=\alpha_2}^{\alpha_l}(F_\alpha \wedge JF_\alpha) = M \dfrac{T \wedge H \wedge F \wedge G \bigwedge_{\alpha=\alpha_2}^{\alpha_l}(F_\alpha \wedge G_\alpha)}{t} \tag{57}$$

with a positive number M. Moreover,

$$U(0) = B^2, \quad U(+\infty) = (l + B)^2. \tag{58}$$

Therefore, the total volume is

$$V_T = \int_{B^2}^{(l+B)^2} Q(U)\, dU. \tag{59}$$

We also see that

$$g_\rho = \dot{h} - \frac{2(n-1)}{t} = \frac{\ddot{U}}{\dot{U}} + \frac{Q'(U)\dot{U}}{Q(U)} - \frac{2n-1}{t}. \tag{60}$$

One can easily check that

$$\left(\frac{\ddot{U}}{\dot{U}} - \frac{1}{t}\right)(0) = 0 \quad \text{and} \quad \left(\frac{\dot{U}}{U - B^2} - \frac{2}{t}\right)(0) = \dot{U}(0) = 0,$$

since g is an odd function and therefore $g_\rho(0) = 0$.

Now, from

$$U = (tg + B)^2 + g^2 = (t^2+1)g^2 + 2Btg + B^2 = (t^2+1)\left(g + \frac{Bt}{t^2+1}\right)^2 + \frac{B^2}{1+t^2}$$

we have

$$\left(g+\frac{Bt}{t^2+1}\right)^2=\frac{1}{(1+t^2)^2}\big((1+t^2)U-B^2\big),$$

which we can solve to get

$$g=-\frac{\sqrt{(1+t^2)U-B^2}+Bt}{1+t^2}.$$

For clarity, we replace t by $\theta=t^2/(1+t^2)$. Then

$$\begin{aligned}tg_\rho&=\Big[\big[\log[U_\theta Q(U)(1-\theta)^2]\big]_\theta 2\theta(1-\theta)-2(n-1)\Big]\\&=\left[2\theta(1-\theta)\left[\frac{U_{\theta\theta}}{U_\theta}+\frac{Q'(U)U_\theta}{Q(U)}\right]-4\theta-2(n-1)\right],\end{aligned}$$

which has the limit $-2(n+1)$ at $\theta=1$, so

$$l_\rho=-2(n+1). \tag{61}$$

Therefore, the Ricci class is $(0,-2(n+1))$.

We also see that

$$U_\rho(1)=l_\rho(B+l)=-2(n+1)(B+l). \tag{62}$$

Now, we have the Kähler–Einstein equation

$$\begin{aligned}\left[2\theta(1-\theta)\left[\frac{U_{\theta\theta}}{U_\theta}+\frac{Q'(U)U_\theta}{Q(U)}\right]-4\theta-2(n-1)\right]=tg&=-\frac{t\sqrt{(1+t^2)U}}{1+t^2}\\&=-\sqrt{\theta U}.\end{aligned} \tag{63}$$

The total scalar curvature is

$$R_T=\int_0^{+\infty}\big[p(U)\dot U+2(U_\rho\dot Q(U))\big]\,dt. \tag{64}$$

From this, we have the average scalar curvature

$$\begin{aligned}R_0=\frac{R_T}{V_T}&=\frac{\int_{B^2}^{(B+l)^2}p(U)\,dU+2(U_\rho Q(U))\big|_{B^2}^{(B+l)^2}}{\int_{B^2}^{(B+l)^2}Q(U)\,dU}\\&=\frac{\int_{B^2}^{(B+l)^2}p(U)\,dU+2l_\rho(B+l)Q\big((B+l)^2\big)}{\int_{B^2}^{(B+l)^2}Q(U)\,dU}.\end{aligned}$$

If $G=S=A_n$ (by [Guan 2009], this is the same as the assumption that the manifold is homogeneous), then $Q=(U-B^2)^{n-1}$ and $p=0$. Therefore,

$$R_0=\frac{l_\rho(B+l)}{n^{-1}\big((B+l)^2-B^2\big)}=2n\frac{Bl_\rho+ll_\rho}{2Bl+l^2}.$$

The equation of constant scalar curvature is $R/V = R_0$. Therefore,

$$2U_\rho Q(U) + \int_{B^2}^{U} p(U)\,dU = R_0 \int_{B^2}^{U} Q(U)\,dU + A_0, \tag{65}$$

with A_0 a constant.

Letting $\theta = 0$, we have

$$0 = 2BB_\rho Q(B^2) = A_0.$$

If we put $\theta = 1$ in, we get the same A_0.

We have

$$U_\rho = \frac{R_0 \int_{B^2}^{U} Q\,dU - \int_{B^2}^{U} p\,dU}{2Q(U)}, \tag{66}$$

where $Q(U) = (U - B^2)^{n-1} Q_1(U)$.

Applying Theorem 6 and integrating by parts, we obtain

$$\begin{aligned} U_\rho &= \frac{R_0 \int_{B^2}^{U} Q\,dU - \int_{B^2}^{U} (U - B^2)^{n-1} P_1\,dU}{2Q} \\ &= \frac{\int_{B^2}^{U} \big(R_0 Q - (U - B^2)^{n-1} P_1\big)\,dU}{2Q} = \frac{R(U)}{2Q_1(U)}, \end{aligned}$$

where $R(U)$ is a polynomial in U. Therefore,

$$g_\rho((t^2+1)g + Bt) = \frac{um(u)}{Q_1(u)},$$

where we let $R(U) = 2um(U)$.

If $G = S = A_n$, we have successively

$$U_\rho = \frac{R_0}{2n}(U - B^2), \quad R(U) = \frac{R_0}{n}(U - B^2), \quad m(u) = \frac{R_0}{2n}.$$

Now, since $tg = -B\theta - \sqrt{\theta(u + B^2\theta)}$, we have

$$(1 + t^2)tg + Bt^2 = -\frac{\sqrt{\theta(u + B^2\theta)}}{1 - \theta},$$

and therefore, if we use $'$ for the derivative with respect to θ, we have

$$\theta(1-\theta)\left[\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right] - 2\theta - n + 1 = -2^{-1}\sqrt{\frac{\theta}{u + B^2\theta}}\,u\,\frac{d}{du}\left(\frac{m(u)}{Q_1(u)}\right). \tag{67}$$

Comparing with (63), we see that

$$m(u) = Q_1(u)$$

if the Kähler metric is in the Ricci class.

If $G = S = A_n$, then $m(u)/Q_1$ is a constant. There is a solution with $u = c\theta$. Actually, if we use $g = lt/(1+t^2)$ in the proof of Theorem 9, we get $u = (2B+l)\, l\theta$, which solves our equation.

From (67), we have $\big(\log(u' Q(u))\big)' = \dfrac{P}{\theta(1-\theta)}$. We also have

$$2\theta + n - 1 - A_{B,l}\theta^{1/2} \le P \le 2\theta + n - 1 + C_{B,l}\theta^{1/2}.$$

for some positive constants $A_{B,l}$, $C_{B,l}$, which depend only on B and l. Since $P(1) = n + 1 + 2^{-1}l_\rho = 0$, we have $A_{B,l} \ge n+1$.

By integration,

$$\frac{a^{n-1}(1-a^{\frac{1}{2}})^{A_{B,l}-n-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+n+1}}{\theta^{n-1}(1-\theta^{\frac{1}{2}})^{A_{B,l}-n-1}(1+a^{\frac{1}{2}})^{A_{B,l}+n+1}} \le \frac{u'(a)u^{n-1}(a)Q_1(u(a))}{u'(\theta)u^{n-1}(\theta)Q_1(u(\theta))} \le \frac{a^{n-1}(1-\theta^{\frac{1}{2}})^{n+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{n+1-C_{B,l}}}{\theta^{n-1}(1-a^{\frac{1}{2}})^{n+1+C_{B,l}}(1+a^{\frac{1}{2}})^{n+1-C_{B,l}}} \tag{68}$$

for $0 < \theta \le a < 1$. We let $V = u^n$ and $x = \theta^n$, and obtain the Harnack inequality

$$\frac{(1-a^{\frac{1}{2}})^{A_{B,l}-n-1}(1+\theta^{\frac{1}{2}})^{A_{B,l}+n+1}}{(1-\theta^{\frac{1}{2}})^{A_{B,l}-n-1}(1+a^{\frac{1}{2}})^{A_{B,l}+n+1}} \le \frac{V_x(a)Q_1(u(a))}{V_x(\theta)Q_1(u(\theta))} \le \frac{(1-\theta^{\frac{1}{2}})^{n+1+C_{B,l}}(1+\theta^{\frac{1}{2}})^{n+1-C_{B,l}}}{(1-a^{\frac{1}{2}})^{n+1+C_{B,l}}(1+a^{\frac{1}{2}})^{n+1-C_{B,l}}}. \tag{69}$$

Arguing as in [Guan 2002], we have:

Theorem 10. *If there is a solution $0 \le u \le l(l+2B)$ of the equation above with $u(0) = 0$ and $u(1) = l(l+2B)$, then there is a Kähler metric with constant scalar curvature in the considered Kähler class.*

Theorem 11. *For any small positive number f, there is a solution $u(0) = 0$, $u(1-f) = l(l+2B)$. This corresponds to a Kähler metric with constant scalar curvature on the manifold with boundary $\theta \le 1 - f$.*

7. Global solutions

In this section, we shall extend our solutions to the hypersurface orbit. We shall let $f \to 0$. As we did in [Guan 2002], we let $\tau = -\log(1-\theta)$ and have

$$\big[\log(u_\tau Q(u))\big]_\tau = \frac{P-\theta}{\theta}.$$

Therefore,

$$
\begin{aligned}
(70)\qquad \left[\log\left(\frac{nu^{n-1}u_\tau}{\theta^{n-1}}Q_1(u)\right)\right]_\tau &= \frac{P-\theta}{\theta} - \frac{(n-1)\theta_\tau}{\theta} \\
&= \frac{P-\theta}{\theta} - (n-1)\left(\frac{1}{\theta} - 1\right) \\
&= \frac{P-n+1+(n-2)\theta}{\theta} \\
&= n - \frac{2^{-1}u}{\sqrt{\theta(u+B^2\theta)}}\frac{\dot{m}(u)}{Q_1(u)} = T(u,\theta) \\
&\to n - \frac{um(u)}{2Q_1(u)\sqrt{u+B^2}} = n-\alpha,
\end{aligned}
$$

when θ turns to 1 and it converges uniformly for $u \geq u_0$ with any $u_0 > 0$.

If ω is in the Ricci class, then $m(u) = Q_1(u)$ and

$$\alpha = 2^{-1}\sqrt{u}.$$

Let u_i be a series of solutions corresponding to $f_i \to 0$. By $P(1) = 0$, for any $e_0 \in (n, n+1)$ there are two numbers $A(e_0) < l(l+2B)$ and $B(e_0) > 0$ such that if $u > A(e_0)$ and $\tau > B(e_0)$, then $\alpha > e_0 > n$ and $T(u, \theta(\tau)) < n - e_0$. Let τ_i be a point of τ such that $u_i(\tau_i) = A(e_0)$, and if we also have $\tau_i > B(e_0)$, then

$$\left[\log\left(\frac{nu_i^{n-1}u_{i,\tau}}{\theta^{n-1}}Q_1(u_i)\right)\right]_\tau = \frac{P-n+1+(n-2)\theta}{\theta} = T(u,\theta) < n - e_0$$

for $\tau \geq \tau_i$.

Let $w = ((nu^{n-1}u')/\theta^{n-1})Q_1(u)$. Then

$$w_i \leq e^{(n-e_0)(\tau-\tau_i)}w_i(\tau_i).$$

If there is no subsequence of τ_i that tends to $+\infty$, then there is a subsequence of τ_i that tends to a finite number τ_0. By the left side of the Harnack inequality (69), we see that $V_{i,x}(\theta(\tau_0))$ must be bounded from above, otherwise $V_{i,x}$ will be bounded from below by a very large number such that V_i will be bigger than $l(l+2B)$ before x reaching the point 1. That is, there is a subsequence of u_i converging to a solution u of our equation with $u(1) > A(e_0)$.

We shall observe that there is no subsequence of τ_i that tends to $+\infty$ under a certain condition given below.

If there is a subsequence of τ_i that tends to $+\infty$, we might as well assume that

$$\lim_{i\to+\infty} \tau_i = +\infty,$$

and $\tau_i > B(e_0)$. To make the things simpler, we should avoid the cases in which $G = S = A_n$. In those cases, the second Betti numbers are 2 and the manifolds are homogeneous. By Calabi's result, all extremal metrics are homogeneous, and therefore they are unique since there is only one invariant metric in the given Kähler class. As we saw immediately after (67), $u = c\theta$ will solve the equations.

Thus, we can assume that $G \neq S$, and therefore there is at least one a_i. From (67), we observe that if

$$u_{i,\tau}(\tau_i)u_i^{n-1}(\tau_i) > 2(l(2B+l))^{n-1}(a_1^2 - B^2)A_{B,l} > 2u^{n-1}(a_1^2 - U)A_{B,l},$$

then

$$\frac{u_{i,\tau}(\tau_i)}{a_1^2 - U(\tau_i)} > 2A_{B,l}$$

and $v_\tau = u_i^{n-1}u_{i,\tau}$ is increasing for $\tau \geq \tau_i$. This cannot happen. Therefore, $u_{i,\tau}(\tau_i)$ is bounded from above.

We shall see that in this circumstance there is a subsequence of

$$\tilde{u}_i(\tau) = u_i(\tau + \tau_i)$$

that converges in C^1 norm to a nonconstant function $\tilde{u}$. For each $\tau \geq 0$, the sequence w_i is decreasing and the $\tilde{u}_{i,\tau}$ are uniformly bounded. For each $\tau < 0$, $-A_{B,l} < [\log w_i]_\tau < n + C_{B,l}$ when i is big enough; that is, the $\tilde{V}_{i,\tau}$ are also bounded uniformly on i over any closed intervals. Therefore, a subsequence of $\tilde{V}_i$ converges in the C^1 norm to a function $\tilde{u}$. Thus, the same thing happens for a subsequence of $\tilde{u}_i$.

To see that $\tilde{u}$ is not a constant, we notice that

$$\frac{nu_i^{n-1}u_{i,\tau}}{\theta^{n-1}} \leq C_i \frac{nu_i^{n-1}(\tau_i)u_{i,\tau}(\tau_i)}{\theta^{n-1}(\tau_i)} e^{(n-e_0)(\tau-\tau_i)}$$

for $\tau \geq \tau_i$, where C_i does not depend on u_i. That is,

$$nu_i^{n-1}u_{i,\tau} \leq Cu_{i,\tau}(\tau_i)e^{(n-e_0)(\tau-\tau_i)}.$$

By integrating both sides,

$$(l(l+2B))^n - A(e_0)^n \leq -\frac{C}{n-e_0}u_{i,\tau}(\tau_i),$$

that is, $u_{i,\tau}(\tau_i)$ is bounded from below. Therefore, the $\tilde{u}_{i,\tau}(0)$ are bounded from below. We have $\tilde{u}_\tau(0) > 0$. This implies that $\tilde{u}$ is not a constant.

Then, $\tilde{u}$ satisfies the equation

$$\left[\log[x^{n-1}x_\tau Q_1(x)]\right]_\tau = -\alpha + n$$

on $(-\infty, +\infty)$. Therefore,

$$[x^{n-1}x_\tau Q_1(x)]_\tau = (-\alpha+n)x^{n-1}Q_1(x)x_\tau.$$

As in [Guan 2002], we integrating and obtain

$$\int_{x(-\infty)}^{x(+\infty)} f_l\,dx = 0,$$

where $f_l = (-\alpha+n)x^{n-1}Q_1(x)$; thus $x(+\infty) = l(l+2B)$.

Lemma 12. *$n-\alpha$ has only one zero.*

Proof. As in [Guan 2002], we may expect that x is related to a Kähler metric of constant scalar curvature on the normal line bundle over the hypersurface orbit. Hence, we may apply the method of counting zeros in [Guan 1995; 2002] to this circumstance. The expression $x^{n-1}x'Q_1(x)$ is proportional to "φQ" in the first of these papers. Therefore, the counting of zeros of $n-\alpha$ should be the same as counting the zeros of the derivative of "φQ" to "U" there.

Set $v=\sqrt{u+B^2}$; then $u=v^2-B^2$ and $a_i^2-u=(-a_i+v)(-a_i-v)$. We observe that $g_l = 2vf_l$ is actually a polynomial in v and should be proportional to the derivative of "φQ" in [Guan 1995]; thus we expect that $y=\frac{2}{l}(-B-v)-1$ corresponds to the "U" in that paper. We let

$$q = 2vQ(v),$$

and observe that q is proportional to the "Q" in [Guan 1995]. We have

$$g_l = nq - m(u)u^n = nq - \frac{R(U)}{2}u^{n-1} = nq - \frac{R_0}{2}\int Q\,dU + \frac{1}{2}\int p\,dU. \tag{71}$$

Letting g_l' be the derivative of g_l to v, we have

$$g_l' = nq' - vR_0Q + vp = nq' + vP_2 - vR_0Q + vP_3 = \Delta - mq, \tag{72}$$

where $P_3 = 2m_1Q$ is the Q term in p and $P_2 = p - P_3$ is a positive linear combination of Q/q_i; further, $\Delta = nq' + vP_2$, and $m = (R_0/2) - m_1$. Therefore,

$$g_l = \int_0^v (\Delta - mq)\,dv.$$

Lemma 13. *The coefficients of Δ are always positive.*

Proof. From Theorem 6, the 1-strings do not contribute to Δ.

The contribution to P_2 of each 2-, 3-, and 4-string of the $U-B^2$ factor is in the first term of $p(U)$ in Theorem 6.

The contribution to P_2 of each 2-, 3-, and 4-string related to the Q_1 factors is $(a_{\rho,i}a_i/q_i)q$.

For the first term of Δ we have $2nQ$, with $2n > 0$. One might call this the v factor term since $Q = q/(2v)$.

Then, we have the $U - B^2$ term

$$2(n-1)v(2nv)(U-B^2)^{n-2}Q_1 = (n-1)v[2n(v-B)+2n(v+B)](U-B^2)^{n-2}Q_1$$

with $2n$ positive.

Similarly, we have the q_s factor of the Q_1 term:

$$2v(-2nv + a_s a_{\rho,s})\frac{Q}{q_s} = v\big((2n - a_{\rho,s})(a_s - v) - (2n + a_{\rho,s})(a_s + v)\big)\frac{Q}{q_s}$$

with coefficients $2n - a_{\rho,s} > 0$ and $-2n - a_{\rho,s}$.

So we need to check that the last coefficient is also positive. There are two ways to do this. One is to check that the coefficients $2n, 2n, 2n$ and $2n - a_{\rho,s}$, $-2n - a_{\rho,s}$ are all positive. We claim that these are the components of the Ricci curvature of the exceptional divisor; then the positivity comes from the positivity of the Ricci curvature of the compact rational homogeneous spaces. The point is that v corresponds to an H in the calculation of the metric and the volume form, and we should prove that the contribution of H to the Ricci curvature is exactly $2n$; that is,

$$(q_{G/P_\infty}, H)_0 = (q_{S/(S\cap P_\infty)}, H)_0 = 2n,$$

where P_∞ is the isotropic group of the exceptional divisor at p_∞. Notice that P_∞ is parabolic.

For $S = A_n$, the semisimple part of $P_{\infty,1}$ is generated by $\alpha_3, \ldots, \alpha_n$ with an orientation $e'_1 = e_1$, $e'_i = e_{i+1}$ $n+1 > i > 1$, $e'_{n+1} = e_2$. Therefore,

$$(q_{S/P_{\infty,1}}, H)_0 = n + n = 2n.$$

This gives a proof of our Lemma 12.

The second way to check the positivity of the last coefficient is a case-by-case analysis. That will also give all the $a_{\rho,s}$ in concrete calculations. This is extremely useful when we check the Fano property of the manifolds and classify the manifolds with higher-codimension ends [Guan 2011a; 2011b; ≥ 2011b]. For example, we can check this:

Proposition 14. *In the affine isotropic case, the manifold is Fano if and only if*

$$-2(n+1) - a_{\rho,s} > 0.$$

We give another proof that the last coefficient $-2n - a_{\rho,s} > 0$. This is a little bit long, since there are many cases. We shall check the last inequality $2n + a_{\rho,s} < 0$ for the cases $G = A_{m+n+k}, B_{m+n+k+1}, C_{m+n+k+1}, D_{m+n+k+1}, G_2$. We will leave the cases $G = F_4, E_8$ for another paper, since the proof is tedious. The cases of $G = E_6, E_7$ will follow from those of E_8.

Case 1: $G = A_{m+n+k}$. In this case

$$\rho_{G/P}(F_{e_l-e_{m+1}}, JF_{e_l-e_{m+1}}) = -(q_{G/P}, -2H_{e_l-e_{m+1}}) = 2(-l_1-l_2+2m+n+2).$$

Also $-2H_{e_l-e_{m+1}} = -2H_{e_l} - H - H_{e_{m+1}+e_{m+2}}$, so

$$a_{\rho,l} = -2(-l_1-l_2+2m+n+2) \leq -2(n+2).$$

The corresponding affine manifolds are Fano.

Case 2: $G = B_{m+n+k+1}$. Here we have $(q_{G/P}, e_l)_0 = -l_1 - l_2 + 2(m+n+k)+3$ in the standard inner product, but we took an inner product such that $(e_l, e_l) = \frac{1}{2}$. Therefore, either

(a) $B_{\rho,l} = 2(l_1+l_2-2(m+n+k)-3)$ if $l_1 \leq l \leq l_2$ and there is an S_1 factor $A_{l_2-l_1}$ or l is not in any S_1 factor, in which case we let $l_1 = l = l_2$; or

(b) $B_{\rho,l} = 0$ if l is in a S_1 factor of type B.

We have 2-strings generated by $e_l - e_{m+1}$, $e_l + e_{m+2}$, e_{m+2} with $l \leq m$, $e_{m+2}+e_i$ with $m+2 < i \leq m+n+1$ and $e_{m+2} \pm e_j$ with $m+n+1 < j \leq m+n+k+1$. The corresponding $a_{\rho,s}$ are

$$\begin{aligned}
-2(-l_1-l_2+2(m+n+k)+3-1-n-2k) &= -2(2m+2+n-l_1-l_2)\\
&\leq -2(n+2),\\
-2(-l_1-l_2+2(m+n+k)+3+1+n+2k) &= -2(2(m+2k+2)+3n-l_1-l_2)\\
&\leq -2(3n+4),\\
-2(1+n+2k) &\leq -2(n+1),\\
-2(2+2n+4k) &\leq -4(n+1),
\end{aligned}$$

and

$$\begin{aligned}
&-2\big(1+n+2k-(l_1+l_2-2(m+n+k)-3)\big) \leq -2(1+n+2k+1) \leq -2(n+4),\\
&-2\big(1+n+2k+(l_1+l_2-2(m+n+k)-3)\big) \leq -2(1+n+2k-2k+1) = -2(n+2)
\end{aligned}$$

in case (a) or

$$-2(1+n+2k) \leq -2(n+1)$$

in case (b). The corresponding manifolds are nef and Fano if and only if $k > 0$.

If $G = B_{m+1}$ and $S = A_1$ is generated by e_{m+1}, then $H = 2H_{e_{m+1}}$. Because $(H, H)_A = 1$, we get $(e_{m+1}, e_{m+1})_A = \frac{1}{4}$. There are 3-strings generated by $e_l - e_{m+1}$, and we have

$$a_{\rho,l} = \frac{B_{\rho,l}}{2} = -2(-l_1-l_2+2m+3) \leq -6 = -2(n+2).$$

The corresponding affine manifold is Fano.

Case 3: $G = C_{m+n+k+1}$. Then either

(a) $B_{\rho,l} = -2(-l_1 - l_2 + 2(m+n+k+2))$ if $l_1 \le l_2$ and there is an S_1 factor $A_{l_2-l_1}$ or l is not in any S_1 factor (in this case $l_1 = l = l_2$); or

(b) $B_{\rho,l} = 0$ if l is an S_1 factor of type C.

We have 2-strings generated by $e_l - e_{m+1}$, $e_l - e_{m+2}$ with $l \le m$, $e_{m+2} + e_i$ with $m+2 < i \le m+n+1$, $e_{m+2} \pm e_l$ with $m+n+1 < l \le m+n+k+1$, and 3-string generated by $2e_{m+2}$. The corresponding $a_{\rho,s}$ are

$$\begin{aligned}
-2(-l_1-l_2+2(m+n+k+2)-2-n-2k) &= -2(-l_1-l_2+2m+n+2)\\
&\le -2(n+2),\\
-2(-l_1-l_2+2(m+n+k+2)+2+n+2k) &= -2(-l_1-l_2+2(m+2k+3)+3n)\\
&\le -6(n+2),\\
-2(2n+4+4k) &\le -4(n+2),\\
-2(n+2+2k-l_1-l_2+2(m+n+k+2)) &\le -2(n+4+2k) \le -2(n+6)\\
(\text{or } -2(n+2+2k) &\le -2(n+2)),\\
-2(n+2+2k+l_1+l_2-2(m+n+k+2)) &\le -2(n+2+2k-2k) = -2(n+2)\\
(\text{or } -2(n+2+2k) &\le -2(n+2)),
\end{aligned}$$

and

$$-2(2n+4+4k) \le -4(n+2).$$

The corresponding affine manifolds are Fano.

If $S = A_1$ and $G = C_{m+1}$, then $\alpha = 2e_{m+1}$. Since $[H_{2e_{m+1}}, F_{2e_{m+1}}] = 4G_{2e_{m+1}}$ and $[H, F] = 2G$, we get $H = \frac{1}{2}H_{2e_{m+1}}$. Since $(H, H)_A = 1$, we have $(e_{m+1}, e_{m+1})_A = 1$. We only need to consider are the 2-strings generated by $e_l - e_{m+1}$; then

$$\begin{gathered}
\omega(F_{e_l-e_{m+1}}, JF_{e_l-e_{m+1}}) = \left(\tfrac{1}{2}aH_{2e_{m+1}} + iB_le_l, -2H_{e_l-e_{m+1}}\right)_A = 2a - 2B_l,\\
a_{\rho,l} = B_{\rho,l} = -(-l_1 - l_2 + 2(m+2)) \le -4 = -2(n+1).
\end{gathered}$$

The corresponding affine manifold is nef but not Fano.

Case 4: $S = A_n$ and $G = D_{m+n+k+1}$. Then either

(a) $B_{\rho,l} = -2(-l_1 - l_2 + 2(n+m+k+1))$ if $l_1 \le l \le l_2$ and there is an S_1 factor $A_{l_2-l_1}$ or l is not related to the Dynkin graph of any S_1 factor ($l_1 = l = l_2$ in this case); or

(b) $B_{\rho,l} = 0$ if l is in an S_1 factor of type D.

There are 2-strings generated by $e_l - e_{m+1}$, $e_l + e_{m+2}$ with $l \le m$, $e_{m+2} + e_i$ with $m+2 < i \le m+n+1$ (if $n > 1$) and $e_{m+2} \pm e_j$ with $m+n+1 < j$. The corresponding $a_{\rho,s}$ are

$$\begin{aligned} -2(-l_1 - l_2 + 2(n+m+k+1) - n - 2k) &= -2(-l_1 - l_2 + 2(m+1) + n) \\ &\le -2(n+2), \\ -2(-l_1 - l_2 + 2(m+n+k+1) + n + 2k) &= -2(-l_1 - l_2 + 2(m+2k+1) + 3n) \\ &\le -2(3n+2), \\ -2(2n+4k) &\le -4n \le -2(n+2), \\ -2(n+2k - l_1 - l_2 + 2(m+n+k+1)) &\le -2(n+2k) \le -2(n+2) \\ (\text{or } -2(n+2k) &\le -2(n+2)), \\ -2(n+2k + l_1 + l_2 - 2(m+n+k+1)) &\le -2(n+2k+2-2k) = -2(n+2) \\ (\text{or } -2(n+2k) &\le -2(n+2)). \end{aligned}$$

The corresponding affine manifolds are Fano.

If $S = A_3$ is generated by $e_{m+1} - e_{m+2}$, $e_{m+2} - e_{m+3}$, $e_{m+2} + e_{m+3}$ in D_{m+3}, let $\alpha = e_{m+2} - e_{m+3}$. We have 2-strings generated by $e_l - e_{m+2}$, $e_l + e_{m+3}$ ($l \le m$) and

$$a_{\rho,l} = B_{\rho,l} = -2(-l_1 - l_2 + 2(m+3)) \le -12 = -2(n+3).$$

The corresponding affine manifold is Fano.

Case 5: $G = G_2$.

If $\alpha = \alpha_1$, then $a_1 = B_1$, $a_2 = 3B_1$, $(aH + cG + B_1 i(3\alpha_1 + 2\alpha_2), -2H_{3\alpha_1+2\alpha_2}) = -6B_1$, and

$$\Big(\sum_{\alpha \in \Delta^+ - \{\alpha_1\}} \alpha, 2(3\alpha_1 + 2\alpha_2) \Big)_0 = (3(\alpha_1 + 2\alpha_2), 2(3\alpha_1 + 2\alpha_1)) = 36.$$

We have $B_{\rho,1} = -6 = -2(n+2)$. The corresponding affine manifold is Fano.

If $\alpha = \alpha_2$, then $H = \frac{1}{3} H_{\alpha_2}$. Since $(H, H)_A = 1$, we get $(H_{\alpha_2}, H_{\alpha_2})_A = 9$. Then $\omega(X, Y) = \big(aH + cG + B_1 i(2\alpha_1 + \alpha_2), [X, Y]\big)$ and $\omega(F_{2\alpha_1+\alpha_2}, JF_{2\alpha_1+\alpha_2}) = -6B_1$. There are two 2-strings generated by α_1 and $3\alpha_1 + \alpha_2$. We have $a_1 = B_1$ and $a_2 = 3B_1$. But we also have

$$\sum_{\alpha \in \Delta^+ - \{\alpha_2\}} \alpha(2(2\alpha_1 + \alpha_2)) = 5(2\alpha_1 + \alpha_2)(2(2\alpha_1 + \alpha_2)) = 20 = -6B_{\rho,1}.$$

Therefore, $B_{\rho,1} = -\frac{10}{3}$, $a_{\rho,1} = -\frac{10}{3}$, and $a_{\rho,2} = -10 < -3 = -2n - 1$. The corresponding manifold is not even nef.

Before we go further, we make an observation. If $G' \subset G$ is a subgroup of G such that the Dynkin graph of G' is a subgraph of that of G and $S \subset G'$ fits with the Dynkin graph of G', then, if the last inequality holds for G, S, it also holds for G', S. Indeed, let β be a positive root in G' that generates a 2-, 3- or 4-string; then

$$(q_{G/P}, \beta) = (q_{G/P_1}, \beta) + (q_{G'/P_2}, \beta) = (q_{G'/P_2}, \beta),$$

where P_1 is the minimal parabolic subgroup of G containing G' and $P_2 = P \cap G'$, since $(q_{G/P_1}, \cdot)$ is trivial on G'. Therefore, once the last inequality is true for E_8, it is also true for both E_6 and E_7. Similarly, the inequality $a_{\rho,s} \leq -2(n+2)$ holds for $G = E_k$, $6 \leq k \leq 8$. Therefore, the last inequality holds for the remaining cases of $G = F_4$, E_6, E_7, E_8, thanks to a further calculation with F_4 and E_8. □

Given Lemma 13, we argue as in [Guan 2002, p. 73]. If $n - \alpha$ has two zeros, $\Delta - mq$ has $\deg q - 3 + 4 = \deg q + 1$ zeros. That contradicts the fact that this polynomial has degree $2 \deg Q + 1$. Thus, we obtain Lemma 12. □

Now, f_l has a unique zero. Therefore, if

$$\int_0^{l(l+2B)} f_l \, dx < 0, \tag{73}$$

we cannot have

$$0 = \int_{x(-\infty)}^{l(1+2B)} f_l \, dx \leq \int_0^{l(l+2B)} f_l \, dx.$$

Otherwise, we have a contradiction.

By choosing $A(e_0)$ close to $l(l+2B)$, we have $u(1) = l(l+2B)$. Arguing as in [Guan 2002], we have $u'(1)$ exists and is finite. Similarly, $u''(0)$ and $u''(1)$ exist and are finite.

Also, if $G = S = A_n$, the manifold is homogeneous and admits a unique extremal metric in any given Kähler class. Therefore we have the following result, whose converse is proved in [Guan $\geq$ 2011a]:

Theorem 15. *There is a Kähler metric of constant scalar curvature in a given Kähler class if the condition (73) is satisfied.*

Corollary 16. *If $G = A_k$ or D_k, then $a_{\rho,s} \leq -2(n+2)$, and therefore the manifolds are Fano.*

We could easily argue as in [Guan 2002] and [2003, p. 273–274] that the right side of (73) is the Ding–Tian generalized Futaki invariant for a (possibly singular) completion of the normal line bundle of the exceptional divisor, although we do not really know that there is an actually analytic degeneration with this completion as the central fiber. Our condition here is stronger than the Ross–Thomas version of Donaldson's version of K-stability; see [Guan $\geq$ 2011b].

8. Type II cases

Now, we consider the case of type II, in other words, the case in which the centralizer of the isotropic group contains a three dimensional simple Lie algebra $\mathscr{A}$. Since most cases are affine and other cases are actually homogeneous, we actually only need to consider the case in which $S = A_1$. We denote the manifold by N.

In that case, the involution induces an involution in $\mathscr{A}$ and $d = 1$. The argument after Theorem 7 and [Guan 2003, Theorem 3.1] tell us that

$$U_\theta(1) = \lim_{t\to+\infty} \frac{(1+t^2)^2 U'}{2t} = 0.$$

We actually have $U_\theta = (1-\theta)h(\theta)$ with $h(1) > 0$. Also $B_\rho = 0 = B,\ k = 1$, and $l_\rho = -4-2 = -6$.

The Kähler–Einstein equation is

$$(1-\theta)\left(\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right) = 2 - 2^{-1}\left(\frac{u}{\theta}\right)^{1/2}.$$

The constant scalar curvature equation is

$$(1-\theta)\left(\frac{u''}{u'} + \frac{Q'(u)u'}{Q(u)}\right) = 2 - 2^{-1}\left(\frac{u}{\theta}\right)^{1/2}\frac{m(u)}{Q(u)} = P\theta^{-1},$$

where $m(u) = (R_0 \int Q\, du - \int p\, du)/u$. Also our $P\theta^{-1}$ here is the P in [Guan 2003].

If $G = S = A_1$, we have $R_0 = 2l_\rho/l = -12/l,\ Q = Q_1 = 1,\ m(u) = l_\rho/l = -6/l$. The equation is

$$(1-\theta)u'' = \left(2 + \frac{3}{l}\left(\frac{u}{\theta}\right)^{1/2}\right)u.$$

We have

$$2(1 - A_l\theta^{-1/2}) \leq P\theta^{-1} \leq 2$$

with a constant $A_l \geq \frac{3}{2}$ since $P(1) = -1$ as in [Guan 2003].

The difference between this case and those in Section 6 can be summarized in the following two theorems:

Theorem 8′. *The metric ω of Section 3 extends to a Kähler metric over the exceptional divisor of N if and only if $\lim_{t\to+\infty} tf = l > a_i$ and $U_\theta(1) = 0$.*

Let $f(t) = 2lt/(1+2t^2)$; then $U = 4l^2\theta(1+\theta)^2$ satisfies the assumption of Theorems 7 and 8′. Actually, one can check that this U is the solution of the equation when $G = S = A_1$.

Therefore:

Theorem 9′. *The Kähler classes on N are in one-to-one correspondence with the elements in the set $\Gamma = \{l|_{0>l>a_i}\}$.*

Also $f_l = (1-\alpha)Q$ with $\alpha = 2^{-1}u^{1/2}m(u)/Q(u)$.

If $G = S = A_1$, then $f_l = 1 + 3l^{-1}u^{1/2}$. The integral always satisfies

$$\int_0^{l^2} (1 + 3l^{-1}u^{1/2})\,du = l^2 - 2l^2 = -l^2 < 0.$$

In general, we have this result, whose converse is proved in [Guan ≥ 2011a]:

Theorem 15′. *A nonaffine type II cohomogeneity one manifold has a Kähler–Einstein metric if it is Fano and*

$$\int_0^{36} (1 - 2^{-1}u^{1/2})Q\,du < 0.$$

It has a Kähler metric of constant scalar curvature if $\displaystyle\int_0^{l^2} f_l\,du < 0.$

9. Kähler–Einstein metrics, Fano properties and further comments

If the Kähler class is the Ricci class, we have

$$B = B_\rho = 0, \quad l = l_\rho = -2(n+1), \tag{74}$$

$$m(u) = Q_1(u), \quad \alpha = 2^{-1}\sqrt{u}. \tag{75}$$

Therefore, $f_l = \big(n - 2^{-1}\sqrt{u}\,\big)u^{n-1}Q_1(u)$.

In this section, we show how we can check the Kähler–Einstein property case by case on the pairs of groups (S, G).

From [Guan 2009] we know that if $S = B_n$ or C_n the manifolds are always Fano.

Now, we consider the case in which $S = A_n$ and $G = A_{m+n+k}$ such that S is generated by $e_{i+1} - e_i$ with $m+1 \le i \le m+n$. We shall see that the manifolds are Fano for the compact affine almost-homogeneous manifolds of cohomogeneity one. For the case of type II manifolds other than the affine case, we shall see that they have numerically effective anticanonical line bundles and are Fano if every e_i that is not in A_n is in some A_l factor in S_1. Here we say that e_i is in an A_l if $e_i - e_j \in A_l$ for some e_j.

By our formula, we have

$$\rho_{G/P}(F_{e_l - e_{m+1}}, JF_{e_l - e_{m+1}}) = 2(-l_1 - l_2 + 2m + n + 2)$$

if $l_1 \le l \le l_2 \le m$ induces an $A_{l_2 - l_1}$ in S_1. Also

$$\begin{aligned}[F_{e_l - e_{m+1}}, JF_{e_l - e_{m+1}}] = [F_{e_l - e_{m+1}}, -G_{e_l - e_{m+1}}] &= -2H_{e_l - e_{m+1}} \\ &= -2H_{e_l} - H - H_{e_{m+1} + e_{m+2}}\end{aligned}$$

and the coefficient of H is -1. Therefore,

$$a_{\rho,l} = -2(-l_1 - l_2 + 2m + n + 2) < l_\rho = -2(n+1)$$

if the manifold is affine. If the manifold is of type II but not affine, then $n = 1$ and $-2(-l_1 - l_2 + n + 2) = -2(-l_1 - l_2 + 2m + 3) \le l_\rho = -6$, with equality only when $l_2 = l_1 = m$. Similarly for $l > m + n$. This yields our claim.

When $k = m = 0$, we have the product of two projective spaces. Therefore, there are Kähler–Einstein metrics. Indeed, one can easily check that

$$K^n_{0,0} = \int_0^{2(n+1)} (2n - v)v^{2n-1}\, dv = \left(v^{2n} - \frac{v^{2n+1}}{2n+1}\right)\Bigg|_0^{2(n+1)} = \left(1 - \frac{2(n+1)}{2n+1}\right)(2(n+1))^{2n} \le 0.$$

When $k = 0$ and $n = 1$ with a maximal parabolic subgroup P, we have the examples M_{m+1} and N_{m+1} in [Guan 2002; 2003].

Similarly, we can consider the general case with the maximal parabolic subgroup, in which $S_1 = A_m A_k$. Then we have the integral

$$K^n_{m,k} = \int_0^{2(n+1)} v(2n - v)v^{2(n-1)}(4(m + n + 1)^2 - v^2)^m (4(k + n + 1)^2 - v^2)^k\, dv$$

for the affine case and

$$K'_{m,k} = \int_0^6 v(2 - v)(4(m + 2)^2 - v^2)^m (4(k + 2)^2 - v^2)^k\, dv$$

for the nonaffine case in which $n = 1$ and $m, k \ne 1$.

For the case of $k = 0$ and $n = 1$, if we let $v = 4x$, we have

$$K^1_{m,0} = \int_0^1 4^2 x \cdot 2(1 - 2x) \cdot 2^m ((m + 2)^2 - 4x^2)^m\, dx.$$

Similarly for $K'_{m,0}$, which is exactly the integrals in [Guan 2003; Guan and Chen 2000] up to multiplication by a constant 2^{m+5}.

Lemma 17. *$K^1_{i,j} < 0$ for $i, j = 0, 1, 2$.*

Proof. By the method in [Guan and Chen 2000] or by performing the integral explicitly (for instance, using Mathematica) for the cases $i = j = 2$, $i = j = 1$ and $i = 1$, $j = 2$. □

We could call the related manifolds $M^n_{m,k}$ and $N_{m,k}$ (not to be confused with similar notation in the previous section).

Theorem 18. *$M^1_{m,k}$ and $N_{m,k}$ are nef. $M^1_{m,k}$ are Kähler–Einstein for all m, k. $N_{m,k}$ are Fano if and only if $m, k \ne 1$, in which case $N_{m,k}$ are Kähler–Einstein.*

Proof. We have $K^1_{m,k} \le C K^1_{2,k}$ if $m \ge 2$, by applying the comparison method we used in [Guan 2009; ≥ 2011b], and reasoning as follows:

We can compare the rate of change of the factor $h(v) = (4(n+m+1)^2 - v^2)^m$. We let

$$t(m) = (\log h)' = m\left(\frac{1}{2n+2m+1+v} - \frac{1}{2n+2m+1-v}\right).$$

Then,

$$t(m+1) - t(m) = \frac{-2v\big(4(n+1)^2 - 4m(m+1) - v^2\big)}{(4(n+m+1)^- v^2)(4(n+m+2)^2 - v^2)} > 0$$

if $m > n$. Therefore, if $K^n_{m,k} \le 0$ with $m > n$, then $K^n_{m+1,k} < 0$. Moreover, $K'_{m,k} < K^1_{m,k} < 0$. □

We have $\lim_{m\to+\infty}(2m)^{-2m} K^n_{m,k} = e^{4(n+1)} K^n_{0,k}$. We shall prove that $K^n_{0,k} < 0$, which implies that

(1) $M^n_{0,k}$ admits Kähler–Einstein metrics, which also generalizes our results in [Guan and Chen 2000], and

(2) for any given n, k there is an integer $N(n,k)$ such that if $m > N(n,k)$ then $M^n_{m,k}$ admits Kähler–Einstein metrics.

Lemma 19. *Let $m = l(n+1)$ and $k = sm$. Then $K^n_{m,k} = -C I_{n,l,s}$, with*

$$\begin{aligned} I_{n,l,s} = \int_0^1 & x^{2n}(1-x)((1+l)^2 - x^2)^{m-1}((1+sl)^2 - x^2)^{k-1} \\ & \big[(1-x^2)\big((1+l+sl)(1-x^2) + l^2(s(2+l+sl)+(1+s)^2) + l(1+s)(1-x)\big) \\ & \qquad + sl^3(1+s)(1-x) + sl^2(sl^2 - 4x)\big]\,dx \end{aligned}$$

and a constant $C > 0$. In particular, $K^n_{0,k} < 0$ and $K^n_{m,k} < 0$ if $mk \ge 4(n+1)^2$. Therefore, $K^n_{m,k} < 0$ if $m \ge 4(n+1)^2$.

Proof. We let $n = l^{-1}m - 1$, $k = sm$ and $v = 2l^{-1}m$. We have

$$\begin{aligned} K^n_{m,k} &= C_1 \int_0^1 x^{2l^{-1}m-3}(l^{-1}m(1-x) - 1)((1+l)^2 - x^2)^m((1+sl)^2 - x^2)^{sm}\,dx \\ &= C_1\Big[l^{-1}m\int_0^1 x^{2n-1}(1-x)\big(((1+l)^2 - x^2)((1+sl)^2 - x^2)^s\big)^m\,dx \\ &\quad - \int_0^1 x^{-3} m \int_0^x \big(y^{2l^{-1}}((1+l)^2 - y^2)((1+sl)^2 - y^2)^s\big)^{m-1} \\ &\quad \big[2l^{-1}y^{2l^{-1}-1}((1+l)^2 - y^2)((1+sl)^2 - y^2)^s - 2y^{2l^{-1}+1}((1+sl)^2 - y^2)^s \\ &\qquad - 2sy^{2l^{-1}+1}((1+l)^2 - y^2)((1+sl)^2 - y^2)^{s-1}\big]\,dy\,dx\Big]. \end{aligned}$$

The right-hand side can be rewritten as

$$C_2\bigg[\int_0^1 x^{2n-1}(1-x)((1+l)^2-x^2)^m((1+sl)^2-x^2)^k\,dx$$
$$-2\int_0^1 y^{2n+1}((1+l)^2-y^2)^{m-1}((1+sl)^2-y^2)^{k-1}$$
$$\big[(1+l)^2(1+sl)^2-(1+l)(1+sl)(2+l+sl)y^2+(1+l+sl)y^4\big]\int_y^1 x^{-3}\,dx\,dy\bigg],$$

which is easily seen to be equivalent to the form claimed. □

This lemma also shows that if l, s are constants and $0 < sl^2 < 4$, then $I_{n,l,s}$ increase with $\lim_{n\to+\infty} I_{n,l,s} > 0$. In particular, $K^n_{n+1,n+1} > 0$ when n is big enough.

Actually, using Mathematica to integrate $v^{2m-3}(m(1-v)-1)(4-v^2)^{2m}$ from 0 to 1, we obtain the following:

Lemma 20. $K^n_{n+1,n+1} > 0$ *if* $n \geq 10$. *Otherwise*, $K^n_{n+1,n+1} < 0$.

Similarly, we can use Mathematica to calculate $K^{11}_{12,13}$, $K^{11}_{12+k,12-k}$ for $1 \leq k \leq 7$, $K^{11}_{13+k,12-k}$ for $1 \leq k \leq 5$, and $K^{11}_{19,k}$ for $k \leq 4$, and obtain this:

Lemma 21. $K^{11}_{12+k,13-k} < 0$, $K^{11}_{19,k} < 0$ *always and* $K^{11}_{12+k,12-k} > 0$ *if* $0 \leq k \leq 6$.

Therefore, we can check that $K^{11}_{m,k} < 0$ for $m \leq 18$ if $m = 1$ and if $k \leq n_m$ or $k \geq N_m$ for $m > 1$ with $n_2 = 2$, $n_k = 1$ for $3 \leq k \leq 11$, $n_l = 2$ for $12 \leq l \leq 15$, $n_{16} = n_{17} = 3$, $n_{18} = 4$; $N_2 = 12$, $N_3 = 16$, $N_4 = 18$, $N_5 = 19$, $N_{6+k} = 19 - k$ for $0 \leq k \leq 12$. One might conjecture that the open set $K^n_{m,k} > 0$ is a convex set with an asymptotic cone $mk \leq 4(n+1)^2$.

Similarly, we check that $K^n_{m,k} < 0$ if $n = 5, 7$ and $K^8_{m,k} < 0$ if $m \leq 2$ or $m \geq 7$. Further, $K^8_{m,k} < 0$ for $3 \leq m \leq 6$ and $k \leq n_m$ or $k \geq N_m$, where $n_3 = 3 = n_6$; $n_4 = 2 = n_5$; $N_3 = 6$; $N_4 = 7 = N_5 = N_6$.

Next, we have $K^9_{m,k} < 0$ if $m \leq 2$ or $m \geq 11$, and $K^9_{m,k} < 0$ when $3 \leq m \leq 10$ and $k \leq n_m$ or $k \geq N_m$, where $n_k = 2$ for $3 \leq k \leq 9$; $n_{10} = 3$; $N_3 = 10 = N_7$; $N_4 = N_5 = N_6 = 11$; $N_8 = 9$; $N_9 = 8$; $N_{10} = 7$.

We can also check that $K^n_{2,2} < 0$ if $n \leq 13$, while $K^{14}_{2,2} > 0$; and that $K^n_{1,n+1} < 0$ if $n \leq 33$ while $K^{34}_{1,35} > 0$, $K^{34}_{1,36} < 0$. Therefore, $K^n_{1,k} < 0$ if $n \leq 33$ and $k \geq n+1$, while $K^{34}_{1,k} < 0$ if $k \geq 36$.

$K^n_{1,1}$ and $K^n_{1,2}$ are always negative. $K^n_{1,3} > 0$ for $25 \leq n \leq 34$. Moreover, $K^{15}_{1,k} < 0$ always.

$K^{10}_{m,k} < 0$ for $k \leq n_m$ or $k \geq N_m$, where $n_2 = 2 = n_i$ for $9 \leq i \leq 12$; $n_i = 1$ for $3 \leq i \leq 8$; $n_{13} = n_{14} = 3$; $N_2 = 9$; $N_3 = 13 = N_8$; $N_i = 15$ for $i = 4, 5, 6$; $N_{6+i} = 15 - i$ for $1 \leq i \leq 8$.

We finally check that $K^n_{k,m} < 0$ for $n = 6, 4, 3, 2$:

Theorem 22. *The $M^n_{k,m}$ are nonhomogeneous Kähler–Einstein manifolds for $n \leq 7$. They admit Kähler–Einstein metric for $8 \leq n \leq 11$ if $k \leq k_n$ or $k \geq K_n$, where $k_8 = 2 = k_9$, $k_{10} = 1 = k_{11}$, and $K_n = 7 + 4(n-8)$. For $k_n < k < K_n$, there are two numbers $m^n_k > k_n$ and $M^n_k < K_n$ such that $M^n_{k,m}$ is Kähler–Einstein for $m \leq m^n_k$ or $m \geq M^n_k$, and $M^n_{k,m}$ is a non-Kähler–Einstein Fano manifold for $m^n_k < m < M^n_k$.*

So far, I could not find any manifolds such that the integral is zero. Otherwise, it might provide a counterexample for being weakly K-stable and Mumford-stable but not Kähler–Einstein.

Our manifolds might not always be Fano in general. For example, if $S = A_n$ and $G = B_{m+n+k+1}$ such that S is determined by e_i, $m+1 \leq i \leq m+n+1$, with the minimal parabolic subgroup P, the manifolds are not Fano when $k = 0$. For example, $a_{\rho,s}$ for the 2-string generated by e_{m+2} is $-2(1+n+2k) = -2(n+1)$ and $l_\rho = -2(n+1)$. Therefore, $a^2_{\rho,s} - v^2 = 0$ at $v = -2(n+1)$. The manifold is not Fano. That is, affine type does not imply Fano in general in the case of $S = A_n$. However, from the proof of Lemma 13, we have $l_\rho - 2 + a_{\rho,s} < 0$; that is, the manifolds are not far from being Fano.

When the manifold is Fano, we notice that in the affine case, the manifold is a $\mathbb{C}P^n$ bundle over a rational projective homogeneous manifold. Let D be the hypersurface line bundle of $\mathbb{C}P^n$. Then $K_F = -(n+1)D$ is just the canonical line bundle of $\mathbb{C}P^n$. We set $K_F = -(n+1)$ and $D = 1$; we let $x = \frac{1}{2}v$ but still denote $Q(v)$ by $Q(x)$. Our integral is proportional to

$$\int_0^{-K_F} (-K_F - D - x)Q(x)\,dx.$$

For the nonaffine Type II case, $F = \mathbb{C}P^2$ as a double-branched quotient of $\mathbb{C}P^1 \times (\mathbb{C}P^1)^*$, the exceptional divisor D is a quadratic. Let H be the hypersurface divisor. Then $K_F = -3H$, $D = 2H$. As above, we denote $K_F = -3$ and $D = 2$. Then the integral is proportional to

$$\int_0^{-K_F} (-K_F - D - x)Q(x)\,dx$$

again. Moreover, by the adjunct formula we have $K_D = K_F + D$ on D, and we write $K_D = K_F + D$ also as numbers.

Combining with [Guan 2006; 2009], we have:

Theorem 23. *If a type II manifold M is Fano, then it admits a Kähler–Einstein metric if and only if*

$$\int_0^{-K_F} (K_F + D + x)Q(x)\,dx = \int_0^{-K_F} (K_D + x)Q(x)\,dx > 0,$$

where $Q(x)\,dx$ is the volume element.

Proof. Let us deal with the integral in [Guan 2006, p. 166] first. If we let $v = \sqrt{u+1} - 1$, the integral is proportional to $\int_0^{3/2} (1-v) Q(v)\, dv$. The open orbit is a $\mathbb{C}^2$ bundle. The manifold is a $\mathbb{C}P^2$ bundle and $K_F = -3$, $D = 1$. Let $v = x/2$. Then the integral is

$$\int_0^3 (2-x) Q(x)\, dx = \int_0^{-K_F} (-K_F - D - x) Q(x)\, dx,$$

as desired. This also confirms that our calculation in [Guan 2006] is correct.

The cases in [Guan 2009] can be found in Theorem 10.2 there. □

Combining with [Guan ≥ 2011b], we have:

Theorem 24. *A cohomogeneity-one, two-orbit Fano manifold with a codimension m closed orbit and a semisimple group action is Kähler–Einstein if and only if*

$$\int_0^{-K_F+m-1} (K_F + D + x) Q(x)\, dx = \int_0^{-K_F+m-1} (K_D + x) Q(x)\, dx > 0,$$

where $Q(x)\, dx$ is the volume element.

Here, we can understand the F to be as the fiber in [Huckleberry and Snow 1982], but not the one in [Akhiezer 1983]. Then K_F is exactly the correspondence of the canonical divisor and D the exceptional divisor.

Combining with Corollary 16, and after some further calculations with exceptional Lie algebras, we have:

Corollary 25. *If the roots of G have the same length, then $a_{\rho,s} \le -2(n+2)$. Therefore, the affine manifolds are Fano and the nonaffine type II manifolds are nef.*

This also provides more Kähler–Einstein metrics.

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