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TOSHIHISA KUBO

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Barchini, Kable, and Zierau constructed a number of conformally invariant systems of differential operators associated to Heisenberg parabolic subalgebras in simple Lie algebras. The construction was systematic, but the existence of such a system was left open in two cases, namely, the Ω_3 system for type A_2 and type D_4. Here, such a system is shown to exist for both cases. The construction of the system may also be interpreted as giving an explicit homomorphism between generalized Verma modules.

1. Introduction

Conformally invariant systems of differential operators on a smooth manifold M on which a Lie algebra $\mathfrak{g}$ acts by first order differential operators were studied by Barchini, Kable, and Zierau in [BKZ08] and [BKZ09]. To recall the definition of the conformally invariant systems from [BKZ09], let $\mathfrak{g}_0$ be a real Lie algebra. A smooth manifold M is a $\mathfrak{g}_0$-*manifold* if there exists a $\mathfrak{g}_0$-homomorphism

$$\Pi_M : \mathfrak{g}_0 \to C^\infty(M) \oplus \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the space of smooth vector fields on M. Given a $\mathfrak{g}_0$-manifold M, write $\Pi_M(X) = \Pi_0(X) + \Pi_1(X)$ with $\Pi_0(X) \in C^\infty(M)$ and $\Pi_1(X) \in \mathfrak{X}(M)$. Let $\mathbb{D}(\mathcal{V})$ denote the space of differential operators on a vector bundle $\mathcal{V} \to M$. A vector bundle $\mathcal{V} \to M$ is a $\mathfrak{g}_0$-*bundle* if there exists a $\mathfrak{g}_0$-homomorphism $\Pi_{\mathcal{V}} : \mathfrak{g}_0 \to \mathbb{D}(\mathcal{V})$ so that in $\mathbb{D}(\mathcal{V})[\Pi_{\mathcal{V}}(X), f] = \Pi_1(X) \bullet f$ for all $X \in \mathfrak{g}_0$ and all $f \in C^\infty(M)$, where the dot $\bullet$ denotes the action of the differential operator $\Pi_1(X)$. We regard any smooth functions f on M as elements in $\mathbb{D}(\mathcal{V})$ by identifying them with the multiplication operator they induce. Then, given a $\mathfrak{g}_0$-bundle $\mathcal{V} \to M$, a list of differential operators $D_1, \ldots, D_m \in \mathbb{D}(\mathcal{V})$ is said to be a *conformally invariant system* on $\mathcal{V}$ with respect to $\Pi_{\mathcal{V}}$ if the following two conditions are satisfied:

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(S1) The list $D_1, \dots, D_m$ is linearly independent at each point of M.

(S2) For each $Y \in \mathfrak{g}_0$ there is an $m \times m$ matrix $C(Y)$ of smooth functions on M so that in $\mathbb{D}(\mathcal{V})$,

$$[\Pi_{\mathcal{V}}(Y), D_j] = \sum_{i=1}^{m} C_{ij}(Y) D_i.$$

By extending the $\mathfrak{g}_0$-homomorphisms Π_M and $\Pi_{\mathcal{V}}$ $\mathbb{C}$-linearly, the definitions of a $\mathfrak{g}_0$-manifold, a $\mathfrak{g}_0$-bundle, and a conformally invariant system can be applied equally well to the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

A general theory of conformally invariant systems is developed in [BKZ09], and examples of such systems of differential operators associated to the Heisenberg parabolic subalgebras of any complex simple Lie algebras are constructed in [BKZ08]. The purpose of this paper is to answer a question left open in [BKZ08] concerning the existence of certain conformally invariant systems of third-order differential operators. This is done by constructing the required systems.

This result may be interpreted as giving an explicit homomorphism between two generalized Verma modules, one of which is nonscalar. See [BKZ09, Section 6] for the general theory. In this paper we describe it explicitly in a less general setting (see the discussion after Lemma 3.4). The problem of constructing and classifying homomorphisms between scalar generalized Verma modules has received a lot of attention. For example, Matumoto [2006] classifies the nonzero $\mathcal{U}(\mathfrak{g})$-homomorphisms between scalar generalized Verma modules associated to maximal parabolics of non-Hermitian symmetric type. For the Hermitian symmetric cases, Boe [1985] solved the existence problem for scalar generalized Verma modules of maximal parabolic type. However, much less is known about maps between generalized Verma modules that are not necessarily scalar.

To explain our main results, we briefly review the results of [BKZ08]. To begin, let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the parabolic subalgebra of Heisenberg type, that is, $\mathfrak{n}$ is a two-step nilpotent algebra with one-dimensional center. Denote by γ the highest root of $\mathfrak{g}$. For each root α let $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ be a corresponding $\mathfrak{sl}(2)$-triple, normalized as in [BKZ08, Section 2]. Then $\mathrm{ad}(H_\gamma)$ on $\mathfrak{g}$ has eigenvalues $-2, -1, 0, 1, 2$, and the corresponding eigenspace decomposition of $\mathfrak{g}$ is denoted by

$$\mathfrak{g} = \mathfrak{z}(\bar{\mathfrak{n}}) \oplus V^- \oplus \mathfrak{l} \oplus V^+ \oplus \mathfrak{z}(\mathfrak{n}). \tag{1.1}$$

Let $\mathbb{D}[\mathfrak{n}]$ be the Weyl algebra of $\mathfrak{n}$, that is, the algebra of partial differential operators on $\mathfrak{n}$ with polynomial coefficients. Then each system of k-th order differential operators constructed in [BKZ08] derives from a $\mathbb{C}$-linear map $\Omega_k : \mathfrak{g}(2-k) \to \mathbb{D}[\mathfrak{n}]$ with $1 \le k \le 4$ and $\mathfrak{g}(2-k)$ the $2-k$ eigenspace of $\mathrm{ad}(H_\gamma)$. Let $\Pi_s : \mathfrak{g} \to \mathbb{D}[\mathfrak{n}]$

be the Lie algebra homomorphism constructed in [BKZ08, Section 4]. Here s is a complex parameter, which indexes line bundles $\mathscr{L}_{-s}$ over a real flag manifold $G_0/\bar{Q}_0$, where G_0 is a real Lie group with Lie algebra $\mathfrak{g}_0$ and $\bar{Q}_0$ is a parabolic subgroup of G_0 with complexified Lie algebra $\bar{\mathfrak{q}}$ opposite to $\mathfrak{q}$. We say that the Ω_k system has special value s_0 when the system is conformally invariant under Π_{s_0}.

In [BKZ08] the special values of s are determined for the Ω_k systems with $k = 1, 2, 4$ for all complex simple Lie algebras, but only exceptional cases are considered for the Ω_3 system. A table in [BKZ08, Section 8.10] lists the special values of s. (Beware that the entries in the columns for the systems Ω_2^{big} and Ω_2^{small} for types B_r and C_r should be transposed.) [BKZ09, Theorem 21] then shows that the Ω_3 system does not exist for A_r with $r \geq 3$, B_r with $r \geq 3$, and D_r with $r \geq 5$. There remain two open cases, namely, the Ω_3 system for type A_2 and type D_4. The aim of this paper is to show that the Ω_3 system does exist for both cases (see Theorem 4.1 and Theorem 5.6). To achieve the result we use several facts from [BKZ08] and [BKZ09]. By using these facts, we significantly reduce the amount of computation to show the existence of the system.

There are two differences between our conventions and those used in [BKZ08]. One is that we choose the parabolic $Q_0 = L_0 N_0$ for the real flag manifold, while the opposite parabolic $\bar{Q}_0 = L_0 \bar{N}_0$ is chosen in [BKZ08]. Because of this, our special values of s are of the form $s = -s_0$, where s_0 are the special values found in [BKZ08, Section 8.10]. The other is that we identify $(V^+)^*$ with V^- by using the Killing form, while $(V^+)^*$ in [BKZ08] is identified with V^+ by using the nondegenerate alternating form $\langle \cdot, \cdot \rangle$ on V^+ defined by $[X_1, X_2] = \langle X_1, X_2 \rangle X_\gamma$ for $X_1, X_2 \in V^+$. Because of this difference the right action R, which will be defined in Section 2, will play the role played by Ω_1 in [BKZ08]. In addition to these notational differences, there are also some methodological differences between [BKZ08] and here. These stem from the fact that we make systematic use of the results of [BKZ09] to streamline the process of proving conformal invariance.

We now outline the remainder of this paper. In Section 2, we review the setting and results of [BKZ09, Section 5], simultaneously specializing them to the situation considered here. In Section 3, we specialize further by taking $\mathfrak{g}$ to be simply laced. We fix a suitable Chevalley basis and define the Ω_3^t system by $\Omega_3^t = \widetilde{\Omega}_3 + tC_3$ for $t \in \mathbb{C}$. A remark on notation might be helpful here. In [BKZ08], a system Ω_3' is initially defined. It is then shown to decompose as a sum of a leading term $\widetilde{\Omega}_3$ and a correction term C_3. These two are recombined with different coefficients to give Ω_3, which is finally shown to be conformally invariant for exceptional algebras. Thus, the Ω_3 system is defined to exist if there exists $t_0 \in \mathbb{C}$ so that the $\Omega_3^{t_0}$ system is conformally invariant.

In Section 4, we take $\mathfrak{g}$ to be of type A_2 and show that the Ω_3 system(s) exists over the line bundle $\mathscr{L}_0$. The Heisenberg parabolic subalgebra coincides with the

Borel subalgebra in this case. Thus V^- decomposes as the direct sum of two one-dimensional $\mathfrak{l}$-submodules. This implies that there will be two Ω_3 systems, each of the operators will be conformally invariant all by itself. The conformal invariance of these operators is shown in Theorem 4.1.

In Section 5, we take $\mathfrak{g}$ to be of type D_4. For type D_4, the data on p. 831 and Theorem 6.1 of [BKZ08] suggest that the complex parameter t_0 for the Ω_3^t system to be conformally invariant is $t_0 = 0$, so that the correction term C_3 is discarded completely. For this reason, we simply proceed to show that $\widetilde{\Omega}_3$ is conformally invariant. This is done in Theorem 5.6.

2. A specialization of the theory

The purpose of this section is to introduce the $\mathfrak{g}$-manifold and the $\mathfrak{g}$-bundle studied in this paper. Let G_0 be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_0$ and complexified Lie algebra $\mathfrak{g}$. Let Q_0 be a parabolic subgroup of G_0 and $Q_0 = L_0N_0$ a Levi decomposition of Q_0. By the Bruhat decomposition, the subset $\bar{N}_0Q_0$ of G_0 is open and dense in G_0, where $\bar{N}_0$ is the nilpotent subgroup of G_0 opposite to N_0. Let $\bar{\mathfrak{n}}$ and $\mathfrak{q}$ be the complexifications of the Lie algebras of $\bar{N}_0$ and Q_0, respectively; we have the direct sum $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{q}$. For $Y \in \mathfrak{g}$, write $Y = Y_{\bar{\mathfrak{n}}} + Y_{\mathfrak{q}}$ for the decomposition of Y in this direct sum. Similarly, write the Bruhat decomposition of $g \in \bar{N}_0Q_0$ as $g = \bar{\mathbf{n}}(g)\mathbf{q}(g)$ with $\bar{\mathbf{n}}(g) \in \bar{N}_0$ and $\mathbf{q}(g) \in Q_0$. For $Y \in \mathfrak{g}_0$, we have

$$Y_{\bar{\mathfrak{n}}} = \frac{d}{dt}\bar{\mathbf{n}}(\exp(tY))\big|_{t=0},$$

and a similar equality holds for $Y_{\mathfrak{q}}$.

Consider the homogeneous space G_0/Q_0. Let $\mathbb{C}_{\chi^{-s}}$ be the one-dimensional representation of L_0 with character χ^{-s} with $s \in \mathbb{C}$, where χ is a real-valued character of L_0. The representation χ^{-s} is extended to a representation of Q_0 by making it trivial on N_0. For any manifold M, denote by $C^\infty(M, \mathbb{C}_{\chi^{-s}})$ the smooth functions from M to $\mathbb{C}_{\chi^{-s}}$. The group G_0 acts on the space

$$\begin{aligned} &C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \\ &\quad = \big\{F \in C^\infty(G_0, \mathbb{C}_{\chi^{-s}}) \;\big|\; F(gq) = \chi^{-s}(q^{-1})F(g) \text{ for all } q \in Q_0 \text{ and } g \in G_0\big\} \end{aligned}$$

by left translation, and the action Π_s of $\mathfrak{g}$ on $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ arising from this action is given by

$$(\Pi_s(Y) \bullet F)(g) = \frac{d}{dt}F(\exp(-tY)g)\big|_{t=0} \tag{2.1}$$

for $Y \in \mathfrak{g}_0$, where the dot $\bullet$ denotes the action of $\Pi_s(Y)$. This action is extended $\mathbb{C}$-linearly to $\mathfrak{g}$ and then naturally to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We use the same symbols for the extended actions.

The restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$ is an injection. Define the action of $\mathcal{U}(\mathfrak{g})$ on the image of the restriction map by $\Pi_s(u)\bullet f = (\Pi_s(u)\bullet F)|_{\overline{N}_0}$ for $u \in \mathcal{U}(\mathfrak{g})$ and $F \in C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}})$ with $f = F|_{\overline{N}_0}$. Define a right action R of $\mathcal{U}(\bar{\mathfrak{n}})$ on $C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(R(X)\bullet f)(\bar{n}) = \frac{d}{dt} f(\bar{n}\exp(tX))\big|_{t=0}$$

for $X \in \bar{\mathfrak{n}}_0$ and $f \in C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$. A direct computation shows that

$$(\mathbf{2.2})\quad (\Pi_s(Y)\bullet f)(\bar{n}) = -s\, d\chi((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}) f(\bar{n}) - (R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}})\bullet f)(\bar{n})$$

for $Y \in \mathfrak{g}$ and f in the image of the restriction map

$$C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}}).$$

Equation (2.2) implies that the representation Π_s extends to a representation of $\mathcal{U}(\mathfrak{g})$ on the whole space $C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$. For all $Y \in \mathfrak{g}$, the linear map $\Pi_s(Y)$ is in $C^\infty(\overline{N}_0) \oplus \mathfrak{X}(\overline{N}_0)$. This property of $\Pi_s(Y)$ makes $\overline{N}_0$ a $\mathfrak{g}$-manifold.

Let $\mathcal{L}_{-s}$ be the trivial bundle of $\overline{N}_0$ with fiber $\mathbb{C}_{\chi^{-s}}$. Then the space of smooth sections of $\mathcal{L}_{-s}$ is identified with $C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$. For $Y \in \mathfrak{g}$ and $f \in C^\infty(\overline{N}_0)$, a computation shows that in $\mathbb{D}(\mathcal{L}_{-s})$,

$$([\Pi_s(Y), f])(\bar{n}) = -(R((\mathrm{Ad}(\bar{n}^{-1})Y)_{\bar{\mathfrak{n}}})\bullet f)(\bar{n}).$$

This verifies that Π_s gives $\mathcal{L}_{-s}$ the structure of a $\mathfrak{g}$-bundle.

Now define

$$\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} = \{D \in \mathbb{D}(\mathcal{L}_{-s}) \mid [\Pi_s(X), D] = 0 \text{ for all } X \in \bar{\mathfrak{n}}\}.$$

Proposition 2.1 [BKZ09, Proposition 13]. *Let $D_1, \ldots, D_m$ be a list of operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Suppose that the list is linearly independent at e and that there is a map $b : \mathfrak{g} \to \mathfrak{gl}(m, \mathbb{C})$ such that*

$$([\Pi_s(Y), D_i]\bullet f)(e) = \sum_{j=1}^{m} b(Y)_{ji}(D_j \bullet f)(e)$$

for all $Y \in \mathfrak{g}$, $f \in C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$, and $1 \le i \le m$. Then $D_1, \ldots, D_m$ is a conformally invariant system on $\mathcal{L}_{-s}$. The structure operator of the system is given by $C(Y)(\bar{n}) = b(\mathrm{Ad}(\bar{n}^{-1})Y)$ for all $\bar{n} \in \overline{N}_0$ and $Y \in \mathfrak{g}$.

As shown in [BKZ09, pp. 801–802] the differential operators in $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ can be described in terms of elements of the generalized Verma module

$$\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s\,d\chi}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbb{C}_{s\,d\chi},$$

where $\mathbb{C}_{sd\chi}$ is the $\mathfrak{q}$-module derived from the Q_0-representation $(\chi^s, \mathbb{C})$. By identifying $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{sd\chi})$ as $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{sd\chi}$, the map $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{\mathfrak{n}})$ given by $u \otimes 1 \mapsto u$ is an isomorphism. The composition

$$\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{sd\chi}) \to \mathcal{U}(\bar{\mathfrak{n}}) \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}} \tag{2.3}$$

is then a vector-space isomorphism, where the map $\mathcal{U}(\bar{\mathfrak{n}}) \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ is given by $u \mapsto R(u)$.

Suppose that $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ and $l \in L_0$. Then define an action of L_0 on $C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$ by

$$(l \cdot f)(\bar{n}) = \chi^{-s}(l) f(l^{-1}\bar{n}l).$$

This action agrees with the action of L_0 by left translation on the image of the restriction map $C^\infty_\chi(G_0/Q_0, \mathbb{C}_{\chi^{-s}}) \to C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. In terms of this action, define an action of L_0 on $\mathbb{D}(\mathcal{L}_{-s})$ by

$$(l \cdot D) \bullet f = l \cdot (D \bullet (l^{-1} \cdot f)).$$

One can check that $l \cdot R(u) = R(\mathrm{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{\mathfrak{n}})$; in particular, this L_0-action stabilizes the subspace $\mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$. Define an action of L_0 on $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{sd\chi})$ by $l \cdot (u \otimes z) = \mathrm{Ad}(l)u \otimes z$. With these actions, the isomorphism (2.3) is L_0-equivariant. For $D \in \mathbb{D}(\mathcal{L}_{-s})$, denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D \bullet f)(\bar{n})$ for $f \in C^\infty(\bar{N}_0, \mathbb{C}_{\chi^{-s}})$. The following result is the specialization of [BKZ09, Theorem 15] to the present situation.

Theorem 2.2. *Suppose that F is a finite-dimensional $\mathfrak{q}$-submodule of the generalized Verma module $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{sd\chi})$. Let $f_1, \dots, f_k$ be a basis of F and define constants $a_{ri}(Y)$ by*

$$Yf_i = \sum_{r=1}^{k} a_{ri}(Y) f_r$$

for $1 \le i \le k$ and $Y \in \mathfrak{q}$. Let $D_1, \dots, D_k \in \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ correspond to the elements $f_1, \dots, f_k \in F$. Then for all $Y \in \mathfrak{g}$, $1 \le i \le k$, and $\bar{n} \in \bar{N}_0$,

$$[\Pi_s(Y), D_i]_{\bar{n}} = \sum_{r=1}^{k} a_{ri}\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)(D_r)_{\bar{n}} - s\, d\chi\big((\mathrm{Ad}(\bar{n}^{-1})Y)_{\mathfrak{q}}\big)(D_i)_{\bar{n}}.$$

3. The Ω_3^t system

Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$ simply laced. In this section we specialize to the situation where G_0 is a real form of G that contains a real parabolic subgroup of Heisenberg type. In this setting, we construct a system of differential operators over the line bundle $\mathcal{L}_{-s}$ and show some technical facts that will be used later sections. We first introduce some notation.

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let Δ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Fix Δ^+ a positive system and denote by S the corresponding set of simple roots. Write ρ for half the sum of the positive roots. Denote the highest root by γ. Let $B_{\mathfrak{g}}$ denote a positive multiple of the Killing form on $\mathfrak{g}$ and denote by $(\cdot\,,\cdot)$ the corresponding inner product induced on $\mathfrak{h}^*$. The normalization of $B_{\mathfrak{g}}$ will be specified below. Write $\|\alpha\|^2 = (\alpha, \alpha)$ for any $\alpha \in \Delta$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to α. For any $\mathrm{ad}(\mathfrak{h})$-invariant subspace $V \subset \mathfrak{g}$, denote by $\Delta(V)$ the set of roots α so that $\mathfrak{g}_\alpha \subset V$.

It is known that we can choose $X_\alpha \in \mathfrak{g}_\alpha$ and $H_\alpha \in \mathfrak{h}$ for each $\alpha \in \Delta$ in such a way that the following conditions (C1)–(C5) hold. Our normalizations are special cases of those used in [BKZ08].

(C1) For any $\alpha \in \Delta^+$, $\{X_{-\alpha}, H_\alpha, X_\alpha\}$ is an $\mathfrak{sl}(2)$-triple. In particular, we have $[X_\alpha, X_{-\alpha}] = H_\alpha$.

(C2) For each $\alpha, \beta \in \Delta$, $[H_\alpha, X_\beta] = \beta(H_\alpha)X_\beta$.

(C3) For $\alpha \in \Delta$ we have $B_{\mathfrak{g}}(X_\alpha, X_{-\alpha}) = 1$. In particular, $(\alpha, \alpha) = 2$.

(C4) For $\alpha, \beta \in \Delta$, we have $\beta(H_\alpha) = (\beta, \alpha)$.

(C5) If $\alpha, \beta, \alpha+\beta \in \Delta$ then there is a nonzero integer $N_{\alpha,\beta}$ so that $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$. For $Z \in \mathfrak{l}$ and $\alpha \in \Delta(V^+)$, define a scalar $M_{\alpha,\beta}(Z)$ by $[Z, X_\alpha] = \sum_{\beta\in\Delta(V^+)} M_{\alpha,\beta}(Z)X_\beta$.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ be the standard parabolic subalgebra of $\mathfrak{g}$ of Heisenberg type with $\mathfrak{l}$ its Levi factor and $\mathfrak{n}$ its nilpotent radical. Then, the action of $\mathrm{ad}(H_\gamma)$ on $\mathfrak{g}$ induces the eigenspace decomposition (1.1) of $\mathfrak{g}$, where γ is the highest root of $\mathfrak{g}$. Since $\mathfrak{z}(\mathfrak{n}) = \mathfrak{g}_\gamma$ is one-dimensional, there is a character χ of L_0 so that $\mathrm{Ad}(l)X_\gamma = \chi(l)X_\gamma$ for all $l \in L_0$. Note that $\mathrm{Ad}(l)X_{-\gamma} = \chi(l)^{-1}X_{-\gamma}$ for all $l \in L_0$, as $\mathfrak{g}_{-\gamma}$ is the $B_{\mathfrak{g}}$-dual space of $\mathfrak{g}_\gamma$. For the rest of this paper, fix χ so that its differential $d\chi$ is $d\chi = \gamma$.

Let $\mathscr{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ be the deleted Dynkin diagram associated to the Heisenberg parabolic $\mathfrak{q}$, that is, the subdiagram of the Dynkin diagram of $(\mathfrak{g}, \mathfrak{h})$ obtained by deleting the node corresponding to the simple root that is not orthogonal to γ, and the edges that involve it.

As in [BKZ08, p. 789] the operator Ω_2 is given in terms of R by

$$\textbf{(3.1)} \qquad \Omega_2(Z) = -\frac{1}{2} \sum_{\alpha,\beta\in\Delta(V^+)} N_{\beta,\beta'} M_{\alpha,\beta'}(Z) R(X_{-\alpha}) R(X_{-\beta})$$

for $Z \in \mathfrak{l}$. One can check that $\Omega_2(\mathrm{Ad}(l)Z) = \chi(l)l \cdot \Omega_2(Z)$ for all $l \in L_0$. This is different from the $\mathrm{Ad}(l)$ transformation law of Ω_2 that appears in [BKZ08], because the parabolic $\mathfrak{q}$ is chosen in this paper, while the opposite parabolic $\bar{\mathfrak{q}}$

is chosen in [BKZ08]. We extend the $\mathbb{C}$-linear maps $d\chi$, R, and Ω_2 to be left $C^\infty(\bar{N}_0)$-linear so that certain relationships can be expressed more easily.

Now for $t \in \mathbb{C}$ define an operator $\Omega_3^t : V^- \to \mathbb{D}(\mathcal{L}_{-s})^{\bar{\mathfrak{n}}}$ by

$$\Omega_3^t(Y) = \widetilde{\Omega}_3(Y) + tC_3(Y),$$

where the operators $\widetilde{\Omega}_3(Y)$ and $C_3(Y)$ are defined in terms of R and Ω_2 by

$$\widetilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(X_{-\epsilon})\Omega_2([X_\epsilon, Y]),$$
$$C_3(Y) = R(Y)R(X_{-\gamma})$$

as in [BKZ08, p. 801].

Lemma 3.1. *Let $W_1, \dots, W_m$ be a basis for V^+ and $W_1^*, \dots, W_m^*$ be the $B_{\mathfrak{g}}$-dual basis of V^-. Then*

$$\widetilde{\Omega}_3(Y) = \sum_{i=1}^m R(W_i^*)\Omega_2([W_i, Y]).$$

Proof. Suppose that $\Delta(V^+) = \{\epsilon_1, \dots, \epsilon_m\}$. Each W_i then may be expressed by $W_i = \sum_{j=1}^m a_{ij} X_{\epsilon_j}$ for $a_{ij} \in \mathbb{C}$. Let $[a_{ij}]$ be the change of basis matrix and set $[b_{ij}] = [a_{ij}]^{-1}$. Then define $W_i^* = \sum_{k=1}^m b_{ki} X_{-\epsilon_k}$ for $i = 1, \dots, m$. Note that $\sum_{s=1}^m a_{is} b_{sj} = \delta_{ij}$ with δ_{ij} the Kronecker delta. Since $B_{\mathfrak{g}}(X_{\epsilon_i}, X_{-\epsilon_j}) = \delta_{ij}$, it follows that $B_{\mathfrak{g}}(W_i, W_j^*) = \delta_{ij}$. So $\{W_1^*, \dots, W_m^*\}$ is the dual basis of $\{W_1, \dots, W_m\}$. Hence,

$$\sum_{i=1}^m R(W_i^*)\Omega_2([W_i, Y]) = \sum_{j,k=1}^m \Big(\sum_{i=1}^m b_{ki} a_{ij}\Big) R(X_{-\epsilon_k})\Omega_2([X_{\epsilon_j}, Y])$$
$$= \sum_{j=1}^m R(X_{-\epsilon_j})\Omega_2([X_{\epsilon_j}, Y]). \qquad \square$$

Lemma 3.2. *For all $l \in L_0$, $Z \in \mathfrak{l}$, and $Y \in V^-$, we have*

$$\Omega_3^t(\mathrm{Ad}(l)Y) = \chi(l) l \cdot \Omega_3^t(Y),$$
$$\Omega_3^t\big([Z, Y]\big) = d\chi(Z)\Omega_3^t(Y) + [\Pi_s(Z), \Omega_3^t(Y)].$$

Proof. To obtain the first equality it suffices to show that $\widetilde{\Omega}_3$ and C_3 have the proposed transformation law. Recall that $l \cdot R(u) = R(\mathrm{Ad}(l)u)$ for $l \in L_0$ and $u \in \mathcal{U}(\bar{\mathfrak{n}})$; in particular, we have $l \cdot R(X_{-\gamma}) = \chi(l)^{-1} R(X_{-\gamma})$. Therefore $\chi(l) l \cdot C_3(Y) = R(\mathrm{Ad}(l)Y)R(X_{-\gamma})$, which is $C_3(\mathrm{Ad}(l)Y)$. Since $\Omega_2(\mathrm{Ad}(l)W) = \chi(l) l \cdot \Omega_2(W)$ for $l \in L_0$ and $W \in \mathfrak{l}$, it follows that

$$\chi(l) l \cdot \widetilde{\Omega}_3(Y) = \sum_{\epsilon \in \Delta(V^+)} R(\mathrm{Ad}(l)X_{-\epsilon})\Omega_2\big([\mathrm{Ad}(l)X_\epsilon, \mathrm{Ad}(l)Y]\big). \tag{3.2}$$

By Lemma 3.1, the value of $\widetilde{\Omega}_3(Y)$ is independent from a choice of a basis for V^+. Thus the righthand side of (3.2) equals $\sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([X_\epsilon, \mathrm{Ad}(l)Y]\big)$, which is $\widetilde{\Omega}_3(\mathrm{Ad}(l)Y)$. The second equality is obtained by differentiating the first. □

Let $\omega_3^t(Y)$ denote the element in $\mathcal{U}(\bar{\mathfrak{n}})\otimes\mathbb{C}_{s\,d\chi}$ that corresponds to $\Omega_3^t(Y)$ under R.

Lemma 3.3. *For $Z \in \mathfrak{l}$ and $Y \in V^-$, we have*

$$\omega_3^t([Z, Y]) = Z\omega_3^t(Y) + (1-s)\,d\chi(Z)\omega_3^t(Y).$$

Proof. Lemma 3.2 shows that $\Omega_3^t(\mathrm{Ad}(l)Y) = \chi(l)l\cdot\Omega_3^t(Y)$ for $l \in L_0$. Thus by [BKZ09, Lemma 18], $\omega_3^t(\mathrm{Ad}(l)Y) = \chi(l)\,\mathrm{Ad}(l)\omega_3^t(Y)$. The formula is then obtained by replacing l by $\exp(tZ)$ with $Z \in \mathfrak{l}_0$, differentiating, and setting at $t = 0$. □

Let E be an irreducible L_0-submodule of V^-. We say that the $\Omega_3|_E$ system exists if there exist $t_0, s_0 \in \mathbb{C}$ so that the list of differential operators $\Omega_3^{t_0}|_E = \Omega_3^{t_0}(X_{\beta_1}), \ldots, \Omega_3^{t_0}(X_{\beta_m})$ with $\Delta(E) = \{\beta_1, \ldots, \beta_m\}$ is conformally invariant over the line bundle $\mathscr{L}_{-s_0}$. Set $F_t(E) = \mathrm{span}_{\mathbb{C}}\{\omega_3^t(Y) \mid Y \in E\}$.

Lemma 3.4. *If the $\Omega_3^t|_E$ system is conformally invariant for $t = t_0$ over $\mathscr{L}_{-s_0}$ then $\mathfrak{n}$ acts on $F_{t_0}(E)$ trivially.*

Proof. Since the $\Omega_3^{t_0}|_E$ system is conformally invariant over the line bundle $\mathscr{L}_{-s_0}$, the space $F_{t_0}(E)$ is a $\mathfrak{q}$-submodule of $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s_0 d\chi})$. By applying Lemma 3.3 with $Z = H_\gamma$, we obtain $H_\gamma\omega_3^{t_0}(Y) = (2s_0-3)\omega_3^{t_0}(Y)$ for all $Y \in E$. For $U \in V^+$ we have $H_\gamma U\omega_3^{t_0}(Y) = (2s_0-2)U\omega_3^{t_0}(Y)$, and $H_\gamma X_\gamma\omega_3^{t_0}(Y) = (2s_0-1)X_\gamma\omega_2(Y)$ for all $Y \in E$. Therefore if $U \in \mathfrak{n}$ then $U\omega_3^{t_0}(Y) = 0$ for all $Y \in E$, because otherwise $U\omega_3^{t_0}(Y)$ would have the wrong H_γ-eigenvalue to lie in $F_{t_0}(E)$. □

By using the transformation law $\omega_3^t(\mathrm{Ad}(l)Y) = \chi(l)\,\mathrm{Ad}(l)\omega_3^t(Y)$ for $l \in L_0$ and $Y \in V^-$, one can check that for any $s \in \mathbb{C}$ the vector space isomorphism

$$E \otimes \mathbb{C}_{(s-1)d\chi} \to F_t(E), \tag{3.3}$$

given by $Y \otimes 1 \mapsto \omega_3^t(Y)$, is L_0-equivariant with respect to the standard action of L_0 on the tensor products $E \otimes \mathbb{C}_{(s-1)d\chi}$ and $F_t(E) \subset \mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{s\,d\chi}$. In particular, the L_0-module $F_{t_0}(E)$ is irreducible. The L_0-action on $F_t(E)$ is given by $l\cdot(u \otimes 1) = \chi^s(l)(\mathrm{Ad}(l)u \otimes 1)$, which is different from the one that is used to establish the L_0-equivariant isomorphism (2.3).

Now suppose that the $\Omega_3^t|_E$ system is conformally invariant for $t = t_0$ over $\mathscr{L}_{-s_0}$. Then $F_{t_0}(E)$ is a $\mathfrak{q}$-submodule of $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{s_0 d\chi}$. Since $F_{t_0}(E)$ is an irreducible L_0-module and $\mathfrak{n}$ acts on it trivially by Lemma 3.4, the inclusion map $F_{t_0}(E) \hookrightarrow \mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s_0 d\chi})$ induces a nonzero $\mathcal{U}(\mathfrak{g})$-homomorphism of generalized Verma modules

$$\mathcal{M}_{\mathfrak{q}}(F_{t_0}(E)) \to \mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s_0 d\chi})$$

that is given by $u \otimes \omega_3^{t_0}(Y) \mapsto u \cdot \omega_3^{t_0}(Y)$. In particular, the two Verma modules $\mathcal{M}_{\mathfrak{q}}(F_{t_0}(E))$ and $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s_0 d\chi})$ have the same infinitesimal characters. Since we choose a character χ so that $d\chi = \gamma$, this implies that if ϖ is the highest weight for E then

$$\|\varpi + (s_0 - 1)\gamma + \rho\|^2 = \|s_0\gamma + \rho\|^2, \tag{3.4}$$

This will restrict the possibility of s_0 for which the Ω_3^t is conformally invariant.

4. The Ω_3 system on $\mathfrak{sl}(3, \mathbb{C})$

We take the complex Lie group G from Section 3 to be $\mathrm{SL}(3, \mathbb{C})$ and show that the Ω_3 system(s) exists over the line bundle $\mathcal{L}_0$. Since the generalized Verma module $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s d\chi})$ is a (ordinary) Verma module in this case, we simply write $\mathcal{M}(\mathbb{C}_{s d\chi}) = \mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{s d\chi})$ throughout this section.

Let α_1 and α_2 be the two simple roots for $\mathfrak{sl}(3, \mathbb{C})$. Then $V^- = \mathbb{C}X_{-\alpha_1} \oplus \mathbb{C}X_{-\alpha_2}$; each of $\mathbb{C}X_{-\alpha_i}$ for $i = 1, 2$ is an L_0-submodule of V^-. A direct computation shows that $\Omega_3^t(X_{-\alpha_i}) = \widetilde{\Omega}_3(X_{-\alpha_i}) + tC_3(X_{-\alpha_i})$ is not identically zero for $i = 1, 2$ and for any $t \in \mathbb{C}$. Then solving (3.4) with $\varpi = -\alpha_i$ for $i = 1, 2$ gives that if $\Omega_3^t(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_{-s_0}$ then the special value s_0 of s must be $s_0 = 0$. Now we show that there exists a unique $t_i \in \mathbb{C}$ so that $\Omega_3^{t_i}(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_0$.

Theorem 4.1. *Let $\mathfrak{g}$ be the complex simple Lie algebra of type A_2, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then for each $i = 1, 2$ the operator $\Omega_3^t(X_{-\alpha_i})$ is conformally invariant over $\mathcal{L}_0$ if and only if $t = \frac{3}{4}$.*

Proof. Fix α_i and denote by α_k the other simple root so that $S = \{\alpha_i, \alpha_k\}$. Observe that $\omega_3^t(X_{-\alpha_i})$ is the element in $\mathcal{M}(\mathbb{C}_0)$ that corresponds to $\Omega_3^t(X_{-\alpha_i})$ in $\mathbb{D}(\mathcal{L}_0)^{\bar{\mathfrak{n}}}$ under the map (2.3). By Theorem 2.2 and Lemma 3.4, it suffices to show that $\mathbb{C}\omega_3^t(X_{-\alpha_i})$ is a $\mathfrak{q}$-submodule of $\mathcal{M}(\mathbb{C}_0)$ with trivial $\mathfrak{n}$ action if and only if $t = \frac{3}{4}$.

A direct computation shows that the element in $\mathcal{M}(\mathbb{C}_0)$ that corresponds to $\widetilde{\Omega}_3(X_{-\alpha_i})$ may be written as

$$-\tfrac{3}{2} N_{\alpha_i,\alpha_k} X_{-\alpha_i}^2 X_{-\alpha_k} \otimes 1 - \tfrac{3}{4} X_{-\alpha_i} X_{-\gamma} \otimes 1.$$

As $C_3(X_{-\alpha_i}) = R(X_{-\alpha_i})R(X_{-\gamma})$, the element in $\mathcal{M}(\mathbb{C}_0)$ corresponding to $C_3(X_{-\alpha_i})$ is $X_{-\alpha_i} X_{-\gamma} \otimes 1$. Thus $\omega_3^t(X_{-\alpha_i})$ is given by

$$\omega_3^t(X_{-\alpha_i}) = -\tfrac{3}{2} N_{\alpha_i,\alpha_k} X_{-\alpha_i}^2 X_{-\alpha_k} \otimes 1 + \left(t - \tfrac{3}{4}\right) X_{-\alpha_i} X_{-\gamma} \otimes 1.$$

One can easily check that $\mathfrak{n}$ acts trivially on $\mathbb{C}X_{-\alpha_1}^2 X_{-\alpha_2} \otimes 1$ and $\mathbb{C}X_{-\alpha_2}^2 X_{-\alpha_1} \otimes 1$ and thus both of them are one-dimensional $\mathfrak{q}$-submodules of $\mathcal{M}(\mathbb{C}_0)$, while it acts nontrivially on $X_{-\alpha_1} X_{-\gamma} \otimes 1$ and $X_{-\alpha_2} X_{-\gamma} \otimes 1$ in $\mathcal{M}(\mathbb{C}_0)$. Therefore $\mathbb{C}\omega_3^t(X_{-\alpha_i})$ is a $\mathfrak{q}$-submodule with trivial $\mathfrak{n}$ action if and only if $t = \frac{3}{4}$. ☐

5. The Ω_3 system on $\mathfrak{so}(8, \mathbb{C})$

We take the complex Lie group G from Section 3 to be $\mathrm{SO}(8, \mathbb{C})$ and show that the $\widetilde{\Omega}_3$ system is conformally invariant over the line bundle $\mathcal{L}_1$.

Since in this case the parabolic $\mathfrak{q}$ is maximal, the $\mathfrak{l}$-module V^- is irreducible with highest weight $-\alpha_\gamma$, where α_γ is the simple root that is not orthogonal to γ. Then by solving (3.4) with $\varpi = -\alpha_\gamma$, one can see that if the Ω_3 system exists then the special value s_0 of s must be $s_0 = -1$. Thus in the rest of this paper the line bundle $\mathcal{L}_{-s}$ is assumed to be $\mathcal{L}_1$, and for simplicity, write Π for the Lie algebra action Π_s defined in (2.1) for $s = -1$. As stated in Section 2, for $D \in \mathbb{D}(\mathcal{L}_{-s})$, denote by $D_{\bar{n}}$ the linear functional $f \mapsto (D \bullet f)(\bar{n})$ for $f \in C^\infty(\overline{N}_0, \mathbb{C}_{\chi^{-s}})$.

Proposition 5.1. *For all $X \in \mathfrak{g}$, $Y \in V^-$, and $\bar{n} \in \overline{N}_0$, we have*

$$[\Pi(X), R(Y)]_{\bar{n}} = R\big([\mathrm{Ad}(\bar{n}^{-1})X, Y]_{V^-}\big)_{\bar{n}} - d\chi\big([\mathrm{Ad}(\bar{n}^{-1})X, Y]_{\mathfrak{l}}\big).$$

Proof. Let F be the subspace of $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$ spanned by $X_{-\alpha} \otimes 1$ and $1 \otimes 1$ with $\alpha \in \Delta(V^+)$. A direct computation shows that F is a $\mathfrak{q}$-submodule of $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$ and that for $Z \in \mathfrak{l}$ and $U \in \mathfrak{n}$ we have

$$\begin{aligned} Z(X_{-\alpha} \otimes 1) &= [Z, X_{-\alpha}] \otimes 1 - d\chi(Z) X_{-\alpha} \otimes 1, \\ U(X_{-\alpha} \otimes 1) &= -d\chi([U, X_{-\alpha}]_{\mathfrak{l}}) 1 \otimes 1. \end{aligned}$$

Then it follows from Theorem 2.2 that if $X \in \mathfrak{g}$ and $(\mathrm{Ad}(\bar{n}^{-1})X)_{\mathfrak{q}} = Z + U$ with $Z \in \mathfrak{l}$ and $U \in \mathfrak{n}$ then for $Y \in V^-$,

$$[\Pi(X), R(Y)]_{\bar{n}} = R([Z, Y])_{\bar{n}} - d\chi([U, Y]).$$

Since $[Z, Y] = [\mathrm{Ad}(\bar{n}^{-1})X, Y]_{V^-}$ and $[U, X_{-\alpha}]_{\mathfrak{l}} = [\mathrm{Ad}(\bar{n}^{-1})X, Y]_{\mathfrak{l}}$, this completes the proof. □

Let $\omega_2(W)$ denote the element in $\mathcal{U}(\bar{\mathfrak{n}}) \otimes \mathbb{C}_{-d\chi}$ that corresponds to $\Omega_2(W)$ under R. Observe that $\Omega_2(\mathrm{Ad}(l)W) = \chi(l) l \cdot \Omega_2(W)$ for all $l \in L_0$; this is the same $\mathrm{Ad}(l)$ transformation law as Ω_3^t (see Lemma 3.2). Then Lemma 3.3 with $s = -1$ implies that for $W, Z \in \mathfrak{l}$, we have

$$\omega_2([Z, W]) = Z\omega_2(W) + 2d\chi(Z)\omega_2(W). \tag{5.1}$$

Proposition 5.2. *For all $X \in \mathfrak{g}$, $W \in \mathfrak{l}$, and $\bar{n} \in \overline{N}_0$, we have*

$$[\Pi(X), \Omega_2(W)]_{\bar{n}} = \Omega_2\big([\mathrm{Ad}(\bar{n}^{-1})X, W]_{\mathfrak{l}}\big)_{\bar{n}} - d\chi\big((\mathrm{Ad}(\bar{n}^{-1})X)_{\mathfrak{l}}\big)\Omega_2(W)_{\bar{n}}.$$

Proof. It follows from [BKZ08, Theorem 5.2] and the data tabulated in [BKZ08, Section 8.10] that each Ω_2 system associated to a singleton component of $\mathcal{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ is conformally invariant on the line bundle $\mathcal{L}_1$. Note here that the special values of our Ω_2 system are of the form $-s_0$ with s_0 the special values of the Ω_2 system given in [BKZ08], as the parabolic $\mathfrak{q}$ is chosen in this paper, while the opposite parabolic $\bar{\mathfrak{q}}$ is chosen in [BKZ08]. Therefore $F \equiv \mathrm{span}_{\mathbb{C}}\{\omega_2(W) \mid W \in \mathfrak{l}\}$ is a $\mathfrak{q}$-submodule of $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$. The same argument for the proof for Lemma 3.4 shows that $\mathfrak{n}$ acts on F trivially. By (5.1), we have

$$Z\omega_2(W) = \omega_2([Z, W]) - 2d\chi(Z)\omega_2(W)$$

for $Z, W \in \mathfrak{l}$. The proposed formula now follows from Theorem 2.2. □

Lemma 5.3. *For $X \in V^+$ and $Y \in V^-$, we have*

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = 2\Omega_2([X, Y]).$$

Proof. Since $\|\epsilon\|^2 = 2$ for all $\epsilon \in \Delta(V^+)$, it follows from [BKZ08, Proposition 2.2] that

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = \frac{1}{2}\sum_{\mathscr{C}} p(D_4, \mathscr{C})\Omega_2\big(\mathrm{pr}_{\mathscr{C}}([X, Y])\big),$$

where $\mathscr{C}$ are the connected components of $\mathscr{D}_\gamma(\mathfrak{g}, \mathfrak{h})$ as in [BKZ08] and $\mathrm{pr}_{\mathscr{C}}([X, Y])$ is the projection of $[X, Y]$ onto $\mathfrak{l}(\mathscr{C})$, the ideal of $[\mathfrak{l}, \mathfrak{l}]$ corresponding to $\mathscr{C}$. (See [BKZ08, Section 2] for further discussion.) From [BKZ08, Section 8.4] we have $p(D_4, \mathscr{C}) = 4$ for all the components $\mathscr{C}$. Then $\Omega_2(H_\gamma) = 0$ shows that

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big) = 2\Omega_2\big([X, Y]\big). \qquad \square$$

Now with the above lemmas and propositions we are ready to show the following key theorem.

Theorem 5.4. *We have $[\Pi(X), \widetilde{\Omega}_3(Y)]_e = 0$ for all $X \in V^+$ and all $Y \in V^-$.*

Proof. Observe that $\widetilde{\Omega}_3(Y) = \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([X_\epsilon, Y]\big)$. Then the commutator $[\Pi(X), \widetilde{\Omega}_3(Y)]$ is a sum of two terms. One of them is given by

$$\textbf{(5.2)} \quad \sum_{\epsilon\in\Delta(V^+)} [\Pi(X), R(X_{-\epsilon})]\Omega_2([X_\epsilon, Y])$$
$$= \sum_{\epsilon\in\Delta(V^+)} R\big([\mathrm{Ad}(\cdot^{-1})X, X_{-\epsilon}]_{V^-}\big)\Omega_2([X_\epsilon, Y])$$
$$- \sum_{\epsilon\in\Delta(V^+)} d\chi([\mathrm{Ad}(\cdot^{-1})X, X_{-\epsilon}]_{\mathfrak{l}})\Omega_2([X_\epsilon, Y]),$$

by Proposition 5.1. At e, the first term is zero, since $[X, X_{-\epsilon}]_{V^-} = 0$ for all $\epsilon \in \Delta(V^+)$. By writing out X as a linear combination of X_α with $\alpha \in \Delta(V^+)$, at the identity the second term in (5.2) evaluates to

$$-\sum_{\epsilon\in\Delta(V^+)} d\chi([X, X_{-\epsilon}])\Omega_2([X_\epsilon, Y])_e = -\Omega_2([X, Y])_e,$$

since $d\chi(H_\alpha) = 1$ for $\alpha \in \Delta(V^+)$. The other term is given by

$$\begin{aligned}
\textbf{(5.3)}\quad \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\big[\Pi(X), \Omega_2([X_\epsilon, Y])\big]
&= \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\Omega_2\big([\mathrm{Ad}(\cdot^{-1})X, [X_\epsilon, Y]]_{\mathfrak{l}}\big) \\
&\quad - \sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})d\chi\big((\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}}\big)\Omega_2([X_\epsilon, Y]),
\end{aligned}$$

by Proposition 5.2. To further evaluate this expression, we make use of a simple general observation. Namely, if D is a first order differential operator, ϕ and ψ are smooth functions, and $\phi(e) = 0$, then $D_e(\phi\psi) = D_e(\phi)\psi(e)$. The map $\bar{n} \mapsto \mathrm{ad}(\mathrm{Ad}(\bar{n}^{-1})X)$ is a smooth function on $\bar{N}_0$. The left $C^\infty(\bar{N}_0)$-linear extension of Ω_2 implies that the first term of the righthand side of (5.3) can be expressed as

$$\sum_{\epsilon\in\Delta(V^+)} R(X_{-\epsilon})\big(\mathrm{ad}(\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}} \cdot \Omega_2([X_\epsilon, Y])\big),$$

where $\mathrm{ad}(\mathrm{Ad}(\cdot^{-1})X)_{\mathfrak{l}}$ denotes the map $Z \mapsto [\mathrm{Ad}(\cdot^{-1})X, Z]_{\mathfrak{l}}$ for $Z \in \mathfrak{g}$. Since

$$\big(R(X_{-\epsilon}) \bullet (\mathrm{Ad}(\cdot^{-1})X)\big)(e) = [X, X_{-\epsilon}],$$

$[X, [X_\epsilon, Y]]_{\mathfrak{l}} = 0$, and $X_{\mathfrak{l}} = 0$, the righthand side of (5.3) then evaluates at the identity to

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big)_e - \sum_{\epsilon\in\Delta(V^+)} d\chi([X, X_{-\epsilon}])\Omega_2([X_\epsilon, Y])_e,$$

which is equivalent to

$$\sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big)_e - \Omega_2([X, Y])_e.$$

Therefore,

$$[\Pi(X), \widetilde{\Omega}_3(Y)]_e = \sum_{\epsilon\in\Delta(V^+)} \Omega_2\big([[X, X_{-\epsilon}], [X_\epsilon, Y]]\big)_e - 2\Omega_2([X, Y])_e.$$

Now it follows from Lemma 5.3 that $[\Pi(X), \widetilde{\Omega}_3(Y)]_e = 0$. □

Proposition 5.5. *For $Y \in V^-$, we have $[\Pi(X_\gamma), \widetilde{\Omega}_3(Y)]_e = 0$.*

Proof. Since $\mathfrak{z}(\mathfrak{n}) = [V^+, V^+]$, it suffices to show that $[\Pi([X_1, X_2]), \widetilde{\Omega}_3(Y)]_e = 0$ for $X_1, X_2 \in V^+$. We have $\Pi([X_1, X_2]) = [\Pi(X_1), \Pi(X_2)]$, so it follows from the Jacobi identity that $[\Pi([X_1, X_2]), \widetilde{\Omega}_3(Y)]$ may be expressed as a sum of two terms. The first is

$$[\Pi(X_1), [\Pi(X_2), \widetilde{\Omega}_3(Y)]] = \Pi(X_1)[\Pi(X_2), \widetilde{\Omega}_3(Y)] - [\Pi(X_2), \widetilde{\Omega}_3(Y)]\Pi(X_1).$$

By (2.2), we have $\Pi(X)_e = 0$ for all $X \in \mathfrak{n}$. This fact and Theorem 5.4 imply $[\Pi(X_1), [\Pi(X_2), \widetilde{\Omega}_3(Y)]]_e = 0$ since $(D_1D_2)_e = (D_1)_e D_2$ for $D_1, D_2 \in \mathbb{D}(\mathscr{L}_1)$. The second term is

$$[\Pi(X_2), [\widetilde{\Omega}_3(Y), \Pi(X_1)]] = \Pi(X_2)[\widetilde{\Omega}_3(Y), \Pi(X_1)] - [\widetilde{\Omega}_3(Y), \Pi(X_1)]\Pi(X_2).$$

By the same argument for the first term, $[\Pi(X_2), [\widetilde{\Omega}_3(Y), \Pi(X_1)]]_e = 0$, which concludes the proof. □

Theorem 5.6. *Let $\mathfrak{g}$ be the complex simple Lie algebra of type D_4 and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the $\widetilde{\Omega}_3$ system is conformally invariant on the line bundle $\mathscr{L}_1$.*

Proof. By Proposition 2.1, it is enough to check the conformal invariance of $[\Pi(X), \widetilde{\Omega}_3(Y)]$ at the identity for all $X \in \mathfrak{g}$ and all $Y \in V^-$. It follows from Lemma 3.2 that

$$[\Pi(Z), \widetilde{\Omega}_3(Y)]_e = \widetilde{\Omega}_3\big([Z, Y]\big)_e - d\chi(Z)\widetilde{\Omega}_3(Y)_e$$

for all $Z \in \mathfrak{l}$. Also Theorem 5.4 and Proposition 5.5 show that $[\Pi(U), \widetilde{\Omega}_3(Y)] = 0$ for all $U \in \mathfrak{n}$. As $\widetilde{\Omega}_3(Y)$ is an element in $\mathbb{D}(\mathscr{L}_1)^{\bar{\mathfrak{n}}}$, it is clear that $[\Pi(\overline{U}), \widetilde{\Omega}_3(Y)]_e = 0$ for all $\overline{U} \in \bar{\mathfrak{n}}$. Since $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{l} \oplus \mathfrak{n}$, this implies that the $\widetilde{\Omega}_3$ system is conformally invariant on $\mathscr{L}_1$. □

Theorem 5.6 implies that $F_0(V^-) = \operatorname{span}_{\mathbb{C}}\{\omega_3^0(Y) \mid Y \in V^-\}$ is a $\mathfrak{q}$-submodule of $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$, where $\omega_3^0(Y)$ is the element in $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$ that corresponds to $\widetilde{\Omega}_3(Y) = \Omega_3^0(Y)$ under R. The argument after Lemma 3.4 then shows that there exists a nonzero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\mathcal{M}_{\mathfrak{q}}(F_0(V^-)) \to \mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi}).$$

It follows from Lemma 3.3 that H_γ acts on $F_0(V^-)$ by -5, while it acts on $\mathbb{C}_{-d\chi}$ by -2; in particular, $F_0(V^-)$ is not equivalent to $\mathbb{C}_{-d\chi}$.

Corollary 5.7. *Let $\mathfrak{g}$ be the complex simple Lie algebra of type D_4, and $\mathfrak{q}$ be the parabolic subalgebra of Heisenberg type. Then the generalized Verma module $\mathcal{M}_{\mathfrak{q}}(\mathbb{C}_{-d\chi})$ is reducible.*

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TOSHIHISA KUBO
DEPARTMENT OF MATHEMATICS
OKLAHOMA STATE UNIVERSITY
STILLWATER OK 74078
UNITED STATES
toskubo@math.okstate.edu

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