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A MEAN CURVATURE ESTIMATE FOR CYLINDRICALLY BOUNDED SUBMANIFOLDS

LUIS J. ALÍAS AND MARCOS DAJCZER

In an earlier article in coauthorship with G. P. Bessa, we obtained an estimate for the mean curvature of a cylindrically bounded proper submanifold in a product manifold where one factor is a Euclidean space. Here we extend this estimate to a general product ambient space endowed with a warped product structure.

Let (L^ℓ, g_L) and (P^n, g_P) be complete Riemannian manifolds of dimension ℓ and n, respectively, where L^ℓ is noncompact. Then let $N^{n+\ell} = L^\ell \times_\rho P^n$ be the product manifold $L^\ell \times P^n$ endowed with the warped product metric $ds^2 = dg_L + \rho^2 dg_P$ for some positive warping function $\rho \in C^\infty(L)$.

Let $B_P(r_0)$ denote the geodesic ball with radius r_0 centered at a reference point $o \in P^n$. Assume that the radial sectional curvatures in $B_P(r_0)$ along the geodesics issuing from o are bounded as $K_P^{\mathrm{rad}} \le b$ for some constant $b \in \mathbb{R}$, and that $0 < r_0 < \min\{\mathrm{inj}_P(o), \pi/2\sqrt{b}\}$, where $\mathrm{inj}_P(o)$ is the injectivity radius at o and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \le 0$. Then the mean curvature of the geodesic sphere $S_P(r_0) = \partial B_P(r_0)$ can be estimated from below by the mean curvature of a geodesic sphere of a space form of curvature b, that is,

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if } b > 0, \\ 1/t & \text{if } b = 0, \\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if } b < 0. \end{cases}$$

This is a direct consequence of the comparison theorems for the Riemannian distance, since the Hessian (respectively, Laplacian) of the distance function is nothing but the second fundamental form (respectively, mean curvature) of the geodesic spheres. A classical reference on this topic is [Greene and Wu 1979]. We also refer the reader to [Petersen 2006] or [Pigola et al. 2008] for a modern approach to the Hessian and Laplacian comparison theorems.

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Keywords: Cylindrically bounded submanifolds, Omori–Yau maximum principle, proper immersions.

By a *cylinder* in the warped space $N^{n+\ell}$, we mean a closed subset of the form

$$\mathscr{C}_{r_0} = \{(x, y) \in N^{n+\ell} : x \in L^\ell \text{ and } y \in B_P(r_0)\}.$$

Since the submanifolds $L^\ell \times \{p_0\} \subset N^{n+\ell}$ are totally geodesic, we have

$$|\rho H_{\mathscr{C}_{r_0}}| \geq \frac{n-1}{\ell+n-1} C_b(r_0),$$

where $H_{\mathscr{C}_{r_0}}$ is the mean curvature vector field of the hypersurface $L^\ell \times S_p(r_0)$.

The following theorem extends the result in [Alías et al. 2009], where the cylinders under consideration are contained in product spaces $\mathbb{R}^\ell \times P^n$. After the statement, we recall from [Alías et al. 2011] the concept of an Omori–Yau pair on a Riemannian manifold and discuss some implications of its existence.

Theorem 1. *Let $f\colon M^m \to L^\ell \times_\rho P^n$ be an isometric immersion, where L^ℓ carries an Omori–Yau pair for the Hessian and the functions ρ and $|\operatorname{grad} \log \rho|$ are bounded. If f is proper and $f(M) \subset \mathscr{C}_{r_0}$, then $\sup_M |H| = +\infty$ or*

$$\sup_M \rho|H| \geq \frac{m-\ell}{m} C_b(r_0), \tag{1}$$

where H is the mean curvature vector field of f.

In the proof, we see that the existence in L^ℓ of an Omori–Yau pair for the Hessian provides conditions, in a function-theoretic form, that guarantee the validity of the Omori–Yau maximum principle on M^m in terms of the corresponding property of L^ℓ and the geometry of the immersion.

Definition 2. The pair of functions (h, γ), for $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma\colon M \to \mathbb{R}_+$, is an *Omori–Yau pair for the Hessian* in M if

(a) $h(0) > 0$ and $h'(t) \geq 0$, for all $t \in \mathbb{R}_+$;

(b) $\limsup_{t\to+\infty} th\big(\sqrt{t}\big)/h(t) < +\infty$;

(c) $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}t}{\sqrt{h(t)}} = +\infty$;

(d) the function γ is proper;

(e) $|\operatorname{grad} \gamma| \leq c\sqrt{\gamma}$ for some $c > 0$ outside a compact subset of M; and

(f) $\operatorname{Hess} \gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

Similarly, the pair (h, γ) is an *Omori–Yau pair for the Laplacian* in M if it satisfies conditions (a)–(e) and

(f′) $\Delta\gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

We say that the *Omori–Yau maximum principle for the Hessian* holds for M if for any function $g \in C^\infty(M)$ bounded from above there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in M such that

(a) $\lim_{k\to\infty} g(p_k) = \sup_M g$,

(b) $|\operatorname{grad} g(p_k)| \le 1/k$,

(c) $\operatorname{Hess} g(p_k)(X, X) \le (1/k) g_M(X, X)$ for all $X \in T_{p_k}M$.

Similarly, the *Omori–Yau maximum principle for the Laplacian* holds for M if these properties are satisfied with (c) replaced by

(c′) $\Delta g(p_k) \le 1/k$.

The following theorem of Pigola, Rigoli, and Setti gives sufficient conditions for an Omori–Yau maximum principle to hold for a Riemannian manifold.

Theorem 3 [Pigola et al. 2005]. *Assume that a Riemannian manifold M carries an Omori–Yau pair for the Hessian* (*resp. Laplacian*). *Then the Omori–Yau maximum principle for the Hessian* (*resp. Laplacian*) *holds in M.*

Example 4. Let M^m be a complete but noncompact Riemannian manifold, and write $r(y) = \operatorname{dist}_M(y, o)$ for some reference point $o \in M^m$. Assume that the radial sectional curvature of M^m satisfies $K^{\mathrm{rad}} \ge -h(r)$, where the smooth function h satisfies (a)–(c) in Definition 2 and is even at the origin, that is, $h^{(2k+1)}(0) = 0$ for $k \in \mathbb{N}$. Then, as shown in [Pigola et al. 2005], the functions (h, r^2) form an Omori–Yau pair for the Hessian. As for the function h, one can choose

$$h(t) = t^2 \prod_{j=1}^{N} (\log^{(j)}(t))^2, \quad t \gg 1,$$

where $\log^{(j)}$ stands for the j-th iterated logarithm.

To conclude this section, we observe that Theorem 1 is sharp. This is clear from (1) by taking as P^n a space-form and as M the hypersurface $L^\ell \times S_P(r_0)$ in $N^{n+\ell}$. In view of Example 4, it also follows that by taking $L^\ell = \mathbb{R}^\ell$ and constant ρ we recover the result in [Alías et al. 2009].

The proof

We first introduce some additional notations, and then recall a few basic facts on warped product manifolds.

Let $\langle\ ,\ \rangle$ denote the metrics in $N^{n+\ell}$, L^ℓ and M^m, while $(\ ,\ )$ stands for the metric in P^n. The corresponding norms are $|\ |$ and $\|\ \|$. In addition, let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections in M^m and $N^{n+\ell}$, respectively, and ∇^L and ∇^P the ones in L^ℓ and P^n.

We always denote vector fields in TL by T, S and in TP by X, Y. Also, we identify vector fields in TL and TP with *basic* vector fields in TN by taking $T(x, y) = T(x)$ and $X(x, y) = X(y)$.

For the Lie-brackets of basic vector fields, we have that $[T, S] \in TL$ and $[X, Y] \in TP$ are basic and that $[X, T] = 0$. Then we have

$$\begin{aligned} \widetilde{\nabla}_S T &= \nabla^L_S T, \\ \widetilde{\nabla}_X T &= \widetilde{\nabla}_T X = T(\varrho)X, \\ \widetilde{\nabla}_X Y &= \nabla^P_X Y - \langle X, Y\rangle \operatorname{grad}^L \varrho, \end{aligned}$$

where the vector fields X, Y and T are basic and $\varrho = \log \rho$.

Our proof follows the main steps in [Alías et al. 2011], where the geometric situation considered differs from ours in that $f(M)$ there is contained in a *cylinder* of the form

$$\{(x, y) \in N^{n+\ell} : x \in B_L(r_0) \text{ and } y \in P^n\}.$$

In fact, a substantial part of the argument is to show that the Omori–Yau pair for the Hessian in L^ℓ induces an Omori–Yau pair for the Laplacian for a noncompact M^m when $|H|$ is bounded. Thus the Omori–Yau maximum principle for the Laplacian holds in M^m, and the proof follows from an application of the latter.

Suppose that M^m is noncompact, and let (h, Γ) be an Omori–Yau pair for the Hessian in L^ℓ. For $p \in M^m$, write $f(p) = (x(p), y(p))$. Set $\tilde{\Gamma}(x, y) = \Gamma(x)$ for $(x, y) \in N^{n+\ell}$ and

$$\gamma(p) = \tilde{\Gamma}(f(p)) = \Gamma(x(p)).$$

We show next that (h, γ) is an Omori–Yau pair for the Laplacian in M^m. First we argue that the function γ is proper. To see this, let $p_k \in M^m$ be a divergent sequence, that is, $p_k \to \infty$ in M^m as $k \to +\infty$. Thus, $f(p_k) \to \infty$ in $N^{n+\ell}$ because f is proper. Because $f(M)$ lies inside a cylinder, $x(p_k) \to \infty$ in L^ℓ. Hence, $\gamma(p_k) \to +\infty$ as $k \to +\infty$ because Γ is proper, and thus γ is proper.

It remains to verify conditions (e) and (f′) in Definition 2. We have from $\tilde{\Gamma}(x, y) = \Gamma(x)$ that

$$\langle \operatorname{grad}^N \tilde{\Gamma}(x, y), X\rangle = 0.$$

Thus

$$\operatorname{grad}^N \tilde{\Gamma}(x, y) = \operatorname{grad}^L \Gamma(x).$$

Since $\gamma = \tilde{\Gamma} \circ f$, we obtain

$$\operatorname{grad}^N \tilde{\Gamma}(f(p)) = \operatorname{grad}^M \gamma(p) + \operatorname{grad}^N \tilde{\Gamma}(f(p))^\perp, \tag{2}$$

where $(\ )^\perp$ denotes taking the normal component to f. Then

$$|\operatorname{grad}^M \gamma(p)| \le |\operatorname{grad}^N \tilde{\Gamma}(f(p))| = |\operatorname{grad}^L \Gamma(x(p))| \le c\sqrt{\Gamma(x(p))} = c\sqrt{\gamma(p)}$$

outside a compact subset of M^m, and thus (e) holds.

We have that

$$\widetilde{\nabla}_T \mathrm{grad}^N \tilde{\Gamma} = \nabla^L_T \mathrm{grad}^L \Gamma.$$

Hence Hess $\tilde{\Gamma}(T, S) =$ Hess $\Gamma(T, S)$ and Hess $\tilde{\Gamma}(T, X) = 0$. Also,

$$\widetilde{\nabla}_X \mathrm{grad}^N \tilde{\Gamma} = \widetilde{\nabla}_X \mathrm{grad}^L \Gamma = \mathrm{grad}^L \Gamma(\varrho) X.$$

Hence

$$\mathrm{Hess}\, \tilde{\Gamma}(X, Y) = \langle \mathrm{grad}^L \Gamma, \mathrm{grad}^L \varrho \rangle \langle X, Y \rangle.$$

For a unit vector $e \in T_p M$, set $e = e^L + e^P$, where $e^L \in T_{x(p)} L$ and $e^P \in T_{y(p)} P$. Then

$$\mathrm{Hess}\, \tilde{\Gamma}(f(p))(e, e) = \mathrm{Hess}\, \Gamma(x(p))(e^L, e^L) + \big\langle \mathrm{grad}^L \Gamma(x(p)), \mathrm{grad}^L \varrho(x(p)) \big\rangle |e^P|^2.$$

Also, an easy computation using (2) yields

$$\mathrm{Hess}\, \gamma(p)(e, e) = \mathrm{Hess}\, \tilde{\Gamma}(f(p))(e, e) + \big\langle \mathrm{grad}^L \Gamma(x(p)), \alpha(p)(e, e) \big\rangle,$$

where α denotes the second fundamental of f with values in the normal bundle. Thus,

$$\begin{aligned} \mathrm{Hess}\, \gamma(p)(e, e) = \mathrm{Hess}\, \Gamma(x(p))(e^L, e^L) &+ \big\langle \mathrm{grad}^L \Gamma(x(p)), \mathrm{grad}^L \varrho(x(p)) \big\rangle |e^P|^2 \\ &+ \big\langle \mathrm{grad}^L \Gamma(x(p)), \alpha(p)(e, e) \big\rangle. \end{aligned}$$

Since Hess $\Gamma \leq d\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some positive constant d outside a compact subset of L^ℓ and the immersion is proper, we have

$$\mathrm{Hess}\, \Gamma(x(p))(e^L, e^L) \leq d\sqrt{\gamma(p) h(\sqrt{\gamma(p)})} |e^L|^2 \leq d\sqrt{\gamma(p) h(\sqrt{\gamma(p)})}$$

outside a compact subset of M^m. From $|\mathrm{grad}^L \Gamma| \leq c\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some c outside a compact subset of L^ℓ and $\sup_L |\mathrm{grad}^L \varrho| < +\infty$, we have

$$\big\langle \mathrm{grad}^L \Gamma(x(p)), \mathrm{grad}^L \varrho(x(p)) \big\rangle |e^P|^2 \leq c' \sqrt{\gamma(p)}$$

for some positive constant c' outside a compact subset of M^m. Since γ is proper and h is unbounded, by (a) and (b) in Definition 2, we have

$$\sqrt{\gamma} \leq \sqrt{\gamma h(\sqrt{\gamma}\,)}$$

outside a compact subset of M^m, because $\gamma \to +\infty$ as $p \to \infty$ and $\lim_{t \to +\infty} h(t) = +\infty$. Thus we obtain

$$\mathrm{Hess}\, \gamma(e, e) \leq d_1 \sqrt{\gamma h(\sqrt{\gamma}\,)} + \big\langle \mathrm{grad}^L \Gamma(x), \alpha(e, e) \big\rangle \tag{3}$$

for some constant $d_1 > 0$, outside a compact subset of M^m.

On the other hand, we may assume that

$$|H| \leq c\sqrt{h(\sqrt{\gamma}\,)} \tag{4}$$

for some constant $c > 0$, outside a compact subset of M^m. Otherwise, there exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that $p_k \to \infty$ as $k \to +\infty$ and

$$|H(p_k)| > k\sqrt{h(\sqrt{\gamma(p_k)})}.$$

With γ being proper and h unbounded from (a) and (b) in Definition 2, we conclude that $\sup_M |H| = +\infty$, in which case we are done with the proof of the theorem.

We obtain from (3) using (4) that $\Delta\gamma \leq c_1\sqrt{\gamma h(\sqrt{\gamma}\,)}$ for some constant $c_1 > 0$ outside a compact subset of M^m, and thus (f′) has been proved.

Consider the distance function $r(y) = \mathrm{dist}_P(y, o)$ in $B_P(r_0)$ and define $\tilde{r} \in C^\infty(N)$ by $\tilde{r}(x, y) = r(y)$. Then

$$\langle \mathrm{grad}^N \tilde{r}(x, y), T\rangle = 0.$$

Thus

$$\rho^2(x)\mathrm{grad}^N \tilde{r}(x, y) = \mathrm{grad}^P r(y).$$

We obtain that

$$\widetilde{\nabla}_T \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_T(\rho^{-2}\mathrm{grad}^P r) = -\rho^{-2}T(\varrho)\mathrm{grad}^P r.$$

Therefore

$$\mathrm{Hess}\, \tilde{r}(T, S) = 0$$

and

$$\mathrm{Hess}\, \tilde{r}(T, X) = -\rho^{-2}T(\varrho)\langle \mathrm{grad}^P r, X\rangle = -T(\varrho)(\mathrm{grad}^P r, X).$$

Also,

$$\widetilde{\nabla}_X \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_X(\rho^{-2}\mathrm{grad}^P r) = \rho^{-2}\big(\nabla^P_X \mathrm{grad}^P r - \langle X, \mathrm{grad}^P r\rangle \mathrm{grad}^L \varrho\big).$$

Hence

$$\mathrm{Hess}\, \tilde{r}(X, Y) = \rho^{-2}\langle \nabla^P_X \mathrm{grad}^P r, Y\rangle = (\nabla^P_X \mathrm{grad}^P r, Y) = \mathrm{Hess}\, r(X, Y).$$

For $e \in TM$, we have

$$\mathrm{Hess}\, \tilde{r}(e, e) = -2\langle \mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + \mathrm{Hess}\, r(e^P, e^P).$$

From the Hessian comparison theorem (see [Pigola et al. 2008, Chapter 2] for a modern approach) we obtain

$$\mathrm{Hess}\, r(e^P, e^P) \geq C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2).$$

Therefore,

$$\mathrm{Hess}\, \tilde{r}(e, e) \geq -2\langle \mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2). \tag{5}$$

We define $u \in C^\infty(M)$ by

$$u(p) = r(y(p)).$$

Thus, $u = \tilde{r} \circ f$ and

$$\operatorname{grad}^N \tilde{r}(f(p)) = \operatorname{grad}^M u(p) + \operatorname{grad}^N \tilde{r}(f(p))^{\perp}. \tag{6}$$

This gives

$$\operatorname{Hess} u(e_i, e_j) = \operatorname{Hess} \tilde{r}(e_i, e_j) + \langle \operatorname{grad}^N \tilde{r}, \alpha(e_i, e_j)\rangle,$$

where $e_1, \ldots, e_m$ is an orthonormal frame of TM. Thus

$$\Delta u = \sum_{j=1}^{m} \operatorname{Hess} \tilde{r}(e_j, e_j) + m\langle \operatorname{grad}^N \tilde{r}, H\rangle. \tag{7}$$

We have from $e_j = e_j^L + e_j^P$ that $1 = \langle e_j, e_j\rangle = \rho^2 \|e_j^P\|^2 + \sum_{k=1}^{\ell} \langle e_j, T_k\rangle^2$, where $T_1, \ldots, T_\ell$ is an orthonormal frame for TL. Hence

$$m = \rho^2 \sum_{j=1}^{m} \|e_j^P\|^2 + \sum_{k=1}^{\ell} |T_k^\top|^2,$$

where $T^\top$ is the tangent component of T. We obtain

$$\sum_{j=1}^{m} \|e_j^P\|^2 \geq (m-\ell)\rho^{-2}. \tag{8}$$

Since $(\operatorname{grad}^P r, e_j^P) = \langle \operatorname{grad}^N \tilde{r}, e_j^P\rangle = \langle \operatorname{grad}^N \tilde{r}, e_j\rangle = \langle \operatorname{grad}^M u, e_j\rangle$, we get from (5) that

$$\operatorname{Hess} \tilde{r}(e_j, e_j) \geq -2\langle \operatorname{grad}^L \varrho, e_j\rangle\langle \operatorname{grad}^M u, e_j\rangle + C_b(u)(\|e_j^P\|^2 - \langle \operatorname{grad}^M u, e_j\rangle^2).$$

Taking the trace and using (8) gives

$$\sum_{j=1}^{m} \operatorname{Hess} \tilde{r}(e_j, e_j) \geq -2\,\langle \operatorname{grad}^L \varrho, \operatorname{grad}^M u\rangle + C_b(u)\big((m-\ell)\rho^{-2} - |\operatorname{grad}^M u|^2\big).$$

Because $\langle \operatorname{grad}^N \tilde{r}, \operatorname{grad}^N \tilde{r}\rangle = \rho^2(\rho^{-2}\operatorname{grad}^P r, \rho^{-2}\operatorname{grad}^P r) = \rho^{-2}$, we have

$$\langle \operatorname{grad}^N \tilde{r}, H\rangle \geq -\rho^{-1}|H|.$$

Using (7), we conclude that

$$\Delta u \geq -2\langle \operatorname{grad}^L \varrho, \operatorname{grad}^M u\rangle + C_b(u)\left((m-\ell)\rho^{-2} - |\operatorname{grad}^M u|^2\right) - m\rho^{-1}|H|.$$

Thus

$$\rho|H| \geq \frac{m-\ell}{m} C_b(u) - \frac{\rho^2}{m}\big(\Delta u + 2|\operatorname{grad}^L \varrho||\operatorname{grad}^M u| + C_b(u)|\operatorname{grad}^M u|^2\big).$$

If M^m is compact, the proof follows easily by computing the inequality at a point of maximum of u. Thus, we may now assume that M^m is noncompact and that (4) holds.

Since $f(M) \subset \mathscr{C}_{r_0}$, we have $u^* = \sup_M u \le r_0 < +\infty$. By the Omori–Yau maximum principle, there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that $u(p_k) > u^* - 1/k$, $|\mathrm{grad}^M u(p_k)| < 1/k$, and $\Delta u(p_k) < 1/k$. By assumption, we have $\sup_L \rho = K_1 < +\infty$ and $\sup_L |\mathrm{grad}^L \varrho| = K_2 < +\infty$. Hence

$$\sup_M \rho|H| \ge \rho(p_k)|H(p_k)| \ge \frac{m-\ell}{m} C_b(u(p_k)) - \frac{K_1^2}{m}\Big(\frac{1+2K_2}{k} + \frac{1}{k^2} C_b(u(p_k))\Big).$$

Letting $k \to +\infty$, we obtain

$$\sup_M \rho|H| \ge \frac{m-\ell}{m} C_b(u^*) \ge \frac{m-\ell}{m} C_b(r_0),$$

and this concludes the proof of the theorem.

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WEYL GROUP MULTIPLE DIRICHLET SERIES OF TYPE C

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We develop the theory of *Weyl group multiple Dirichlet series* for root systems of type C. For a root system of rank r and a positive integer n, these are Dirichlet series in r complex variables with analytic continuation and functional equations isomorphic to the associated Weyl group. They conjecturally arise as Whittaker coefficients of Eisenstein series on a metaplectic group with cover degree n. For type C and n odd, we construct an infinite family of Dirichlet series and prove they satisfy the above analytic properties in many cases. The coefficients are exponential sums built from Gelfand–Tsetlin bases of certain highest weight representations. Previous attempts to define such series by Brubaker, Bump, and Friedberg required n sufficiently large, so that coefficients were described by Weyl group orbits. We demonstrate that these two radically different descriptions match when both are defined. Moreover, for $n = 1$, we prove our series are Whittaker coefficients of Eisenstein series on $\mathrm{SO}(2r+1)$.

1. Introduction

Let Φ be a reduced root system of rank r. *Weyl group multiple Dirichlet series* (associated to Φ) are Dirichlet series in r complex variables which initially converge on a cone in $\mathbb{C}^r$, possess analytic continuation to a meromorphic function on the whole complex space, and satisfy functional equations whose action on $\mathbb{C}^r$ is isomorphic to the Weyl group of Φ.

For various choices of Φ and a positive integer n, infinite families of Weyl group multiple Dirichlet series defined over any number field F containing the $2n$-th roots of unity were introduced in [Chinta and Gunnells 2007; 2010; Brubaker et al. 2007; 2008]. The coefficients of these Dirichlet series are intimately related to the n-th power reciprocity law in F. It is further expected that these families are related to metaplectic Eisenstein series as follows. If one considers the split, semisimple, simply connected algebraic group G over F whose Langlands L-group

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Keywords: Weyl group multiple Dirichlet series, metaplectic group, Eisenstein series, Gelfand–Tsetlin pattern.

has root system Φ, then it is conjectured that the families of multiple Dirichlet series associated to n and Φ (or the dual root system, depending on n) are precisely the Fourier–Whittaker coefficients of minimal parabolic Eisenstein series on the n-fold metaplectic cover of G. See Remark 3 for more details.

In light of this suggested relationship with Eisenstein series, one should be able to provide definitions of multiple Dirichlet series for any reduced root system Φ and any positive integer n having the desired analytic properties. However a satisfactory theory of the connections between various Dirichlet series and their relation to metaplectic Eisenstein series has only recently emerged for type A. This paper improves the current theory by developing some of the corresponding results for type C, suggesting that such representations of Eisenstein series should hold in great generality.

We begin by describing the basic shape of the Weyl group multiple Dirichlet series, which can be done uniformly for any reduced root system Φ of rank r. Given a number field F containing the $2n$-th roots of unity and a finite set of places S of F (chosen with certain restrictions described in Section 2.2), let $\mathcal{O}_S$ denote the ring of S-integers in F and $\mathcal{O}_S^\times$ the units in this ring. Then to any r-tuple of nonzero $\mathcal{O}_S$ integers $\boldsymbol{m} = (m_1, \dots, m_r)$, we associate a *Weyl group multiple Dirichlet series* in r complex variables $\boldsymbol{s} = (s_1, \dots, s_r)$ of the form

$$Z_\Psi(s_1, \dots, s_r; m_1, \dots, m_r) = Z_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \sum_{\substack{\boldsymbol{c}=(c_1,\dots,c_r)\\ \in(\mathcal{O}_S/\mathcal{O}_S^\times)^r}} \frac{H^{(n)}(\boldsymbol{c}; \boldsymbol{m})\Psi(\boldsymbol{c})}{|c_1|^{2s_1}\cdots|c_r|^{2s_r}}, \tag{1}$$

where the coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ carry the main arithmetic content. The function $\Psi(\boldsymbol{c})$ guarantees the numerator of our series is well-defined up to $\mathcal{O}_S^\times$ units and is defined precisely in Section 2.3. Finally $|c_i| = |c_i|_S$ denotes the norm of the integer c_i as a product of local norms in $F_S = \prod_{v\in S} F_v$.

The coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ are not multiplicative, but nearly so and (as we will demonstrate in (17) and (19) of Section 2.4) can nevertheless be reconstructed from coefficients of the form

$$H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) := H^{(n)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}), \tag{2}$$

where p is a fixed prime in $\mathcal{O}_S$ and $k_i = \mathrm{ord}_p(c_i)$, $l_i = \mathrm{ord}_p(m_i)$.

There are two approaches to defining these prime-power contributions. Chinta and Gunnells [2007; 2010] use a remarkable action of the Weyl group to define the coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ as an average over elements of the Weyl group for any root system Φ and any integer $n \geq 1$, from which functional equations and analytic continuation of the series Z follow. By contrast, for Φ of type A and any $n \geq 1$, Brubaker, Bump, and Friedberg [2007] define the prime-power coefficients as a sum over basis vectors in a highest weight representation for $\mathrm{GL}(r+1,\mathbb{C})$ associated to the fixed r-tuple $\boldsymbol{l}$ in (2). They subsequently prove functional equations and analytic continuation for the multiple Dirichlet series via intricate combinatorial arguments in [Brubaker et al. 2009; 2011b]. It is therefore natural to ask whether a definition in the mold of [Brubaker et al. 2007] exists for the prime-power coefficients $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ for every root system Φ.

For Φ of type C, we present a positive answer to this question, in the form of the following conjecture and its subsequent proof in many special cases.

Conjecture. *For $\Phi = C_r$ for any r and for n odd, the Dirichlet series $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ described in* (1), *with coefficients of the form $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ as defined in Section 3, has the following properties*:

(I) *$Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ possesses analytic continuation to a meromorphic function on $\mathbb{C}^r$ and satisfies a group of functional equations isomorphic to $W(\mathrm{Sp}(2r))$, the Weyl group of* $\mathrm{Sp}(2r)$, *of the form* (24), *where the W action on $\mathbb{C}^r$ is as given in* (21).

(II) *$Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ is the Whittaker coefficient of a minimal parabolic Eisenstein series on an n-fold metaplectic cover of* $\mathrm{SO}(2r+1,\, F_S)$.

Part (II) of this conjecture would imply part (I) according to the general Langlands–Selberg theory of Eisenstein series extended to metaplectic covers as in [Mœglin and Waldspurger 1995]. In practice, other methods to prove part (I) have resulted in sharp estimates for the scattering matrix involved in the functional equations that would be difficult to obtain from the general theory; see, for example, [Brubaker et al. 2006] .

In this paper, we make progress toward this general conjecture by proving the following two results, which will be restated more precisely in later sections once careful definitions have been given.

Theorem 1. *For n sufficiently large* (*as given in* (41)), *$Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ matches the multiple Dirichlet series defined in [Brubaker et al. 2008] for the root system $\Phi = C_r$. Therefore, for such odd n, the multiple Dirichlet series possess the analytic properties cited in part* (I) *of the Conjecture.*

Theorem 2. *For $n = 1$, $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ is a multiplicative function whose prime-power coefficients match those of the Casselman–Shalika formula for* $\mathrm{Sp}(2r)$. *Hence $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ agrees with the minimal parabolic* (*nonmetaplectic*) *Eisenstein series for* $\mathrm{SO}(2r+1,\, F_S)$. *Thus both parts of the Conjecture hold for $n = 1$.*

These theorems are symplectic analogs of those proven for type A in [Brubaker et al. 2007; 2008]. Theorem 2 is proved using a combinatorial identity from [Hamel and King 2002]. Theorem 1, our main result, also has a combinatorial proof using rather subtle connections between the Weyl group and Gelfand–Tsetlin patterns (henceforth GT-patterns) that parametrize basis vectors for highest weight representations of $\mathrm{Sp}(2r, \mathbb{C})$, the Langlands dual group of $\mathrm{SO}(2r+1)$.

Remark 3. The restriction that n must be odd is natural in light of earlier work by Savin [1988] showing that the structure of the Iwahori–Hecke algebra depends on the parity of the metaplectic cover and by Bump, Friedberg, and Ginzburg [2006] on conjectural dual groups for metaplectic covers. Indeed, though the construction of the Dirichlet series we propose in Section 3 makes sense for any n, attempts to prove functional equations for n even and $\boldsymbol{m}$ fixed using the techniques of [Beineke et al. 2010] suggest the coefficients have the wrong shape. In view of this evidence, we expect a similar combinatorial definition to hold for n even, but making use of the highest weight representation theory for $\mathrm{SO}(2r+1, \mathbb{C})$ (in contrast with the case n odd, and weights from $\mathrm{Sp}(2r, \mathbb{C})$ as in the Conjecture and the two subsequent theorems).

As noted above, the analog of the Conjecture is known for type A for any $n \geq 1$. Its proof, completed in [Brubaker et al. 2009; 2011b], makes critical use of the outer automorphism of the Dynkin diagram for type A. Thus mimicking the proof techniques to obtain results for type C is not possible. However, given any fixed $\boldsymbol{m}$ and n, one can verify the functional equations and meromorphic continuation with a finite amount of checking. See [Beineke et al. 2010] for the details of this argument in a small rank example.

The type A analog of part (II) of the Conjecture is proved in [Brubaker et al. 2011a] by computing the Fourier–Whittaker coefficients of Eisenstein series directly by inducing from successive maximal parabolics. The result is essentially a complicated recursion involving exponential sums and lower rank Eisenstein series. Then one checks the definition given in [Brubaker et al. 2007] satisfies the recursion. A similar approach should be possible in type C, and this will be the subject of future work. Such an approach depends critically on having a proposed solution to satisfy the recursion, so the methods of this paper are a necessary first step.

The precise definition of the prime-power coefficients (2) for type C is somewhat complicated, so we postpone it until Section 3. As alluded to earlier, coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ will be described in terms of basis vectors for highest weight representations of $\mathrm{Sp}(2r, \mathbb{C})$ with highest weight corresponding to $\boldsymbol{l}$. As noted in Remark 6, the definition produces Gauss sums which encode subtle information about Kashiwara raising/lowering operators in the crystal graph associated to the highest weight representation. As such, this paper offers the first evidence that mysterious connections between metaplectic Eisenstein series and crystal bases may hold in much greater generality, persisting beyond the type A theory in [Brubaker et al. 2007; 2011a; 2011b]. These connections may not be properly understood until a general solution to our problem for all root systems Φ is obtained.

Finally, the results of this paper give infinite classes of Dirichlet series with analytic continuation. One can then use standard Tauberian techniques to extract mean-value estimates for families of number-theoretic quantities appearing in the numerator of the series (or the numerator of polar residues of the series). For the n-cover of A_r, this method yielded the mean-value results of [Chinta 2005] for $r = 5$, $n = 2$ and [Brubaker and Bump 2006b] for $r = 3$, $n = 3$. It would be interesting to explore similar results in type C (remembering that our Conjecture may be verified for any given example with n, r, and $\boldsymbol{m}$ fixed with only a finite amount of checking, as sketched in [Beineke et al. 2010]).

Note. Since the initial submission of this paper, Chinta and Offen [2009] have given a proof in type A that the multiple Dirichlet series constructed by Chinta and Gunnells is in fact a metaplectic Whittaker coefficient. This argument has been extended in great generality by McNamara [2011]. Further, Ivanov [2010] has used the results of this paper to give an alternate definition of the prime-power coefficients (2) in terms of two-dimensional lattice models defined by Kuperberg [2002]. In the case $n = 1$, his methods give an alternate proof of Theorem 2. All of these results make a resolution of the Conjecture given above more desirable.

2. Definition of the multiple Dirichlet series

In this section, we present general notation for root systems and the corresponding Weyl group multiple Dirichlet series.

2.1. ***Root systems.*** Let Φ be a reduced root system contained in V, a real vector space of dimension r. The dual vector space $V^\vee$ contains a root system $\Phi^\vee$ in bijection with Φ, where the bijection switches long and short roots. Writing the dual pairing

$$V \times V^\vee \to \mathbb{R}, \quad (x, y) \mapsto B(x, y), \tag{3}$$

then $B(\alpha, \alpha^\vee) = 2$. Moreover, the simple reflection $\sigma_\alpha : V \to V$ corresponding to α is given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\alpha.$$

Note that σ_α preserves Φ. Similarly, define a dual reflection $\sigma_{\alpha^\vee} : V^\vee \to V^\vee$ by $\sigma_{\alpha^\vee}(x) = x - B(\alpha, x)\alpha^\vee$ with $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

For our purposes, without loss of generality, we may take Φ to be irreducible (that is, there do not exist orthogonal subspaces Φ_1, Φ_2 with $\Phi_1 \cup \Phi_2 = \Phi$). Then set $\langle \cdot, \cdot \rangle$ to be the Euclidean inner product on V and $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm, where we normalize so that $2\langle \alpha, \beta \rangle$ and $\|\alpha\|^2$ are integral for all $\alpha, \beta \in \Phi$.

With this notation,

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \quad \text{for any } \alpha, \beta \in \Phi. \tag{4}$$

Partition Φ into positive roots Φ^+ and negative roots Φ^- and denote by $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ the subset of simple positive roots. Further, denote the fundamental dominant weights by ϵ_i for $i = 1, \ldots, r$ satisfying

$$\frac{2\langle \epsilon_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker delta. Any dominant weight λ is expressible in terms of the ϵ_i, and a distinguished role in the theory is played by the Weyl vector ρ, defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{r} \epsilon_i. \tag{6}$$

2.2. ***Algebraic preliminaries.*** Keeping with the established foundations on Weyl group multiple Dirichlet series (see [Brubaker et al. 2006; 2008]), we define our Dirichlet series as indexed by integers rather than ideals. By using this approach, the coefficients of the Dirichlet series will closely resemble classical exponential sums, but some care needs to be taken to ensure the resulting series remains well-defined up to units.

Given a fixed positive odd integer n, let F be a number field containing the $2n$-th roots of unity, and let S be a finite set of places containing all ramified places over $\mathbb{Q}$, all archimedean places, and enough additional places so that the ring of S-integers $\mathbb{O}_S$ is a principal ideal domain. Recall that the $\mathbb{O}_S$ integers are defined as

$$\mathbb{O}_S = \{a \in F \mid a \in \mathbb{O}_v \ \forall v \notin S\}$$

and can be embedded diagonally in

$$F_S = \prod_{v \in S} F_v.$$

There exists a pairing

$$(\cdot\,,\cdot)_S : F_S^\times \times F_S^\times \to \mu_n \text{ defined by } (a, b)_S = \prod_{v \in S} (a, b)_v,$$

where the $(a, b)_v$ are local Hilbert symbols associated to n and v.

Further, to any $a \in \mathbb{O}_S$ and any ideal $\mathfrak{b} \subseteq \mathbb{O}_S$, we may associate the n-th power residue symbol $(\frac{a}{\mathfrak{b}})_n$ as follows. For prime ideals $\mathfrak{p}$, the expression $(\frac{a}{\mathfrak{p}})_n$ is the unique n-th root of unity satisfying the congruence

$$\left(\frac{a}{\mathfrak{p}}\right)_n \equiv a^{(N(\mathfrak{p})-1)/n} \pmod{\mathfrak{p}}.$$

Extend the symbol to arbitrary ideals $\mathfrak{b}$ by multiplicativity, with the convention that the symbol is 0 whenever a and $\mathfrak{b}$ are not relatively prime. Since $\mathcal{O}_S$ is a principal ideal domain by assumption, we will write

$$\left(\frac{a}{b}\right)_n = \left(\frac{a}{\mathfrak{b}}\right)_n \quad \text{for } \mathfrak{b} = b\mathcal{O}_S$$

and often drop the subscript n on the symbol when the power is understood from context.

Then if a, b are coprime integers in $\mathcal{O}_S$, we have the n-th power reciprocity law (see [Neukirch 1999, Theorem 6.8.3])

$$\left(\frac{a}{b}\right) = (b, a)_S \left(\frac{b}{a}\right), \tag{7}$$

which, in particular, implies that if $\epsilon \in \mathcal{O}_S^\times$ and $b \in \mathcal{O}_S$, then

$$\left(\frac{\epsilon}{b}\right) = (b, \epsilon)_S.$$

Finally, for a positive integer t and $a, c \in \mathcal{O}_S$ with $c \neq 0$, we define the Gauss sum $g_t(a, c)$ as follows. First, choose a nontrivial additive character ψ of F_S trivial on the $\mathcal{O}_S$ integers (see [Brubaker and Bump 2006a] for details). Then the n-th power Gauss sum is given by

$$g_t(a, c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)_n^t \psi\left(\frac{ad}{c}\right), \tag{8}$$

where we have suppressed the dependence on n in the notation on the left. The Gauss sum g_t is not multiplicative, but rather satisfies

$$g_t(a, cc') = \left(\frac{c}{c'}\right)_n^t \left(\frac{c'}{c}\right)_n^t g_t(a, c) g_t(a, c') \tag{9}$$

for any relatively prime pair $c, c' \in \mathcal{O}_S$.

2.3. *Kubota's rank-1 Dirichlet series.* Many of the definitions for Weyl group multiple Dirichlet series are natural extensions of those from the rank-1 case, so we begin with a brief description of these.

A subgroup $\Omega \subset F_S^\times$ is said to be *isotropic* if $(a, b)_S = 1$ for all $a, b \in \Omega$. In particular, $\Omega = \mathcal{O}_S(F_S^\times)^n$ is isotropic (where $(F_S^\times)^n$ denotes the n-th powers in $F_S^\times$). Let $\mathcal{M}_t(\Omega)$ be the space of functions $\Psi : F_S^\times \to \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon c) = (c, \epsilon)_S^{-t} \Psi(c) \quad \text{for any } \epsilon \in \Omega, c \in F_S^\times. \tag{10}$$

For $\Psi \in \mathcal{M}_t(\Omega)$, consider the generalization of Kubota's Dirichlet series:

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathbb{O}_S/\mathbb{O}_S^\times} \frac{g_t(a, c)\Psi(c)}{|c|^{2s}}. \tag{11}$$

Here $|c|$ is the order of $\mathbb{O}_S/c\mathbb{O}_S$, $g_t(a, c)$ is as in (8) and the term $g_t(a, c)\Psi(c)|c|^{-2s}$ is independent of the choice of representative c, modulo S-units. Standard estimates for Gauss sums show that the series is convergent if $\Re(s) > \frac{3}{4}$. Our functional equation computations will hinge on the functional equation for this Kubota Dirichlet series. Before stating this result, we require some additional notation. Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-2} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{12}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s, \Psi, a) = \boldsymbol{G}_m(s)^{[F:\mathbb{Q}]/2} \zeta_F(2ms - m + 1) \mathcal{D}_t(s, \Psi, a), \tag{13}$$

where $m = n/\gcd(n, t)$, $\frac{1}{2}[F : \mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathbb{O}_v/\mathcal{P}_v$, where $\mathbb{O}_v$ is the local ring in F_v and $\mathcal{P}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}. If $\Psi \in \mathcal{M}_t(\Omega)$ and $\eta \in F_S^\times$, denote

$$\widetilde{\Psi}_\eta(c) = (\eta, c)_S \, \Psi(c^{-1}\eta^{-1}). \tag{14}$$

Then we have the next result, which follows from [Brubaker and Bump 2006a].

Theorem [Brubaker et al. 2008, Theorem 1]. *Let $\Psi \in \mathcal{M}_t(\Omega)$ and $a \in \mathbb{O}_S$. Let $m = n/\gcd(n, t)$. Then $\mathcal{D}_t^*(s, \Psi, a)$ has meromorphic continuation to all s, analytic except possibly at $s = 1/2 \pm 1/(2m)$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ depending only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}_t^*(s, \Psi, a) = |a|^{1-2s} \sum_{\eta \in F_S^\times/(F_S^\times)^n} P_{a\eta}^t(s) \mathcal{D}_t^*(1 - s, \widetilde{\Psi}_\eta, a). \tag{15}$$

This result, based on ideas of Kubota [1969], relies on the theory of Eisenstein series. The case $t = 1$ is handled in [Brubaker and Bump 2006a]; the general case follows as discussed in the proof of [Brubaker et al. 2006, Proposition 5.2]. Notably, the factor $|a|^{1-2s}$ is independent of the value of t.

2.4. ***The form of higher rank multiple Dirichlet series.*** We now begin explicitly defining the multiple Dirichlet series, retaining our previous notation. By analogy with the rank-1 definition in (10), given an isotropic subgroup Ω, let $\mathcal{M}(\Omega^r)$ be the space of functions $\Psi : (F_S^\times)^r \to \mathbb{C}$ that satisfy the transformation property

$$\Psi(\boldsymbol{\epsilon c}) = \left(\prod_{i=1}^{r} (\epsilon_i, c_i)_S^{\|\alpha_i\|^2} \prod_{i<j} (\epsilon_i, c_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \right) \Psi(\boldsymbol{c}) \tag{16}$$

for all $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in \Omega^r$ and all $\boldsymbol{c} = (c_1, \dots, c_r) \in (F_S^\times)^r$.

Recall from the introduction that, given a reduced root system Φ of fixed rank r, an integer $n \geq 1$, $\boldsymbol{m} \in \mathcal{O}_S^r$, and $\Psi \in \mathcal{M}(\Omega^r)$, we consider a function of r complex variables $\boldsymbol{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi(s_1, \dots, s_r; m_1, \dots, m_r) = Z_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \sum_{\substack{\boldsymbol{c}=(c_1,\dots,c_r) \\ \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r}} \frac{H^{(n)}(\boldsymbol{c}; \boldsymbol{m}) \Psi(\boldsymbol{c})}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}}.$$

The function $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ carries the main arithmetic content. It is not defined as a multiplicative function, but rather a "twisted multiplicative" function. For us, this means that for S-integer vectors $\boldsymbol{c}, \boldsymbol{c}' \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r$ with $\gcd(c_1 \cdots c_r, c_1' \cdots c_r') = 1$,

$$H^{(n)}(c_1 c_1', \dots, c_r c_r'; \boldsymbol{m}) = \mu(\boldsymbol{c}, \boldsymbol{c}') H^{(n)}(\boldsymbol{c}; \boldsymbol{m}) H^{(n)}(\boldsymbol{c}'; \boldsymbol{m}), \tag{17}$$

where $\mu(\boldsymbol{c}, \boldsymbol{c}')$ is an n-th root of unity depending on $\boldsymbol{c}, \boldsymbol{c}'$. It is given precisely by

$$\mu(\boldsymbol{c}, \boldsymbol{c}') = \prod_{i=1}^{r} \left(\frac{c_i}{c_i'} \right)_n^{\|\alpha_i\|^2} \left(\frac{c_i'}{c_i} \right)_n^{\|\alpha_i\|^2} \prod_{i<j} \left(\frac{c_i}{c_j'} \right)_n^{2\langle \alpha_i, \alpha_j \rangle} \left(\frac{c_i'}{c_j} \right)_n^{2\langle \alpha_i, \alpha_j \rangle} \tag{18}$$

where $(\frac{\cdot}{\cdot})_n$ is the n-th power residue symbol defined in Section 2.2. In the special case $\Phi = A_1$, the twisted multiplicativity in (17) and (18) agrees with the identity for Gauss sums in (9) in accordance with the numerator for the rank-1 case in (11).

Remark 4. We often think of twisted multiplicativity as the appropriate generalization of multiplicativity for the metaplectic group. In particular, for $n = 1$ we reduce to the usual multiplicativity on relatively prime coefficients. Moreover, many of the global properties of the Dirichlet series follow (upon careful analysis of the twisted multiplicativity and associated Hilbert symbols) from local properties, for example, functional equations as in [Brubaker et al. 2006; 2008]. For more on this perspective, see [Friedberg 2010].

The transformation property of functions in $\mathcal{M}(\Omega^r)$ in (16) is motivated by the identity

$$H^{(n)}(\boldsymbol{\epsilon c}; \boldsymbol{m}) \Psi(\boldsymbol{\epsilon c}) = H^{(n)}(\boldsymbol{c}; \boldsymbol{m}) \Psi(\boldsymbol{c}) \quad \text{for all } \boldsymbol{\epsilon} \in \mathcal{O}_S^r, \ \boldsymbol{c}, \boldsymbol{m} \in (F_S^\times)^r.$$

The proof can be verified using the n-th power reciprocity law from Section 2.2.

Now, given any $\boldsymbol{m}, \boldsymbol{m}', \boldsymbol{c} \in \mathcal{O}_S^r$ with $\gcd(m_1' \cdots m_r', c_1 \cdots c_r) = 1$, let

$$H^{(n)}(\boldsymbol{c}; m_1 m_1', \dots, m_r m_r') = \prod_{i=1}^{r} \left(\frac{m_i'}{c_i}\right)_n^{-\|\alpha_i\|^2} H^{(n)}(\boldsymbol{c}; \boldsymbol{m}). \tag{19}$$

The definitions in (17) and (19) imply that it is enough to specify the coefficients $H^{(n)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r})$ for any fixed prime p with $l_i = \mathrm{ord}_p(m_i)$ in order to completely determine $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ for any pair of S-integer vectors $\boldsymbol{m}$ and $\boldsymbol{c}$. These prime-power coefficients are described in terms of data from highest-weight representations associated to $(l_1, \dots, l_r)$ and will be given precisely in Section 3.

2.5. *Weyl group actions.* In order to precisely state a functional equation for the Weyl group multiple Dirichlet series, we require an action of the Weyl group W of Φ on the complex parameters $(s_1, \dots, s_r)$. This arises from the linear action of W, realized as the group generated by the simple reflections $\sigma_{\alpha^\vee}$, on $V^\vee$. From the perspective of Dirichlet series, it is more natural to consider this action shifted by $\rho^\vee$, half the sum of the positive coroots. Then each $w \in W$ induces a transformation $V_{\mathbb{C}}^\vee = V^\vee \otimes \mathbb{C} \to V_{\mathbb{C}}^\vee$ (still denoted by w) if we require that

$$B\big(w\alpha, w(\boldsymbol{s}) - \tfrac{1}{2}\rho^\vee\big) = B\big(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee\big).$$

We introduce coordinates on $V_{\mathbb{C}}^\vee$ using simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ as follows. Define an isomorphism $V_{\mathbb{C}}^\vee \to \mathbb{C}^r$ by

$$\boldsymbol{s} \mapsto (s_1, s_2, \dots, s_r), \quad s_i = B(\alpha_i, \boldsymbol{s}). \tag{20}$$

This action allows us to identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$, and so the complex variables s_i that appear in the definition of the multiple Dirichlet series may be regarded as coordinates in either space. It is convenient to describe this action more explicitly in terms of the s_i, and it suffices to consider simple reflections which generate W. Using the action of the simple reflection σ_{α_i} on the root system Φ given in (4) in conjunction with (20) above gives:

Proposition 5. *The action of σ_{α_i} on $\boldsymbol{s} = (s_1, \dots, s_r)$ defined implicitly in* (20) *is given by*

$$s_j \mapsto s_j - \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\big(s_i - \tfrac{1}{2}\big), \quad j = 1, \dots, r. \tag{21}$$

In particular, $\sigma_{\alpha_i} : s_i \mapsto 1 - s_i$.

2.6. *Normalizing factors and functional equations.* The multiple Dirichlet series must also be normalized using Gamma and zeta factors in order to state precise functional equations. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}, \quad \alpha \in \Phi^+.$$

For example, if $\Phi = C_r$ and we normalize short roots to have length 1, this implies that $n(\alpha) = n$ unless α is a long root and n is even (in which case $n(\alpha) = n/2$). By analogy with the zeta factor appearing in (13), for any $\alpha \in \Phi^+$, let

$$\zeta_\alpha(\boldsymbol{s}) = \zeta\left(1 + 2n(\alpha)B(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee)\right),$$

where ζ is the Dedekind zeta function attached to the number field F. Further, for $\boldsymbol{G}_n(s)$ as in (12), we may define

$$\boldsymbol{G}_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\tfrac{1}{2} + B(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee)\right). \tag{22}$$

Then for any $\boldsymbol{m} \in \mathbb{O}_S^r$, the normalized multiple Dirichlet series is given by

$$Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \left(\prod_{\alpha \in \Phi^+} \boldsymbol{G}_\alpha(\boldsymbol{s})\zeta_\alpha(\boldsymbol{s})\right) Z_\Psi(\boldsymbol{s}, \boldsymbol{m}). \tag{23}$$

By considering the product over all positive roots, we guarantee that the other zeta and Gamma factors are permuted for each simple reflection $\sigma_i \in W$, and hence for all elements of the Weyl group.

Given any fixed n, $\boldsymbol{m}$ and root system Φ, we seek to define $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ (or equivalently, given twisted multiplicativity, to define H at prime-power coefficients) so that $Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m})$ satisfies functional equations of the form

$$Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m}) = |m_i|^{1-2s_i} Z^*_{\sigma_i \Psi}(\sigma_i \boldsymbol{s}; \boldsymbol{m}) \tag{24}$$

for all simple reflections $\sigma_i \in W$. Here, $\sigma_i \boldsymbol{s}$ is as in (21) and the function $\sigma_i \Psi$, which essentially keeps track of the rather complicated scattering matrix in this functional equation, is defined as in [Brubaker et al. 2008, (37)]. As noted in [Brubaker et al. 2008, Section 7], given functional equations of this type, one can obtain analytic continuation to a meromorphic function of $\mathbb{C}^r$ with an explicit description of polar hyperplanes.

3. Definition of the prime-power coefficients

In this section, we give a precise definition of the coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ needed to complete the description of the multiple Dirichlet series for root systems of type C_r and n odd. All the previous definitions are stated in sufficient generality for application to multiple Dirichlet series for any reduced root system Φ and any positive integer n. Only the prime-power coefficients require specialization to our particular root system $\Phi = C_r$, though this remains somewhat complicated. We summarize the definition at the end of the section.

The vector $\boldsymbol{l} = (l_1, l_2, \ldots, l_r)$ appearing in $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ can be associated to a dominant weight for $\mathrm{Sp}_{2r}(\mathbb{C})$ of the form

$$\lambda = (l_1 + l_2 + \cdots + l_r, \ldots, l_1 + l_2, l_1). \tag{25}$$

The contributions to $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ will then be parametrized by basis vectors of the highest weight representation of highest weight $\lambda+\rho$, where ρ is the Weyl vector for C_r defined in (6), so that

$$\lambda+\rho = (l_1+l_2+\cdots+l_r+r, \ldots, l_1+l_2+2, l_1+1) =: (L_r, \ldots, L_1). \tag{26}$$

In [Brubaker et al. 2007], prime-power coefficients for multiple Dirichlet series of type A were attached to Gelfand–Tsetlin patterns, which parametrize highest weight vectors for $\mathrm{SL}_{r+1}(\mathbb{C})$ (see [Gelfand and Tsetlin 1950]). Here, we use an analogous basis for the symplectic group, according to branching rules given in [Zhelobenko 1962]. We will continue to refer to the objects comprising this basis as Gelfand–Tsetlin patterns, or GT-patterns.

More precisely, a GT-pattern P has the form

$$P = \begin{array}{cccccccccc} a_{0,1} & & a_{0,2} & & \cdots & & a_{0,r} & & \\ & b_{1,1} & & b_{1,2} & \cdots & b_{1,r-1} & & & b_{1,r} \\ & & a_{1,2} & & \cdots & & a_{1,r} & & \\ & & & \ddots & & \ddots & & & \vdots \\ & & & & & & a_{r-1,r} & & \\ & & & & & & & & b_{r,r} \end{array} \tag{27}$$

where the $a_{i,j}, b_{i,j}$ are nonnegative integers and the rows of the pattern interleave. That is, for all $a_{i,j}, b_{i,j}$ in the pattern P above,

$$\begin{aligned} \min(a_{i-1,j}, a_{i,j}) \geq b_{i,j} \geq \max(a_{i-1,j+1}, a_{i,j+1}), \\ \min(b_{i+1,j-1}, b_{i,j-1}) \geq a_{i,j} \geq \max(b_{i+1,j}, b_{i,j}). \end{aligned}$$

The set of all patterns with top row $(a_{0,1}, \ldots, a_{0,r}) = (L_r, \ldots, L_1)$ form a basis for the highest weight representation with highest weight $\lambda+\rho$. Hence, we will consider GT-patterns with top row $(L_r, \ldots, L_1)$ as in (26), and refer to this set of patterns as $\mathrm{GT}(\lambda+\rho)$.

The contributions to each $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ with both $\boldsymbol{k}$ and $\boldsymbol{l}$ fixed come from a single weight space corresponding to $\boldsymbol{k} = (k_1, \ldots, k_r)$ in the highest weight representation $\lambda+\rho$ corresponding to $\boldsymbol{l}$. We first describe how to associate a weight vector to each GT-pattern. Let

$$s_a(i) := \sum_{m=i+1}^{r} a_{i,m} \quad \text{and} \quad s_b(i) := \sum_{m=i}^{r} b_{i,m} \tag{28}$$

be the row sums for the respective rows of a's and b's in P. (Here we understand that $s_a(r) = 0$ corresponds to an empty sum.) Then define the weight vector $\mathrm{wt}(P) = (\mathrm{wt}_1(P), \ldots, \mathrm{wt}_r(P))$ by

$$\mathrm{wt}_i = \mathrm{wt}_i(P) = s_a(r-i) - 2s_b(r+1-i) + s_a(r+1-i), \quad i = 1, \ldots, r. \tag{29}$$

As the weights are generated in turn, we begin at the bottom of the pattern P and work our way up to the top. Our prime-power coefficients will then be supported at $(p^{k_1}, \dots, p^{k_r})$ with

$$k_1 = \frac{1}{2}\sum_{j=1}^{r} \mathrm{wt}_j + L_j, \quad k_i = \sum_{j=i}^{r} \left(\mathrm{wt}_j + L_j\right), \quad i = 2, \dots, r, \tag{30}$$

so that in particular, the k_i are nonnegative integers.

In terms of the GT-pattern P, the reader may check that

$$k(P) = (k_1(P), k_2(P), \dots, k_r(P)),$$

with

$$\begin{aligned} k_1(P) &= s_a(0) - \sum_{m=1}^{r} \left(s_b(m) - s_a(m)\right), \\ k_i(P) &= s_a(0) - 2\sum_{m=1}^{r+1-i} \left(s_b(m) - s_a(m)\right) - s_a(r+1-i) + \sum_{m=1}^{r+1-i} a_{0,m} \end{aligned} \tag{31}$$

for $1 < i \le r$.

Then we define

$$H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = H^{(n)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \sum_{\substack{P \in \mathrm{GT}(\lambda+\rho) \\ k(P) = (k_1, \dots, k_r)}} G(P), \tag{32}$$

where the sum is over all GT-patterns P with top row $(L_r, \dots, L_1)$ as in (26) satisfying the condition $\boldsymbol{k}(P) = (k_1, \dots, k_r)$ and $G(P)$ is a weighting function whose definition depends on the following elementary quantities. To each pattern P, define the corresponding data

$$v_{i,j} = \sum_{m=i}^{j} (a_{i-1,m} - b_{i,m}), \quad w_{i,j} = \sum_{m=j}^{r} (a_{i,m} - b_{i,m}), \quad u_{i,j} = v_{i,r} + w_{i,j}, \tag{33}$$

where we understand the entries $a_{i,j}$ or $b_{i,j}$ to be 0 if they do not appear in the pattern P.

Remark 6. The integers $u_{i,j}$ and $v_{i,j}$ have representation-theoretic meaning in terms of Kashiwara raising and lowering operators in the crystal graph associated to the highest weight representation of highest weight $\lambda + \rho$ for $U_q(\mathfrak{sp}(2r))$, the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sp}(2r)$. See [Littelmann 1998] for details, particularly Corollary 2 of Section 6. See also [Brubaker et al. 2011a; 2011b] for a more complete description in crystal language, focusing mainly on type A. We find this interpretation quite striking in light of the connection to Whittaker models on the metaplectic group. Ultimately, this can be seen as another instance of connections between quantum groups and principal series representations in the spirit of [Lusztig 2003]. This is not a perspective we emphasize here, but this line of inquiry is discussed further in [Beineke et al. 2010].

To each entry $b_{i,j}$ in P, associate

$$\gamma_b(i,j) = \begin{cases} g_{\delta_{jr}+1}(p^{v_{i,j}-1}, p^{v_{i,j}}) & \text{if } b_{i,j} = a_{i-1,j+1}, \\ \phi(p^{v_{i,j}}) & \text{if } a_{i-1,j} < b_{i,j} < a_{i-1,j+1}, n \mid v_{i,j} \cdot (\delta_{jr}+1), \\ 0 & \text{if } a_{i-1,j} < b_{i,j} < a_{i-1,j+1}, \ n \nmid v_{i,j} \cdot (\delta_{jr}+1), \\ q^{v_{i,j}} & \text{if } b_{i,j} = a_{i-1,j}, \end{cases} \tag{34}$$

where $g_t(p^\alpha, p^\beta)$ is an n-th power Gauss sum as in (8), $\phi(p^a)$ is the Euler phi function for $\mathbb{O}_S/p^a\mathbb{O}_S$, $q = |\mathbb{O}_S/p\mathbb{O}_S|$, and δ_{jr} is the Kronecker delta function. These cases may be somewhat reduced, using elementary properties of Gauss sums, to

$$\gamma_b(i,j) = \begin{cases} q^{v_{i,j}} & \text{if } b_{i,j} = a_{i-1,j}, \\ g_{\delta_{jr}+1}(p^{v_{i,j}+b_{i,j}-a_{i-1,j+1}-1}, p^{v_{i,j}}) & \text{else.} \end{cases} \tag{35}$$

To each entry $a_{i,j}$ in P, with $i \geq 1$, we may associate

$$\gamma_a(i,j) = \begin{cases} g_1(p^{u_{i,j}-1}, p^{u_{i,j}}) & \text{if } a_{i,j} = b_{i,j-1}, \\ \phi(p^{u_{i,j}}) & \text{if } b_{i,j} < a_{i,j} < b_{i,j-1}, \ n \mid u_{i,j}, \\ 0 & \text{if } b_{i,j} < a_{i,j} < b_{i,j-1}, \ n \nmid u_{i,j}, \\ q^{u_{i,j}} & \text{if } a_{i,j} = b_{i,j}, \end{cases} \tag{36}$$

which can similarly be compacted to

$$\gamma_a(i,j) = \begin{cases} q^{u_{i,j}} & \text{if } a_{i,j} = b_{i,j}, \\ g_1(p^{u_{i,j}-a_{i,j}+b_{i,j-1}-1}, p^{u_{i,j}}) & \text{else.} \end{cases} \tag{37}$$

We introduce terminology to describe relationships between elements in a pattern P:

Definition 7. A GT-pattern P is *minimal at $b_{i,j}$* if $b_{i,j} = a_{i-1,j}$. It is *maximal at $b_{i,j}$* if $1 \leq j < r$ and $b_{i,j} = a_{i-1,j+1}$, or if $b_{i,r} = 0$. If none of these equalities holds, we say P is *generic at $b_{i,j}$*.

Likewise, P is *minimal at $a_{i,j}$* if $a_{i,j} = b_{i,j}$, and *maximal at $a_{i,j}$* if $a_{i,j} = b_{i,j-1}$. If neither equality holds, we say P is *generic at $a_{i,j}$*.

Definition 8. A GT-pattern P is *strict* if its entries are strictly decreasing across each horizontal row.

Define the coefficients

$$G(P) = \begin{cases} \prod_{1\le i\le j\le r} \gamma_a(i,j)\gamma_b(i,j) & \text{if } P \text{ is strict,} \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

where we again understand $\gamma_a(r, r)$ to be 1 since $a_{r,r}$ is not in the pattern P. Combining these definitions gives a definition of the prime-power coefficients in the series:

Definition 9 (summary of definitions for H). Given any prime p, define

$$H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \sum_{\substack{P\in \mathrm{GT}(\lambda+\rho)\\ k(P)=\boldsymbol{k}}} G(P), \tag{39}$$

where the sum is over all GT-patterns with top row corresponding to $\lambda+\rho$ and row sums fixed according to (31), and $G(P)$ is given as in (38) above with $\gamma_a(i, j)$ and $\gamma_b(i, j)$ of (37) and (35), respectively, defined in terms of $v_{i,j}$ and $u_{i,j}$ in (33).

In the right-hand side of (39), we have suppressed the dependence on n. This is appropriate since the expressions in (35) and (37) are given in terms of Gauss sums, which are defined uniformly for all n.

The coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ appearing in (1) are now implicitly defined by (39) together with the twisted multiplicativity given in (17) and (19). The resulting multiple Dirichlet series $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is initially absolutely convergent for $\Re(s_i)$ sufficiently large. Indeed, if a pattern P has weight $\boldsymbol{k} = (k_1, \dots, k_r)$, then

$$|G(P)| < q^{k_1+\cdots+k_r},$$

and the number of patterns in a given weight space is bounded as a function of $\boldsymbol{m}$ corresponding to the highest weight vector.

4. Comparison in the stable case

We now compare our multiple Dirichlet series, having p-th-power coefficients as defined in (39), with the multiple Dirichlet series defined for arbitrary root systems Φ in [Brubaker et al. 2008], when n is sufficiently large. In this section, we determine the necessary lower bound on n explicitly, according to a *stability assumption* introduced in [Brubaker et al. 2006]. With this lower bound, we can then prove that for n odd, the two prescriptions agree.

Let $\boldsymbol{m} = (m_1, \dots, m_r)$ be a fixed r-tuple of nonzero $\mathbb{O}_S$ integers. To any fixed prime p in $\mathbb{O}_S$, set $l_i = \mathrm{ord}_p(m_i)$ for $i = 1, \dots, r$. Then define λ_p as in (25), so that in terms of the fundamental dominant weights ϵ_i, we have

$$\lambda_p = \sum_{i=1}^{r} l_i \epsilon_i.$$

Then we may define the function d_{λ_p} on the set of positive roots Φ^+ by

$$d_{\lambda_p}(\alpha) = \frac{2\langle \lambda_p + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle}. \tag{40}$$

For ease of computation in the results that follow, normalize the inner product $\langle\ ,\ \rangle$ so that $\|\alpha\|^2 = \langle \alpha, \alpha\rangle = 1$ if α is a short root, while $\|\alpha\|^2 = 2$ if α is a long root.

Stability Assumption. Let $\alpha = \sum_{i=1}^{r} t_i \alpha_i$ be the largest positive root in the partial ordering for Φ. Then for every prime p, we require that the positive integer n satisfies

$$n \geq \gcd(n, \|\alpha\|^2) \cdot d_{\lambda_p}(\alpha) = \gcd(n, \|\alpha\|^2) \cdot \sum_{i=1}^{r} t_i (l_i + 1). \tag{41}$$

When the Stability Assumption holds, we say we are “in the stable case.” This is well-defined since $l_i = 0$ for all $i = 1, \ldots, r$ for all but finitely many primes p. For the remainder of this section, we work with a fixed prime p, and so write λ in place of λ_p when no confusion can arise.

For $\Phi = C_r$, let α_1 denote the long simple root, so the largest positive root is $\alpha_1 + \sum_{i=2}^{r} 2\alpha_i$. Moreover if n is odd, the condition (41) becomes

$$n \geq l_1 + 1 + \sum_{i=2}^{r} 2(l_i + 1). \tag{42}$$

For any $w \in W(\Phi)$, define the set $\Phi_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$. Following [Brubaker et al. 2006; 2008], the p-th-power coefficients of the multiple Dirichlet series in the stable case are given by

$$H_{\text{st}}^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}), \tag{43}$$

where the dependence on n occurs only in the n-th-power residue symbol in the Gauss sums. In [Brubaker et al. 2008], it was established that the above definition of $H_{\text{st}}^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ produces a Weyl group multiple Dirichlet series $Z^*(\boldsymbol{s}, \boldsymbol{m})$ with analytic continuation and functional equations (of the form in the Conjecture) provided the Stability Assumption on n holds. The proof works for any reduced root system Φ. In this section, we demonstrate that our definition $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ in terms of GT-patterns as in (39) matches that in (43) for n satisfying the (41) of the Stability Assumption.

Definition 10. If $P \in \mathrm{GT}(\lambda + \rho)$ is a GT-pattern and $G(P)$ is defined as in (38), then P is said to be *stable* if $G(P) \neq 0$ for some (odd) n satisfying (41) of the Stability Assumption.

As we will see in the following result, if P is stable for one such n, then $G(P)$ is nonzero for all n satisfying (41). These are the relevant patterns we must consider in establishing the equivalence of the two definitions $H_{\mathrm{st}}^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ and $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ in the stable case, and we begin by characterizing all such patterns.

Proposition 11. *A pattern $P \in \mathrm{GT}(\lambda + \rho)$ is stable if and only if, in each pair of rows in P with index i (that is, pattern entries $\{b_{i,j}, a_{i,j}\}_{j=i}^{r}$), the ordered set*

$$\{b_{i,i}, b_{i,i+1}, \dots, b_{i,r}, a_{i,r}, a_{i,r-1}, \dots, a_{i,i+1}\}$$

has an initial string in which all elements are minimal (as in Definition 7) *and all remaining elements are maximal.*

Proof. If any element $a_{i,j}$ or $b_{i,j}$ in the pattern P is neither maximal nor minimal, that is, is "generic" in the sense of Definition 7, then $\gamma_a(i, j)$ (or $\gamma_b(i, j)$, respectively) is nonzero if and only if $n \mid u_{i,j}$ according to (36) (or $n \mid v_{i,j}(\delta_{jr} + 1)$ according to (34), respectively). But one readily checks that n is precisely chosen in the Stability Assumption so that $n > \max_{i,j}\{u_{i,j}, (\delta_{jr}+1)v_{i,j}\}$ and hence neither divisibility condition can be satisfied. Therefore all entries of any stable P must be maximal or minimal. The additional necessary condition that P be strict (as in Definition 8) so that $G(P)$ is not always zero according to (38) guarantees that neighboring entries in the ordered set can never be of the form (maximal,minimal), which gives the result. □

The number of stable patterns P is thus $2^r r! = |W(C_r)|$, the order of the Weyl group of C_r.

4.1. ***Action of W on Euclidean space.*** In demonstrating the equality of the two prime-power descriptions, it was necessary to use an explicit coordinatization of the root system embedded in $\mathbb{R}^r$; it would be desirable to find a coordinate-free proof. Let $\boldsymbol{e}_i$ be the standard basis vector (1 in i-th component, 0 elsewhere) in $\mathbb{R}^r$. Choose the following coordinates for the simple roots of C_r:

$$\alpha_1 = 2\boldsymbol{e}_1, \quad \alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_1, \quad \dots, \quad \alpha_r = \boldsymbol{e}_r - \boldsymbol{e}_{r-1}. \tag{44}$$

Consider an element $w \in W(C_r)$, the Weyl group of C_r. As an action on $\mathbb{R}^r$, this group is generated by all permutations σ of the basis vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_r$ and all reflections $\boldsymbol{e}_i \mapsto -\boldsymbol{e}_i$ for $i = 1, \dots, r$. Thus we may describe the action explicitly using $\varepsilon_w^{(i)} \in \{+1, -1\}$ for $i = 1, 2, \dots, r$ so that

$$w(t_1, t_2, \dots, t_r) = (\varepsilon_w^{(1)} t_{\sigma^{-1}(1)}, \varepsilon_w^{(2)} t_{\sigma^{-1}(2)}, \dots, \varepsilon_w^{(r)} t_{\sigma^{-1}(r)}). \tag{45}$$

In the following proposition, we associate a unique Weyl group element w with each GT-pattern P that is stable. In this result, and in the remainder of this section, it will be convenient to refer to the rows of P beginning at the *bottom* rather than the top. We will therefore discuss rows a_{r-i}, for $1 \le i \le r$, for instance.

Proposition 12. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1} \cdots L_1$, hence with associated dominant weight vector $\lambda = \sum_{i=1}^r \ell_i \varepsilon_i$. Let nonnegative integers $k_1(P), \ldots, k_r(P)$ be defined as in* (31)*, and let $k_{r+1}(P) = 0$. Then there exists a unique element $w \in W(C_r)$ such that*

$$\lambda + \rho - w(\lambda + \rho) = (2k_1 - k_2, k_2 - k_3, \ldots, k_{r-1} - k_r, k_r) = \sum_{i=1}^r k_i \alpha_i. \tag{46}$$

In fact, for $i = 2, \ldots, r$,

$$k_{i+1} - k_i + L_i = -\,\mathrm{wt}_i = \varepsilon_w^{(i)} L_{\sigma^{-1}(i)}, \tag{47}$$

where $L_{\sigma^{-1}(i)}$ is the unique element in row a_{r-i} that is not in row a_{r+1-i}, and the weight coordinate wt_i *is as in* (29)*. Similarly,*

$$k_2 - 2k_1 + L_1 = -\,\mathrm{wt}_1 = \varepsilon_w^{(1)} L_{\sigma^{-1}(1)}, \tag{48}$$

where $L_{\sigma^{-1}(1)}$ is the unique element in row a_{r-1} that is not in row a_r.

Proof. The definitions for ρ and λ give $\lambda + \rho = (L_1, \ldots, L_r)$ in Euclidean coordinates. Compute the coordinates of $(\lambda + \rho) - \sum_{i=1}^r k_i \alpha_i$ using (31) gives

$$L_1 + k_2 - 2k_1 = -\big(s_a(r-1) - 2s_b(r) + s_a(r)\big) = -\,\mathrm{wt}_1 \tag{49}$$

and similarly, for $i = 2, \ldots, r$,

$$L_i + k_{i+1} - k_i = -\big(s_a(r-i) - 2s_b(r+1-i) + s_a(r+1-i)\big) = -\,\mathrm{wt}_i, \tag{50}$$

so that

$$\lambda + \rho - \sum_{i=1}^r k_i \alpha_i = -(\mathrm{wt}_1, \mathrm{wt}_2, \ldots, \mathrm{wt}_r). \tag{51}$$

Each pattern P has a unique weight vector. Since P is a stable pattern, it is easy to see that the i-th weight consists of the unique entry that is in row a_{r-i} but not in row a_{r+1-i}, with a negative sign if this entry is present in row b_{r+1-i}, or a positive sign if not. Thus the weight vector is simply a permutation of the entries in the top row, with a choice of sign in each entry. We may find a unique w (whose action is described above), for which

$$w(\lambda + \rho) = (\varepsilon_w^{(1)} L_{\sigma^{-1}(1)}, \ldots, \varepsilon_w^{(r)} L_{\sigma^{-1}(r)}) = -(\mathrm{wt}_1, \mathrm{wt}_2, \ldots, \mathrm{wt}_r). \tag{52}$$

Thus $L_{\sigma^{-1}(i)}$ is the unique element in row a_{r-i} that is not present in row a_{r+1-i}. □

Corollary 13. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1} \cdots L_1$. For $1 \le i \le r$, the set of elements in row a_{r-i} satisfies*

$$\{a_{r-i,r+1-i}, a_{r-i,r+2-i}, \ldots, a_{r-i,r}\} = \{L_{\sigma^{-1}(i)}, L_{\sigma^{-1}(i-1)}, \ldots, L_{\sigma^{-1}(1)}\}. \tag{53}$$

Proof. From Proposition 12, $L_{\sigma^{-1}(j)}$ is the unique element in row a_{r-j} that is not in row a_{r+1-j}. Working downwards, eliminate these elements for $j = i, i+1, \dots, r$, in order to reach row a_{r-j}. This leaves the remaining set. □

4.2. ***Agreement of the multiple Dirichlet series.***

Theorem 1. *Let* $\Phi = C_r$ *and choose a positive integer* n *such that* (41) *of the Stability Assumption holds.*

(i) *Let* P *be a stable strict* GT*-pattern, and let* $G(P)$ *be the product of Gauss sums defined in* (38) *in Section 2. Let* w *be the Weyl group element associated to* P *as in Proposition 12. Then*

$$G(P) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}),$$

matching the definition given in (43), *with* $d_\lambda(\alpha)$ *as defined in* (40).

(ii) $H_{\mathrm{st}}(c_1, \dots, c_r; m_1, \dots m_r) = H^{(n)}(c_1, \dots, c_r; m_1, \dots m_r)$.

That is, the Weyl group multiple Dirichlet series in the twisted stable case is identical to the series defined by the Gelfand–Tsetlin description for n *sufficiently large.*

Remark 14. The Conjecture presented in the introduction states that n should be odd. In fact, the proof of Theorem 1 works for any n satisfying the Stability Assumption, regardless of parity. However, we believe this is an artifact of the relative combinatorial simplicity of the "stable" coefficients. As noted in Remark 3, one expects a distinctly different combinatorial recipe than the one presented in this paper to hold uniformly for all even n.

Proof. It is clear that part (i) implies part (ii), since both coefficients are obtained from their prime-power parts by means of twisted multiplicativity.

In proving part (i), let P be the GT-pattern with top row $L_r\, L_{r-1} \cdots L_1$ associated to w by Proposition 12. Since P is stable, we have $u_{i,j} = 0$ if P is minimal at $a_{i,j}$, and $v_{i,j} = 0$ if P is minimal at $b_{i,j}$. Thus

$$G(P) = \prod_{a_{i,j} \text{ maximal}} g_1(p^{u_{i,j}-1}, p^{u_{i,j}}) \prod_{b_{i,j} \text{ maximal}} g_{\delta_{jr}+1}(p^{v_{i,j}-1}, p^{v_{i,j}}).$$

It suffices to show that the set of Gauss sum exponents $u_{i,j}$ and $v_{i,j}$ at maximal entries in P coincides with the set of $d_\lambda(\alpha)$ as α runs over Φ_w. (In fact, we show a slightly sharper statement, which matches Gauss sum exponents at maximal entries in pairs of rows of P with values of $d_\lambda(\alpha)$ as α runs over certain subsets of Φ_w.)

The number of maximal elements in a pair of rows b_{r+1-i} and a_{r+1-i} is described in the next result. First, we say that (i, j) is an i*-inversion* for w^{-1} if $j < i$ and $\sigma^{-1}(j) > \sigma^{-1}(i)$. The number of these pairs, as well as the number of those for which the inequality is preserved rather than inverted, will play an important role in counting Gauss sums. To this end, define the quantities

$$\begin{aligned} \mathrm{inv}_i(w^{-1}) &= \#\{(i,j) \mid \sigma^{-1}(j) > \sigma^{-1}(i) \text{ and } j < i\}, \\ \mathrm{pr}_i(w^{-1}) &= \#\{(i,j) \mid \sigma^{-1}(j) < \sigma^{-1}(i) \text{ and } j < i\}. \end{aligned} \tag{54}$$

Proposition 15. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1} \cdots L_1$, and let $w \in W$ be the Weyl group element associated to P as in Proposition 12. Let* $\mathrm{inv}_i(w)$ *and* $\mathrm{pr}_i(w)$ *be as defined in* (54), *and let $m_i(P)$ denote the number of maximal entries in rows b_{r+1-i} and a_{r+1-i} together. Then,*

$$m_i(P) = \begin{cases} \mathrm{inv}_i(w^{-1}) & \text{if } \varepsilon_w^{(i)} = +1, \\ i + \mathrm{pr}_i(w^{-1}) & \text{if } \varepsilon_w^{(i)} = -1. \end{cases} \tag{55}$$

Proof. Recall from our means of associating w to P that $\varepsilon_w^{(i)}$ is opposite in sign from the i-th Gelfand–Tsetlin weight. Consider row b_{r+1-i} together with the rows immediately above and below:

$$\begin{array}{ccccccccc} a_{r-i,r+1-i} & & a_{r-1,r+2-i} & & \cdots & & \cdots & a_{r-i,r} & \\ & b_{r+1-i,r+1-i} & & \cdots & & \cdots & & & b_{r+1-i,r} \\ & & a_{r+1-i,r+2-i} & & \cdots & & \cdots & a_{r+1-i,r} & \end{array}$$

Suppose $\varepsilon_w^{(i)} = +1$, so $L_{\sigma^{-1}(i)}$ is missing from row a_{r+1-i} but present in row b_{r+1-i}. Then there are no maximal entries in row b_{r+1-i}, and m_i maximal entries in row a_{r+1-i}, so

$$b_{r+1-i,r+j-i} = a_{r-i,r+j-i} \quad \text{for } 1 \le j \le i, \tag{56}$$

$$a_{r+1-i,r+(j+1)-i} = \begin{cases} b_{r+1-i,r+j-i} & \text{for } 1 \le j \le m_i, \\ b_{r+1-i,r+(j+1)-i} & \text{for } m_i + 1 \le j \le i. \end{cases} \tag{57}$$

Moreover, the entry $L_{\sigma^{-1}(i)}$ in row b_{r+1-i} marks the switch from maximal to minimal as we move from left to right in row a_{r+1-i}. That is, all entries in row a_{r+1-i} to the left of $L_{\sigma^{-1}(i)}$ are maximal, while all those to the right are minimal. By Corollary 13, row a_{r+1-i} consists of the elements in the set $\{L_{\sigma^{-1}(j)} \mid j < i\}$. Since the rows of P are strictly decreasing, this means the maximal entries in row a_{r+1-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) > \sigma^{-1}(i)\}.$$

This set clearly has order $\mathrm{inv}_i(w^{-1})$.

Now suppose $\varepsilon_w^{(i)} = -1$, so that $L_{\sigma^{-1}(i)}$ is missing from both row a_{r+1-i} and row b_{r+1-i}. Then all entries in row a_{r+1-i} are maximal, and the last $m_i - i + 1$

entries in row b_{r+1-i} are maximal, so

$$a_{r+1-i,r+(j+1)-i} = b_{r+1-i,r+j-i} \quad \text{for } 1 \le j \le i-1, \tag{58}$$

$$b_{r+1-i,r+j-i} = \begin{cases} a_{r-i,r+j-i} & \text{for } 1 \le j \le 2i-1-m_i, \\ a_{r-i,r+(j+1)-i} & \text{for } 2i-m_i \le j \le i-1, \\ 0 & \text{for } j = i. \end{cases} \tag{59}$$

The entry $L_{\sigma^{-1}(i)}$ in row a_{r-i} marks the switch from minimal to maximal as we move to the right in row b_{r+1-i}. That is, all entries below and to the left of $L_{\sigma^{-1}(i)}$ are minimal, while those below and to the right are maximal. Since rows b_{r+1-i} and a_{r+1-i} are identical, the entries of row b_{r+1-i} are $\{L_{\sigma^{-1}(j)} \mid j < i\}$, by Corollary 13. Moreover, since rows are strictly decreasing, the maximal entries in row b_{r+1-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) < \sigma^{-1}(i)\} \cup \{0\}.$$

This set has order $\mathrm{pr}_i(w^{-1}) + 1$. Counting maximal entries in both rows, we obtain $m_i = (i-1) + \mathrm{pr}_i(w^{-1}) + 1 = i + \mathrm{pr}_i(w^{-1})$. □

Next, we establish a finer characterization of $\Phi_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$. For $\Phi = C_r$, the roots in Φ^+ take different forms; the positive long roots are $2\boldsymbol{e}_\ell$ for $1 \le \ell \le r$, while the positive short roots are $\boldsymbol{e}_m \pm \boldsymbol{e}_\ell$ for $1 \le \ell < m \le r$. We will express Φ_w as a disjoint union of subsets indexed by $i \in \{1, 2, \ldots, r\}$. To this end, let i be fixed, and let j be any positive integer such that $j < i$. Consider positive roots of the following three types:

Type L: $\alpha_{i,w} := 2\boldsymbol{e}_{\sigma^{-1}(i)}$.

Type S$^+$: $\alpha^+_{i,j,w} := \boldsymbol{e}_{\sigma^{-1}(j)} + \boldsymbol{e}_{\sigma^{-1}(i)}$.

Type S$^-$: $\alpha^-_{i,j,w} := \begin{cases} \boldsymbol{e}_{\sigma^{-1}(j)} - \boldsymbol{e}_{\sigma^{-1}(i)} & \text{if } \sigma^{-1}(j) > \sigma^{-1}(i), \\ \boldsymbol{e}_{\sigma^{-1}(i)} - \boldsymbol{e}_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i). \end{cases}$

Clearly we encounter each positive root exactly once as i and j vary as indicated. Let $\Phi_w^{(i)} \subseteq \Phi_w$ denote the set of all $\alpha_{i,w}$, $\alpha^+_{i,j,w}$, $\alpha^-_{i,j,w}$ belonging to Φ_w. The next lemma completely characterizes $\Phi_w^{(i)}$.

Lemma 16. *Let $i \in \{1, 2, \ldots, r\}$ be fixed, let j be any positive integer with $j < i$, and let $\Phi_w^{(i)}$ be as defined above.*

(1) *$\alpha_{i,w} \in \Phi_w^{(i)}$ if and only if $\varepsilon_w^{(i)} = -1$.*

(2) *$\alpha^-_{i,j,w} \in \Phi_w^{(i)}$ if and only if $\sigma^{-1}(j) < \sigma^{-1}(i)$ and $\varepsilon_w^{(i)} = -1$, or $\sigma^{-1}(j) > \sigma^{-1}(i)$ and $\varepsilon_w^{(i)} = +1$.*

(3) *$\alpha^+_{i,j,w} \in \Phi_w^{(i)}$ if and only if $\varepsilon_w^{(i)} = -1$.*

Consequently, $|\Phi_w^{(i)}| = m_i(P)$, as defined in Proposition 15.

Proof. As defined in (45), w acts on a basis vector $\boldsymbol{e}_\ell$ simply as $w(\boldsymbol{e}_\ell) = \varepsilon_w^{(\ell)}\, \boldsymbol{e}_{\sigma(\ell)}$, and this action extends linearly to each of the roots. Part (1) is immediate from the definition of Φ_w.

For part (2), if $\sigma^{-1}(j) < \sigma^{-1}(i)$ then

$$w(\alpha^-_{i,j,w}) = \varepsilon_w^{(i)} \boldsymbol{e}_i - \varepsilon_w^{(j)} \boldsymbol{e}_j.$$

If $\varepsilon_w^{(i)} = +1$, then since $j < i$, we have $w(\alpha^-_{i,j,w}) \in \Phi^+$ regardless of the value of $\varepsilon_w^{(j)}$. Thus $\alpha^-_{i,j,w} \notin \Phi_w^{(i)}$. Similarly, if $\varepsilon_w^{(i)} = -1$, then since $j < i$, we have $w(\alpha^-_{i,j,w}) \in \Phi^-$ regardless of the value of $\varepsilon_w^{(j)}$. Thus $\alpha^-_{i,j,w} \in \Phi_w^{(i)}$.

On the other hand, if $\sigma^{-1}(j) > \sigma^{-1}(i)$ then

$$w(\alpha^-_{i,j,w}) = \varepsilon_w^{(j)} \boldsymbol{e}_j - \varepsilon_w^{(i)} \boldsymbol{e}_i.$$

Considering the cases $\varepsilon_w^{(i)} = +1, -1$ in turn, we find that regardless of the value of $\varepsilon_w^{(j)}$, we have $w(\alpha^-_{i,j,w}) \in \Phi_w^{(i)}$ if and only if $\varepsilon_w^{(i)} = +1$.

For part (3), we have

$$w(\alpha^+_{i,j,w}) = \varepsilon_w^{(j)} \boldsymbol{e}_j + \varepsilon_w^{(i)} \boldsymbol{e}_i.$$

Using a similar argument, we see that independently of the value of $\varepsilon_w^{(j)}$, $w(\alpha^+_{i,j,w})$ is a negative root when $\varepsilon_w^{(i)}$ is negative, and a positive root otherwise.

Finally, we count elements in $\Phi_w^{(i)}$. If $\varepsilon_w^{(i)} = +1$, the conditions yield $\mathrm{inv}_i(w^{-1})$ elements of type S^-, and zero elements of types L and S^+. On the other hand, if $\varepsilon_w^{(i)} = -1$, there is one element of type L, $i-1$ elements of type S^+, and $\mathrm{pr}_i(w^{-1})$ elements of type S^-. In either case, $|\Phi_w^{(i)}| = m_i(P)$. $\square$

For each of the roots in $\Phi_w^{(i)}$, we compute the corresponding d_λ (as defined in (40)) below.

Lemma 17. *With the notation as above, we have*

(1) $d_\lambda(\alpha_{i,w}) = L_{\sigma^{-1}(i)}$.

(2) $d_\lambda(\alpha^-_{i,j,w}) = \begin{cases} L_{\sigma^{-1}(j)} - L_{\sigma^{-1}(i)} & \text{if } \sigma^{-1}(j) > \sigma^{-1}(i), \\ L_{\sigma^{-1}(i)} - L_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i). \end{cases}$

(3) $d_\lambda(\alpha^+_{i,j,w}) = L_{\sigma^{-1}(j)} + L_{\sigma^{-1}(i)}$.

Proof. First, we compute $d_\lambda(\alpha_{i,w}) = d_\lambda(2\boldsymbol{e}_{\sigma^{-1}(i)})$. Using (44), we have

$$\alpha_{i,w} = \alpha_1 + \sum_{k=2}^{\sigma^{-1}(i)} 2\alpha_k, \tag{60}$$

where we regard the sum to be 0 if $\sigma^{-1}(i)=1$. Since $\langle\alpha_{i,w},\alpha_{i,w}\rangle=\langle\alpha_1,\alpha_1\rangle=2$ and $\langle\alpha_k,\alpha_k\rangle=1$ for $k=2,\dots,r$, we have

$$d_\lambda(\alpha_{i,w})=\frac{2\langle\lambda+\rho,\alpha_{i,w}\rangle}{\langle\alpha_{i,w},\alpha_{i,w}\rangle}=\sum_{m=1}^{r}(l_m+1)\sum_{k=1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}=L_{\sigma^{-1}(i)}. \tag{61}$$

Next, we compute $d_\lambda(\alpha^-_{i,j,w})=d_\lambda(\boldsymbol{e}_{\sigma^{-1}(i)}-\boldsymbol{e}_{\sigma^{-1}(j)})$ if $\sigma^{-1}(j)<\sigma^{-1}(i)$. (The computations if $\sigma^{-1}(j)>\sigma^{-1}(i)$ are analogous.) In this case, (44) gives

$$\alpha^-_{i,j,w}=\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\alpha_k, \tag{62}$$

where the sum is nonempty as $\sigma^{-1}(j)<\sigma^{-1}(i)$. Since $\langle\alpha^-_{i,j,w},\alpha^-_{i,j,w}\rangle=1$,

$$d_\lambda(\alpha^-_{i,j,w})=\sum_{m=1}^{r}(l_m+1)\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}=L_{\sigma^{-1}(i)}-L_{\sigma^{-1}(j)}. \tag{63}$$

Finally, we compute $d_\lambda(\alpha^+_{i,j,w})=d_\lambda(\boldsymbol{e}_{\sigma^{-1}(i)}+\boldsymbol{e}_{\sigma^{-1}(j)})$. Here, (44) gives

$$\alpha^+_{i,j,w}=\alpha_1+\sum_{k=2}^{\sigma^{-1}(j)}2\alpha_k+\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\alpha_k, \tag{64}$$

where the first sum is 0 if $\sigma^{-1}(j)=1$. Since $\langle\alpha^+_{i,j,w},\alpha^+_{i,j,w}\rangle=1$ as well, we have

$$\begin{aligned} d_\lambda(\alpha^+_{i,j,w})&=\sum_{m=1}^{r}(l_m+1)\left(\sum_{k=1}^{\sigma^{-1}(j)}\frac{4\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}+\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}\right)\\ &=L_{\sigma^{-1}(i)}+L_{\sigma^{-1}(j)}, \end{aligned} \tag{65}$$

which completes the proof. □

Now let $D_i=\{d_\lambda(\alpha)\mid\alpha\in\Phi^{(i)}_w\}$. By Lemmas 16 and 17, we see that if $\epsilon^{(i)}_w=+1$, then

$$D_i=\{L_{\sigma^{-1}(j)}-L_{\sigma^{-1}(i)}\mid j<i\text{ and }\sigma^{-1}(j)>\sigma^{-1}(i)\}, \tag{66}$$

while if $\epsilon^{(i)}_w=-1$, then

$$\begin{aligned} D_i=\{L_{\sigma^{-1}(i)}\}\cup\{L_{\sigma^{-1}(j)}+L_{\sigma^{-1}(i)}\mid j<i\}\\ \cup\{L_{\sigma^{-1}(i)}-L_{\sigma^{-1}(j)}\mid j<i\text{ and }\sigma^{-1}(j)<\sigma^{-1}(i)\}. \end{aligned} \tag{67}$$

Now we examine the Gauss sums obtained from the GT-pattern P with top row $L_r\ L_{r-1}\cdots L_1$ associated to w. Suppose there are $m_i=m_i(P)$ maximal entries in rows b_{r+1-i} and a_{r+1-i} combined. First, suppose there are no maximal entries in row b_{r+1-i}. Then the first m_i entries in row a_{r+1-i} (reading from the left) are maximal. Since there are $i-1$ entries in row a_{r+1-i}, in this case we have $m_i < i$. We may apply (56) and (57) to compute the sums defining $u_{k,\ell}$ and $v_{k,\ell}$. These sums telescope, and we have

$$v_{r+1-i,r+j-i} = 0 \quad \text{for } 1 \le j \le i-1,$$
$$u_{r+1-i,r+(j+1)-i} = \begin{cases} 0 & \text{for } m_i + 1 \le j \le i, \\ a_{r-i,r+j-i} - b_{r+1-i,r+(m_i+1)-i} & \text{for } 1 \le j \le m_i. \end{cases}$$

By Proposition 12, $b_{r+1-i,r+(m_i+1)-i} = L_{\sigma^{-1}(i)}$, so to compute $u_{r+1-i,r+(j+1)-i}$ as j varies, we must determine the set of values for $a_{r-i,r+j-i}$ with $1 \le j \le m_i$. Recall that by Corollary 13, the entries in row a_{r-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid 1 \le j \le i\}. \tag{68}$$

The rows are strictly decreasing, so the entries appearing left of $a_{r-1,r+(m_i+1)-i} = L_{\sigma^{-1}(i)}$ have an index greater than $\sigma^{-1}(i)$. That is,

$$\{a_{r-i,r+j-i} \mid 1 \le j \le m_i\} = \{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) > \sigma^{-1}(i)\}. \tag{69}$$

Thus the nonzero Gauss sum exponents for rows b_{r+1-i} and a_{r+1-i} are given by $u_{r+1-i,r+(j+1)-i} = L_{\sigma^{-1}(j)} - L_{\sigma^{-1}(i)}$ with $j < i$ and $\sigma^{-1}(j) > \sigma^{-1}(i)$. Finally, $\varepsilon_w^{(i)} = +1$, since there are no maximal entries in row b_{r+1-i} in this case. Thus our set of nonzero Gauss sum exponents matches the set D_i as given in (66).

Second, suppose there are maximal entries in row b_{r+1-i}. Consequently, all entries in row a_{r+1-i} are maximal, so there are $n_i := m_i - i + 1$ maximal entries in row b_{r+1-i}. We may apply (58) and (59) to compute the sums defining $u_{k,\ell}$ and $v_{k,\ell}$. These sums telescope, and we have

$$v_{r+1-i,r+j-i} = \begin{cases} 0 & \text{for } 1 \le j \le i - n_i, \\ a_{r-i,r+1-n_i} - a_{r-i,r+(j+1)-i} & \text{for } i+1-n_i \le j \le i-1, \\ a_{r-i,r+1-n_i} & \text{for } j = i, \end{cases}$$
$$u_{r+1-i,r+(j+1)-i} = a_{r-i,r+1-n_i} + a_{r+1-i,r+(j+1)-i} \quad \text{for } 1 \le j \le i-1.$$

By Proposition 12, $a_{r+1-i,r+1-n_i} = L_{\sigma^{-1}(i)}$, and thus $v_{r+1-i,r} = L_{\sigma^{-1}(i)}$. To compute the remaining exponents $v_{r+1-i,r+j-i}$ as j varies, we again appeal to (68). Since the rows are strictly decreasing, the entries appearing to the right of $L_{\sigma^{-1}(i)}$ in row a_{r-1} must have an index smaller than $\sigma^{-1}(i)$. That is,

$$\{a_{r-i,r+(j+1)-i} \mid i+1-n_i \le j \le i-1\} = \{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

Thus $v_{r+1-i,r+j-i} = L_{\sigma^{-1}(i)} - L_{\sigma^{-1}(j)}$ with $i+1-n_i \le j < i$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$.

To compute the exponents $u_{r+1-i,r+(j+1)-i}$, we note that by Corollary 13, the entries in row a_{r+1-i} are the $L_{\sigma^{-1}(j)}$ for which $1 \le j \le i-1$. Thus

$$u_{r+1-i,r+(j+1)-i} = L_{\sigma^{-1}(i)} + L_{\sigma^{-1}(j)}, \tag{70}$$

with $1 \le j \le i-1$. Finally, $\varepsilon_w^{(i)} = -1$, since there are maximal entries in row b_{r+1-i}. Combining the cases above, we match the set D_i given in (67).

This completes the proof of Theorem 1. □

5. Comparison with the Casselman–Shalika formula

The main focus of this section is the proof of Theorem 2, using a generating function identity given in [Hamel and King 2002]. This identity may be regarded as a deformation of the Weyl character formula for $\mathrm{Sp}(2r)$, though it is stated in the language of symplectic, shifted tableaux (whose definition we will soon recall) so we postpone the precise formulation. Recall that our multiple Dirichlet series take the form

$$Z_\Psi(\boldsymbol{s};\boldsymbol{m}) = \sum_{\boldsymbol{c}=(c_1,\dots,c_r)\in(\mathbb{O}_S/\mathbb{O}_S^\times)^r} \frac{H^{(n)}(\boldsymbol{c};\boldsymbol{m})\Psi(\boldsymbol{c})}{|c_1|^{2s_1}\cdots|c_r|^{2s_r}}.$$

In brief, we show that for $n=1$ our formulas for the prime-power supported contributions of $Z_\Psi(\boldsymbol{s},\boldsymbol{m})$ match one side of Hamel and King's identity, while the other side of the identity is given in terms of a character of a highest weight representation for $\mathrm{Sp}(2r)$. By combining the Casselman–Shalika formula with Hamel and King's result, we will establish Theorem 2.

5.1. ***Specialization of the multiple Dirichlet series for $n = 1$.*** Many aspects of the definition Z_Ψ are greatly simplified when $n=1$. First, we may take Ψ to be constant, since the Hilbert symbols appearing in the definition (16) are trivial for $n=1$. Moreover, the coefficients $H^{(n)}(\boldsymbol{c};\boldsymbol{m})$ for $n=1$ are perfectly multiplicative in both $\boldsymbol{c}$ and $\boldsymbol{m}$. That is, according to (18),

$$H^{(1)}(\boldsymbol{c}\cdot\boldsymbol{c}';\boldsymbol{m}) = H^{(1)}(\boldsymbol{c};\boldsymbol{m})H^{(1)}(\boldsymbol{c}';\boldsymbol{m}) \quad \text{when } \gcd(c_1\cdots c_r, c_1'\cdots c_r') = 1,$$

and according to (19),

$$H^{(1)}(\boldsymbol{c};\boldsymbol{m}\cdot\boldsymbol{m}') = H^{(1)}(\boldsymbol{c};\boldsymbol{m}) \quad \text{when } \gcd(m_1'\cdots m_r', c_1\cdots c_r) = 1.$$

Hence the global definition of $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ for fixed $\boldsymbol{m}$ is easily recovered from its prime-power supported contributions as follows:

$$Z_\Psi(\boldsymbol{s};\boldsymbol{m}) = \prod_{p\in\mathbb{O}_S}\Bigg(\sum_{\boldsymbol{k}=(k_1,\dots,k_r)} \frac{H^{(1)}(p^{\boldsymbol{k}};p^{\boldsymbol{l}})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}}\Bigg), \tag{71}$$

with $\boldsymbol{l} = (l_1, \dots, l_r)$ given by $\mathrm{ord}_p(m_i) = l_i$ for $i = 1, \dots, r$. The sum on the right-hand side runs over the finite number of vectors $\boldsymbol{k}$ for which $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ has nonzero support for fixed $\boldsymbol{l}$ according to (39).

We now simplify our formulas for $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ when $n = 1$. As before, set $q = |\mathbb{O}_S/p\mathbb{O}_S|$. With definitions as given in (34) and (36), let

$$\tilde{\gamma}_a(i, j) := q^{-u_{i,j}} \gamma_a(i, j) \quad \text{and} \quad \tilde{\gamma}_b(i, j) := q^{-v_{i,j}} \gamma_b(i, j).$$

Then by analogy with the definitions (38) and (39), define

$$\widetilde{G}(P) := \prod_{1 \le i \le j \le r} \tilde{\gamma}_a(i, j) \tilde{\gamma}_b(i, j),$$
$$\widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \widetilde{H}^{(1)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) := \sum_{k(P)=(k_1,\dots,k_r)} \widetilde{G}(P),$$

where again the sum is taken over GT-patterns P with fixed top row $(L_r, \dots, L_1)$ as in (26). By elementary properties of Gauss sums, when $n = 1$ we have, for a strict GT-pattern P,

$$\tilde{\gamma}_a(i, j) = \begin{cases} 1 & \text{if } P \text{ is minimal at } a_{i,j}, \\ 1 - 1/q & \text{if } P \text{ is generic at } a_{i,j}, \\ -1/q & \text{if } P \text{ is maximal at } a_{i,j}, \end{cases} \tag{72}$$

recalling the language of Definition 7 and similarly,

$$\tilde{\gamma}_b(i, j) = \begin{cases} 1 & \text{if } P \text{ is minimal at } b_{i,j}, \\ 1 - 1/q & \text{if } P \text{ is generic at } a_{i,j}, \\ -1/q & \text{if } P \text{ is maximal at } b_{i,j}. \end{cases} \tag{73}$$

When P is generic at $a_{i,j}$ (respectively $b_{i,j}$), the condition $n \mid u_{i,j}$ (respectively $n \mid v_{i,j}$) is trivially satisfied, since $n = 1$.

We claim that

$$H^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})\, q^{k_1+\cdots+k_r}. \tag{74}$$

This equality follows from the definitions of $H^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ and $\widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$, after matching powers of q on each side by applying the following combinatorial lemma.

Lemma 18. *For each* GT-*pattern* P,

$$\sum_{i=1}^{r} k_i(P) = \sum_{i=1}^{r} \Bigl(\sum_{j=i}^{r} v_{i,j} + \sum_{j=i+1}^{r} u_{i,j} \Bigr). \tag{75}$$

Proof. We proceed by expanding each side in terms of the entries $a_{i,j}$ and $b_{i,j}$ in the GT-pattern P, using the definitions above. Applying (31), we have

$$\sum_{i=1}^{r} k_i(P) = \left(r\, s_a(0) + \sum_{m=1}^{r-1} s_a(m) + \sum_{i=2}^{r} \sum_{m=1}^{r+1-i} (2\, s_a(m) + a_{0,m}) - \sum_{i=2}^{r} s_a(r+1-i) \right) - \left(\sum_{m=1}^{r} s_b(m) + \sum_{i=2}^{r} \sum_{m=i}^{r+1-i} 2\, s_b(m) \right).$$

Since $\sum_{i=2}^{r} s_a(r+1-i) = \sum_{m=1}^{r-1} s_a(m)$, the corresponding terms in the first parentheses cancel. After interchanging order of summation and evaluating sums over i, we obtain

$$\sum_{i=1}^{r} k_i(P) = r\, s_a(0) + \sum_{m=1}^{r} (r-m)\, a_{0,m} + \sum_{m=1}^{r-1} 2(r-m)\, s_a(m) - \sum_{m=1}^{r} (1+2(r-m))\, s_b(m).$$

Finally, applying (28) and combining the first two terms, we conclude that

$$\sum_{i=1}^{r} k_i(P) = \sum_{m=1}^{r} (2r-m)\, a_{0,m} + \sum_{m=1}^{r-1} \sum_{\ell=m+1}^{r} 2(r-m)\, a_{m,\ell} - \sum_{m=1}^{r} \sum_{\ell=m}^{r} (1+2(r-m))\, b_{m,\ell}. \tag{76}$$

On the other hand, from (33), after recombining terms we have

$$\begin{aligned}\sum_{i=1}^{r} \left(\sum_{j=i}^{r} v_{i,j} + \sum_{j=i+1}^{r} u_{i,j} \right) &= -\sum_{i=1}^{r} \left(b_{i,i} + \sum_{j=i+1}^{r} \left(b_{i,j} + 2 \sum_{m=i}^{r} b_{i,m} \right) \right) \\ &\quad + \sum_{i=1}^{r} \left(a_{i-1,i} + \sum_{j=i+1}^{r} \left(\sum_{m=i}^{j} 2\, a_{i-1,m} + \sum_{m=j+1}^{r} a_{i-1,m} + \sum_{m=j}^{r} a_{i,m} \right) \right).\end{aligned}$$

After interchanging order of summation and evaluating sums on j, this equals

$$\sum_{i=1}^{r} \left((1+2(r-i))\, a_{i-1,i} + \sum_{m=i+1}^{r} (2r+1-(i+m))\, a_{i-1,m} + \sum_{m=i+1}^{r} (m-i)\, a_{i,m} \right) - \sum_{i=1}^{r} \sum_{m=i}^{r} (1+2(r-i))\, b_{i,m}.$$

The $i = 1$ terms from the first two summands in the big parentheses evaluate to $\sum_{m=1}^{r}(2r-m)\,a_{0,m}$, the first term in (76). After reindexing, the remaining terms in the parentheses give $\sum_{i=1}^{r-1}\sum_{m=i+1}^{r}2(r-i)\,a_{i,m}$. Relabeling indices where needed gives the result. $\square$

We now manipulate the prime-power supported contributions to the multiple Dirichlet series as in (71). Setting $y_i = |p|^{-2s_i}$ for $i = 1, \ldots, r$ and using (74) gives

$$\sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \frac{H^{(1)}(p^{k_1}, \ldots, p^{k_r})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}} = \sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1}, \ldots, p^{k_r})\,(qy_1)^{k_1}\cdots(qy_r)^{k_r}. \tag{77}$$

After making the change of variables

$$qy_1 \mapsto x_1^2, \quad qy_2 \mapsto x_1^{-1}x_2, \quad \ldots, \quad qy_r \mapsto x_{r-1}^{-1}x_r,$$

the right-hand side of (77) becomes

$$\sum_{(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1}, \ldots, p^{k_r})\,x_1^{2k_1}(x_1^{-1}x_2)^{k_2}\cdots(x_{r-1}^{-1}x_r)^{k_r}.$$

By the relationship between the coordinates k_i and the weight coordinates wt_i given in (30), this is just

$$x_1^{L_1}\cdots x_r^{L_r}\sum_{(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1}, \ldots, p^{k_r})\,x_1^{\mathrm{wt}_1}\cdots x_r^{\mathrm{wt}_r},$$

where the L_i relate to l_i as in (26). Finally, letting

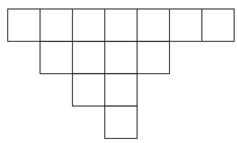
$$\mathrm{gen}(P) = \#\{\text{generic entries in } P\} \quad \text{and} \quad \max(P) = \#\{\text{maximal entries in } P\}$$

and using the simplifications for $n = 1$ in (72) and (73) for $\widetilde{H}^{(1)}$ in terms of $\widetilde{G}(P)$, then

$$\sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \frac{H^{(1)}(p^{k_1}, \ldots, p^{k_r})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}} = x_1^{L_1}\cdots x_r^{L_r}\sum_{(k_1,\ldots,k_r)}\left(\frac{-1}{q}\right)^{\max(P)}\left(1-\frac{1}{q}\right)^{\mathrm{gen}(P)}x_1^{\mathrm{wt}_1}\cdots x_r^{\mathrm{wt}_r}, \tag{78}$$

with the x_i given in terms of $|p|^{-2s_i}$ by the composition of the above changes of variables. The right-hand side of (78) is now amenable to comparison with the identity of Hamel and King.

5.2. *Symplectic shifted tableaux.* In order to state the needed identity of Hamel and King, we introduce some additional terminology. To each strict GT-pattern P, we may associate an $\mathrm{Sp}(2r)$-*standard shifted tableau* of shape $\lambda+\rho$. Below, we follow [Hamel and King 2002], specializing Definition 2.5 to our circumstances. Consider the partition μ of $\lambda+\rho$, whose parts are given by $\mu_i = l_1 + \cdots l_i + r - i + 1$ for $i = 1, \dots, r$. (These are simply the entries in the top row of the pattern P in $\mathrm{GT}(\lambda+\rho)$.) Such a partition defines a *shifted Young diagram* constructed as follows: $|\mu|$ boxes are arranged in r rows of lengths $\mu_1, \mu_2, \dots, \mu_r$, and the rows are left-adjusted along a diagonal line. For instance, if $\mu = (7, 4, 2, 1)$, then our tableau has the following shape:



It remains to define how the tableau is to be filled. The alphabet will consist of the set $A = \{1, 2, \dots, r\} \cup \{\bar{1}, \bar{2}, \dots \bar{r}\}$, with ordering $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r$. We place an entry from A in each of the boxes of the tableau so that the entries are: (1) *weakly increasing* from left to right across each row and from top to bottom down each column, and (2) *strictly increasing* from top-left to bottom-right along each diagonal.

An explicit correspondence between $\mathrm{Sp}(2r)$-standard shifted tableaux and strict GT-patterns is given in [Hamel and King 2002, Definition 5.2]. Below we describe the prescription for determining S_P, the tableau corresponding to a given GT-pattern P, with notation as in (27).

(1) For $j = i, \dots, r$, the entries $a_{i-1,j}$ of P count, respectively, the number of boxes in the $(j-i+1)$–st row of S_P whose entries are less than or equal to the value $r - i + 1$.

(2) For $j = i, \dots, r$, the entries $b_{i,j}$ of P count, respectively, the number of boxes in the $(j-i+1)$–st row of S_P whose entries are less than or equal to the value $\overline{r - i + 1}$.

An example of this bijection is given in Figure 1.

We also associate the following statistics to any symplectic shifted tableau S:

(1) $\mathrm{wt}(S) = (\mathrm{wt}_1(S), \mathrm{wt}_2(S), \dots, \mathrm{wt}_r(S))$ for $\mathrm{wt}_i(S) = \#(i \text{ entries}) - \#(\bar{\imath} \text{ entries})$.

(2) $\mathrm{con}_k(S)$ is the number of connected components of the ribbon strip of S consisting of all the entries k.

(3) $\mathrm{row}_k(S)$ is the number of rows of S containing an entry k, and similarly $\mathrm{row}_{\bar{k}}(S)$ is the number of rows of S containing an entry $\bar{k}$.

$$
\begin{array}{cccccccccccccc}
9 & & 6 & & 5 & & 3 & & 2 & & & & & \\
& 7 & & 6 & & 5 & & 3 & & 2 & & & & \\
& & 7 & & 5 & & 4 & & 2 & & & & & \\
& & & 5 & & 4 & & 3 & & 1 & & & & \\
& & & & 5 & & 3 & & 1 & & & & & \\
& & & & & 4 & & 2 & & 1 & & & & \\
& & & & & & 4 & & 2 & & & & & \\
& & & & & & & 3 & & 2 & & & & \\
& & & & & & & & 3 & & & & & \\
& & & & & & & & & 1 & & & &
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 \\
\hline
 & \bar{2} & \bar{2} & 3 & \bar{4} & 4 & \bar{5} & & \\
\hline
 & & \bar{3} & \bar{4} & \bar{4} & 4 & \bar{5} & & \\
\hline
 & & & \bar{4} & 4 & \bar{5} & & & \\
\hline
 & & & & \bar{5} & \bar{5} & & & \\
\hline
\end{array}
$$

Figure 1. The bijection between GT-patterns and symplectic shifted tableaux.

(4) $\mathrm{str}(S)$ is the total number of connected components of all ribbon strips of S.

(5) $\mathrm{bar}(S)$ is the total number of barred entries in S.

(6) $\mathrm{hgt}(S) = \sum_{k=1}^{r}(\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\bar{k}}(S))$.

It is easy to see that the weights associated with the tableaux S_P are identical to the previously defined weights associated with the pattern P.

Theorem [Hamel and King 2002, Theorem 1.2]. *Let λ be a partition into at most r parts, and let $\rho = (r, r-1, \dots, 1)$. Then defining*

$$D_{\mathrm{Sp}(2r)}(\boldsymbol{x}; t) = \prod_{i=1}^{r} x_i^{r-i+1} \prod_{i=1}^{r}(1+tx_i^{-2}) \prod_{1 \le i < j \le r}(1+tx_i^{-1}x_j)(1+tx_i^{-1}x_j^{-1}), \tag{79}$$

and letting $\mathrm{sp}_\lambda(\boldsymbol{x}) := \mathrm{sp}_\lambda(x_1, \dots, x_r)$ be the character of the highest weight representation of $\mathrm{Sp}(2r)$ with highest weight λ, we have

$$D_{\mathrm{Sp}(2r)}(t\boldsymbol{x}; t)\, \mathrm{sp}_\lambda(\boldsymbol{x}) = \sum_{S \in \mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))} t^{\mathrm{hgt}(S)+r(r+1)/2}(1+t)^{\mathrm{str}(S)-r}\boldsymbol{x}^{\mathrm{wt}(S)}, \tag{80}$$

where $\mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))$ denotes the set of all $\mathrm{Sp}(2r)$-standard shifted tableaux of shape $\lambda + \rho$.

Remark 19. The identity appears in the theorem cited in the form

$$D_{\mathrm{Sp}(2r)}(\boldsymbol{x}; t)\, \mathrm{sp}_\lambda(\boldsymbol{x}; t) = \sum_{S \in \mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))} t^{\mathrm{hgt}(S)+2\,\mathrm{bar}(S)}(1+t)^{\mathrm{str}(S)-r}\boldsymbol{x}^{\mathrm{wt}(S)}, \tag{81}$$

where $\mathrm{sp}_\lambda(\boldsymbol{x}; t)$ is a simple deformation of the usual symplectic character given in [Hamel and King 2002, (1.13)]. To relate (81) to (80), put $x_i \to tx_i$ for each $i = 1, \dots, r$, which introduces a factor of $t^{\sum \mathrm{wt}_i(S)}$ on the right-hand side. From the definition of $\mathrm{wt}(S)$ and the correspondence with P, we see that

$$\sum_{i=1}^{r} \mathrm{wt}_i(S) = \frac{r(r+1)}{2} - 2\,\mathrm{bar}(S) + \sum_{i=1}^{r}(r-i+1)l_i. \tag{82}$$

Moreover, it is a simple exercise to show that

$$\mathrm{sp}_\lambda(t\boldsymbol{x};t) = t^{\sum(r-i+1)l_i}\ \mathrm{sp}_\lambda(\boldsymbol{x}). \tag{83}$$

Applying the previous two identities to (81) gives (80).

We now show that the right-hand side of (80) may be expressed in terms of the right-hand side of (78), leading to an expression for the generating function for $H(p^{k_1},\dots,p^{k_r})$ in terms of a symplectic character. The following lemma relates the exponents in this equation back to our GT-pattern P and the statistics of (78).

Lemma 20. *Let P be a strict* GT*-pattern of rank r and S_P its associated standard shifted tableau. Then we have the following relationships*:

(a) $\mathrm{gen}(P) = \mathrm{str}(S_P) - r$.

(b) $\max(P) = \mathrm{hgt}(S_P) + r(r+1)/2$.

This is stated without proof implicitly in [Hamel and King 2002, Corollary 5.3], using slightly different notation. The proof is elementary, but we include it in the next section for completeness. Assuming the lemma, letting $t = -1/q$ in (80), and using (78) with $|p| = q$, we see that

$$\begin{aligned}\sum_{(k_1,\dots,k_r)} & H(p^{k_1},\dots,p^{k_r})q^{-2k_1s_1}\cdots q^{-2k_rs_r} \\ &= x_1^{L_1}\cdots x_r^{L_r}\, D_{\mathrm{Sp}(2r)}(-x_1/q,\dots,-x_r/q;-1/q)\ \mathrm{sp}_\lambda(x_1,\dots,x_r),\end{aligned} \tag{84}$$

with the identification

$$q^{1-2s_1} = x_1^2,\quad q^{1-2s_2} = x_1^{-1}x_2,\quad \dots,\quad q^{1-2s_r} = x_{r-1}^{-1}x_r. \tag{85}$$

One checks by induction on the rank r that, with x_i assigned as above,

$$x_1x_2^2\cdots x_r^r D_{\mathrm{Sp}(2r)}(-x_1/q,\dots,-x_r/q;-1/q) = \prod_{\alpha\in\Phi^+}(1-q^{-(1+2B(\alpha,\boldsymbol{s}-(1/2)\rho^\vee))})$$

with $B(\alpha,\boldsymbol{s}-\frac{1}{2}\rho^\vee)$ as defined in (3). Moving this product to the left-hand side of (84), we can rewrite that equality as

$$\begin{aligned}\prod_{\alpha\in\Phi^+}(1-q^{-(1+2B(\alpha,\boldsymbol{s}-(1/2)\rho^\vee))})^{-1} & \sum_{(k_1,\dots,k_r)} H(p^{k_1},\dots,p^{k_r})q^{-2k_1s_1}\cdots q^{-2k_rs_r} \\ &= x_1^{L_1-1}\cdots x_r^{L_r-r}\ \mathrm{sp}_\lambda(x_1,\dots,x_r).\end{aligned} \tag{86}$$

The terms in the product are precisely the Euler factors for the normalizing zeta factors of $Z_\Psi^*(\boldsymbol{s}; \boldsymbol{m})$ defined in (23) for the case $n = 1$. Hence, the terms on the left-hand side of (86) constitute the complete set of terms in the multiple Dirichlet series $Z_\Psi^*(\boldsymbol{s}; \boldsymbol{m})$ supported at monomials of the form $|p|^{-k_1 s_1 - \cdots - k_r s_r}$.

Finally, we can restate and prove our second main result.

Theorem 2. *Let $\boldsymbol{m} = (m_1, \ldots, m_r) \in \mathbb{O}_S$ with m_i nonzero for all i. For each prime $p \in \mathbb{O}_S$, let $\operatorname{ord}_p(m_i) = l_i$. Let $H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r})$ with $n = 1$ be defined as in Section 5.1. Then the resulting multiple Dirichlet series $Z_\Psi^*(\boldsymbol{s}; \boldsymbol{m})$ agrees with the $(m_1, \ldots, m_r)$-th Fourier–Whittaker coefficient of a minimal parabolic Eisenstein series on $\mathrm{SO}_{2r+1}(F_S)$.*

Proof. In the case $n = 1$, the multiple Dirichlet series $Z_\Psi^*(\boldsymbol{s}; \boldsymbol{m})$ is Eulerian. Indeed, the power residue symbols used in the definition of twisted multiplicativity in (17) and (19) are all trivial. Hence it suffices to check that the Euler factors for Z_Ψ^* match those of the corresponding minimal parabolic Eisenstein series at each prime $p \in \mathbb{O}_S$.

The Euler factors for the minimal parabolic Eisenstein series can be computed using the Casselman–Shalika formula [1980, Theorem 5.4]. We briefly recall the form of this expression for a split, reductive group G over a local field F_v with usual Iwasawa decomposition $G = ANK = BK$. Let χ be an unramified character of the split maximal torus A and consider the induced representation $\operatorname{ind}_B^G(\chi)$. Given an unramified additive character ψ of the unipotent $N^-(F_v)$, opposite the unipotent N of B, there is an associated Whittaker functional

$$W_\psi(\phi) = \int_{N^-(F_v)} \phi(\bar{n})\psi(\bar{n})\, d\bar{n}, \tag{87}$$

where $\phi(ank) := \chi(a)\delta_B(a)^{1/2}$ is the normalized spherical vector with δ_B is the modular quasicharacter. The associated Whittaker function is given by setting $\mathcal{W}_\phi(g) := W(g\phi)$ and is determined by its value on $\pi^{-\lambda}$ for $\lambda \in X_*$, the coweight lattice and π a uniformizer for F_v. Then the Casselman–Shalika formula states that $\mathcal{W}_\phi(\pi^{-\lambda}) = 0$ unless λ is dominant, in which case

$$\delta_B(\pi^{-\lambda})^{1/2}\mathcal{W}_\phi(\pi^{-\lambda}) = \left(\prod_{\alpha \in \Phi^+} (1 - q^{-1}\boldsymbol{t}^{-\alpha^\vee})\right)\operatorname{ch}_\lambda(\boldsymbol{t}), \tag{88}$$

where $\operatorname{ch}_\lambda$ is the character of the irreducible representation of the Langlands dual group $G^\vee$ with highest weight λ and $\boldsymbol{t}$ denotes a diagonal representative of the semisimple conjugacy class in $G^\vee$ associated to $\operatorname{ind}_B^G(\chi)$ by Langlands via the Satake isomorphism (see [Borel 1979] for details). In the special case $G = \mathrm{SO}(2r+1)$, for relations with the above multiple Dirichlet series, we determine $\boldsymbol{t} = (x_1, \ldots, x_r)$ according to (85) where $|\pi|_v^{-1} = q$. Since $G^\vee = \mathrm{Sp}(2r)$ in this case, the character $\operatorname{ch}_\lambda(\boldsymbol{t})$ in (88) is just $\operatorname{sp}_\lambda(x_1, \ldots, x_r)$ as in the right-hand side of (86). Furthermore, the product over positive roots in (88) matches the Euler factors for the normalizing zeta factors of Z_Ψ^* appearing on the left-hand side of (86).

While the Casselman–Shalika formula is stated for principal series over a local field, because the global Whittaker coefficient is Eulerian, there is no obstacle to obtaining the analogous global result for F_S from the local result via passage to the adele group. Moreover, the minimal parabolic Eisenstein series Whittaker functional

$$\int_{N(\mathbb{A})/N(F)} E_\phi(ng)\psi_{\underline{m}}(n)\,dn = \int_{N(\mathbb{A})/N(F)} \sum_{\gamma\in B(F)\backslash G(F)} \phi(\gamma ng)\psi_{\underline{m}}(n)\,dn$$

can be shown to match the integral in (87) with $\psi = \psi_{\underline{m}}$ by the usual Bruhat decomposition for $G(F)$ and a standard unfolding argument.

Hence according to (86), the Euler factor for $Z_\Psi^*(\boldsymbol{s};\boldsymbol{m})$ matches that of the Fourier–Whittaker coefficient except possibly up to a monomial in the $|p|^{-2s_i}$ with $i = 1,\ldots,r$. This disparity arises from the fact that the Whittaker functions in the Casselman–Shalika formula are normalized by the modular quasicharacter $\delta_B^{1/2}$, whereas our multiple Dirichlet series should correspond to unnormalized Whittaker coefficients in accordance with the functional equations σ_i as in (21) sending $s_i \mapsto 1 - s_i$. Hence, to check that the right-hand side of (86) exactly matches the unnormalized Whittaker coefficient of the Eisenstein series, it suffices to verify that

$$x_1^{L_1-1}\cdots x_r^{L_r-r}\,\mathrm{sp}_\lambda(x_1,\ldots,x_r)$$

satisfies a local functional equation σ_j given in (21) as Dirichlet polynomials in $|p|^{-2s_i}$ for $i = 1,\ldots,r$. $\square$

5.3. *Proof of Lemma 20.* For part (a) of the lemma, we induct on the rank. When $r = 2$, there are at most six connected components among all the ribbon strips of S_P, since 1 and $\bar{1}$ may only appear in the top row. Moreover, since P is strict there must be at least two connected components. Thus $0 \le \mathrm{str}(S_P) - 2 \le 4$. At each of the four entries in P below the top row, one shows that if the given entry is generic, it increases the count $\mathrm{str}(S_P)$ by 1.

Suppose that for a GT-pattern of rank $r-1$, each of the r^2 entries below the top row increases the count $\mathrm{str}(P)$ by 1. Then consider a GT-pattern P of rank r, and consider the collection of entries $a_{i,j}$, $b_{i,j}$ below the double line. These entries control the number of connected components consisting of copies of $\bar{1}, 1, \ldots, \overline{r-1}$, and $r-1$ in P, in precisely the same way as the full collection of entries below the top row in a pattern of rank $r-1$. Thus inductively, for each generic entry $a_{i,j}$ with $2 \le i \le r-1$ and $3 \le j \le r$ or $b_{i,j}$ with $2 \le i, j \le r$, the count $\mathrm{str}(P)$ is increased by 1. Finally, for $i = 1$, one easily checks that the value of $\mathrm{str}(S_P)$ is increased by 1 for every generic $a_{1,j}$ or $b_{1,j}$.

For (b), we first establish the correct range for $\mathrm{hgt}(S_P)+r(r+1)/2$. For each k, it is clear that $0 \le \mathrm{row}_k(S_P)-\mathrm{con}_k(S_P) \le k-1$ and $0 \le \mathrm{row}_{\bar{k}}(S_P) \le k$. Combining these inequalities and summing over k, we have $0 \le \mathrm{hgt}(S_P)+r(r+1)/2 \le r^2$. We proceed by showing that each of the maximal entries increases the count $\mathrm{hgt}(S)$ by 1. The cases are as follows.

(1) If $a_{i,j}$ is maximal, then $a_{i,j} = b_{i,j-1}$, hence there are no $\overline{r+1-i}$ entries in row $j-i$ of the tableau. This decreases $\sum_{k=1}^{r} \mathrm{row}_{\bar{k}}(S_P)$ by 1, hence increasing $\mathrm{hgt}(S_P)$ by 1.

(2) If $b_{i,r}$ is maximal, then $b_{i,r} = 0$, which implies there are no $\overline{r+1-i}$ entries in row $r-i+1$. This similarly increases $\mathrm{hgt}(S_P)$ by 1.

(3) If $b_{i,j}$ is maximal with $1 \le j \le r-1$, then $b_{i,j} = a_{i-1,j+1}$. Since P is a strict pattern, it must follow that $b_{i,j} < a_{i-1,j}$ and $b_{i,j+1} < a_{i-1,j+1}$. By these strict inequalities, $r+1-i$ occurs in both row $j+1-i$ and row $j+2-i$. However, by the equality defining $b_{i,j}$ as maximal, the occurrences in each of these two rows form one connected component. (See, for instance, the $\bar{4}$ component in the example in Figure 1.) This decreases $\sum_{k=1}^{r} \mathrm{con}_k(S_P)$ by 1, hence increasing $\mathrm{hgt}(S_P)$ by 1. □

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MILNOR OPEN BOOKS OF LINKS OF SOME RATIONAL SURFACE SINGULARITIES

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We determine Legendrian surgery diagrams for the canonical contact structures of links of rational surface singularities that are also small Seifert fibered 3-manifolds. Moreover, we describe an infinite family of Milnor fillable contact 3-manifolds so that, for each member of this family, the Milnor genus and Milnor norm are strictly greater than the support genus and support norm of the canonical contact structure. For some of these contact structures we construct supporting Milnor open books.

1. Introduction

The link of a normal complex surface singularity carries a canonical contact structure ξ_{can} (also known as the Milnor fillable contact structure) which is supported by any Milnor open book on this link [Caubel et al. 2006]. The canonical contact structure ξ_{can} is known to be Stein fillable [Bogomolov and de Oliveira 1997] and therefore it is tight [Eliashberg and Gromov 1991]. In fact, ξ_{can} is universally tight, that is, the pullback to the universal cover is tight [Lekili and Ozbagci 2010].

Etnyre and Ozbagci [2008] defined three numerical invariants of contact structures in terms of open books supporting the contact structures. These invariants are the support genus $\text{sg}(\xi)$ (the minimal genus of a page of a supporting open book for ξ), the binding number $\text{bn}(\xi)$ (the minimal number of binding components of a supporting open book for ξ with minimal genus pages) and the support norm $\text{sn}(\xi)$ (minus the maximal Euler characteristic of a page of a supporting open book for ξ).

Altınok and Bhupal [2008] derived a new set of invariants specifically for the canonical contact structure ξ_{can} on the link of a complex surface singularity by restricting the set of open books to only Milnor open books on the link at hand. In this article we will call these invariants the Milnor genus $\text{Mg}(\xi_{\text{can}})$, the Milnor

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binding number $\mathrm{Mb}(\xi_{\mathrm{can}})$, and the Milnor norm $\mathrm{Mn}(\xi_{\mathrm{can}})$ of the canonical contact structure ξ_{can}. The well-known *Milnor number* corresponds to the first Betti number of the page of the Milnor open book in our context.

It follows by definition that $\mathrm{sg}(\xi_{\mathrm{can}}) \leq \mathrm{Mg}(\xi_{\mathrm{can}})$ and $\mathrm{sn}(\xi_{\mathrm{can}}) \leq \mathrm{Mn}(\xi_{\mathrm{can}})$, since the set of Milnor open books is a subset of all open books on the link of a surface singularity. (No such inequality exists in general, however, between $\mathrm{bn}(\xi_{\mathrm{can}})$ and $\mathrm{Mb}(\xi_{\mathrm{can}})$.) In Section 9, we show that for each positive integer k, there is a rational surface singularity whose canonical contact structure ξ_{can} satisfies

$$k \leq \mathrm{Mg}(\xi_{\mathrm{can}}) - \mathrm{sg}(\xi_{\mathrm{can}}) \text{ and } k \leq \mathrm{Mn}(\xi_{\mathrm{can}}) - \mathrm{sn}(\xi_{\mathrm{can}}).$$

An immediate consequence is the existence of links of surface singularities carrying open books that are not isomorphic to Milnor open books. As another consequence, we deduce that Milnor open books are neither norm- nor genus-minimizing, although our aim originally was to show that the support genus of a Milnor fillable contact structure is realized by a Milnor open book. We find this result interesting since there are other instances in geometric topology where the "complex representatives" are minimizers. Most notably, the link of a complex *plane* curve singularity bounds a smooth complex curve of genus equal to its Seifert genus.

The aforementioned examples are the canonical contact structures on links of some rational surface singularities which are also small Seifert fibered 3-manifolds. In Section 8, we identify the canonical contact structures on *all* such manifolds via their Legendrian surgery diagrams. A Legendrian surgery diagram is perhaps the most efficient way of describing a contact structure from a topological point of view, since it also allows one to calculate many invariants of the contact structure (for example, the Euler class of the underlying oriented plane field) by easily converting the diagram into a smooth handlebody diagram.

2. Open books and contact structures

A complete exposition of the correspondence between open books and contact structures can be found in the lecture notes of Etnyre [2006]. In this section, we recall some basic definitions.

Suppose that for an oriented link B in a closed and oriented 3-manifold Y, the complement $Y \setminus B$ fibers over the circle as $p\colon Y \setminus B \to S^1$ such that $p^{-1}(t) = \Sigma_t$ is the interior of a compact surface with $\partial\Sigma_t = B$ for all $t \in S^1$. Then (B, p) is called an *open book decomposition* (or just an *open book*) of Y. For each $t \in S^1$, the surface Σ_t is called a *page*, while B is referred to as the *binding* of the open book.

The monodromy of the fibration p is defined as the diffeomorphism of a fixed page which is given by the first return map of a flow that is transverse to the pages and meridional near the binding. The isotopy class of this diffeomorphism is independent of the chosen flow and is called the *monodromy* of the open book decomposition. In order to describe the monodromy of an open book explicitly, one usually writes it as a product of Dehn twists along some curves on the page. In this paper, we will denote a right-handed (respectively left-handed) Dehn twist along a curve α as α (respectively α^{-1}), for simplicity.

An open book can also be described as follows. First consider the mapping torus

$$\Sigma_\phi = \big([0,1] \times \Sigma\big)/\big((1,x) \sim (0,\phi(x))\big),$$

where Σ is a compact oriented surface with r boundary components and ϕ is an element of the mapping class group Γ_Σ of Σ. Since ϕ is the identity map on $\partial\Sigma$, the boundary $\partial\Sigma_\phi$ of the mapping torus Σ_ϕ can be canonically identified with r copies of $T^2 = S^1 \times S^1$, where the first S^1 factor is identified with $[0,1]/(0 \sim 1)$ and the second one comes from a component of $\partial\Sigma$. Now we glue in r copies of $D^2 \times S^1$ to cap off Σ_ϕ so that ∂D^2 is identified with $S^1 = [0,1]/(0 \sim 1)$ and the S^1 factor in $D^2 \times S^1$ is identified with a boundary component of $\partial\Sigma$. Thus we get a closed 3-manifold $Y = \Sigma_\phi \cup_r D^2 \times S^1$ equipped with an open book decomposition whose binding is the union of the core circles of the copies of $D^2 \times S^1$ that we glue to Σ_ϕ to obtain Y. In conclusion, an element $\phi \in \Gamma_\Sigma$ determines a 3-manifold together with an *abstract* open book decomposition on it. By conjugating the monodromy ϕ of an open book on a 3-manifold Y by an element in Γ_Σ, we get an isomorphic open book on a 3-manifold Y' which is diffeomorphic to Y.

It has been known for a long time that every closed and oriented 3-manifold admits an open book decomposition. A new interest in open books on 3-manifolds arose recently from their connection to contact structures, which we will describe very briefly.

Recall that a (positive) contact structure ξ on an oriented 3-manifold is locally the kernel of a 1-form α such that $\alpha \wedge d\alpha > 0$. In this paper we assume ξ is coorientable, that is, α is a global 1-form.

Definition 2.1. An open book decomposition (B, p) of a 3-manifold Y is said to *support* a contact structure ξ on Y if ξ can be represented by a contact form α such that α evaluates positively on B and $d\alpha$ is a symplectic form on every page.

Thurston and Winkelnkemper [1975] associated a contact structure to every open book. This contact structure is in fact supported by the underlying open book. (Definition 2.1 was not available at the time.) To state the converse we need a little digression.

Suppose that an open book decomposition with page Σ is specified by $\phi \in \Gamma_\Sigma$. Attach a 1-handle to the surface Σ connecting two points on $\partial\Sigma$ to obtain a new surface Σ'. Let γ be a closed curve in Σ' going over the new 1-handle exactly once. Define a new open book decomposition with

$$\phi' = \phi \circ t_\gamma \in \Gamma_{\Sigma'},$$

where t_γ denotes the right-handed Dehn twist along γ. The resulting open book decomposition is called a *positive stabilization* of the one defined by ϕ. Although the resulting monodromy depends on the chosen curve γ, the 3-manifold specified by (Σ', ϕ') is diffeomorphic to the 3-manifold specified by (Σ, ϕ). A converse to Thurston and Winkelnkemper's result is:

Theorem 2.2 [Giroux 2002]. *Every contact structure on a* 3*-manifold is supported by an open book. Two open books supporting the same contact structure admit a common positive stabilization. Moreover two contact structures supported by the same open book are isotopic.*

3. Legendrian surgery diagrams

Recall that a knot in a contact 3-manifold is called Legendrian if it is everywhere tangent to the contact planes. In order to have a better understanding of the topological constructions in the later sections, we discuss a standard way to visualize Legendrian knots in $\mathbb{R}^3$ (and therefore S^3) equipped with the standard contact structure $\xi_{\text{st}} = \ker(dz + x\,dy)$.

Now consider a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{\text{st}})$ and take its *front* projection, that is, its projection to the yz-plane. This projection has no vertical tangencies since $-dz/dy = x \neq \infty$, and for the same reason, at a crossing the strand with smaller slope is in front. It turns out that L can be C^2-approximated by a Legendrian knot for which the projection has only transverse double points and *cusp* singularities (see [Geiges 2008], for example). Conversely, a knot projection with these properties gives rise to a unique Legendrian knot in $(\mathbb{R}^3, \xi_{\text{st}})$ by defining x from the projection as $-dz/dy$. Since any projection can be isotoped to satisfy the above properties, every knot in S^3 can be isotoped (nonuniquely) to a Legendrian knot.

Two classical invariants of a Legendrian knot L are the Thurston–Bennequin number $\text{tb}(L)$ and the rotation number $\text{rot}(L)$. Recall that $\text{tb}(L)$ is the contact framing of L (measured with respect to the Seifert framing in S^3), which can be easily computed from a front projection of L. Define the *writhe* $w(L)$ of L as the sum of signs of the double points. For this to make sense, we need to fix an orientation on the knot, but the result is independent of this choice. If $c(L)$ is the number of cusps, then $\text{tb}(L) = w(L) - \frac{1}{2}c(L)$.

The rotation number $\text{rot}(L)$ is defined by trivializing ξ_{st} along a Seifert surface Σ whose oriented boundary is L and then taking the winding number of TL with respect to this trivialization. For this invariant to make sense we need to orient L, and the result changes sign by reversing the orientation. Since $H^2(S^3; \mathbb{Z}) = 0$, this number is independent of the chosen trivialization. If $c_d(L)$ and $c_u(L)$ denote the number of down and up cusps in the projection, then $\mathrm{rot}(L) = \frac{1}{2}(c_d(L) - c_u(L))$.

To describe the Stein fillable contact structures dealt with in this paper, we use Legendrian knots (actually their front projections) as follows: Consider the standard Stein 4-ball B^4 with the induced standard contact structure on its boundary. Then attach Weinstein 2-handles [1991] along an arbitrary Legendrian link in $\partial B^4 = S^3$ to this ball. By [Eliashberg 1990], the Stein structure on B^4 extends over the 2-handles as long as the attaching framing of each 2-handle is one less than the Thurston–Bennequin number. The resulting Stein domain has an induced contact structure on its boundary which can be represented by the front projection of the Legendrian link along which we attach the 2-handles. Such a front projection is called a *Legendrian surgery diagram* (see [Gompf 1998] for a thorough discussion). Legendrian surgery is equivalent to performing contact (-1)-surgery along the given Legendrian link in the standard contact S^3 [Ding and Geiges 2004]. To describe all Stein fillable contact structures in general, one needs 1-handles as well, but those will not appear in our discussion.

4. Milnor open books and canonical contact structures

Let (X, x) be an isolated normal complex surface singularity (see [Némethi 1999]). Fix a local embedding of (X, x) in $(\mathbb{C}^N, 0)$. Then a small sphere $S_\epsilon^{2N-1} \subset \mathbb{C}^N$ centered at the origin intersects X transversely, and the complex hyperplane distribution ξ_{can} on $M = X \cap S_\epsilon^{2N-1}$ induced by the complex structure on X is called the *canonical* contact structure. It is known that, for sufficiently small radius ϵ, the contact manifold is independent of ϵ and the embedding, up to isomorphism. The 3-manifold M is called the link of the singularity and (M, ξ_{can}) is called the *contact boundary* of (X, x). While Y denotes a general 3-manifold, we use M for those which are Milnor-filled.

Definition 4.1. A contact manifold (Y, ξ) is said to be *Milnor fillable* and the germ (X, x) is called a *Milnor filling* of (Y, ξ) if (Y, ξ) is isomorphic to the contact boundary (M, ξ_{can}) of some isolated complex surface singularity (X, x). In addition, we say that a closed and oriented 3-manifold Y is Milnor fillable if it carries a contact structure ξ so that (Y, ξ) is Milnor fillable. Such a contact structure ξ is called a Milnor fillable contact structure.

By a theorem in [Mumford 1961], if a contact 3-manifold is Milnor fillable, then it can be obtained by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix. Conversely, it follows from a well-known theorem of [Grauert 1962] that any 3-manifold that is given by plumbing oriented circle bundles over surfaces according to a weighted graph with negative definite intersection matrix is Milnor fillable. As for the uniqueness of Milnor fillable contact structures, we have the fundamental result:

Theorem 4.2 [Caubel et al. 2006]. *Any closed and oriented* 3*-manifold has at most one Milnor fillable contact structure up to* isomorphism.

In summary, Milnor fillability of a closed and oriented 3-manifold Y is determined entirely by its topology and if Y is Milnor fillable, then it carries a canonical contact structure ξ_{can} which is unique up to isomorphism.

Since the groundbreaking result (Theorem 2.2) of [Giroux 2002], the geometry of contact structures is often studied via their topological counterparts, namely open book decompositions. In the realm of surface singularities this fits nicely with [Milnor 1968].

Definition 4.3. Given an analytic function $f\colon (X, x) \to (\mathbb{C}, 0)$ vanishing at x, with an isolated singularity at x, the open book decomposition $\mathcal{OB}_f$ of the link M of (X, x) with binding $L = M \cap f^{-1}(0)$ and projection $\pi = f/|f|\colon M \setminus L \to S^1 \subset \mathbb{C}$ is called the *Milnor open book* induced by f.

Such functions f exist and one can talk about many Milnor open books on the singularity link M. Therefore, there are many Milnor open books on any given Milnor fillable contact 3-manifold (Y, ξ), since, by definition, it is isomorphic to the link (M, ξ_{can}) of some isolated complex surface singularity (X, x).

A Milnor open book on a Milnor fillable 3-manifold Y has two essential features as shown in [Caubel et al. 2006]:

(i) It supports the canonical contact structure ξ_{can}.

(ii) It is horizontal when considered on the plumbing description of Y.

Suppose that the 3-manifold Y is obtained by plumbing oriented circle bundles $M_i \to S_i$, for $i = 1, 2, \ldots, r$. For any r-tuple of nonnegative integers $\underline{n} = (n_1, n_2, \ldots, n_r)$, a vertical link of type $\underline{n}$ consists of a disjoint union of n_i generic fibers from each bundle $M_i \to S_i$. An open book on Y is called *horizontal* if its binding is a vertical link and its pages are transverse to the fibers. We also require that the orientation induced on the binding by the pages coincides with the orientation of the fibers induced by the fibration.

5. Rational surface singularities

Let (X, x) be a germ of a normal complex surface having a singularity at x. Recall that (X, x) is called rational [Artin 1966] if the geometric genus $p_g := \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is equal to zero, where $\widetilde{X} \to X$ is a resolution of the singular point $x \in X$. This definition does not depend on the resolution.

Now fix a resolution $\pi : \widetilde{X} \to X$ and denote the irreducible components of the exceptional divisor $E = \pi^{-1}(x)$ by $\bigcup_{i=1}^r E_i$. The *fundamental cycle* of E is by definition the componentwise smallest nonzero effective divisor $Z = \sum z_i E_i$ satisfying $Z \cdot E_i \leq 0$ for all $1 \leq i \leq r$.

The singularity (X, x) is rational if each irreducible component E_i of the exceptional divisor E is isomorphic to $\mathbb{C}P^1$ and

$$Z \cdot Z + \sum_{i=1}^{r} z_i(-E_i^2 - 2) = -2,$$

where $Z = \sum z_i E_i$ is the fundamental cycle of E. Once a dual resolution graph of a surface singularity is given, then the so-called Laufer algorithm [1972] can be applied to calculate the corresponding fundamental cycle. Therefore the criterion in the equation above makes it particularly simple to identify the given singularity as rational.

Suppose that (X, x) is a germ of a normal complex surface having a rational singularity at x.

Theorem 5.1 [Altınok and Bhupal 2008]. *Both the page-genus and the page-genus plus the number of binding components of the Milnor open book $\mathcal{OB}_f$ are minimized when f is taken to be the restriction of a generic linear form on $\mathbb{C}^N$ to (X, x) for some local embedding of (X, x) in $(\mathbb{C}^N, 0)$.*

If $\mathcal{OB}_{\min}$ denotes the Milnor open book given by taking the restriction of a generic linear form on $\mathbb{C}^N$ to (X, x) for some local embedding of (X, x) in $(\mathbb{C}^N, 0)$, then Theorem 5.1 implies that $\mathrm{Mg}(\xi_{\mathrm{can}}) = g(\mathcal{OB}_{\min})$ and $\mathrm{Mb}(\xi_{\mathrm{can}}) = \mathrm{bc}(\mathcal{OB}_{\min})$, where $g(\mathcal{OB})$ (respectively $\mathrm{bc}(\mathcal{OB})$) denotes the page-genus (respectively the number of binding components) of the open book $\mathcal{OB}$. We will call $\mathcal{OB}_{\min}$ the *minimal* Milnor open book. For the Milnor norm, from the definition,

$$\mathrm{Mn}(\xi_{\mathrm{can}}) = \min\bigl(2g(\mathcal{OB}) - 2 + \mathrm{bc}(\mathcal{OB})\bigr),$$

where the minimum is taken over all supporting Milnor open books $\mathcal{OB}$. Hence it also follows from Theorem 5.1 that

$$\mathrm{Mn}(\xi_{\mathrm{can}}) = 2g(\mathcal{OB}_{\min}) - 2 + \mathrm{bc}(\mathcal{OB}_{\min}) = 2\,\mathrm{Mg}(\xi_{\mathrm{can}}) - 2 + \mathrm{Mb}(\xi_{\mathrm{can}}).$$

Remark 5.2. The equation $\mathrm{sn}(\xi) = 2\,\mathrm{sg}(\xi) - 2 + \mathrm{bn}(\xi)$ is not necessarily true for an arbitrary contact structure ξ, as illustrated in [Baldwin and Etnyre 2011; Etgü and Lekili 2010].

Suppose that $\pi : \widetilde{X} \to X$ is a good resolution of (X, x) and let $E_1, \ldots, E_r$ denote the irreducible components of the exceptional divisor E. Given an analytic function $f : (X, x) \to (\mathbb{C}, 0)$ vanishing at x, with an isolated singularity at x, the open book decomposition $\mathcal{OB}_f$ is a horizontal open book with binding a vertical link of type $\underline{n} = (n_1, \ldots, n_r)$, where the n_i are defined as follows: Consider the decomposition $(f \circ \pi) = (f \circ \pi)_e + (f \circ \pi)_s$ of the divisor $(f \circ \pi) \in \mathrm{Div}(\widetilde{X})$ into its exceptional and strict parts such that $(f \circ \pi)_e$ is supported on E and $\dim(|(f \circ \pi)_s \cap E|) < 1$. Then n_i is the number of components of $(f \circ \pi)_s$ which cut E_i. It is known that the r-tuple $\underline{n} = (n_1, \ldots, n_r)$ satisfies

$$I(\Gamma(\pi))\underline{m}^t = -\underline{n}^t \tag{5-1}$$

for some r-tuple $\underline{m} = (m_1, \ldots, m_r)$ of positive integers, where $I(\Gamma(\pi))$ denotes the intersection matrix of the dual resolution graph $\Gamma(\pi)$ associated to π and t is used for transpose. In [Altınok and Bhupal 2008, Lemma 3.1] it is proved that

$$g(\mathcal{OB}_f) = 1 + \sum_{i=1}^{r} \frac{(v_i - 2)m_i + (m_i - 1)n_i}{2}, \tag{5-2}$$

where v_i denotes the number of irreducible curves E_j, $j \neq i$, in E intersecting E_i for $i = 1, \ldots, r$. Also

$$\mathrm{bc}(\mathcal{OB}_f) = \sum_{i=1}^{r} n_i.$$

On the other hand, it follows from [Artin 1966] that for any r-tuple $\underline{n}$ of nonnegative integers which satisfies (5-1) for some r-tuple $\underline{m}$ of positive integers there is a Milnor open book decomposition of the boundary of (X, x) whose binding is equivalent to a vertical link of type $\underline{n}$.

The upshot is that if $Z = \sum_{i=1}^{r} z_i E_i$ is the fundamental cycle of the resolution π, then the above construction for the r-tuple $\underline{m} = (z_1, \ldots, z_r)$ gives the minimal Milnor open book $\mathcal{OB}_{\min}$, which we will also denote by $\mathcal{OB}(\underline{m})$.

Remark 5.3. Némethi and Tosun [2011] (see also [Némethi 2008]) give a generalization of Theorem 5.1 for all Milnor fillable rational homology 3-spheres and prove

$$\mathrm{Mg}(\xi_{\mathrm{can}}) = Z \cdot E - Z \cdot Z \quad \text{and} \quad \mathrm{Mb}(\xi_{\mathrm{can}}) = -Z \cdot E.$$

6. Tight contact structures on small Seifert fibered 3-manifolds

A *small* Seifert fibered 3-manifold Y is a closed and oriented 3-manifold which admits a Seifert fibration over S^2 with at most three singular fibers. Equivalently, such a manifold $Y = Y(e_0; r_1, r_2, r_3)$ can be described by the rational surgery diagram depicted in Figure 1, where $e_0 \in \mathbb{Z}$ and $r_i \in (0, 1) \cap \mathbb{Q}$, for $i = 1, 2, 3$.

One can also obtain an integral surgery description of Y as follows. Consider the continued fraction expansion of $-1/r_i$:

$$-\frac{1}{r_i} = a_1^{(i)} - \cfrac{1}{a_2^{(i)} - \cfrac{1}{\ddots - \cfrac{1}{a_{n_i}^{(i)}}}}, \qquad i = 1, 2, 3,$$

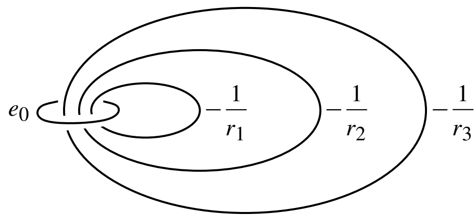

Figure 1. Rational surgery diagram for the small Seifert fibered 3-manifold $Y(e_0; r_1, r_2, r_3)$.

for some uniquely determined integers $a_1^{(i)}, \dots, a_{n_i}^{(i)} \le -2$. Let $K_0^{(i)}$ denote the unknot with framing e_0 in Figure 1, for $i = 1, 2, 3$. Now replace the unknot with coefficient $-1/r_i$, by a chain of unknots $K_1^{(i)}, \dots, K_{n_i}^{(i)}$ with integral framings $a_1^{(i)}, \dots, a_{n_i}^{(i)}$, respectively, so that

$$\mathrm{lk}(K_j^{(i)}, K_k^{(i)}) = \begin{cases} \pm 1 & \text{if } |j-k| = 1, \\ 0 & \text{otherwise}, \end{cases}$$

for $0 \le j, k \le n_i$ and $i = 1, 2, 3$. As discussed in the next paragraph, this "star-shaped" integral surgery presentation is very convenient in terms of describing Stein fillable contact structures on Y.

Wu [2006] classified all tight contact structures on Y up to isotopy under the assumption that $e_0 \le -3$. They are all Stein fillable and can be represented by Legendrian surgery diagrams which are obtained by all possible Legendrian realizations (without double points) of the unknots imposed by the surgery coefficients in the integral surgery description of Y. Moreover, Ghiggini [2008] showed that the same classification scheme works for the case $e_0 = -2$, as long as Y is assumed to be an L-space (a rational homology sphere whose Heegaard Floer homology is as simple as possible, that is, $\mathrm{rk}\,\widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$).

The link of any rational surface singularity is an L-space by a theorem in [Némethi 2005]. A necessary condition for the 3-manifold $Y = Y(e_0; r_1, r_2, r_3)$ to be the link of a rational singularity is that $e_0 \le -2$; however, it is not sufficient.

7. Planar Milnor open books

By a weighted plumbing graph we mean a graph such that each vertex is decorated by some integer "weight". Such a graph naturally represents a closed oriented 3-manifold called a graph manifold, which can be described as follows. For each vertex of the graph, take an oriented circle bundle over S^2 whose Euler number is equal to the weight of that vertex and plumb these bundles together according to the given graph. In other words, if there is an edge connecting two vertices in Γ, then plumb the circle bundles corresponding to these vertices. More precisely, first remove a neighborhood of a circle fiber on each circle bundle which is given by the preimage of a disk on the base sphere. The resulting boundary torus on each circle bundle can be identified with $S^1 \times S^1$ using the natural trivialization of the circle fibration over the disk that is removed. Now glue these bundles together using the diffeomorphism that exchanges the two circle factors on the boundary tori.

If the graph is a tree, in particular, then an integral surgery presentation of the 3-manifold is readily available by replacing each vertex by an unknot framed by the weight of that vertex such that any two of these unknots are linked once if there is an edge between the vertices they represent and they are unlinked otherwise.

Recall that the degree of a vertex in a graph is the number of edges emanating from that vertex. A vertex in a weighted plumbing graph is called a bad vertex if the sum of the weight (the Euler number) and the degree of that vertex is positive.

Proposition 7.1. *If Y is the link of a rational surface singularity presented by a plumbing tree without any bad vertices, then Y carries a planar Milnor open book.*

Proof. Let Γ be a plumbing tree for Y with r vertices $v_1, \dots, v_r$ so that v_i has Euler number e_i and degree d_i. Suppose that Γ has no bad vertices, that is, $e_i + d_i \leq 0$ for $i = 1, \dots, r$. Etgü and Ozbagci [2006] constructed an explicit planar horizontal open book $\mathcal{OB}$ with binding a vertical link of type

$$\underline{n} = (-e_1 - d_1, -e_2 - d_2, \dots, -e_r - d_r)$$

on such a graph manifold Y. It is easy to check that for $\underline{m} = (1, 1, \dots, 1)$, we have

$$I(\Gamma)\underline{m}^t = -\underline{n}^t,$$

where $I(\Gamma)$ denotes the intersection matrix of the tree Γ which defines Y. Suppose that Y is the link of some rational surface singularity (X, x). Then $\underline{m}$ corresponds to the fundamental cycle of the minimal resolution of (X, x). As we indicated in the paragraph preceding Remark 5.3, the binding of the minimal Milnor open book $\mathcal{OB}_{\min}$ is a vertical link of type $\underline{n}$ as well. Since the open books $\mathcal{OB}$ and $\mathcal{OB}_{\min}$ on the rational homology 3-sphere Y have equivalent bindings, by [Caubel and Popescu-Pampu 2004] it follows that $\mathcal{OB}$ is isotopic to $\mathcal{OB}_{\min}$. This proves that Y carries a planar Milnor open book, that is, $\mathrm{Mg}(\xi_{\mathrm{can}}) = 0$. Moreover, since the binding of $\mathcal{OB}$ is a vertical link of type $\underline{n}$, we have $\mathrm{Mb}(\xi_{\mathrm{can}}) = -\sum_{i=1}^{r}(e_i + d_i)$ and hence $\mathrm{Mn}(\xi_{\mathrm{can}}) = \mathrm{Mb}(\xi_{\mathrm{can}}) - 2$. $\square$

In the proof of Proposition 7.1 we showed that the horizontal open book $\mathcal{OB}$ constructed in [Etgü and Ozbagci 2006] is isotopic to a Milnor open book, which implies that the horizontal contact structure supported by $\mathcal{OB}$ is isomorphic to ξ_{can} on such rational singularity links. On the other hand, Legendrian surgery diagrams of such horizontal contact structures (which are known to be Stein fillable) were studied in [Ozbagci 2008]. A lens space $L(p, q)$, for example, is given by a linear plumbing diagram without any bad vertices. Equivalently, an integral surgery diagram of $L(p,q)$ is given by a chain of unknots so that the linking number between every two consecutive unknots is equal to ± 1 and the framing of every unknot is less than or equal to -2.

Proposition 7.2 [Ozbagci 2008, Proposition 3.2]. *Orient the unknots in the linear integral surgery diagram giving $L(p,q)$ so that the linking number is $+1$ between any two consecutive unknots. A Legendrian surgery diagram for ξ_{can} on $L(p,q)$ is obtained by Legendrian realizing each unknot with maximum possible rotation number imposed by its surgery coefficient.*

A Legendrian realization of an unknot with maximum possible rotation number imposed by its surgery coefficient is given by a front projection without any double points and with a single up cusp or a single down cusp, depending on the chosen orientation. Once we orient any knot in the chain describing $L(p,q)$, the orientations of the other knots are determined uniquely, by the hypothesis in Proposition 7.2. Hence there are two choices of overall orientations inducing two Legendrian surgery diagrams for ξ_{can} which are mirror images of each other. In other words, ξ_{can} is represented by a Legendrian surgery diagram where all the zigzags of all the Legendrian unknots are on the left or all on the right (see [Ozbagci 2008, Figure 4]). Nevertheless, these two diagrams induce isomorphic contact structures, where the underlying plane fields are obtained from each other by simply reversing the orientations.

There are two key properties used to prove Proposition 7.2: (i) the linear plumbing diagram of $L(p,q)$ does not have any bad vertices and (ii) all the tight contact structures on $L(p,q)$ are Stein fillable and given by all possible Legendrian realizations (without double points) of the unknots in the plumbing diagram. In Section 6, we described a star-shaped plumbing diagram of a small Seifert fibered 3-manifold $Y = Y(e_0; r_1, r_2, r_3)$ which does not include any bad vertices as long as $e_0 \leq -3$. Therefore it is straightforward to generalize Proposition 7.2 to all rational singularity links which are small Seifert fibered spaces with $e_0 \leq -3$, using the methods in [Ozbagci 2008] coupled with Wu's classification [2006] of tight contact structures on such manifolds. This generalization is included in the statement of Theorem 8.1 for which we present a more conceptual proof.

8. Legendrian surgery diagrams for canonical contact structures

Theorem 8.1. *Let $Y = Y(e_0; r_1, r_2, r_3)$ be a small Seifert fibered 3-manifold that is diffeomorphic to the link of some rational surface singularity. Orient the unknots in the star-shaped integral surgery diagram giving Y, so that the linking number is $+1$ between any two consecutive unknots in every chain. A Legendrian surgery diagram for ξ_{can} on Y is obtained by Legendrian realizing each unknot with maximum possible rotation number imposed by its surgery coefficient.*

Proof. Suppose that (Y, ξ_{can}) is diffeomorphic to the link of the rational surface singularity (X, x). Then the minimal resolution $\pi : \widetilde{X} \to X$ provides a holomorphic filling (W, J) of (Y, ξ_{can}). In particular, W is a regular neighborhood of the exceptional divisor $E = \bigcup E_j$ of π. Since the curves E_j are holomorphic, by the adjunction formula, we have

$$\langle c_1(J), [E_j]\rangle = E_j \cdot E_j - 2\,\mathrm{genus}(E_j) + 2 = E_j \cdot E_j + 2.$$

Recall that in Section 6 we discussed the classification of tight contact structures on a small fibered 3-manifold Y under the assumption that Y is an L-space for the case $e_0 = -2$, which is satisfied for a rational singularity link. For each such 3-manifold Y, there are finitely many tight contact structures $\xi_1, \dots, \xi_m$, all of which are Stein fillable. Moreover, for $i = 1, \dots, m$, a Stein filling (W^i, J^i) of ξ_i is given by taking a Legendrian surgery diagram, obtained from the plumbing tree describing Y, with the zigzags chosen in a certain way. Denote by U^i_j a component of the corresponding Legendrian link and let S^i_j denote the associated surface in the Stein filling (W^i, J^i) obtained by pushing a Seifert surface for U^i_j into the 4-ball and capping off by the core of the corresponding 2-handle (see [Gompf 1998]). Each W^i is diffeomorphic to W by a diffeomorphism which carries S^i_j to E_j for each j.

Now, using the well-known identities

$$S^i_j \cdot S^i_j = \mathrm{tb}(U^i_j) - 1, \quad \langle c_1(J^i), [S^i_j]\rangle = \mathrm{rot}(U^i_j)$$

(see [Gompf 1998] for the second), observe that $\langle c_1(J^i), [S^i_j]\rangle = S^i_j \cdot S^i_j + 2$ precisely when $\mathrm{rot}(U^i_j) = \mathrm{tb}(U^i_j) + 1$. Since the latter equality holds exactly when all the cusps of U^i_j except one are up cusps, it follows that $\langle c_1(J), [E_j]\rangle = \langle c_1(J^i), [S^i_j]\rangle$ for each j precisely when all the extra zigzags are chosen so the additional cusps are all up cusps, that is, when all the extra zigzags are chosen on the same fixed side (which is determined by the orientation of the Legendrian unknots). In the finite list of tight contact structures on Y there is only one such Stein fillable contact structure up to isomorphism [Wu 2006; Ghiggini 2008], which completes the proof. □

9. Milnor versus support genus

In this section, we describe an infinite family of Milnor fillable contact 3-manifolds whose canonical contact structure has support genus (respectively norm) *strictly* less than its Milnor genus (respectively norm).

Consider the small Seifert fibered 3-manifold

$$Y_p = Y\Big(-2; \frac{1}{3}, \frac{2}{3}, \frac{p}{p+1}\Big),$$

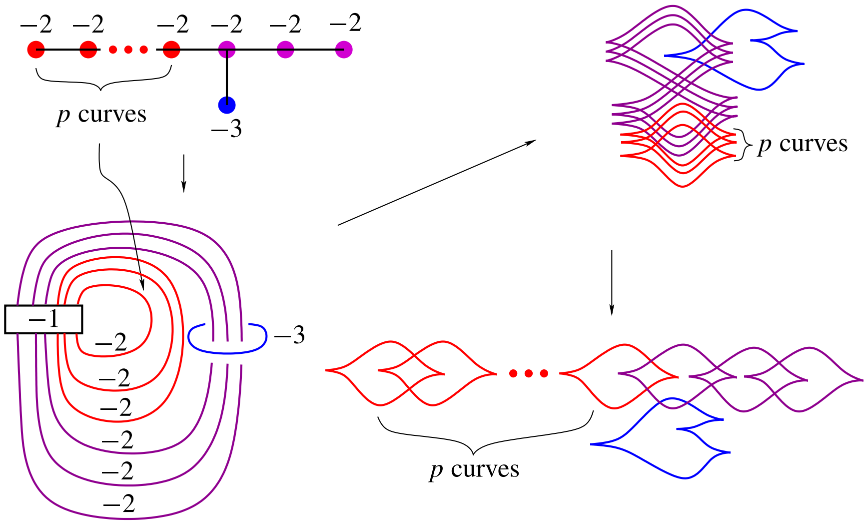

Figure 2. The contact structure $\xi \cong \xi_1 \cong \xi_{\text{can}}$ on Y_p.

for $p \geq 2$. Observe that Y_p, whose dual resolution graph Γ_p is shown in Figure 2, is the link of a rational complex surface singularity. By the classification of the tight contact structures on Y_p given in [Ghiggini 2008], there are exactly two non-isotopic tight contact structures ξ_1 and ξ_2 on Y_p, both of which are Stein fillable. A Legendrian surgery diagram of ξ_1 is depicted on the bottom right in Figure 2. By putting the extra zigzag on the blue curve on the opposite side, we get a diagram for ξ_2.

Proposition 9.1. *For* $i = 1, 2$, *we have* $\operatorname{sg}(\xi_i) \leq 1$ *and* $\operatorname{sn}(\xi_i) = 2$.

Proof. We first construct a supporting elliptic (that is, genus one) open book with two binding components for some Stein fillable contact structure ξ on Y_p following the recipe in [Etnyre and Ozbagci 2006]. Start from the plumbing diagram on the top left in Figure 2 (which is equivalent to a smooth surgery diagram including only unknots linked according to the given tree) and "roll up" this diagram by appropriately sliding handles to obtain the surgery diagram of Y_p on the bottom left. Next, Legendrian realize the given surgery curves as depicted on the top right in Figure 2 to obtain the Legendrian surgery diagram for some Stein fillable contact structure ξ on Y_p, which is isomorphic to ξ_1 depicted on the bottom right. Refer to [Etnyre and Ozbagci 2006] for the justification of such statements.

In order to construct an open book of Y_p supporting ξ, start from an open book of S^3 and then embed the surgery curves onto the pages as depicted on the left in Figure 3 (the colors make it easier to follow how the surgery curves are embedded on the page). The initial page is a torus with one boundary component, and the

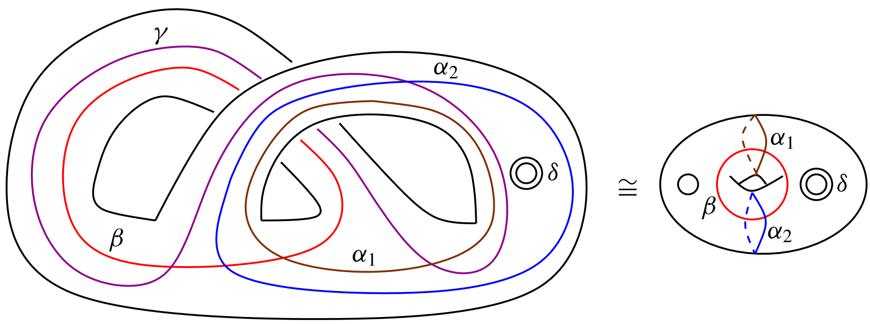

Figure 3. The page of an open book compatible with ξ.

monodromy of this open book of S^3 before the surgery is given by $\beta\alpha_1$, where α_1 and β generate the first homology group of the page. Now apply Legendrian surgeries along the given curves to get an open book of Y_p with monodromy $\phi_p = \alpha_2\gamma^3\beta^p\beta\alpha_1\delta$, where δ is parallel to the small puncture on the torus, which occurs as a result of stabilizing the page appropriately. Next move β over γ^3 to the left and use the fact that $\gamma\beta = \beta\alpha_1$ to get $\phi_p = \alpha_2\beta\alpha_1^3\beta^p\alpha_1\delta$. Then use the well-known braid relations and some simple overall conjugations to obtain a more symmetrical presentation of the monodromy as

$$\phi_p = (\alpha_2\beta)^2(\alpha_1\beta)^2\beta^{p-2}\delta.$$

This describes an *abstract* open book which is compatible with ξ, where the page is a torus with two boundary components and monodromy is ϕ_p. Note that ξ is isomorphic to ξ_1 (which is isomorphic to ξ_2, since one cannot distinguish the abstract open books corresponding to ξ_1 and ξ_2). It follows that $\mathrm{sg}(\xi_i) \leq 1$, since we have already constructed a genus one open book compatible with ξ.

Using the handlebody diagram of Y_p depicted on the bottom left in Figure 2, it is straightforward to calculate that

$$H_1(Y_p, \mathbb{Z}) = \begin{cases} \mathbb{Z}_3 \oplus \mathbb{Z}_3 & p = 2 \bmod 3, \\ \mathbb{Z}_9 & \text{otherwise.} \end{cases}$$

Moreover, one can show that the Poincaré dual $\mathrm{PD}(e(\xi_i)) \in H_1(Y_p, \mathbb{Z})$ of the Euler class $e(\xi_i)$ is a generator of one of the $\mathbb{Z}_3$-factors when p is congruent to 2 mod 3. Similarly $\mathrm{PD}(e(\xi_i))$ is a generator of $H_1(Y_p, \mathbb{Z})$ when p is not congruent to 2 mod 3. Therefore the contact structure ξ_i cannot be compatible with an elliptic open book with connected binding by [Etnyre and Ozbagci 2008, Lemma 6.1], since $e(\xi_i) \neq 0$. Note that $e(\xi_1) = -e(\xi_2)$, which implies that ξ_1 is not homotopic to ξ_2 as oriented plane fields, although they are isomorphic to each other. In fact, ξ_2 is obtained from ξ_1 by reversing the orientation of the underlying plane field.

Now we claim that $\mathrm{sn}(\xi_i) = 2$. To prove this, we need to exclude the possibility that ξ_i is compatible with a planar open book with less than four binding components. Suppose that ξ_i is compatible with a planar open book, that is, $\mathrm{sg}(\xi_i) = 0$. If $\mathrm{bn}(\xi_i) \leq 2$, then ξ_i is the unique tight contact structure on the lens space $L(n, n-1)$ for some $n \geq 0$ (see [Etnyre and Ozbagci 2008]) which is indeed impossible since Y_p is not a lens space.

Next we rule out the possibility that $\mathrm{bn}(\xi_i) = 3$. Let Σ be the planar surface with three boundary components. Any diffeomorphism of Σ is determined by three numbers q, r, s that give the number of Dehn twists on curves τ_1, τ_2, τ_3 parallel to each boundary component. It is easy to see that the 3-manifold determined by the open book with page Σ and monodromy given by $\tau_1^q \tau_2^r \tau_3^s$ is the Seifert fibered 3-manifold $Y(0, -\frac{1}{q}, -\frac{1}{r}, -\frac{1}{s})$. The first homology group of $Y(0, -\frac{1}{q}, -\frac{1}{r}, -\frac{1}{s})$ has order $qr + qs + rs$.

Suppose that ξ_i is compatible with an open book with page Σ and monodromy $\tau_1^q \tau_2^r \tau_3^s$. The tightness of ξ_i implies that the integers q, r, and s are all nonnegative, because otherwise $\tau_1^q \tau_2^r \tau_3^s$ is not right-veering [Honda et al. 2007]. Moreover, since the order of the first homology group of Y_p is 9, for all $p \geq 2$, we conclude that (q, r, s) is equal to either $(0, 1, 9)$, $(0, 3, 3)$ or $(1, 1, 4)$. Hence Y_p is diffeomorphic to either $L(9, 8)$, $L(3, 2) \# L(3, 2)$ or $L(9, 4)$, which is a contradiction. Hence, $\mathrm{bn}(\xi_i) \geq 4$. This finishes the proof of our claim that $\mathrm{sn}(\xi_i) = 2$. □

One can ask whether or not $\mathrm{sg}(\xi_i) = 1$, although it is not essential for the purposes of this paper. There are two known methods for finding obstructions to the planarity of a contact structure; see [Etnyre 2004; Ozsváth et al. 2005]. Unfortunately, both fail in our case, because Y_p is an L-space and it is not an integral homology sphere.

Proposition 9.2. *For the canonical contact structure ξ_{can} on the rational singularity link Y_p, we have* $\mathrm{Mg}(\xi_{\mathrm{can}}) = 2$ *and* $\mathrm{Mn}(\xi_{\mathrm{can}}) = 3$.

Proof. Enumerate the vertices of the plumbing graph of Y_p depicted in Figure 2 from left to right along the top row with the bottom vertex coming last. It is then easy to check that the $(p+4)$-tuple of positive integers $\underline{m}$ corresponding to the fundamental cycle of the minimal resolution of the singularity of which Y_p is the link is given by $\underline{m} = (1, 2, 3, 3, \ldots, 3, 3, 2, 1, 1)$. The construction in [Bhupal 2009] now gives an open book decomposition $\mathcal{OB}(\underline{m}) = \mathcal{OB}_{\min}$ of Y_p with binding a vertical link of type $\underline{n} = (0, 0, 1, 0, \ldots, 0)$, where $\underline{m}$ and $\underline{n}$ are related by

$$I(\Gamma_p)\underline{m}^t = -\underline{n}^t.$$

Using formula (5-2) with $r = p + 4$ gives $\mathrm{Mg}(\xi_{\mathrm{can}}) = g(\mathcal{OB}(\underline{m})) = 2$ for the canonical contact structure ξ_{can} on Y_p. Also, $\mathrm{Mb}(\xi_{\mathrm{can}}) = \mathrm{bc}(\mathcal{OB}(\underline{m})) = \sum_{i=1}^{p+4} n_i = 1$ and therefore $\mathrm{Mn}(\xi_{\mathrm{can}}) = 3$. □

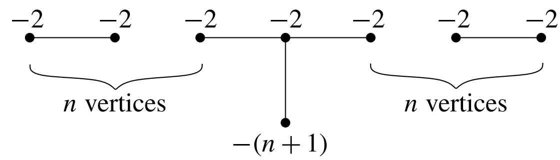

Figure 4. The plumbing graph for P_n.

Corollary 9.3. *For $p \geq 2$, we have $\mathrm{sg}(\xi_{\mathrm{can}}) < \mathrm{Mg}(\xi_{\mathrm{can}})$ and $\mathrm{sn}(\xi_{\mathrm{can}}) < \mathrm{Mn}(\xi_{\mathrm{can}})$ for the canonical contact structure ξ_{can} on the singularity link Y_p.*

Proof. Since any Milnor fillable contact structure is Stein fillable, ξ_{can} is isomorphic to ξ_i by Ghiggini's classification [2008]. (It does not make sense to distinguish ξ_1 and ξ_2 here since they are isomorphic to each other.) Thus Proposition 9.1 coupled with Proposition 9.2 clearly implies the corollary. □

Remark 9.4. In contrast, $\mathrm{Mb}(\xi_{\mathrm{can}}) = 1$ while $\mathrm{bn}(\xi_{\mathrm{can}}) \geq 2$, which shows that the binding number is not necessarily less than or equal to the Milnor binding number.

We can improve Corollary 9.3 as follows.

Theorem 9.5. *For each positive integer k, there exists a Milnor fillable contact 3-manifold such that $\mathrm{Mg}(\xi_{\mathrm{can}}) - \mathrm{sg}(\xi_{\mathrm{can}}) \geq k$ and $\mathrm{Mn}(\xi_{\mathrm{can}}) - \mathrm{sn}(\xi_{\mathrm{can}}) \geq k$.*

Proof. The small Seifert fibered 3-manifold $P_n = Y(-2; \frac{1}{n+1}, \frac{n}{n+1}, \frac{n}{n+1})$ for $n \geq 2$ is a rational singularity link, whose dual resolution graph is depicted in Figure 4. There are exactly n nonisotopic tight contact structures $\xi_1, \dots, \xi_n$ on P_n, each of which is Stein fillable [Ghiggini 2008]. On the other hand, ξ_i is supported by an elliptic open book with n binding components for $1 \leq i \leq n$ [Etnyre and Ozbagci 2006]. This proves that $\mathrm{sg}(\xi_{\mathrm{can}}) \leq 1$ and $\mathrm{sn}(\xi_{\mathrm{can}}) \leq n$ since the canonical contact structure ξ_{can} on P_n is isomorphic to ξ_i for some $1 \leq i \leq n$.

Now enumerate the vertices of the graph in Figure 4 from left to right along the top row with the bottom vertex coming last and consider the $(2n+2)$-tuple $\underline{m} = (1, 2, 3, \dots, n-1, n, n+1, n, n-1, \dots, 3, 2, 1, 1)$ of positive integers. This corresponds to the fundamental cycle of the minimal resolution of the singularity of which P_n is the link. It follows that $\mathrm{Mg}(\xi_{\mathrm{can}}) = n$ and $\mathrm{Mn}(\xi_{\mathrm{can}}) = 2n-1$. Taking $k = n-1$ now proves the theorem. □

10. Final remarks

The minimal Milnor open book $\mathcal{OB}_{\min}$ on $Y = Y(e_0; r_1, r_2, r_3)$ realizes $\mathrm{Mg}(\xi_{\mathrm{can}})$, $\mathrm{Mb}(\xi_{\mathrm{can}})$ and $\mathrm{Mn}(\xi_{\mathrm{can}})$. In fact, it follows from the proof of Theorem 5.1 given in [Altınok and Bhupal 2008] that $\mathcal{OB}_{\min}$ is the unique Milnor open book that realizes $\mathrm{Mg}(\xi_{\mathrm{can}})$, $\mathrm{Mb}(\xi_{\mathrm{can}})$ and $\mathrm{Mn}(\xi_{\mathrm{can}})$. Thus any other Milnor open book on Y that realizes $\mathrm{Mg}(\xi_{\mathrm{can}})$ cannot realize $\mathrm{Mb}(\xi_{\mathrm{can}})$ and $\mathrm{Mn}(\xi_{\mathrm{can}})$. For example, consider

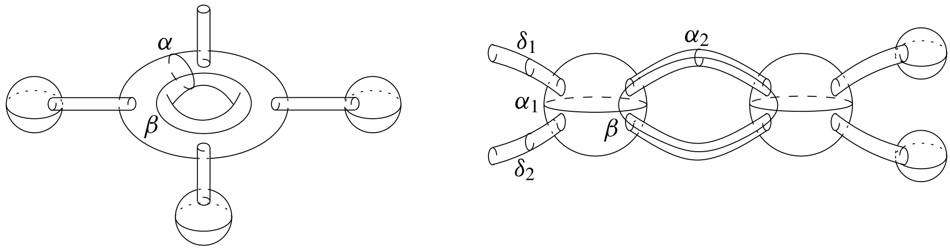

Figure 5. Pages of two different Milnor open books on $Y(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

the 3-manifold $Y = Y(-2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, which is the link of the singularity D_4. The pages of two Milnor open books on Y are given in Figure 5. The left pictures the minimal Milnor open book $\mathcal{OB}_{\min} = \mathcal{OB}((1, 2, 1, 1))$ with page a once-punctured torus and monodromy $\phi = (\alpha\beta)^3$; the right pictures the Milnor open book $\mathcal{OB} = \mathcal{OB}((2, 2, 1, 1))$ with page a twice-punctured torus and monodromy ψ satisfying $\psi^2 = \delta_1\delta_2\alpha_2^2$. Using the uniqueness result from [Bonatti and Paris 2009] and the two-holed torus relation, one can check that $\psi = \alpha_1\alpha_2\beta\alpha_2^2\beta\alpha_2$. It is easy to see that $\mathcal{OB}$ is related to $\mathcal{OB}_{\min}$ by a single positive stabilization.

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SIMPLE CLOSED CURVES, WORD LENGTH, AND NILPOTENT QUOTIENTS OF FREE GROUPS

KHALID BOU-RABEE AND ASAF HADARI

We consider the fundamental group π of a surface of finite type equipped with the infinite generating set consisting of all simple closed curves. We show that every nilpotent quotient of π has finite diameter with respect to the word metric given by this set. This is in contrast with a result of Danny Calegari that shows that π has infinite diameter with respect to this set. We also give a general criterion for a finitely generated group equipped with a generating set to have this property.

1. Introduction

A surface of finite type is a surface whose fundamental group is finitely generated. Given such a surface, there is no canonical choice of generating set. If one wishes to define a suitably canonical generating set of a geometric nature, then it becomes necessary to consider infinite generating sets. One such set is the set of all elements whose conjugacy class can be represented by a simple closed curve. These are in some sense the simplest elements of the fundamental group, and are thus a natural choice for a generating set.

Benson Farb posed the question whether the fundamental group, endowed with the word metric given by this set, has finite diameter. This question was answered negatively by Danny Calegari [2008]. In this paper, our goal is to investigate the same question for some quotients of the fundamental group. In contrast with Calegari's result, we find the following.

Theorem 1.1. *Let Σ be a surface of finite type, $\pi = \pi_1(\Sigma)$, and let $\mathcal{S} \subset \pi$ be any generating set containing at least one element in each conjugacy class that is represented by a nonseparating simple closed curve. Let $\rho : \pi \to N$ be a homomorphism into any nilpotent group. Then $\rho(\pi)$ has finite diameter in the word metric with respect to the set $\rho(\mathcal{S})$.*

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In surfaces of genus greater than 1, π has many nilpotent quotients of every degree of nilpotency. Furthermore, it is residually nilpotent; that is, for every $x \in \pi$, there is some nilpotent quotient $q : \pi \to N$ such that $q(x) \neq 1$.

We say that a group G is *nilpotent-bounded with respect to the set S* if any nilpotent quotient of G has finite diameter with respect to the word metric given by the image of S. As part of the proof, we prove the following more general result.

Theorem 1.2. *Let G be a finitely generated group, and let $S \subset G$ be a generating set such that $G/[G, G]$ has finite diameter with respect to the word metric given by S. Then G is nilpotent-bounded with respect to S.*

2. Nilpotent groups and lower central series

Given a group Γ, we define a decreasing sequence of subgroups of Γ called the *lower central series of* Γ by the following rule:

$$\Gamma_0 = \Gamma, \quad \Gamma_{n+1} = [\Gamma, \Gamma_n].$$

A group is nilpotent if $\Gamma_n = \langle 1 \rangle$ for some n. A group is called n-step nilpotent if $\Gamma_n = 1$ and $\Gamma_{n-1} \neq 1$. For every n, the group $L_n := \Gamma/\Gamma_n$ is a nilpotent group. These groups have the property that any nilpotent quotient of G factors through one of the projections $\Gamma \to L_n$.

Put $A_n := \Gamma_{n-1}/\Gamma_n$. It is a standard fact that $A_n < Z(L_n)$, the center of L_n. Also, if Γ is finitely generated, then A_n is also finitely generated. Given a generating set S of Γ, the group A_n is generated by the images of elements of the form $[a_1, \dots, a_n]$, where $a_1, \dots, a_n \in S$ and $[a_1, \dots, a_n]$ denotes a generalized commutator, that is,

$$[a_1, \dots, a_n] = [\dots[a_1, a_2], a_3], \dots, a_n].$$

In the course of the proof, we require the following technical lemma about generalized commutators in nilpotent groups.

Lemma 2.1. *Let Γ be any group, let $n, k \in \mathbb{N}$, and let $a_1, \dots, a_n \in \Gamma$. Then*

$$[a_1, \dots, a_n]^k \equiv_{n+1} ([a_1^k, \dots, a_n]),$$

where $\equiv_i$ is understood as having equal images in L_i.

Proof. First, recall that $A_n < Z(L_{n+1})$. Let $x \in \Gamma_{n-1}$ and $y \in \Gamma$. Note that $[x, y] \in \Gamma_n$. Thus we have that

$$[x^k, y] \equiv_{n+1} x^k y x^{-k} y^{-1} \equiv_{n+1} x^k y [x, y]^k y^{-1} x^{-k} \equiv_{n+1} [x, y]^k.$$

The last equality stems from the fact that $[x, y]^k$ is central in L_{n+1}, and thus is invariant under conjugation. This proves the claim for the case $n = 1$. We now proceed by induction.

By the case $n = 1$, we have that:

$$[a_1, \dots, a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}], a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}]^k, a_n].$$

By induction, we can write:

$$[a_1, \dots, a_{n-1}]^k \equiv_{n+1} [[a_1, \dots, a_{n-2}]^k, a_{n-1}]\gamma_n,$$

where $\gamma_n \in \Gamma_n$. Since the image of Γ_n is central in L_{n+1}, we have that

$$[[a_1, \dots, a_{n-1}]^k \gamma_n^{-1}, a_n] \equiv_{n+1} [a_1, \dots, a_{n-1}]^k, a_n].$$

Proceeding similarly, we get the claim of the lemma. □

3. Proof of the main theorems

Lemma 3.1. *Let $n \in \mathbb{N}$ and let $e_1, \dots, e_{2n}$ be the standard basis for $\mathbb{Z}^{2n}$. Then the set $\mathcal{S} = \mathrm{Sp}_{2n}(\mathbb{Z}) \cdot e_1$ generates $\mathbb{Z}^{2n}$ with finite diameter.*

Proof. We prove this fact first for $n = 1$. In this case, $\mathrm{Sp}_{2n}(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$. Given a vector $v = \binom{a}{b} \in \mathbb{Z}^2$ such that $\gcd(a, b) = 1$, there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. In this case,

$$A = \begin{pmatrix} a & -y \\ b & x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and $A \cdot e_1 = v$, and thus $v \in \mathcal{S}$. For a general vector $v = \binom{a}{b}$, notice that

$$v = \begin{pmatrix} a-1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ b-1 \end{pmatrix}$$

and that $\gcd(1, a-1) = \gcd(1, b-1) = 1$, and thus $v \in \mathcal{S} + \mathcal{S}$.

Now consider the case $n > 1$. In this case, we have that $D < \mathrm{Sp}_{2n}(\mathbb{Z})$, where $D \cong \prod_{i=1}^n \mathrm{SL}_2(\mathbb{Z})$ is the group of matrices containing n copies of $\mathrm{SL}_2(\mathbb{Z})$ along the diagonal and zeroes in all other entries. Also, $\hat{e} = e_1 + e_3 + \cdots + e_{2n-1}$ is in $\mathcal{S}$. Given $\binom{a_i}{b_i}_{i=1}^n \in \mathbb{Z}^{2n}$, by the case $n = 1$ there are $2n$ matrices $A_1, \dots, A_n, B_1, \dots B_n \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$A_i \cdot e_1 = \begin{pmatrix} a_i - 1 \\ 1 \end{pmatrix}, \quad B_i \cdot e_1 = \begin{pmatrix} 1 \\ b_i - 1 \end{pmatrix}.$$

Let $A = \mathrm{diag}(A_1, \dots, A_n)$ and $B = \mathrm{diag}(B_1, \dots, B_n)$. Then

$$v = A \cdot \hat{e} + B \cdot \hat{e}.$$

Thus $\mathbb{Z}^{2n}$ is generated by $\mathcal{S}$ with finite diameter. □

Lemma 3.2. *Let Γ be a finitely generated group, and let $n \in \mathbb{N}$. Suppose that $\mathcal{S} \subset \Gamma$ generates Γ and generates L_n with finite diameter. Then $\mathcal{S}$ generates L_{n+1} with finite diameter.*

Proof. By assumption, there exists an N_0 such that for any $w \in \Gamma$, there exist $s_1, \dots s_m \in \mathcal{S}$ (with $m < N_0$) such that

$$(s_1 \dots s_m)^{-1} w \in \Gamma_n.$$

Thus, it is enough to show that the image of $\mathcal{S}$ in L_{n+1} generates A_n with finite diameter. The group A_n is a finitely generated abelian group that is generated by elements of the form $[s_1, \dots, s_n]$, where $s_1, \dots s_n, \in \mathcal{S}$. Choose such a generating set: $\gamma_1, \dots, \gamma_p$. Consider $\gamma_1 = [s_1, \dots, s_n]$. Given any $k \in \mathbb{N}$, by Lemma 2.1, we have that $\gamma_1^k \equiv_{n+1} [s_1^k, \dots, s_n]$. Further, there exist elements $\sigma_1, \dots, \sigma_m \in \mathcal{S}$ with $m < N_0$ and an element $\gamma \in \Gamma_n$ such that

$$s_1^k = \sigma_1 \cdots \sigma_m \gamma.$$

The elements $\sigma_1, \dots, \sigma_m, \gamma$ depend on γ_1 and k, but their number does not. Thus

$$\gamma_1^k \equiv_{n+1} [\sigma_1 \cdots \sigma_m \gamma, \dots, s_n] \equiv_{n+1} [\sigma_1 \cdots \sigma_m, \dots, s_n],$$

where the last equality stems from the centrality of Γ_n. The last expression is a word in the elements of $\mathcal{S}$, whose length is bounded from above by a number that does not depend on k. This is true not just for γ_1, but for $\gamma_2, \dots, \gamma_p$. Since the group A_n is abelian, and every element in it can be written as a product of powers of $\gamma_1, \dots, \gamma_p$, we get that A_n is generated by $\mathcal{S}$ with finite diameter, as required. □

Proof of Theorem 1.2. It is a direct consequence of Lemma 3.2 and induction. □

Proof of Theorem 1.1. Let $H = H_1(S, \mathbb{Z})$. There exists a simple closed curve in π that is mapped to e_1 under this mapping. The mapping class group acts on H, and it is well-known that this action induces a surjective homomorphism onto $\mathrm{Sp}_{2g}(\mathbb{Z})$ [Farb and Margalit 2012, Proposition 8.4]. Furthermore, the nonseparating simple closed curves form a single mapping class group orbit. Thus, by Lemma 3.1 and Theorem 1.2, π is nilpotent-bounded with respect to $\mathcal{S}$. □

4. Finding smaller generating sets

Using Theorem 1.2, it is possible to find smaller generating sets for which π is nilpotent-bounded. We give one such set here, but it is relatively simple to find many of them. In order to do so, we need a simple corollary.

Corollary 4.1. *Let G be a finitely generated group. Let $H = H_1(G, \mathbb{Z}) \cong G/[G, G]$. Suppose that $H \cong H_1 \oplus \cdots \oplus H_k$, and that for each $i = 1, \dots, k$ we are given a set $S_i \subset \Sigma$ whose projection to H is contained in H_i and generates H_i with finite diameter. Then G is nilpotent-bounded with respect to $S_1 \cup \cdots \cup S_k$.*

Proof of Corollary 4.1. This is a direct result of Theorem 1.2 and the fact that any element of $x \in H$ can be written as $x = h_1 + \cdots + h_k$ with $h_i \in H_i$. □

An example of an application of Corollary 4.1 is the following. Let Σ be an orientable surface of genus $g > 1$. It is common to choose a generating set for $\pi = \pi_1(\Sigma)$ of the form $S' = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, where all of the above are represented by simple closed curves, the geometric intersection number of α_i and β_i is one, and they can be realized disjointly from all the other curves. Let $\Gamma_i = \langle \alpha_i, \beta_i \rangle$. The group Γ_i is the fundamental group of an embedded torus with one boundary component. Let $H = H_1(\Sigma)$, and let H_i be the projection to H of Γ_i. Then $H = H_1 \oplus \cdots \oplus H_g$. Thus, if we let $\mathcal{S}$ be any set containing at least one representative in each conjugacy class of a simple closed curve that lies in one of the g tori described above, then π is nilpotent-bounded with respect to $\mathcal{S}$.

5. Further questions

The contrast between the result in this paper and Calegari's result that π has infinite diameter with respect to $\mathcal{S}$ gives rise to several questions.

Question 1. Recall that $L_n = \pi/\pi_n$. By Theorem 1.1, L_n has finite diameter with respect to $\mathcal{S}$. Call this diameter d_n. The sequence $\{d_n\}_{n=1}^{\infty}$ is nondecreasing. Is this sequence bounded? If so, by what value? If not, what is its asymptotic growth rate?

If the sequence $\{d_n\}_{n=1}^{\infty}$ were indeed unbounded, that would imply that π has infinite diameter with respect to $\mathcal{S}$. However, the converse implication is not necessarily true. One way to see this is to consider the following example: Suppose that π is a free group. Choose a free generating set for π, and let $|\,.\,|$ be the word metric given by this set. The set $\bigcup_{i=1}^{\infty} L_i$ is countable. Choose an enumeration of all of its elements: $\{\ell_i\}_{i=1}^{\infty}$. Each of the ℓ_i's is a coset of an infinite subgroup of π. For each i, choose an element $l_i \in \ell_i$ such that $|l_{i+1}| > 2^{|l_i|}$. Let $\mathcal{L} = \{l_i\}_{i=1}^{\infty}$. The group π is nilpotent-bounded with respect to the set $\mathcal{L}$. Indeed, by construction, $\mathcal{L}$ surjects onto every nilpotent quotient, and thus generates each nilpotent quotient with diameter 1. However, by using the triangle inequality for $|\,.\,|$, it is simple to see that $\mathcal{L}$ cannot generate π with finite diameter.

Question 2. The lower central series is but one of the important series of nested subgroups of π. Another such series is the derived series, whose elements are quotients of surjections onto solvable groups. This sequence is defined by

$$\Gamma^{(0)} = \Gamma, \quad \Gamma^{(n+1)} = [\Gamma^{(n)}, \Gamma^{(n)}].$$

Is the conclusion of Theorem 1.1 true if we replace the word nilpotent with the word solvable?

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STRONG SUBMODULES OF ALMOST PROJECTIVE MODULES

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The structure of almost projective modules can be better understood in the case when the following Condition (P) holds: The union of each countable pure chain of projective modules is projective. We prove this condition, and its generalization to pure-projective modules, for all countable rings, using the new notion of a strong submodule of the union.

However, we also show that Condition (P) fails for all Prüfer domains of finite character with uncountable spectrum, and in particular, for the polynomial ring $K[x]$, where K is an uncountable field. One can even prescribe the Γ-invariant of the union. Our results generalize earlier work of Hill, and complement recent papers by Macías-Díaz, Fuchs, and Rangaswamy.

By a classic theorem of Kaplansky, the structure theory of projective modules over an arbitrary ring reduces to that of countably generated ones. In stark contrast, *almost projective modules* (modules possessing a rich supply of small projective submodules) generally have a very complex structure. Perhaps the most successful invariant measuring their complexity is the Γ-invariant. A projective module has a trivial Γ-invariant [Eklof 1993; Eklof and Mekler 2002].

There are additional conditions on almost projective modules that guarantee projectivity. In his work on Whitehead groups, Hill [1970] discovered a remarkable condition in the particular case of abelian groups: if A is the union of a countable pure chain of (arbitrarily large) projective groups, then A is projective. Here, we call the analogous property for modules over an arbitrary ring *Condition (P)*.

In the past decade, several authors have attempted to extend Hill's result and establish Condition (P) for large classes of rings, notably for commutative domains and noetherian rings [Fuchs and Rangaswamy 2011; Fuchs and Salce 2001]. So far, Macías-Díaz [2010] has obtained the strongest result, that Prüfer domains with countable spectrum have Condition (P).

Section 1 of our paper gives more motivation for considering Condition (P), by showing its role in relating various notions of almost projectivity appearing in the

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literature. In Section 2, we prove Condition (P) and some of its generalizations for all countable rings, using the new notion of a strong submodule.

However, in Section 3, we show that Condition (P) fails completely for all Prüfer domains of finite character with uncountable spectrum (and thus, for example, for the polynomial ring $K[x]$, where K is any uncountable field). Here, "completely" refers to the fact that there are essentially no restrictions on the Γ-invariant of A.

In what follows, R denotes a ring (that is, an associative ring with 1), and the term *module* means a right R-module.

1. Almost projective modules

The following definition is the analogue of [Eklof and Mekler 2002, IV.1.1] for general rings, with "free" replaced by "projective".

Definition 1.1. Let R be a ring and κ a regular uncountable cardinal. A module M is called *κ-projective* if there exists a set $\mathcal{S}$ consisting of $< \kappa$-generated projective submodules of M such that

(i) each subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{S}$, and

(ii) $\mathcal{S}$ is closed under unions of well-ordered chains of length $< \kappa$.

We recall some other relevant notions for the study of almost projectivity (see, for example, [Eklof and Mekler 2002, IV.1; Trlifaj 1995]).

Definition 1.2. Let R be a ring and κ a regular uncountable cardinal. A module M is called *weakly κ-projective* if each subset of M of cardinality $< \kappa$ is contained in a pure submodule N of M that is $< \kappa$-generated and projective.

Recall that a module M is *flat* if the functor $M \otimes_R -$ is exact, and that M is *Mittag-Leffler* if the canonical map

$$M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I} (M \otimes_R Q_i)$$

is monic for each family of left R-modules $(Q_i \mid i \in I)$.

Lemma 1.3 [Raynaud and Gruson 1971; Herbera and Trlifaj 2009]. *Let R be a ring and M a module. Then the following conditions are equivalent*:

(i) *M is $\aleph_1$-projective.*

(ii) *M is weakly $\aleph_1$-projective.*

(iii) *Each finite subset of M is contained in a projective, countably generated and pure submodule of M.*

(iv) *M is flat Mittag-Leffler.*

Also, if κ is a regular uncountable cardinal and M is κ-projective, then M is $\aleph_1$-projective.

Proof. The equivalence of (i), (ii) and (iii) is proved in [Raynaud and Gruson 1971] (see also [Drinfeld 2006]), while (i) and (iv) are equivalent by [Herbera and Trlifaj 2009, Theorem 2.9(i)] (see also [Rothmaler 1994; 1997]). The last statement is [Herbera and Trlifaj 2009, Theorem 2.9(ii)]. □

The implication (i) $\Rightarrow$ (ii) extends to arbitrary regular uncountable cardinals κ:

Lemma 1.4. *Let R be a ring, M a module, and κ an infinite cardinal.*

(i) *Assume that M is $\aleph_1$-projective. Then each subset of M of cardinality $\leq \kappa$ is contained in a $\leq \kappa$-generated pure submodule of M.*

(ii) *Assume that κ is regular uncountable and M is κ-projective. Then M is weakly κ-projective.*

Proof. (i) We prove the claim by induction on κ. The case of $\kappa = \aleph_0$ follows by Lemma 1.3.

Assume $\kappa \geq \aleph_1$, and let $X = \{x_\alpha \mid \alpha < \kappa\}$ be a subset of M of cardinality κ. For each $\alpha < \kappa$, let $X_\alpha = \{x_\beta \mid \beta < \alpha\}$. By induction on α, we define an increasing chain $(P_\alpha \mid \alpha < \kappa)$ of $< \kappa$-generated pure submodules of M as follows: $P_0 = 0$, $P_{\alpha+1}$ is a $< \kappa$-generated pure submodule of M containing $X_\alpha \cup P_\alpha$ (which exists by the inductive premise), and $P_\alpha = \bigcup_{\beta<\alpha} P_\beta$ when $\alpha < \kappa$ is a limit ordinal. Then $P = \bigcup_{\alpha<\kappa} P_\alpha$ is a $\leq \kappa$-generated pure submodule of M containing X.

(ii) Let $\mathcal{S}$ be as in Definition 1.1, and let X be a subset of M of cardinality $< \kappa$. By condition (i) of Definition 1.1, X is contained in a $< \kappa$-generated projective submodule $P_0 \in \mathcal{S}$. By the last statement of Lemma 1.3 and by Lemma 1.4(i), P_0 is contained in a $< \kappa$-generated pure submodule Q_0 of M. Proceeding similarly, we obtain a countable chain

$$P_0 \subseteq Q_0 \subseteq P_1 \subseteq Q_1 \subseteq \cdots \subseteq P_n \subseteq Q_n \subseteq \cdots,$$

where $P_n \in \mathcal{S}$, so P_n is $< \kappa$-generated and projective, and Q_n is $< \kappa$-generated and pure in M, for all $n < \omega$. Let $P = \bigcup_{n<\omega} P_n = \bigcup_{n<\omega} Q_n$. Then $P \in \mathcal{S}$ by condition (ii) of Definition 1.1, and P is pure in M. □

Whatever the cardinality of the ring R, Lemma 1.4(i) makes it possible to purify a submodule without increasing the number of generators. So in the particular case when R is a right hereditary ring, κ-projectivity and weak κ-projectivity are equivalent (to the property that each $< \kappa$-generated submodule is projective). However, the converse of Lemma 1.4(ii) fails in general:

Example 1.5. Let $\kappa > \aleph_1$ be a regular cardinal, let K be a field, and let R denote the endomorphism ring of a κ-dimensional K-linear space modulo its maximal ideal. Then there exists a κ-generated right ideal I in R such that I is weakly κ-projective, but not κ-projective [Trlifaj 1995, Theorem 8].

Another relevant property is the following (where a chain $(P_n \mid n < \omega)$ is a *pure chain* if P_n is a pure submodule of P_{n+1} for each $n < \omega$):

Definition 1.6. Let R be a ring. Then R satisfies *Condition* (P) if for each pure chain $(P_n \mid n < \omega)$ consisting of projective modules, the module $P = \bigcup_{n<\omega} P_n$ is projective.

Condition (P) yields a characterization of weak κ-projectivity:

Proposition 1.7. *Let R be a ring satisfying Condition* (P). *Let M be a module and κ a regular infinite cardinal. Then M is weakly κ-projective if and only if there exists a set $\mathcal{S}$ consisting of $<\kappa$-generated projective submodules of M such that*

(i) *each subset of M of cardinality $<\kappa$ is contained in an element of $\mathcal{S}$, and*

(ii) *$\mathcal{S}$ is closed under unions of countable chains.*

Proof. By [Herbera and Trlifaj 2009, Corollary 2.3], assumptions (i) and (ii) assure $\aleph_1$-projectivity of M, so the "if" implication is proved as in Lemma 1.4. For the "only if", let $\mathcal{S}$ be the set of all $<\kappa$-generated projective and pure submodules of M. Then (i) holds by the assumption. If $M_0 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$ is a countable chain of elements of $\mathcal{S}$, then $M_\omega = \bigcup_{n<\omega} M_n$ is projective by Condition (P), so $M_\omega \in \mathcal{S}$. $\square$

Condition (P) holds for $R = \mathbb{Z}$. This was shown by Hill [1970], who proved thus the singular compactness of almost free abelian groups of cardinality $\aleph_\alpha$, where α has cofinality ω.

More generally, Condition (P) is known to hold for all Prüfer domains with countably many maximal ideals [Macías-Díaz 2010, Corollary 15], and hence for all valuation domains. In Theorem 2.5 below, we prove it for all countable rings.

However, attempts to prove Condition (P) for arbitrary domains in [Fuchs and Salce 2001, XVI.1.4] and [Fuchs and Rangaswamy 2011, Theorem 1.3] have gaps; in fact, as we see in Theorem 3.1, Condition (P) fails even for $R = K[x]$, where K is an uncountable field.

The main goal of the next section is to prove Condition (P), and hence the equivalence in Proposition 1.7, for all countable rings. Before proceeding to that point, we note that under additional assumptions on R and M, the equivalence holds even without assuming Condition (P):

Proposition 1.8. *Let κ be an infinite cardinal, and let R be a ring that is either a domain or is right $<\kappa$-noetherian* (*that is, every right ideal is $<\kappa$-generated*). *Let M be a module of projective dimension ≤ 1. Then M is weakly κ-projective if and only if there exists a set $\mathcal{S}$ consisting of $<\kappa$-generated projective submodules of M such that*

(i) *each subset of M of cardinality $< \kappa$ is contained in an element of $\mathcal{S}$, and*

(ii) *$\mathcal{S}$ is closed under unions of countable chains.*

The following lemma helps in finding projective submodules:

Lemma 1.9. *Let M be a module of projective dimension at most* 1. *Let N be a tight submodule; that is, let M/N have also projective dimension at most* 1. *If N is contained in a projective submodule of M, then N is projective.*

Proof. Let P be a projective module such that $N \subseteq P \subseteq M$. We can estimate the projective dimensions of various modules built from N, P and M using the long exact sequence for Ext as follows:

$$\text{proj.dim}\, M/P \leq \max\{\text{proj.dim}\, M, \text{proj.dim}\, P + 1\} \leq 2, \tag{1}$$

$$\text{proj.dim}\, P/N \leq \max\{\text{proj.dim}\, M/N, \text{proj.dim}\, M/P - 1\} \leq 1, \tag{2}$$

$$\text{proj.dim}\, N \leq \max\{\text{proj.dim}\, P, \text{proj.dim}\, P/N - 1\} \leq 0. \tag{3}$$

The last line shows that N is projective. □

Proof of Proposition 1.8. As in the proof of Proposition 1.7, the conditions (i) and (ii) of Proposition 1.8 imply that M is κ-projective (because neither Condition (P) nor any of our additional assumptions are needed there).

For the other direction, we note that by the assumptions on R, there is a Hill family consisting of tight submodules of M: when R is a domain, this follows by [Fuchs and Salce 2001, Proposition VI.5.1] and [Göbel and Trlifaj 2006, 4.2.6], and when R is $< \kappa$-noetherian, we apply [Göbel and Trlifaj 2006, 4.1.11 and 4.2.6].

Let $\mathcal{S}$ be the subfamily of the $< \kappa$-generated members of this family. Conditions (i) and (ii) automatically hold. Finally, the assumption of weak κ-projectivity and Lemma 1.9 imply that $\mathcal{S}$ consists of projective modules. □

2. Hill families of strong submodules

We start this section by considering a general version of Condition (P), where the chain $(P_n \mid n < \omega)$ is not necessarily pure, and the modules P_n $(n < \omega)$ are direct sums of modules from a given class $\mathscr{C}$ consisting of countably presented modules or modules of countable rank. The relevant notion here is that of a strong submodule. It is introduced in the following definition, where, for a class of modules $\mathscr{C}$, we denote by Sum $(\mathscr{C})$ the class of all direct sums of copies of modules from $\mathscr{C}$.

Definition 2.1. Let R be a ring and $\mathscr{C}$ a class of modules.

Let $(P_n \mid n < \omega)$ be a countable increasing chain of modules, and suppose $P = \bigcup_{n<\omega} P_n$. Assume that $P_n \in$ Sum $(\mathscr{C})$ for each $n < \omega$; that is, there exists a decomposition $P_n = \bigoplus_{\alpha<\kappa_n} P_{n,\alpha}$, where $P_{n,\alpha}$ is isomorphic to an element of $\mathscr{C}$ for each $\alpha < \kappa_n$.

We fix these decompositions, and for each $n < \omega$ and each subset $S \subseteq \kappa_n$, define $P(n, S) = \bigoplus_{\alpha \in S} P_{n,\alpha}$. So, in particular, $P_n = P(n, \kappa_n)$.

A submodule N of P is called *strong* if there exist $(A_n \mid n < \omega)$ such that $A_n \subseteq \kappa_n$ and $N \cap P_n = P(n, A_n)$ for each $n < \omega$. The sequence $(A_n \mid n < \omega)$ is then uniquely determined by N; it is the *witnessing sequence* for N.

In this section, P denotes the union $\bigcup_{n<\omega} P_n$, where $(P_n \mid n < \omega)$ is a countable increasing chain of modules, as in Definition 2.1.

In the case when $\mathscr{C}$ is the class of all countably presented projective modules, Definition 2.1 covers the setting of Condition (P), because by a classic theorem of Kaplansky, each projective module is a direct sum of modules in $\mathscr{C}$.

Note that 0 and P are strong submodules of P. Also, unions of chains of strong submodules are strong, and so are arbitrary intersections of strong submodules. Indeed, in Theorem 2.9, we prove that strong submodules are abundant.

If N is strong in P and the chain $(P_n \mid n < \omega)$ is pure, then N is a pure submodule of P, because $N = \bigcup_{n<\omega} N \cap P_n$ and $N \cap P_n$ is a direct summand in the pure submodule P_n of P for each $n < \omega$.

For the next lemma, we recall that a ring R is right $\aleph_0$-*noetherian* provided that each right ideal of R is countably generated. For example, all right noetherian rings, and all countable rings, are right $\aleph_0$-noetherian. It is easy to see that a ring R is right $\aleph_0$-noetherian if and only if each submodule of a countably generated module is countably generated.

Lemma 2.2. *Assume that R is right $\aleph_0$-noetherian and $\mathscr{C}$ consists of countably presented modules, or that R is a commutative domain and $\mathscr{C}$ consists of torsion-free modules of countable rank, respectively. Let N be a strong submodule of P with witnessing sequence $(A_n \mid n < \omega)$. Let C be a countable subset of P or a subset of P such that $\langle C \rangle$ has countable rank, respectively.*

Then there is a strong submodule N' of P such that $N \cup C \subseteq N'$, the witnessing sequence $(A'_n \mid n < \omega)$ for N' satisfies $A_n \subseteq A'_n$, and $A'_n \setminus A_n$ is countable for each $n < \omega$.

Proof. We simultaneously and recursively construct chains $(C_{n,i} : i < \omega)$ of subsets of κ_n.

As a start, for each $n < \omega$, put $A_n \subseteq C_{n,0} \subseteq \kappa_n$, with $C_{n,0} \setminus A_n$ countable and $C \cap P_n \subseteq P(n, C_{n,0})$.

For $i \geq 0$, let $C_{n,i} \subseteq C_{n,i+1} \subseteq \kappa_n$, with $C_{n,i+1} \setminus C_{n,i}$ countable and

$$P(m, C_{m,i}) \cap P_n \subseteq P(n, C_{n,i+1})$$

for all m.

Finally, we define $A'_n = \bigcup_{i<\omega} C_{n,i}$ for each $n < \omega$. Then $A_n \subseteq A'_n \subseteq \kappa_n$, and $A'_n \setminus A_n$ is countable for each $n < \omega$. Let $N' = \bigcup_{n<\omega} P(n, A'_n) = \bigcup_{n,i<\omega} P(n, C_{n,i})$.

The $P(n, C_{n,i})$ form an upper directed system of submodules, so their union is a submodule.

Recall that $P(m, C_{m,i}) \cap P_n \subseteq P(n, C_{n,i+1})$ for all $m, n, i < \omega$, and hence $N' \cap P_n = P(n, A'_n)$. All in all, N' is a strong submodule of P with witnessing sequence $(A'_n \mid n < \omega)$.

Since $C \cap P_n \subseteq P(n, A'_n)$ for each $n < \omega$, we conclude that $N \cup C \subseteq N'$. □

Lemma 2.2 serves as inductive step for proving the following:

Proposition 2.3. *Assume either that R is right $\aleph_0$-noetherian and $\mathscr{C}$ consists of countably presented modules, or that R is a commutative domain and $\mathscr{C}$ consists of torsion-free modules of countable rank, respectively.*

Then P is the union of a continuous increasing chain $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ of strong submodules of P, such that for each $\alpha < \lambda$, there is a countably generated or countable-rank submodule N_α of P, respectively, with $M_{\alpha+1} = M_\alpha + N_\alpha$.

Proof. Let $\{p_\alpha \mid \alpha < \lambda\}$ be an R-generating subset of P. Since $M_0 = 0$ is strong, and the union of a chain of strong submodules is strong, it remains to perform the nonlimit step of the construction. However, applying Lemma 2.2 for $N = M_\alpha$ and $C = \{p_\alpha\}$, we can take $N_\alpha = \sum_{n<\omega} P(n, A'_n \setminus A_n)$ and $M_{\alpha+1} = N'$. □

We can prove more in the particular case of countable rings. We consider a class of modules $\mathscr{C}$ to have *Property* (C) if for each increasing pure chain of modules $(Q_n \mid n < \omega)$ such that $Q_n \in \operatorname{Sum}(\mathscr{C})$ for all $n < \omega$, and each countably presented pure submodule C of $\bigcup_{n<\omega} Q_n$, the module C is $\mathscr{C}$-filtered. Also, $\mathscr{C}$ has *Property* (C+) if the same assumptions yield the stronger conclusion of $C \in \operatorname{Sum}(\mathscr{C})$.

For example, the class of all countably presented modules and the class of all projective modules have Property (C+), because the union of a pure chain of projective modules is always $\aleph_1$-projective, by Lemma 1.3.

Lemma 2.4. *Let R be a countable ring. Let $\mathscr{C}$ be a class of countably presented modules that has Property (C). Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \operatorname{Sum}(\mathscr{C})$ for all $n < \omega$, and let $P = \bigcup_{n<\omega} P_n$. Then P is $\mathscr{C}$-filtered.*

Also, if $\mathscr{C}$ has Property (C+), then P is the union of a continuous increasing chain $\mathcal{M} = (M_\alpha \mid \alpha < \lambda)$ consisting of strong submodules of P such that $M_{\alpha+1}/M_\alpha \in \operatorname{Sum}(\mathscr{C})$.

Proof. Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules with $P_n \in \operatorname{Sum}(\mathscr{C})$ for all $n < \omega$. Since R is countable, the continuous chain $\mathcal{M}$ from Proposition 2.3 can be taken with the additional property of $M_\alpha + P_n$ being pure in P for all $n < \omega$ and $\alpha < \kappa$. This is arranged by improving Lemma 2.2 for countable R: when for the strong submodule N, all the submodules $N + P_n$ are pure, then N' can be chosen with the $N' + P_n$ also pure.

It follows that for each $\alpha < \kappa$, the factor $Q = P/M_\alpha$ is the union of the pure chain $(Q_n \mid n < \omega)$, where $Q_n = (M_\alpha + P_n)/M_\alpha$. Also, $Q_n \cong P_n/(P_n \cap M_\alpha) \in \mathrm{Sum}\,(\mathscr{C})$, because M_α is strong. Similarly, the countably presented submodule $C = M_{\alpha+1}/M_\alpha$ is pure in Q, so C is $\mathscr{C}$-filtered by Property (C). Then $P = \bigcup_{\alpha<\kappa} M_\alpha$ is $\mathscr{C}$-filtered as well.

Also, if $\mathscr{C}$ has Property (C+), then $C = M_{\alpha+1}/M_\alpha \in \mathrm{Sum}\,(\mathscr{C})$. □

The assumptions of Lemma 2.4 are satisfied for R countable and $\mathscr{C}$ the class of all countably generated projective modules. Since in this case $\mathscr{C}$-filtered is the same as projective, we get:

Theorem 2.5. *Let R be a countable ring. Then R satisfies Condition* (*P*).

As another consequence, we obtain the general version of Condition (P) for the case when R is countable, $\mathscr{C}$ has Property (C+), and $\mathscr{C}$ consists of finitely presented modules:

Corollary 2.6. *Let R be a countable ring*, *and let $\mathscr{C}$ be a class of finitely presented modules that has Property* (*C*+). *Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \mathrm{Sum}\,(\mathscr{C})$ for all $n < \omega$ and that $P = \bigcup_{n<\omega} P_n$. Then $P \in \mathrm{Sum}\,(\mathscr{C})$.*

Proof. By Lemma 2.4, P is the union of a continuous increasing chain

$$\mathcal{M} = (M_\alpha \mid \alpha < \lambda),$$

consisting of strong submodules of P such that $M_{\alpha+1}/M_\alpha \in \mathrm{Sum}\,(\mathscr{C})$. In particular, M_α is pure in $M_{\alpha+1}$ for each $n < \omega$. As $\mathscr{C}$ consists of finitely presented modules, $M_{\alpha+1}/M_\alpha$ is pure-projective, and the embedding $M_\alpha \hookrightarrow M_{\alpha+1}$ splits. This proves that $P \in \mathrm{Sum}\,(\mathscr{C})$. □

A variation of Corollary 2.6 gives the version of Condition (P) for pure-projective modules over countable rings.

Theorem 2.7. *Let R be a countable ring*, *$(P_n \mid n < \omega)$ be an increasing pure chain of pure-projective modules*, *and $P = \bigcup_{n<\omega} P_n$. Then P is pure-projective.*

Proof. By [Raynaud and Gruson 1971, Seconde partie, Corollaire 2.2.2], a countably presented module is pure-projective if and only if it is Mittag-Leffler, and the latter property is clearly inherited by pure submodules. As in Lemma 2.4, we infer that P is the union of a continuous chain $\mathcal{M}$ consisting of strong submodules of P such that all consecutive factors in $\mathcal{M}$ are pure-projective, and hence P is pure-projective as well. □

Alternatively, we can deduce Theorem 2.5 from Theorem 2.7, because projective = flat + pure-projective.

Of course, the union of a nonpure countable chain of projective modules need not be projective even for countable rings: just consider $R = \mathbb{Z}$ and $\mathbb{Q}$ as the union of the chain of free groups $(1/n! \cdot \mathbb{Z} \mid n < \omega)$.

Also, the general version of Condition (P) for pure chains consisting of modules from Sum $(\mathscr{C})$ may fail even for countable rings and $\mathscr{C}$ having Property (C). That is, even though P is $\mathscr{C}$-filtered by Lemma 2.4, $P \notin$ Sum $(\mathscr{C})$ in general:

Example 2.8. Let R be a simple, countable von Neumann regular ring that is not artinian — for example, let R be the directed union of the full matrix rings $M_{2^n}(\mathbb{Q})$ $(n < \omega)$ with the block diagonal embeddings

$$\mathbb{Q} \subseteq M_2(\mathbb{Q}) \subseteq M_4(\mathbb{Q}) \subseteq \cdots \subseteq M_{2^n}(\mathbb{Q}) \subseteq M_{2^{n+1}}(\mathbb{Q}) \subseteq \cdots .$$

Consider a simple nonprojective module S, and let $\mathscr{C}$ be the class of all finitely $\{S\}$-filtered modules. Then $\mathscr{C}$ is a class of countable modules and has Property (C).

Define a chain of finite length modules $(P_n \mid n < \omega)$ such that $P_0 = S$ and that P_{n+1} fits in a nonsplit short exact sequence $0 \to P_n \subseteq P_{n+1} \to S \to 0$ for each $n < \omega$. This is possible by [Trlifaj 1996, Proposition 3.3]. This chain is pure because R is von Neumann regular, so all R-modules are flat.

Let $P = \bigcup_{n<\omega} P_n$. Then $P_n \in \mathscr{C}$ for all $n < \omega$, and P is $\mathscr{C}$-filtered, but $P \notin$ Sum $(\mathscr{C})$. Indeed, $S = P_0$ is an essential submodule of P, so P is uniform, and hence indecomposable.

Returning to the general setting and using an idea by Hill, we can extend the chain $\mathscr{M}$ from Proposition 2.3 further, to a large family of strong submodules:

Theorem 2.9. *Assume that R is right $\aleph_0$-noetherian and $\mathscr{C}$ consists of countably presented modules, or that R is a commutative domain and $\mathscr{C}$ consists of torsion-free modules of countable rank, respectively. Let $\mathscr{M} = (M_\alpha : \alpha < \lambda)$ be a continuous increasing chain of strong submodules of P as in Proposition 2.3. There is a family $\mathscr{H}$ of strong submodules of P such that*:

(i) $\mathscr{M} \subseteq \mathscr{H}$.

(ii) *$\mathscr{H}$ is closed under arbitrary sums and intersections*; *in fact*, *$\mathscr{H}$ is a complete distributive sublattice of the modular lattice of all submodules of P.*

(iii) *Let $N, N' \in \mathscr{H}$ be such that $N \subseteq N'$. Then there exists a continuous increasing chain $(N_\beta \mid \beta \leq \tau)$ consisting of elements of $\mathscr{H}$ such that $\tau \leq \lambda$, $N_0 = N$, $N_\tau = N'$, and for each $\beta < \tau$ there is $\alpha < \kappa$ such that $N_{\beta+1}/N_\beta$ is isomorphic to $M_{\alpha+1}/M_\alpha$.*

(iv) *Let $N \in \mathscr{H}$, and let X be a countable subset of P* (*a subset of P such that $\langle X \rangle$ has countable rank*, *respectively*). *Then there are $N' \in \mathscr{H}$ and a submodule $Y \subseteq P$ such that Y is countably generated* (*of countable rank*, *respectively*) *and $N \cup X \subseteq N' = N + Y$.*

Proof. For all $\alpha < \lambda$ and $n < \omega$, let $D_{\alpha,n} = A_{\alpha+1,n} \setminus A_{\alpha,n}$, where $(A_{\alpha,n} \mid n < \omega)$ and $(A_{\alpha+1,n} \mid n < \omega)$ are the witnessing sequences for M_α and $M_{\alpha+1}$. By the assumption on the chain $\mathcal{M}$, all the sets $D_{\alpha,n}$ are countable. Let $\Delta_\alpha = \sum_{n<\omega} P(n, D_{\alpha,n})$. Then Δ_α is countably generated, and $M_{\alpha+1} = M_\alpha + \Delta_\alpha$ for each $\alpha < \lambda$.

As in [Göbel and Trlifaj 2006, §4.2], we call a subset S of σ *closed* when

$$\Delta_\alpha \cap M_\alpha \subseteq \sum_{\beta<\alpha,\ \beta\in S} \Delta_\beta$$

for each $\alpha \in S$. We define $\mathcal{H} = \{\sum_{\alpha\in S} \Delta_\alpha \mid S \text{ is closed in } \lambda\}$.

Since each ordinal $\sigma \leq \lambda$ is closed, $\mathcal{M} \subseteq \mathcal{H}$, and (i) holds. Properties (ii) and (iii) are proved in [Göbel and Trlifaj 2006, 4.2.6]. If R is $\aleph_0$-noetherian, then (iv) is proved in [Göbel and Trlifaj 2006, 4.2.6], while in the domain case, (iv) follows by [Göbel and Trlifaj 2006, 4.2.8].

It remains to show that all modules in $\mathcal{H}$ are strong. Let S be a closed subset of λ, and let $N = \sum_{\alpha\in S} \Delta_\alpha$ and $B_n = \bigcup_{\alpha\in S} D_{\alpha,n}$. It suffices to prove that $N \cap P_n = P(n, B_n)$ for each $n < \omega$. The inclusion $\supseteq$ is clear from the definitions above.

Assume there exists $x \in (N \cap P_n) \setminus P(n, B_n)$. Then there is one of the form $x = x_{\alpha_1} + \cdots + x_{\alpha_i}$, where $\alpha_1 < \cdots < \alpha_i$ are elements of S, and $x_{\alpha_k} \in \Delta_{\alpha_k} \setminus P(n, B_n)$ for all $1 \leq k \leq i$. Without loss of generality, we can assume that $\alpha = \alpha_i$ is minimal. Since $x \in M_{\alpha+1} \cap P_n = P(n, A_{\alpha+1,n})$, we also have $x = y_{\beta_1} + \cdots + y_{\beta_j}$, where $\beta_1 < \cdots < \beta_j$ are elements of $A_{\alpha+1,n}$ and $y_{\beta_l} \in P_{n,\beta_l}$ for each $1 \leq l \leq j$. If $\beta_l \in D_{\alpha,n}$ for some $1 \leq l \leq j$, then $x_\alpha - y_{\beta_l} \in \Delta_\alpha \setminus P(n, B_n)$. Possibly replacing x by $x - y_{\beta_l}$, we can assume that $\beta_l \in A_{\alpha,n}$ for all $1 \leq l \leq j$. But then $x_\alpha \in \Delta_\alpha \cap M_\alpha \subseteq \sum_{\beta<\alpha,\beta\in S} \Delta_\beta$, in contradiction with the minimality of α.

This proves that N is strong in P. □

We can now improve the second part of Lemma 2.4:

Corollary 2.10. *Let R be a countable ring. Let $\mathcal{C}$ be a class of countably presented modules that has Property (C+). Let $(P_n \mid n < \omega)$ be an increasing pure chain of modules such that $P_n \in \mathrm{Sum}\,(\mathcal{C})$ for all $n < \omega$, and let $P = \bigcup_{n<\omega} P_n$.*

Then P is the union of a continuous increasing pure chain $\mathcal{N} = (N_\alpha \mid \alpha < \aleph_1)$ consisting of strong submodules of P such that $N_{\alpha+1}/N_\alpha \in \mathrm{Sum}\,(\mathcal{C})$ for all $\alpha < \aleph_1$.

Proof. Let $\mathcal{M}$ be the chain constructed in the second part of Lemma 2.4, and consider the corresponding family $\mathcal{H}$, as in Theorem 2.9. By [Šťovíček 2012], one can select from $\mathcal{H}$ an increasing continuous chain $\mathcal{N} = (N_\alpha \mid \alpha < \aleph_1)$ of length no greater than $\aleph_1$, such that $N_{\alpha+1}/N_\alpha$ is isomorphic to a direct sum of some of the successive factors of the original chain $\mathcal{M}$, for all $\alpha < \aleph_1$. By Lemma 2.4, all these factors are in $\mathrm{Sum}\,(\mathcal{C})$. Since $\mathcal{H}$ consists of strong (and hence pure) submodules of P, so does $\mathcal{N}$. □

3. The failure of Condition (P)

In this section we prove that Condition (P) fails for Prüfer domains of finite character with uncountable spectrum, and, notably, for every principal ideal domain with an uncountable spectrum. We adopt [Eklof and Mekler 2002, Theorem VII.1.4] to illustrate that failure of Condition (P) has little, if any, restriction on the Γ-invariant of even large almost-projective modules.

Recall from [Fuchs and Salce 2001, Chapter III, Lemma 2.7] that in a Prüfer domain of finite character, every maximal ideal contains a finitely generated ideal, which is not contained in any other maximal ideal. Selecting one for every maximal ideal, we obtain a system of pairwise coprime proper invertible ideals. In fact, all we need is such a system of ideals:

Theorem 3.1. *Let R be a commutative domain with uncountably many pairwise coprime invertible proper ideals. Let κ be a regular uncountable cardinal, and E be a nonreflecting stationary subset of κ, all of whose elements have cofinality ω. Then there is a κ-projective κ-generated R-module M with Γ-invariant $\widetilde{E}$ that is a union of a countable pure chain of projective submodules.*

Before proving Theorem 3.1, we follow the suggestion of the referee and present a simple particular case of the construction.

Example 3.2. Let R be a principal ideal domain with uncountably many maximal ideals (p_α) for $0 < \alpha < \aleph_1$.

We define our module via generators and relations:

$$P := \langle e_{\alpha,n} : \alpha < \omega_1, n < \omega \bigm| p_\alpha e_{\alpha,n+1} = e_{\alpha,n} + e_{0,n+1} : \alpha > 0 \rangle. \tag{4}$$

(This is an example for the theorem with $\kappa = \aleph_1$, and $E = \{\alpha < \aleph_1 \mid \mathrm{cf}(\alpha) = \aleph_0\}$.)

We leave it to the reader to verify that for every $0 < \alpha \leq \aleph_1$ and $i < \omega$, the submodule

$$N_{\alpha,i} = \langle e_{\beta,j} : j \leq i, \beta < \alpha \rangle \tag{5}$$

is actually free, with a basis formed by the $e_{0,j}$ for $j \leq i$ and the $e_{\beta,i}$ for $0 < \beta < \alpha$. Since

$$N_{\alpha,i+1}/N_{\alpha,i} \cong \langle R, p_\beta^{-1} : 0 < \beta < \alpha \rangle$$

(with $e_{0,i+1}$ corresponding to 1 and $e_{\beta,i+1}$ corresponding to p_β^{-1}) is torsion-free, $N_{\alpha,i}$ is a pure submodule of $N_{\alpha,i+1}$. Hence, P is a union of a pure chain $P_i = N_{\aleph_1,i}$ of projective submodules.

On the other hand, P is a union of a continuous chain

$$N_\alpha = \bigcup_{i<\omega} N_{\alpha,i} = \langle e_{\beta,i} : \beta < \alpha, i < \omega \rangle, \quad 0 < \alpha < \aleph_1 \tag{6}$$

of (strong) submodules with nonprojective factors $N_{\alpha+1}/N_\alpha \cong R[p_\alpha^{-1}]$ (with $e_{\alpha,i}$ corresponding to p_α^{-i}), and hence is not projective.

The proof of Theorem 3.1 is mostly the same as that of [Eklof and Mekler 2002, Theorems VII.1.3–4], so we present only the differences. To include the sequence of submodules in the structure, we work in the category of ω-filtered modules, that is, modules M together with an increasing sequence $(M(n) : n < \omega)$ of submodules satisfying $\bigcup_{n=0}^{\infty} M(n) = M$. A *filtered submodule* of M is a submodule N together with the filtration $N(n) := M(n) \cap N$. Note that M/N is also a filtered module, with the filtration $(M(n)/N(n) \cong (M(n)+N)/N : n < \omega)$.

For the free module $R^{(\lambda\times\omega)}$, we always use the filtration $(R^{(\lambda\times n)} : n < \omega)$.

For a module N, let $N[n]$ denote the filtered module

$$N[n](m) := \begin{cases} 0, & m < n, \\ N, & m \geq n. \end{cases} \tag{7}$$

For example, $R^{(\lambda\times\omega)} = \displaystyle\bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]$ as filtered modules.

Proof of Theorem 3.1. We distinguish the cases $\kappa > \aleph_1$ and $\kappa = \aleph_1$. To avoid repetition, we first provide the common part of both cases, and then fill out the missing parts separately.

We build a continuous increasing chain of ω-filtered modules $(M_\mu : \mu < \kappa)$ whose filtrations consist of pure and projective submodules. By "increasing", we mean that M_ν is a filtered submodule of M_μ for $\mu < \nu$.

The union M of the chain is our κ-projective module with Γ-invariant $\widetilde{E}$.

To ensure that all the $M_\mu(n)$ and $M(n)$ are projective, we make the filtrations of the $M_{\mu+1}/M_\mu$ consist of projective modules.

We fix an infinite cardinal $\lambda < \kappa$. For $\mu \notin E$, let

$$M_{\mu+1} := M_\mu \oplus P_\mu, \quad P_\mu := R^{(\lambda\times\omega)} = \bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]e_{\mu,n}.$$

For the case $\mu \in E$, we select a template as in [Eklof and Mekler 2002, Corollary VII.1.2], that is, a nonprojective λ-generated module N_μ with an ω-filtration by projective modules. By adding a projective module, we may assume that the filtration consists of λ-generated free modules; that is, $N_\mu(n) \cong R^{(\lambda)}$. The filtration induces a short exact sequence

$$0 \to K_\mu \to F_\mu \to N_\mu \to 0$$

of ω-filtered modules, where

$$F_\mu := \bigoplus_{n=0}^{\infty} N_\mu(n)[n]e_n, \quad K_\mu := \bigoplus_{n=0}^{\infty} N_\mu(n)[n+1]e_n \cong R^{(\lambda\times\omega)}. \tag{8}$$

The embedding of K_μ into F_μ maps xe_n into $xe_{n+1} - xe_n$, and the homomorphism $F_\mu \to N_\mu$ maps xe_n into x for any $x \in N_\mu(n)$ and natural number n. In particular, the filtrations of K_μ and F_μ consist of direct summands, and hence of pure and projective submodules. We see that the modules

$$F_\mu/(K_\mu(n)) = \bigoplus_{m=n}^{\infty} N_\mu(m)$$

are projective for all N.

We define $M_{\mu+1}$ as the pushout of the inclusion $K_\mu \subseteq F_\mu$ by a suitable embedding $K_\mu \to M_\mu$ identifying K_μ with the direct summand

$$\bigoplus_{n=0}^{\infty} R^{(\lambda)}[n+1]e_{\mu_n,n}$$

of the filtered submodule $\bigoplus_{n=0}^{\infty} P_{\mu_n}$ for an increasing sequence of successor ordinals μ_n with supremum μ. Then

$$M_{\mu+1}/M_\mu \cong N_\mu$$

as filtered modules, and therefore $M_{\mu+1}/M_\mu$ is filtered by projective submodules.

The rest of [Eklof and Mekler 2002, Theorems VII.1.3–4] apply to show that M is a κ-free module of Γ-invariant $\widetilde{E}$. The filtration of M consists of projective submodules by construction.

All that is left is to find λ and the N_μ and to verify that the filtration of M actually consists of *pure* submodules.

When $\kappa > \aleph_1$, we choose $\lambda = \aleph_1$, and let N_μ be an $\aleph_1$-generated nonprojective module with an ω-filtration by pure and projective submodules. (We may choose all the N_μ the same.) Such an N_μ exists by the $\kappa = \aleph_1$ case. Since the filtration of N_μ is by pure submodules, it follows that all the $M_\mu(n)$ and $M(n)$ are pure submodules.

When $\kappa = \aleph_1$, we let $\lambda = \aleph_0$. Let $(I_\alpha : \alpha < \aleph_1)$ be a collection of pairwise coprime invertible proper ideals of R. We define the N_μ as submodules of the quotient field of R:

$$N_\mu(n) := I_\mu^{-n}, \quad N_\mu := I_\mu^{-\infty}. \tag{9}$$

Clearly, N_μ is nonprojective and its filtration is by projective submodules.

To show that the filtration of the M_μ is pure, we show that its localization by any maximal ideal Q is pure. When $I_\mu \nsubseteq Q$ and $\mu \in E$, then $N_{\mu,Q} = R_Q[0]$, so the short exact sequence $K_{\mu,Q} \to F_{\mu,Q} \to N_{\mu,Q}$ of filtered modules splits, and hence $M_{\mu+1,Q} = M_{\mu,Q} \oplus N_{\mu,Q}$ as filtered modules.

There is at most one $\mu \in E$ with $I_\mu \subseteq Q$. Hence, by the previous paragraph, if there is such a μ, then $M_{\nu,Q}$ is a direct summand of $M_{\nu+1,Q}$ as filtered modules, for all $\nu < \mu$. So

$$M_{\mu,Q} \cong \bigoplus_{\nu<\mu} M_{\nu+1,Q}/M_{\nu,Q},$$

with an arbitrary choice of split preimages of the $M_{\nu+1,Q}/M_{\nu,Q}$. Recall that $K_{\mu,Q}$ is a direct summand of a sum of some of these preimages, so it is actually a direct summand of $M_{\mu,Q}$; that is, $M_{\mu,Q} = K_{\mu,Q} \oplus H_\mu$. It follows that

$$M_{\mu+1,Q} = F_{\mu,Q} \oplus H_\mu.$$

These decompositions of filtered modules show that the filtrations of H_μ and $M_{\mu+1,Q}$ consist of pure submodules. $\square$

We finish with:

Problem 3.3. Characterize the rings R satisfying Condition (P).

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INTERLACING LOG-CONCAVITY OF THE BOROS–MOLL POLYNOMIALS

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We say a sequence $\{P_m(x)\}_{m\geq 0}$ of polynomials of degree m with positive coefficients is interlacingly log-concave if the ratios of consecutive coefficients of $P_m(x)$ interlace the ratios of consecutive coefficients of $P_{m+1}(x)$ for any $m \geq 0$. Interlacing log-concavity of a sequence of polynomials is stronger than log-concavity of the polynomials themselves. We show that the Boros–Moll polynomials are interlacingly log-concave. Furthermore, we give a sufficient condition for interlacing log-concavity which implies that some classical combinatorial polynomials are interlacingly log-concave.

1. Introduction

Let $\{P_m(x)\}_{m\geq 0}$ be a sequence of polynomials, where

$$P_m(x) = \sum_{i=0}^{m} a_i(m)x^m$$

is a polynomial of degree m. Let

$$r_i(m) = \frac{a_i(m)}{a_{i+1}(m)}.$$

We say that the sequence of polynomials $\{P_m(x)\}_{m\geq 0}$ is *interlacingly log-concave* if the ratios $r_i(m)$ interlace the ratios $r_i(m+1)$, that is,

$$\begin{aligned} r_0(m+1) \leq r_0(m) \leq r_1(m+1) \leq r_1(m) \\ \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1). \end{aligned}$$

Recall that a sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

It is obvious that interlacing log-concavity implies log-concavity.

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The main objective of this paper is to prove the interlacing log-concavity of the Boros–Moll polynomials. For the background on these polynomials, see [Boros and Moll 1999a; 1999b; 1999c; 2001; 2004; Moll 2002; Amdeberhan and Moll 2009]. From now on, we use $P_m(x)$ to denote the Boros–Moll polynomial given by

$$P_m(x)=\sum_{j,k}\binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(x+1)^j(x-1)^k}{2^{3(k+j)}}. \tag{1}$$

Boros and Moll [1999b] derived the following formula for the coefficient $d_i(m)$ of x^i in $P_m(x)$:

$$d_i(m)=2^{-2m}\sum_{k=i}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{2}$$

In [Boros and Moll 1999c], they showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ is unimodal and that the maximum element appears in the middle. In other words,

$$d_0(m)<d_1(m)<\cdots<d_{[m/2]}(m)>d_{[m/2]-1}(m)>\cdots>d_m(m). \tag{3}$$

They also established the unimodality by a different approach [Boros and Moll 1999a]; see also [Alvarez et al. 2001].

Moll [2002] conjectured that the sequence $\{d_i(m)\}_{0\le i\le m}$ is log-concave. Kauers and Paule [2007] proved this conjecture based on recurrence relations found by using a computer algebra approach. Chen and Xia [2009] showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the ratio monotone property which implies log-concavity and the spiral property. A combinatorial proof of the log-concavity of $P_m(x)$ was found by Chen, Pang and Qu [$\ge$ 2011].

In addition to the Boros–Moll polynomials, we study polynomials whose coefficients satisfy triangular recurrence relations. It is easy to show that the binomial coefficients, the Narayana numbers and the Bessel numbers are interlacingly log-concave. We also give a sufficient condition for the interlacing log-concavity of a sequence of polynomials and prove that the rising factorials, the Bell polynomials and the Whitney polynomials are interlacingly log-concave.

2. The interlacing log-concavity of $d_i(m)$

In this section, we show that for $m\ge 2$, the Boros–Moll polynomials $P_m(x)$ are interlacingly log-concave.

Theorem 2.1. *For $m\ge 2$ and $0\le i\le m$, we have*

$$d_i(m)d_{i+1}(m+1)>d_{i+1}(m)d_i(m+1), \tag{4}$$

$$d_i(m)d_i(m+1)>d_{i-1}(m)d_{i+1}(m+1). \tag{5}$$

The proof relies on recurrence relations derived in [Kauers and Paule 2007]:

$$d_i(m+1) = \frac{m+i}{m+1}\, d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}\, d_i(m), \qquad 0 \le i \le m+1, \tag{6}$$

$$\begin{aligned} d_i(m+1) = {} & \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}\, d_i(m) \\ & - \frac{i(i+1)}{(m+1)(m+1-i)}\, d_{i+1}(m), \quad 0 \le i \le m, \end{aligned} \tag{7}$$

$$\begin{aligned} d_i(m+2) = {} & \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) \\ & - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \quad 0 \le i \le m+1, \end{aligned} \tag{8}$$

and for $0 \le i \le m+1$,

$$(m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0. \tag{9}$$

Moll [2007] independently derived the recurrence relations (6) and (9) from which the other two relations can be easily deduced.

To prove Theorem 2.1(4), we need the following lemma.

Lemma 2.2. *Assume that $m \ge 2$. For $0 \le i \le m-2$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{(4m+2i+3)d_{i+1}(m)}{(4m+2i+7)d_{i+2}(m)}. \tag{10}$$

Proof. We proceed by induction on m. When $m = 2$, it is easy to check that the result holds. Assume that the lemma is valid for n, namely,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{(4n+2i+3)d_{i+1}(n)}{(4n+2i+7)d_{i+2}(n)}, \quad 0 \le i \le n-2. \tag{11}$$

We aim to show that (10) holds for $n+1$, that is,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{(4n+2i+7)d_{i+1}(n+1)}{(4n+2i+11)d_{i+2}(n+1)}, \qquad 0 \le i \le n-1. \tag{12}$$

From the recurrence relation (6), we deduce that, for $0 \le i \le n-1$,

$$\begin{aligned} & (2i+4n+7)d_{i+1}^2(n+1) - (2i+4n+11)d_i(n+1)d_{i+2}(n+1) \\ & \quad = (2i+4n+7)\left(\frac{i+n+1}{n+1}d_i(n) + \frac{2i+4n+5}{2(n+1)}d_{i+1}(n)\right)^2 \\ & \qquad - (2i+4n+11)\left(\frac{i+n+2}{n+1}d_{i+1}(n) + \frac{2i+4n+7}{2(n+1)}d_{i+2}(n)\right) \\ & \qquad \times \left(\frac{n+i}{n+1}d_{i-1}(n) + \frac{2i+4n+3}{2(n+1)}d_i(n)\right) \end{aligned}$$

$$= \frac{A_1(n,i)+A_2(n,i)+A_3(n,i)}{4(n+1)^2},$$

where $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are given by

$$\begin{aligned}
A_1(n,i) &= 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&\qquad -4(n+i)(2i+4n+11)(i+n+2)d_{i+1}(n)d_{i-1}(n), \\
A_2(n,i) &= (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&\qquad -(2i+4n+3)(2i+4n+11)(2i+4n+7)d_i(m)d_{i+2}(n), \\
A_3(n,i) &= (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8) \\
&\qquad \cdot d_{i+1}(n)d_i(n)-2(n+i)(2i+4n+11)(2i+4n+7)d_{i+2}(n)d_{i-1}(n).
\end{aligned}$$

We will show that $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are all positive for $0 \le i \le n-2$. By the induction hypothesis (11), we find that for $0 \le i \le n-2$,

$$\begin{aligned}
A_1(n,i) &> 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\
&\qquad -4(n+i)(2i+4n+11)(i+n+2)\frac{(4n+2i+1)}{(4n+2i+5)}\, d_i^2(n) \\
&= 4\,\frac{35+96n+72i+64ni+40n^2+28i^2}{2i+4n+5}\, d_i^2(n), \\
A_2(n,i) &> (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\
&\qquad -(2i+4n+3)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)}{(4n+2i+7)}\, d_{i+1}^2(n) \\
&= (40i+80n+76)d_{i+1}^2(n),
\end{aligned}$$

which are both positive. Also by the induction hypothesis (11), we see that

$$d_i(n)d_{i+1}(n) > \frac{(2i+4n+5)(2i+4n+7)}{(2i+4n+3)(2i+4n+1)} d_{i-1}(n)d_{i+2}(n), \tag{13}$$

for $0 \le i \le n-2$. This implies that

$$\begin{aligned}
&A_3(n,i) \\
&> (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8)d_{i+1}(n)d_i(n) \\
&\qquad -2(n+i)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)(4n+2i+1)}{(4n+2i+5)(4n+2i+7)}\, d_{i+1}(n)d_i(n) \\
&= 8\,\frac{5+22n+30i+44ni+24n^2+16i^2}{2i+4n+5}\, d_{i+1}(n)d_i(n),
\end{aligned}$$

which is positive for $0 \le i \le n-2$. Hence the inequality (12) holds for $0 \le i \le n-2$. It remains to show that (12) is true for $i = n-1$, that is,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} < \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}. \tag{14}$$

From (2) it follows that

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1}, \tag{15}$$

$$d_{n+1}(n+1) = \frac{1}{2^{n+1}}\binom{2n+2}{n+1}, \tag{16}$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}. \tag{17}$$

Consequently,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} = \frac{n(4n^2+10n+7)}{2(2n+1)(2n+3)} < \frac{(2n+3)(6n+5)}{2(6n+9)} = \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}.$$

This completes the proof. □

We can now prove Theorem 2.1(4). In fact, we shall prove a stronger inequality.

Lemma 2.3. *Assume that* $m \ge 2$. *For* $0 \le i \le m-1$, *we have*

$$\frac{d_i(m)}{d_{i+1}(m)} > \frac{(2i+4m+5)d_i(m+1)}{(2i+4m+3)d_{i+1}(m+1)}. \tag{18}$$

Proof. By Lemma 2.2, we have for $0 \le i \le m-1$,

$$d_i^2(m) > \frac{2i+4m+5}{2i+4m+1}\, d_{i-1}(m)d_{i+1}(m). \tag{19}$$

From (19) and the recurrence relation (6), for $0 \le i \le m-1$,

$$\begin{aligned}
&d_{i+1}(m+1)d_i(m) - \frac{2i+4m+5}{2i+4m+3}\, d_{i+1}(m)d_i(m+1)\\
&\quad= \frac{2i+4m+5}{2(m+1)}\, d_{i+1}(m)d_i(m) + \frac{i+m+1}{m+1}\, d_i(m)^2\\
&\qquad\qquad - \frac{2i+4m+5}{2i+4m+3}\left(\frac{2i+4m+3}{2(m+1)} d_i(m)d_{i+1}(m) + \frac{i+m}{m+1} d_{i-1}(m)d_{i+1}(m)\right)\\
&\quad= \frac{i+m+1}{m+1} d_i^2(m) - \frac{(4m+2i+5)(m+i)}{(4m+2i+3)(m+1)}\, d_{i-1}(m)d_{i+1}(m)\\
&\quad> \left(\frac{m+1+i}{m+1} - \frac{(4m+2i+1)(m+i)}{(4m+2i+3)(m+1)}\right) d_i^2(m)\\
&\quad= \frac{6m+4i+3}{(4m+2i+3)(m+1)}\, d_i^2(m) > 0,
\end{aligned}$$

which yields (18). □

We now turn to the proof of Theorem 2.1(5).

Lemma 2.4. *Assume that $m \geq 2$. For $0 \leq i \leq m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{d_{i+1}(m+1)}{d_{i+2}(m+1)}. \tag{20}$$

Proof. We proceed by induction on m. It is easy to check the lemma holds for $m = 2$. Assume that the lemma is true for $n \geq 2$, that is,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{d_{i+1}(n+1)}{d_{i+2}(n+1)}, \quad 0 \leq i \leq n-1. \tag{21}$$

It will be shown that the theorem holds for $n+1$, that is,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{d_{i+1}(n+2)}{d_{i+2}(n+2)}, \quad 0 \leq i \leq n. \tag{22}$$

Recall that the sequence $\{d_i(n+1)\}_{0 \leq i \leq n+1}$ is unimodal. Furthermore, from (3) or the ratio monotone property [Chen and Xia 2009], the maximum element appears in the middle, namely, $d_i(n+1) < d_{i+1}(n+1)$ when $0 \leq i \leq [(n+1)/2]-1$ and $d_i(n+1) > d_{i+1}(n+1)$ when $[(n+1)/2] \leq i \leq n$.

Showing (22) for $0 \leq i \leq n-1$ breaks into two cases.

The first case is $d_i(n+1) < d_{i+1}(n+1)$, namely, $0 \leq i \leq [(n+1)/2]-1$. From the recurrence relation (6), we find that for $0 \leq i \leq [(n+1)/2]-1$,

$$\begin{aligned}
&d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) + \frac{i+n+2}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\qquad\qquad - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) - \frac{i+n+3}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad = \frac{2i+4n+9}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1) \\
&\qquad\qquad - \frac{1}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\quad > \frac{2i+4n+7}{2(n+2)} d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)} d_i(n+1)d_{i+2}(n+1),
\end{aligned}$$

which is positive by Lemma 2.2. It follows that for $0 \leq i \leq [(n+1)/2]-1$,

$$d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) > 0. \tag{23}$$

This completes the proof of the first case.

The second case is when $[(n+1)/2] \leq i \leq n-1$. From the recurrence relations (6) and (7), it follows that for $[(n+1)/2] \leq i \leq n-1$,

$$
\begin{aligned}
&d_{i+1}(n+2)d_{i+1}(n+1)-d_{i+2}(n+2)d_i(n+1)\\
&\quad=\left(\frac{(4n-2i+5)(n+i+3)}{2(n+2)(n+1-i)}\,d_{i+1}(n+1)-\frac{(i+1)(i+2)}{(n+2)(n+1-i)}\,d_{i+2}(n+1)\right)\\
&\qquad\times\left(\frac{n+1+i}{n+1}d_i(n)+\frac{4n+2i+5}{2(n+1)}d_{i+1}(n)\right)\\
&\qquad\quad-\left(\frac{n+3+i}{n+2}\,d_{i+1}(n+1)+\frac{4n+2i+11}{2(n+2)}d_{i+2}(n+1)\right)\\
&\qquad\qquad\times\left(\frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)}\,d_i(n)-\frac{i(i+1)}{(n+1)(n+1-i)}\,d_{i+1}(n)\right)\\
&\quad=B_1(n,i)d_{i+1}(n+1)d_i(n)+B_2(n,i)d_{i+1}(n+1)d_{i+1}(n)\\
&\qquad\qquad+B_3(n,i)d_{i+2}(n+1)d_i(n)+B_4(n,i)d_{i+2}(n+1)d_{i+1}(n),
\end{aligned}
$$

where $B_1(n,i)$, $B_2(n,i)$, $B_3(n,i)$ and $B_4(n,i)$ are given by

$$B_1(n,i)=\frac{(n+i+3)(n+1+i)}{(n+2)(n+1-i)(n+1)}, \tag{24}$$

$$B_2(n,i)=\frac{(n+i+3)(16n^2+40n+25+4i)}{4(n+2)(n+1-i)(n+1)}, \tag{25}$$

$$B_3(n,i)=-\frac{(n+1+i)(41+16n^2+56n-4i)}{4(n+2)(n+1-i)(n+1)}, \tag{26}$$

$$B_4(n,i)=-\frac{(i+1)(4n+5-i)}{(n+2)(n+1-i)(n+1)}. \tag{27}$$

Since $[(n+1)/2]\le i\le n-1$, it follows from (3) that $d_{i+1}(n+1)>d_{i+2}(n+1)$ and $d_i(n)>d_{i+1}(n)$. Thus we get

$$d_{i+1}(n+1)d_i(n)>d_{i+1}(n+1)d_{i+1}(n), \tag{28}$$

$$d_{i+1}(n+1)d_{i+1}(n)>d_{i+2}(n+1)d_{i+1}(n). \tag{29}$$

Observe that $B_1(n,i)$ and $B_2(n,i)$ are positive, and $B_3(n,i)$ and $B_4(n,i)$ are negative. By the induction hypothesis (21) and inequalities (28) and (29), we find that, for $[(n+1)/2]\le i\le n-1$,

$$
\begin{aligned}
&d_{i+1}(n+2)d_{i+1}(n+1)-d_{i+2}(n+2)d_i(n+1)\\
&\qquad>\left(B_1(n,i)+B_2(n,i)+B_3(n,i)+B_4(n,i)\right)d_{i+1}(n+1)d_{i+1}(n)\\
&\qquad=\frac{24n+10n^2-8ni+8i^2+13}{2(n+2)(n+1-i)(n+1)}d_{i+1}(n+1)d_{i+1}(n)>0.
\end{aligned} \tag{30}
$$

From the inequalities (23) and (30), it follows that (22) holds for $0\le i\le n-1$.

It is still necessary to show that (22) is true for $i = n$, that is,

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} < \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)}. \tag{31}$$

For the recurrence relation (9), setting $i = n+2$, we find that

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} = \frac{2n+3}{2} < \frac{2n+5}{2} = \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)},$$

as desired. Hence the proof is complete by induction. □

Lemmas 2.3 and 2.4 immediately imply the interlacing log-concavity of the Boros–Moll polynomials.

3. Polynomials with triangular relations on coefficients

Many combinatorial polynomials admit triangular relations on the coefficients. The log-concavity of polynomials of this kind has been extensively studied. We show that many classical polynomials of this kind are also interlacingly log-concave. For example, it is easy to check that the binomial coefficients, the Narayana numbers

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

and the Bessel numbers

$$B(n,k) = \frac{(2n-k-1)!}{2^k(n-k)!(k-1)!}$$

are interlacingly log-concave. Moreover, we give a criterion that applies to many combinatorial sequences such as the signless Stirling numbers of the first kind, the Stirling numbers of the second kind and the Whitney numbers.

Theorem 3.1. *Suppose that for any $n \geq 0$,*

$$G_n(x) = \sum_{k=0}^{n} T(n,k)x^k$$

is a polynomial of degree n which has only real zeros, and suppose that the coefficients $T(n,k)$ satisfy a recurrence relation of the form

$$T(n,k) = f(n,k)T(n-1,k) + g(n,k)T(n-1,k-1).$$

If

$$\frac{(n-k)k}{(n-k+1)(k+1)} f(n+1,k+1) \leq f(n+1,k) \leq f(n+1,k+1), \tag{32}$$

$$g(n+1,k+1) \leq g(n+1,k) \leq \frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1), \tag{33}$$

then the polynomials $G_n(x)$ are interlacingly log-concave.

Proof. Since the polynomial $G_n(x)$ has only real zeros, by Newton's inequality,

$$k(n-k)T(n,k)^2 \geq (k+1)(n-k+1)T(n,k-1)T(n,k+1).$$

Hence

$$\begin{aligned}
&T(n,k)T(n+1,k+1)-T(n+1,k)T(n,k+1)\\
&\quad= f(n+1,k+1)T(n,k)T(n,k+1)+g(n+1,k+1)T(n,k)^2\\
&\qquad\qquad -f(n+1,k)T(n,k)T(n,k+1)-g(n+1,k)T(n,k-1)T(n,k+1)\\
&\quad\geq (f(n+1,k+1)-f(n+1,k))\,T(n,k)T(n,k+1)\\
&\qquad +\left(\frac{(n-k+1)(k+1)}{(n-k)k}g(n+1,k+1)-g(n+1,k)\right)T(n,k-1)T(n,k+1),
\end{aligned}$$

which is positive by (32) and (33). It follows that

$$\frac{T(n,k)}{T(n,k+1)} \geq \frac{T(n+1,k)}{T(n+1,k+1)}. \tag{34}$$

On the other hand, we have

$$\begin{aligned}
&T(n,k+1)T(n+1,k+1)-T(n,k)T(n+1,k+2)\\
&\quad= f(n+1,k+1)T(n,k+1)^2+g(n+1,k+1)T(n,k)T(n,k+1)\\
&\qquad\qquad -f(n+1,k+2)T(n,k)T(n,k+2)-g(n+1,k+2)T(n,k+1)T(n,k)\\
&\quad\geq \left(f(n+1,k+1)-\frac{(n-k-1)(k+1)}{(n-k)(k+2)}f(n+1,k+2)\right)T(n,k+1)^2\\
&\qquad\qquad +(g(n+1,k+1)-g(n+1,k+2))T(n,k+1)T(n,k).
\end{aligned}$$

It follows from (32) that

$$\frac{T(n,k)}{T(n,k+1)} \leq \frac{T(n+1,k+1)}{T(n+1,k+2)}. \tag{35}$$

This completes the proof. □

Employing Theorem 3.1, we can show that many combinatorial polynomials which have only real zeros are interlacingly log-concave, for example,

(1) the polynomials

$$x(x+1)(x+2)\cdots(x+n-1),$$

whose coefficients are the signless Stirling numbers of the first kind, which satisfy the recurrence relation

$$c(n,k)=(n-1)c(n-1,k)+c(n-1,k-1);$$

(2) the Bell polynomials whose coefficients are the Stirling numbers of the second kind $S(n, k)$, which satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k);$$

(3) the Whitney polynomials

$$W_n(x) = \sum_{k=0}^{n} W_m(n, k)x^k,$$

which have only real zeros; see [Benoumhani 1997; 1999]. The coefficients $W_m(n, k)$ satisfy the recurrence relation

$$W_m(n, k) = (1+mk)W_m(n-1, k) + W_m(n-1, k-1).$$

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SCHWARZIAN NORMS AND TWO-POINT DISTORTION

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An analytic function f with Schwarzian norm $\|\mathcal{S}f\| \le 2(1+\delta^2)$ is shown to satisfy a pair of two-point distortion conditions, one giving a lower bound and the other an upper bound for the deviation. Conversely, each of these conditions is found to imply that $\|\mathcal{S}f\| \le 2(1+\delta^2)$. Analogues of the lower bound are also developed for curves in $\mathbb{R}^n$ and for canonical lifts of harmonic mappings to minimal surfaces.

1. Introduction

A well known theorem of Nehari [16] states that if the Schwarzian derivative $\mathcal{S}f = (f''/f')' - \frac{1}{2}(f''/f')^2$ of an analytic locally univalent function f satisfies the inequality

$$|\mathcal{S}f(z)| \le \frac{2}{(1-|z|^2)^2} \tag{1}$$

for all points z in the unit disk $\mathbb{D}$, then f is univalent in $\mathbb{D}$. The result is the best possible, since for any $\delta > 0$ the weaker condition

$$|\mathcal{S}f(z)| \le \frac{2(1+\delta^2)}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{2}$$

admits functions f with infinite valence. However, such functions are uniformly locally univalent in the sense that any two distinct points where f assumes equal values are uniformly separated in the hyperbolic metric

$$d(\alpha, \beta) = \frac{1}{2}\log\frac{1+\rho(\alpha,\beta)}{1-\rho(\alpha,\beta)}, \quad \text{where } \rho(\alpha,\beta) = \left|\frac{\alpha-\beta}{1-\bar{\alpha}\beta}\right|.$$

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More precisely, if f satisfies the inequality (2) for some constant $\delta > 0$, then $d(\alpha, \beta) \geq \pi/\delta$ for any pair of points α and β in $\mathbb{D}$ where $f(\alpha) = f(\beta)$ but $\alpha \neq \beta$. Moreover, the separation constant π/δ is best possible. This result is essentially due to B. Schwarz [17]. A proof and further discussion can be found in [6]. Generalizations to Nehari functions other than $p(x) = (1 - x^2)^{-2}$ are given in [6] and [8].

The Schwarzian norm of an analytic locally univalent function f is defined by

$$\|\mathcal{S}f\| = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}f(z)|.$$

Thus Nehari's theorem says that f is univalent if $\|\mathcal{S}f\| \leq 2$, whereas the theorem of Schwarz says it is uniformly locally univalent if $\|\mathcal{S}f\| \leq 2(1 + \delta^2)$ for some constant $\delta > 0$.

Chuaqui and Pommerenke [4] gave a quantitative version of Nehari's theorem by showing that the condition $\|\mathcal{S}f\| \leq 2$ implies that f has the two-point distortion property

$$\Delta_f(\alpha, \beta) = \frac{|f(\alpha) - f(\beta)|}{\{(1 - |\alpha|^2)|f'(\alpha)|\}^{1/2} \{(1 - |\beta|^2)|f'(\beta)|\}^{1/2}} \geq d(\alpha, \beta) \tag{3}$$

for all points $\alpha, \beta \in \mathbb{D}$. Conversely, they found that if f satisfies (3), then $\|\mathcal{S}f\| \leq 2$. Thus the distortion property (3) actually characterizes functions in the Nehari class.

In the present paper we show more generally that for any $\delta > 0$ the analytic functions with Schwarzian norm $\|\mathcal{S}f\| \leq 2(1 + \delta^2)$ are characterized by the local distortion property

$$\Delta_f(\alpha, \beta) \geq \frac{1}{\delta} \sin(\delta\, d(\alpha, \beta)), \quad \alpha, \beta \in \mathbb{D},\ d(\alpha, \beta) \leq \frac{\pi}{\delta}. \tag{4}$$

The lower bound equals zero, as it must, when $d(\alpha, \beta) = 0$ or π/δ. Also, as $\delta \to 0$, the inequality (4) reduces to (3).

We also show that for any constant $\delta > 0$ an analytic function f has Schwarzian norm $\|\mathcal{S}f\| \leq 2(1 + \delta^2)$ if and only if

$$\Delta_f(\alpha, \beta) \leq \frac{1}{\sqrt{2+\delta^2}} \sinh\bigl(\sqrt{2 + \delta^2}\, d(\alpha, \beta)\bigr), \quad \alpha, \beta \in \mathbb{D}. \tag{5}$$

As a corollary, we can draw the rather surprising conclusion that for any constant $\delta > 0$ and any analytic function f, the upper bound (5) holds for all points $\alpha, \beta \in \mathbb{D}$ if and only if the lower bound (4) holds for all $\alpha, \beta \in \mathbb{D}$ with $d(\alpha, \beta) \leq \pi/\delta$. Also, an analytic function f satisfies $\Delta_f(\alpha, \beta) \leq (1/\sqrt{2}) \sinh\bigl(\sqrt{2}\, d(\alpha, \beta)\bigr)$ for all $\alpha, \beta \in \mathbb{D}$ if and only if f is univalent and $\|\mathcal{S}f\| \leq 2$.

The final section of the paper develops a generalization of the lower bound (4) for canonical lifts of harmonic mappings to minimal surfaces.

2. A basic lemma

The proofs make essential use of a comparison lemma for solutions of differential equations.

Comparison Lemma. *Let $Q(x)$ be continuous and $Q(x) > 0$ for $x \in [0, 1)$. Let $v(x)$ and $w(x)$ be defined as the solutions of*

$$v''(x) + Q(x)v(x) = 0, \quad v(0) = 0, \ v'(0) = 1,$$
$$w''(x) - Q(x)w(x) = 0, \quad w(0) = 0, \ w'(0) = 1,$$

respectively. Suppose that $v(x) > 0$ in an interval $(0, \xi)$, where $0 < \xi \le 1$. Let $p(z)$ be analytic and satisfy $|p(z)| \le Q(|z|)$ for all $z \in \mathbb{D}$. Then the solution of

$$u''(z) + p(z)u(z) = 0, \quad u(0) = 0, \ u'(0) = 1$$

satisfies the inequalities

$$v(|z|) \le |u(z)| \quad \text{for } |z| < \xi, \qquad |u(z)| \le w(|z|) \quad \text{for all } z \in \mathbb{D}.$$

It is clear that $w(x) > 0$ for all $x \in (0, 1)$, since the differential equation implies that $w''(x) \ge 0$. On the other hand, $v''(x) \le 0$ and so it is possible that $v(x) = 0$ for some $x \in (0, 1)$.

The upper inequality $|u(z)| \le w(|z|)$ was proved and applied by Essén and Keogh [12]. Herold [13] had previously obtained a more general result for differential equations of higher order. The lower inequality is essentially contained in [3], and a proof is sketched in [4]. For completeness we include detailed proofs of both inequalities here.

Proof of the Comparison Lemma. After rotation, the problem reduces to proving the inequalities for points z in the real interval $0 \le z < 1$. (Let $U(r) = u(re^{i\theta})$ for fixed θ.) To prove the upper inequality $|u(x)| \le w(x)$ for $0 \le x < 1$, we convert the differential equation and initial conditions to an integral equation. Integration gives

$$u'(x) = 1 - \int_0^x p(t)u(t)\,dt,$$
$$u(x) = x - \int_0^x \int_0^y p(t)u(t)\,dt\,dy.$$

Reversing the order of integration, we have

$$u(x) = x - \int_0^x (x-t)p(t)u(t)\,dt,$$

so that

$$|u(x)| \le x + \int_0^x (x-t)Q(t)|u(t)|\,dt, \quad 0 \le x < 1.$$

A similar analysis gives

$$w(x) = x + \int_0^x (x-t)Q(t)w(t)\,dt, \quad 0 \le x < 1.$$

Subtraction now shows that $h(x) = |u(x)| - w(x)$ satisfies

$$h(x) \le \int_0^x (x-t)Q(t)h(t)\,dt, \quad 0 \le x < 1.$$

To infer that $h(x) \le 0$, fix an arbitrary point $x_0 \in (0, 1)$ and let

$$s_0 = \sup\{s \in [0, 1) : h(x) \le 0 \text{ for all } x \in [0, s]\}.$$

If $s_0 < x_0$, let M be the maximum value of $Q(x)$ for $0 \le x \le x_0$ and choose $x_1 \in (s_0, x_0)$ such that $M(x_1 - s_0) < 1$. Let μ be the maximum value of $h(x)$ for $s_0 \le x \le x_1$, so that $\mu = h(x_2) > 0$ for some $x_2 \in (s_0, x_1]$. Then

$$\begin{aligned}\mu = h(x_2) \le \int_0^{x_2} (x_2 - t)Q(t)h(t)\,dt &\le \int_{s_0}^{x_2} (x_2 - t)Q(t)h(t)\,dt \\ &\le \int_{s_0}^{x_2} (x_2 - t)Q(t)\mu\,dt \le M(x_1 - s_0)\mu < \mu,\end{aligned}$$

a contradiction. This shows that $s_0 \ge x_0$, which proves $h(x) \le 0$ or $|u(x)| \le w(x)$ in $[0, x_0)$, hence in $[0, 1)$, since the point x_0 was chosen arbitrarily in $(0, 1)$. Thus $|u(z)| \le w(|z|)$ for all $z \in \mathbb{D}$.

Now consider the lower bound $v(|z|) \le |u(z)|$ for $|z| < \xi$. Again it suffices to carry out the proof for $z \in [0, 1)$. Let $\varphi(x) = |u(x)|$, so that $\varphi^2 = u\bar{u}$, and calculate

$$\varphi(x)\varphi'(x) = \tfrac{1}{2}(u'(x)\overline{u(x)} + u(x)\overline{u'(x)}) = \operatorname{Re}\{u'(x)\overline{u(x)}\}.$$

Hence $|\varphi'(x)| \le |u'(x)|$ wherever $u(x) \ne 0$. Another differentiation gives

$$\varphi(x)\varphi''(x) + \varphi'(x)^2 = \operatorname{Re}\{u''(x)\overline{u(x)}\} + |u'(x)|^2,$$

from which we infer that

$$\varphi(x)\varphi''(x) \ge \operatorname{Re}\{u''(x)\overline{u(x)}\} = -\operatorname{Re}\{p(x)\}\varphi(x)^2,$$

in view of the differential equation for u. Consequently, since $\varphi(x) = |u(x)| \ge 0$ and $|p(x)| \le Q(x)$, we arrive at the differential inequality

$$\varphi''(x) + Q(x)\varphi(x) \ge 0, \quad 0 \le x < 1.$$

On the other hand, the function v satisfies the differential equation

$$v''(x) + Q(x)v(x) = 0, \quad 0 \le x < 1.$$

Since $v(0) = \varphi(0) = 0$ and $v'(0) = \varphi'(0) > 0$, it now follows from the Sturm comparison theorem that $\varphi(x) \geq v(x)$ up to the first zero of v. Thus $|u(x)| \geq v(x)$ for $0 \leq x < \xi$, and so $|u(z)| \geq v(|z|)$ for $|z| < \xi$. □

3. Distortion of analytic functions

We turn now to the main result of this paper. It will be convenient to employ the notation $\Delta_f(\alpha, \beta)$ defined by (3), where f is analytic and locally univalent in the disk and $\alpha, \beta \in \mathbb{D}$. It is important that this quantity is invariant under both precomposition and postcomposition with Möbius transformations. Specifically, if σ is any Möbius automorphism of the disk, then

$$\Delta_{f\circ\sigma}(\alpha, \beta) = \Delta_f(\sigma(\alpha), \sigma(\beta)), \quad \alpha, \beta \in \mathbb{D},$$

as can be seen by direct calculation using the identity

$$\frac{|\sigma'(z)|}{1 - |\sigma(z)|^2} = \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}. \tag{6}$$

To show that

$$\Delta_{T\circ f}(\alpha, \beta) = \Delta_f(\alpha, \beta)$$

for every Möbius transformation T, it suffices to verify by simple calculation that $\Delta_{1/f}(\alpha, \beta) = \Delta_f(\alpha, \beta)$, since the relation clearly holds for every affine mapping T.

Theorem 1. *Let f be analytic and locally univalent in $\mathbb{D}$ and suppose that the bound $\|\mathcal{S}f\| \leq 2(1+\delta^2)$ holds for some $\delta > 0$. Then*

$$\Delta_f(\alpha, \beta) \geq \frac{1}{\delta} \sin(\delta\, d(\alpha, \beta)) \tag{7}$$

for all $\alpha, \beta \in \mathbb{D}$ with hyperbolic separation $d(\alpha, \beta) \leq \pi/\delta$, and

$$\Delta_f(\alpha, \beta) \leq \frac{1}{\sqrt{2+\delta^2}} \sinh\bigl(\sqrt{2+\delta^2}\, d(\alpha, \beta)\bigr) \tag{8}$$

for all $\alpha, \beta \in \mathbb{D}$. Each of the inequalities (7) *and* (8) *is sharp*; *for each pair of points α and β in the specified range*, *equality occurs for some function f with $\|\mathcal{S}f\| \leq 2(1+\delta^2)$. Equality holds in* (7) *precisely for $f = T \circ F \circ \sigma$ and in* (8) *for $f = T \circ G \circ \sigma$*, *where F and G are defined by*

$$F(z) = \left(\frac{1+z}{1-z}\right)^{i\delta} \quad \text{and} \quad G(z) = \left(\frac{1+z}{1-z}\right)^{\sqrt{2+\delta^2}}, \tag{9}$$

σ is the Möbius automorphism of $\mathbb{D}$ with $\sigma(\alpha) = 0$ and $\sigma(\beta) > 0$, and T is an arbitrary Möbius transformation. For each such function f, equality holds along the entire (*admissible portion of the*) *hyperbolic geodesic through α and β. Conversely*, *if either inequality holds for all points α and β in the specified range*, *then $\|\mathcal{S}f\| \leq 2(1+\delta^2)$.*

Proof. The strategy is to establish the inequalities first in the special case where $\alpha = 0$, then to derive them in the general case by Möbius invariance. Suppose that

$$|\mathcal{S}f(z)| \leq \frac{2(1+\delta^2)}{(1-|z|^2)^2}, \quad z \in \mathbb{D},$$

for some $\delta > 0$, and assume without loss of generality that $f(0) = 0$ and $f'(0) = 1$. Define

$$g(z) = -\frac{1}{f(z)}, \quad \text{so that} \quad g'(z) = \frac{f'(z)}{f(z)^2}.$$

Then the function

$$u(z) = (g'(z))^{-1/2} = z + c_2 z^2 + \cdots$$

is analytic in $\mathbb{D}$, with $u(0) = 0$ and $u'(0) = 1$, and it satisfies the differential equation

$$u'' + (\tfrac{1}{2}\mathcal{S}f)\, u = 0,$$

since $\mathcal{S}g = \mathcal{S}f$. Define the functions $v(x)$ and $w(x)$ by

$$v''(x) + \frac{1+\delta^2}{(1-x^2)^2}\, v(x) = 0, \quad v(0) = 0, \; v'(0) = 1,$$
$$w''(x) - \frac{1+\delta^2}{(1-x^2)^2}\, w(x) = 0, \quad w(0) = 0, \; w'(0) = 1.$$

Suppose $v(x) > 0$ in the interval $(0, \xi)$, where $0 < \xi \leq 1$. Then in view of the hypothesis that $\left|\frac{1}{2}\mathcal{S}f(z)\right| \leq (1+\delta^2)(1-|z|^2)^{-2}$ in $\mathbb{D}$, by the Comparison Lemma $|u(z)| \leq w(|z|)$ for all $z \in \mathbb{D}$, and $v(|z|) \leq |u(z)|$ for all $z \in \mathbb{D}$ with $|z| < \xi$.

The solutions $v(x)$ and $w(x)$ are

$$v(x) = \frac{1}{\delta}\sqrt{1-x^2}\, \sin\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right), \tag{10}$$

$$w(x) = \frac{\sqrt{1-x^2}}{\sqrt{2+\delta^2}}\, \sinh\left(\frac{\sqrt{2+\delta^2}}{2}\log\frac{1+x}{1-x}\right). \tag{11}$$

These explicit formulas can be found with reference to Kamke [14], or by means of the substitution

$$y(t) = \frac{v(x)}{\sqrt{1-x^2}}, \quad \text{where } t = \frac{1}{2}\log\frac{1+x}{1-x},$$

which reduces the first differential equation to $y''(t) + \delta^2 y(t) = 0$. Similarly, the second equation reduces to $y''(t) - (2+\delta^2)y(t) = 0$ through the same substitution with w in place of v.

The first positive zero of $v(x)$ occurs at the point $\xi = \tanh(\pi/\delta)$. Since

$$u(z) = (g'(z))^{-1/2} = f(z)(f'(z))^{-1/2},$$

the inequality $|u(z)| \geq v(|z|)$ obtained from the Comparison Lemma reduces to

$$\frac{|f(z)|^2}{|f'(z)|} \geq \frac{1}{\delta^2}(1-|z|^2)\sin^2\Bigl(\frac{\delta}{2}\,\log\frac{1+|z|}{1-|z|}\Bigr), \tag{12}$$

or

$$\Delta_f(0,z) \geq \frac{1}{\delta}\,\sin\left(\delta\, d(0,z)\right) \quad \text{for } d(0,z) \leq \frac{\pi}{\delta}.$$

Now let α and β be arbitrary points in the unit disk and define

$$f_1(z) = \frac{f(\sigma(z)) - f(\alpha)}{(1-|\alpha|^2)f'(\alpha)}, \quad \text{where } \sigma(z) = \frac{z+\alpha}{1+\bar{\alpha}z}. \tag{13}$$

This function has the form $f_1 = T \circ f \circ \sigma$, where T is a Möbius transformation, so

$$\Delta_{f_1}(0,z) = \Delta_{f\circ\sigma}(0,z) = \Delta_f(\sigma(0),\sigma(z)) = \Delta_f(\alpha,\sigma(z)).$$

On the other hand, $\mathcal{S}f_1 = \mathcal{S}(f\circ\sigma) = \bigl((\mathcal{S}f)\circ\sigma\bigr)\sigma'^2$, so that

$$|\mathcal{S}f_1(z)| = |\mathcal{S}f(\sigma(z))||\sigma'(z)|^2 \leq \frac{2(1+\delta^2)|\sigma'(z)|^2}{(1-|\sigma(z)|^2)^2} = \frac{2(1+\delta^2)}{(1-|z|^2)^2}.$$

Since $\|\mathcal{S}f_1\| \leq 2(1+\delta^2)$ and $f_1(0)=0$, $f_1'(0)=1$, it follows from what has already been proved that

$$\Delta_{f_1}(0,z) \geq \frac{1}{\delta}\,\sin\left(\delta\, d(0,z)\right), \quad d(0,z) \leq \frac{\pi}{\delta}.$$

Therefore, if z is chosen so that $\sigma(z) = \beta$, we have

$$\Delta_f(\alpha,\beta) = \Delta_{f_1}(0,z) \geq \frac{1}{\delta}\,\sin\left(\delta\, d(\sigma(0),\sigma(z)))\ \right) = \frac{1}{\delta}\,\sin\left(\delta\, d(\alpha,\beta)\right)$$

for $d(\alpha,\beta) \leq \pi/\delta$, by the invariance of the hyperbolic metric under Möbius automorphisms of $\mathbb{D}$. The proof of the lower bound (7) is now complete.

The upper bound is derived in similar fashion. The Comparison Lemma gives $|u(z)| \leq w(|z|)$ for all $z \in \mathbb{D}$, which reduces to

$$\Delta_f(0,z) \leq \frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(0,z)\bigr).$$

It then follows as before that

$$\Delta_f(\alpha,\beta) \leq \frac{1}{\sqrt{2+\delta^2}}\,\sinh\bigl(\sqrt{2+\delta^2}\,d(\alpha,\beta)\bigr), \quad \alpha,\beta \in \mathbb{D},$$

by choosing $z = \sigma^{-1}(\beta)$. This proves (8).

In order to prove the sharpness of (7), we now show that for each pair of points $\alpha, \beta \in \mathbb{D}$ with $0 < d(\alpha,\beta) < \pi/\delta$, there is a function f with $\|\mathcal{S}f\| \leq 2(1+\delta^2)$ such that $\Delta_f(\alpha,\beta) = (1/\delta)\ \sin(\delta\, d(\alpha,\beta))$. By Möbius invariance, it is equivalent to show that $\Delta_F(0,b) = (1/\delta)\ \sin(\delta\, d(0,b))$, where $F = f\circ\sigma^{-1}$ and σ is the Möbius automorphism of the disk for which $\sigma(\alpha)=0$ and $\sigma(\beta)=b>0$. This will be the case if and only if $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$, which is the requirement for equality in the Comparison Lemma (see [3]). Thus the general form of the extremal function is $f = T \circ F \circ \sigma$, where F is a particular function (as given by (9), for instance) with Schwarzian $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$, σ is the Möbius automorphism defined above, and T is an arbitrary Möbius transformation. Similarly, for each pair of distinct points $\alpha, \beta \in \mathbb{D}$, equality occurs in (8) precisely for functions of the form $f = T \circ G \circ \sigma$, where G is a particular function (as defined by (9), for instance) with $\mathcal{S}G(z) = -2(1+\delta^2)(1-z^2)^{-2}$, σ is the Möbius automorphism with $\sigma(\alpha) = 0$ and $\sigma(\beta) > 0$, and T is an arbitrary Möbius transformation (see [12]).

Conversely, we want to show that either of the two-point distortion conditions (7) or (8) implies the bound $\|\mathcal{S}f\| \le 2(1+\delta^2)$ on the Schwarzian norm. The proofs follow an argument given by Chuaqui and Pommerenke [4] to show that the condition (3) implies $\|\mathcal{S}f\| \le 2$. It will suffice to carry out the details only for the condition (8), because the proof for (7) is quite similar. In view of the Möbius invariance, no information is lost if we take $\alpha = 0$. Without loss of generality, we may assume that $f(0) = 0$ and $f'(0) = 1$, so that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots .$$

The condition (8) then reduces to

$$\frac{|f(z)|^2}{|f'(z)|} \le \frac{1-|z|^2}{2+\delta^2} \sinh^2\!\left(\sqrt{2+\delta^2}\, d(0,z)\right), \quad z \in \mathbb{D}. \tag{14}$$

In order to conclude from (14) that $\|\mathcal{S}f\| \le 2(1+\delta^2)$, it will suffice to show that $|\mathcal{S}f(0)| \le 2(1+\delta^2)$, because of the Möbius invariance. Indeed, for the function f_1 defined by (13) we have

$$(1-|z|^2)^2 |\mathcal{S}f_1(z)| = (1-|\sigma(z)|^2)^2 |\mathcal{S}f(\sigma(z))|,$$

and so $|\mathcal{S}f_1(0)| = (1-|\alpha|^2)^2 |\mathcal{S}f(\alpha)|$. But $\mathcal{S}f(0) = 6(a_3 - a_2^2)$, so the problem reduces to showing that $|a_3 - a_2^2| \le \frac{1}{3}(1+\delta^2)$. Straightforward calculations give

$$\frac{f(z)^2}{f'(z)} = z^2\left(1 + (a_2^2 - a_3)z^2 + \cdots\right),$$

$$\frac{1-|z|^2}{2+\delta^2} \sinh^2\!\left(\sqrt{2+\delta^2}\, d(0,z)\right) = r^2\left(1 + \tfrac{1}{3}(1+\delta^2)r^2 + \cdots\right), \quad r = |z|.$$

Therefore, the inequality (14) implies

$$\left|1 + (a_2^2 - a_3)z^2 + O(r^3)\right|^2 \le \left|1 + \tfrac{1}{3}(1+\delta^2)r^2 + O(r^3)\right|^2,$$

which reduces to

$$1 + 2\,\mathrm{Re}\left\{(a_2^2 - a_3)z^2 + O(r^3)\right\} \le 1 + \tfrac{2}{3}(1+\delta^2)r^2 + O(r^3).$$

From this we infer that

$$\mathrm{Re}\big\{(a_2^2 - a_3)e^{2i\theta}\big\} \le \tfrac{1}{3}(1+\delta^2)$$

by setting $z = re^{i\theta}$ for fixed θ and letting $r \to 0$. Since the angle θ can be chosen arbitrarily, we conclude that $|a_3 - a_2^2| \le \frac{1}{3}(1+\delta^2)$, as desired.

Essentially the same calculations show that if the inequality (12) holds for all $z \in \mathbb{D}$ with $d(0,z) \le \pi/\delta$ (or equivalently for $|z| \le \tanh(\pi/\delta)$), then $|\mathcal{S}f(0)| \le 2(1+\delta^2)$ and so $\|\mathcal{S}f\| \le 2(1+\delta^2)$. $\square$

Similar results are obtained under the hypothesis $\|\mathcal{S}f\| \le 2(1-\delta^2)$ for $0<\delta<1$. Then the relevant functions v and w of the Comparison Lemma are obtained by replacing δ by $i\delta$ in the formulas (10) and (11). Specifically,

$$v(x) = \frac{1}{\delta}\sqrt{1-x^2}\,\sinh\left(\frac{\delta}{2}\log\frac{1+x}{1-x}\right),$$
$$w(x) = \frac{\sqrt{1-x^2}}{\sqrt{2-\delta^2}}\,\sinh\left(\frac{\sqrt{2-\delta^2}}{2}\log\frac{1+x}{1-x}\right).$$

The inequalities $v(|z|) \le |u(z)| \le w(|z|)$ now reduce to

$$\frac{1}{\delta}\sinh(\delta\, d(0,z)) \le \Delta_f(0,z) \le \frac{1}{\sqrt{2-\delta^2}}\sinh\big(\sqrt{2-\delta^2}\, d(0,z)\big), \quad z \in \mathbb{D},$$

whereupon the same argument based on Möbius invariance gives

$$\frac{1}{\delta}\sinh(\delta\, d(\alpha,\beta)) \le \Delta_f(\alpha,\beta) \le \frac{1}{\sqrt{2-\delta^2}}\sinh\big(\sqrt{2-\delta^2}\, d(\alpha,\beta)\big) \tag{15}$$

for all $\alpha, \beta \in \mathbb{D}$. Conversely, if either of the inequalities in (15) holds for some $\delta \in (0,1)$ and for all α and β in $\mathbb{D}$, calculations similar to the above lead to the conclusion that $\|\mathcal{S}f\| \le 2(1-\delta^2)$.

Theorem 1 was essentially proved by Mejía [15] and was discovered independently in joint work by Chuaqui, Duren, and Osgood.

4. Distortion of harmonic mappings

By a similar method, the lower bound (7) can be extended to harmonic mappings, or rather to their canonical lifts to minimal surfaces. The result will generalize a theorem in [9] in the case of the extremal Nehari function $p(x) = (1-x^2)^{-2}$. As in [9], we begin with a distortion theorem for curves in $\mathbb{R}^n$.

Let $\varphi : (-1,1) \to \mathbb{R}^n$ be a mapping of class C^3 with $\varphi'(x) \ne 0$. The Ahlfors Schwarzian of φ is defined by

$$S_1\varphi = \frac{\langle\varphi', \varphi'''\rangle}{|\varphi'|^2} - 3\frac{\langle\varphi', \varphi''\rangle^2}{|\varphi'|^4} + \frac{3}{2}\frac{|\varphi''|^2}{|\varphi'|^2},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the Euclidean inner product and $|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x}\rangle$ for $\mathbf{x} \in \mathbb{R}^n$. As Ahlfors [1] observed, S_1 is invariant under postcomposition with Möbius transformations of $\mathbb{R}^n$. Chuaqui and Gevirtz [2] used it to give an injectivity criterion for curves. Here is a special case of their theorem.

Theorem A. *Let $\varphi : (-1, 1) \mapsto \mathbb{R}^n$ be a curve of class C^3 with tangent vector $\varphi'(x) \neq 0$. If $S_1\varphi(x) \leq 2(1-x^2)^{-2}$, then φ is injective.*

Chuaqui and Gevirtz also showed that the arclength $s = s(x)$ of the curve φ has Schwarzian

$$\mathcal{S}s(x) = S_1\varphi(x) - \tfrac{1}{2}|\varphi'(x)|^2\kappa(x)^2 \leq S_1\varphi(x), \tag{16}$$

where $\kappa = \kappa(x)$ is the curvature of φ.

Our next theorem extends Theorem A to a criterion for uniform local injectivity, in the manner of B. Schwarz's extension of Nehari's theorem. Moreover, it expresses the local injectivity in quantitative form as a two-point distortion result analogous to the lower bound (7) in Theorem 1. In terms of the curve $\varphi(x)$, we define

$$\Delta_\varphi(a, b) = \frac{|\varphi(a) - \varphi(b)|}{\{(1-a^2)|\varphi'(a)|\}^{1/2}\,\{(1-b^2)|\varphi'(b)|\}^{1/2}}, \quad a, b \in (-1, 1).$$

Theorem 2. *Let $\varphi : (-1, 1) \mapsto \mathbb{R}^n$ be a curve of class C^3 with $\varphi'(x) \neq 0$. If*

$$S_1\varphi(x) \leq \frac{2(1+\delta^2)}{(1-x^2)^2} \quad \textit{for some } \delta > 0,$$

then the inequality

$$\Delta_\varphi(a, b) \geq \frac{1}{\delta}\,\sin(\delta\, d(a, b)) \tag{17}$$

holds for all $a, b \in (-1, 1)$ with $d(a, b) \leq \pi/\delta$.

Proof. First, the quantity $\Delta_\varphi(a, b)$ is Möbius invariant. If σ is any Möbius automorphism of the disk that preserves the real segment $(-1, 1)$, or equivalently if σ is a Möbius automorphism with real coefficients, then

$$\Delta_{\varphi\circ\sigma}(a, b) = \Delta_\varphi(\sigma(a), \sigma(b)), \quad a, b \in (-1, 1).$$

If T is any Möbius transformation of $\mathbb{R}^n$, then $\Delta_{T\circ\varphi}(a, b) = \Delta_\varphi(a, b)$. The proofs for curves are essentially the same as for analytic functions.

As in the proof of Theorem 1, we will derive the inequality (17) first for $a = 0$, then deduce the general result by Möbius invariance. Because of Möbius invariance, we may assume without loss of generality that $\varphi(0) = 0$ and $|\varphi'(0)| = 1$. Consider the inverted curve

$$\Phi(x) = \frac{\varphi(x)}{|\varphi(x)|^2}, \quad \text{with } |\Phi'(x)| = \frac{|\varphi'(x)|}{|\varphi(x)|^2},$$

as a straightforward calculation of $|\Phi'(x)|^2$ shows. By Möbius invariance, $S_1\Phi = S_1\varphi$. Recall that if $g(x)$ is a real-valued function with $g'(x) > 0$, the function $u(x) = g'(x)^{-1/2}$ satisfies the differential equation $u'' + \frac{1}{2}(\mathcal{S}g)u = 0$. Thus if $g(x) = s(x)$, the arclength function along the curve $\Phi(x)$, then the function

$$u(x) = |\Phi'(x)|^{-1/2} = \frac{|\varphi(x)|}{|\varphi'(x)|^{1/2}}$$

satisfies $u'' + \frac{1}{2}(\mathcal{S}s)u = 0$ and has initial data $u(0) = 0$ and $u'(0) = 1$, since $\varphi(0) = 0$ and $|\varphi'(0)| = 1$. But

$$\mathcal{S}s(x) \le S_1\Phi(x) = S_1\varphi(x) \le \frac{2(1+\delta^2)}{(1-x^2)^2},$$

so by the Sturm comparison theorem $u(x) \ge v(x)$ for $0 \le x \le \tanh(\pi/\delta)$, where $v(x)$ is the function given in (10). In terms of the hyperbolic metric, this last inequality takes the form

$$\Delta_\varphi(0, x) \ge \frac{1}{\delta}\,\sin(\delta\, d(0, x)), \quad d(0, x) \le \pi/\delta,$$

which is the desired result (17) for $a = 0$. The general inequality (17) is deduced from this special case by Möbius invariance. $\square$

With the help of Theorem 2, we can now derive a two-point distortion inequality for the canonical lift of a harmonic mapping to a minimal surface. A harmonic mapping is a complex-valued harmonic function $f(z) = u(z) + iv(z)$ for $z = x + iy$ in the unit disk $\mathbb{D}$ of the complex plane. Such a mapping has a canonical decomposition $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. The basic properties of harmonic mappings are described in [11].

According to the Weierstrass–Enneper formulas, a harmonic mapping $f = h + \bar{g}$ with $|h'(z)| + |g'(z)| \ne 0$ lifts locally to a minimal surface described by conformal parameters if and only if its dilatation $\omega = g'/h'$ has the form $\omega = q^2$ for some meromorphic function q. The Cartesian coordinates (U, V, W) of the surface are then given by

$$U(z) = \mathrm{Re}\big\{f(z)\big\}, \quad V(z) = \mathrm{Im}\big\{f(z)\big\}, \quad W(z) = 2\ \mathrm{Im}\left\{\int_0^z h'(\zeta)q(\zeta)\,d\zeta\right\}.$$

We use the notation $\tilde{f}(z) = \big(U(z), V(z), W(z)\big)$ for the lifted mapping from $\mathbb{D}$ to the minimal surface. The first fundamental form of the surface is $ds^2 = \lambda^2|dz|^2$, where the conformal metric is $\lambda = |h'| + |g'|$.

For a harmonic mapping $f = h + \bar{g}$ with $\lambda(z) = |h'(z)| + |g'(z)| > 0$, whose dilatation is the square of a meromorphic function, the Schwarzian derivative is defined by the formula

$$\mathcal{S}f = 2(\sigma_{zz} - \sigma_z^2), \quad \sigma = \log\lambda.$$

If f is analytic, it is easily verified that $\mathcal{S}f$ reduces to the classical Schwarzian.

In [7], the following criterion was given for the lift of a harmonic mapping to be univalent.

Theorem B. *Let $f = h + \bar{g}$ be a harmonic mapping of the unit disk, with $\lambda(z) = |h'(z)| + |g'(z)| > 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\tilde{f}$ denote the Weierstrass–Enneper lift of f to a minimal surface with Gauss curvature $K = K(\tilde{f}(z))$ at the point $\tilde{f}(z)$. Suppose that the inequality*

$$|\mathcal{S}f(z)| + \lambda(z)^2 |K(\tilde{f}(z))| \le \frac{2}{(1-|z|^2)^2}$$

holds for all $z \in \mathbb{D}$. Then $\tilde{f}$ is univalent in $\mathbb{D}$.

If f is analytic, its associated minimal surface is the complex plane itself, with Gauss curvature $K = 0$, and the result reduces to Nehari's theorem.

In [9], Theorem B was sharpened to express the univalence in the form of a two-point distortion condition. It was shown in [6] that if the bound $2(1-|z|^2)^{-2}$ is weakened to $2(1+\delta^2)(1-|z|^2)^{-2}$, then $\tilde{f}$ is uniformly locally univalent, the analogue of B. Schwarz's extension of Nehari's theorem. We now express the uniform local univalence in quantitative form, thus obtaining a harmonic analogue of the lower bound (7) in Theorem 1. Let

$$\Delta_{\tilde{f}}(\alpha, \beta) = \frac{|\tilde{f}(\alpha) - \tilde{f}(\beta)|}{\{(1-|\alpha|^2)\lambda(\alpha)\}^{1/2}\{(1-|\beta|^2)\lambda(\beta)\}^{1/2}}, \qquad \alpha, \beta \in \mathbb{D},$$

where λ is the conformal metric of the minimal surface.

Theorem 3. *Let $f = h + \bar{g}$ be a harmonic mapping of the unit disk, with $\lambda(z) = |h'(z)| + |g'(z)| > 0$ and dilatation $g'/h' = q^2$ for some meromorphic function q. Let $\tilde{f}$ denote the canonical lift of f to a minimal surface. Suppose that*

$$|\mathcal{S}f(z)| + \lambda(z)^2 |K(\tilde{f}(z))| \le \frac{2(1+\delta^2)}{(1-|z|^2)^2}, \qquad z \in \mathbb{D}. \tag{18}$$

Then

$$\Delta_{\tilde{f}}(\alpha, \beta) \ge \frac{1}{\delta} \sin(\delta\, d(\alpha, \beta)) \tag{19}$$

for all $\alpha, \beta \in \mathbb{D}$ with hyperbolic separation $d(\alpha, \beta) \le \pi/\delta$. For each pair of points α, β with $0 < d(\alpha, \beta) < \pi/\delta$, equality occurs in (19) *only for harmonic mappings of the form $f = h + c\,\bar{h}$, with c a constant of modulus $|c| < 1$ and $h = T \circ F \circ \sigma$, where F is defined by* (9), *σ is the Möbius automorphism of $\mathbb{D}$ for which $\sigma(\alpha) = 0$ and $\sigma(\beta) > 0$, and T is an arbitrary Möbius transformation. The corresponding minimal surface is then a plane.*

Proof. The proof will apply Theorem 2. The canonical lift $\tilde{f}$ onto a minimal surface Σ defines a curve $\tilde{f} : (-1, 1) \to \Sigma \subset \mathbb{R}^3$. As shown in [7], the Ahlfors Schwarzian of this curve satisfies

$$\begin{aligned} S_1 \tilde{f}(x) &= \mathrm{Re}\{\mathcal{S} f(x)\} + \tfrac{1}{2}\lambda(x)^2 \kappa_e(\tilde{f}(x))^2 + \tfrac{1}{2}\lambda(x)^2 |K(\tilde{f}(x))| \\ &\leq \mathrm{Re}\{\mathcal{S} f(x)\} + \lambda(x)^2 |K(\tilde{f}(x))| \\ &\leq |\mathcal{S} f(x)| + \lambda(x)^2 |K(\tilde{f}(x))|, \quad -1 < x < 1, \end{aligned} \tag{20}$$

where $\kappa_e(\tilde{f}(x))$ denotes the normal curvature of the curve at the point $\tilde{f}(x)$. Thus the hypothesis (18) tells us that $S_1 \tilde{f}(x) \leq 2(1+\delta^2)(1-x^2)^{-2}$, and so by Theorem 2 we have the inequality

$$\Delta_{\tilde{f}}(a, b) \geq \frac{1}{\delta} \sin(\delta\, d(a, b)) \tag{21}$$

for all $a, b \in (-1, 1)$ with $d(a, b) \leq \pi/\delta$, since $|\tilde{f}'(x)| = \lambda(x)$.

To extend the inequality (21) to arbitrary points $\alpha, \beta \in \mathbb{D}$, we appeal again to Möbius invariance. First, the quantity $\Delta_{\tilde{f}}(\alpha, \beta)$ is invariant under precomposition with Möbius automorphisms of the disk. Indeed, if σ is any such automorphism, the composition $F = f \circ \sigma$ is a harmonic mapping with canonical lift $\tilde{F} = \tilde{f} \circ \sigma$ and conformal metric $\Lambda(z) = \lambda(\sigma(z))|\sigma'(z)|$. Combining this with the identity (6), we see that $\Delta_{\tilde{F}}(\alpha, \beta) = \Delta_{\tilde{f}}(\sigma(\alpha), \sigma(\beta))$. Given any pair of points $\alpha, \beta \in \mathbb{D}$, choose σ so that $\sigma(a) = \alpha$ and $\sigma(b) = \beta$ for some $a, b \in (-1, 1)$. In view of (6), the hypothesis (18) is also Möbius invariant, and so $\Delta_{\tilde{F}}(a, b) \geq (1/\delta) \sin(\delta\, d(a, b))$, by what we have already proved. But $d(a, b) = d(\alpha, \beta)$ by Möbius invariance of the hyperbolic metric, whereas

$$\Delta_{\tilde{F}}(a, b) = \Delta_{\tilde{f}}(\sigma(a), \sigma(b)) = \Delta_{\tilde{f}}(\alpha, \beta).$$

Therefore, the inequality (19) holds for all points $\alpha, \beta \in \mathbb{D}$ with $d(\alpha, \beta) \leq \pi/\delta$.

We now turn to the case of equality in (19) for two distinct points $\alpha, \beta \in \mathbb{D}$ with $d(\alpha, \beta) < \pi/\delta$. After precomposing with an automorphism of the disk, we may assume that $\alpha = 0$ and $\beta = r$ with $0 < r < \pi/\delta$. More precisely, if σ is the automorphism with $\sigma(\alpha) = 0$ and $\sigma(\beta) = r > 0$, we need only consider equality for functions $f_1 = f \circ \sigma^{-1}$ at the points 0 and r. Let $\varphi(x) = \widetilde{f_1}(x)$ denote the lifted curve on the corresponding minimal surface Σ. With the notation in the proof of Theorem 2, we see that equality in (19), namely $\Delta_{\widetilde{f_1}}(0, r) = (1/\delta) \sin(\delta\, d(0, r))$, is equivalent to $u(r) = (1/\delta) \sin(\delta\, d(0, r))$, which by the Sturm comparison theorem can occur only if

$$\mathcal{S} s(x) = S_1 \varphi(x) = \frac{2(1+\delta^2)}{(1-x^2)^2} \quad \text{for all } x \in [0, r]. \tag{22}$$

But in view of (16), the equality $\mathcal{S}s(x) = S_1\varphi(x)$ implies that the curvature $\kappa(x)$ of the curve φ vanishes for all $x \in [0, r]$, and so that portion of the curve is a straight line in space. On the other hand, because of (20) and the hypothesis (18), the equality $S_1\varphi(x) = 2(1+\delta^2)(1-x^2)^{-2}$ implies that the normal curvature has the property $\kappa_e(\varphi(x))^2 \equiv |K(\varphi(x))|$ on $[0, r]$, so that the corresponding portion of the curve is a line of curvature of Σ. (Here we use the fact that Σ is a minimal surface, with zero mean curvature.) But by uniqueness in the Björling problem (see [10]), a minimal surface containing a straight line segment as a line of curvature must reduce to a plane. Therefore, as shown in [5], the harmonic mapping f_1 has the form $h_1 + c\,\overline{h_1}$ for some locally univalent analytic function h_1 and some constant c with $|c| < 1$. It is then easily seen that $\mathcal{S}f_1 = \mathcal{S}h_1$. Furthermore, since the surface Σ is a plane, it has Gauss curvature $K = 0$, and so (22) combines with (20) and (18) to show that

$$\mathcal{S}h_1(x) = \mathcal{S}f_1(x) = S_1\widetilde{f}_1(x) = \frac{2(1+\delta^2)}{(1-x^2)^2} \quad \text{for all } x \in [0, r].$$

But $\mathcal{S}h_1$ is an analytic function, so this implies that $\mathcal{S}h_1(z) = 2(1+\delta^2)(1-z^2)^{-2}$ for all $z \in \mathbb{D}$. Therefore, $h_1 = T \circ F$, where T is a Möbius transformation and F is a particular function (as given by (9), for instance) with Schwarzian $\mathcal{S}F(z) = 2(1+\delta^2)(1-z^2)^{-2}$. Hence $f = f_1 \circ \sigma = h + c\,\overline{h}$, where $h = T \circ F \circ \sigma$, as claimed. The argument also shows, as in Theorem 1, that the same functions f give equality along the entire hyperbolic geodesic through α and β. □

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THE PRINCIPLE OF STATIONARY PHASE FOR THE FOURIER TRANSFORM OF D-MODULES

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We show that the formal germ at the infinity of the Fourier transform of a holonomic D-module depends only on the formal germ of the D-module at its singular points and at the infinity.

1. Introduction

The stationary phase approximation is a basic principle of asymptotic analysis, exemplified by the oscillatory integral

$$I(t') = \int g(t)e^{it'f(t)}dt.$$

If the derivative of $f(t)$ does not vanish at any point in $\mathrm{Supp}(f)$, then $I(t')$ is rapidly decreasing at ∞. If $f(t)$ has only finitely many critical points in $\mathrm{Supp}(f)$, the major contribution to the value of the integral $I(t')$ for large t' comes from neighborhoods of those critical points. More generally, consider the integral

$$I(t') = \int_{a(t')}^{b(t')} g(t,t')e^{if(t,t')}dt,$$

where all the functions are real-valued. Under certain conditions, for $t' \to \infty$,

$$I(t') = \sum_{f_t(t,t')=0} \left(g(t,t')\sqrt{\frac{2\pi}{|f_{tt}(t,t')|}}e^{if(t,t')+\frac{i\pi}{4}\operatorname{sgn} f_{tt}(t,t')} + o\left(\frac{g(t,t')}{\sqrt{|f_{tt}(t,t')|}}\right)\right).$$

The classical principle of stationary phase outlined above relates to the real Fourier transform. To study Deligne's ℓ-adic Fourier transform, Gérard Laumon [1987] introduced a corresponding principle of stationary phase and the local ℓ-adic Fourier transform. (See [Katz 1988] for a good exposition.) We are interested in the D-module case.

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We fix a field k of characteristic 0 and use the following notations:

(1) Let $p_1,\ p_2$ be the projections $\operatorname{Spec} k[t,t'] = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$, and let $\bar{p}_1,\ \bar{p}_2$ be the projections $\mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$. Let $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\mu : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ be the inclusions.

(2) For any $x \in k$, let $t_x = t - x$ and $t'_x = t' - x$. Let $t_\infty = 1/t = z$, $t'_\infty = 1/t' = z'$ and $\eta' = \operatorname{Spec} k(t')$. For any $x \in k \cup \{\infty\}$, let $\eta_x = \operatorname{Spec} k((t_x))$, $\eta'_x = \operatorname{Spec} k((t'_x))$.

(3) For any $x,\ y \in k \cup \{\infty\}$, let $k((t_x, t'_y))$ be the field of the formal Laurent series $\sum_{i,j \gg -\infty} a_{ij} t_x^i t_y'^j$, $a_{ij} \in k$. For any $k((t_x))$-vector space M, let
$$M((t'_y)) = M \otimes_{k((t_x))} k((t_x, t'_y)).$$

(4) Denote by $\mathscr{L}$ the rank-one connection $(\mathscr{O}_{\mathbb{A}^1_k}, d + dt)$ on $\mathbb{A}^1_k$. Then $\mathscr{L}$ corresponds to the D-module $\mathscr{O}_{\mathbb{A}^1_k} \cdot e^t$ on $\mathbb{A}^1_k$. So $\mathscr{L}$ is a substitute of e^{it} in classical Fourier analysis. Let X be a scheme. Any section $f \in \mathscr{O}(X)$ defines a morphism $\phi : X \to \mathbb{A}^1_k$ and let $\mathscr{L}_f = \phi^* \mathscr{L}$.

Let $\mathscr{M}$ be a vector bundle with a connection ∇ on a nonempty open subscheme U of $\mathbb{A}^1_k$ and let $i : U \hookrightarrow \mathbb{A}^1$ and $j : U \to \mathbb{P}^1_k$ be the inclusions. The connection ∇ on $\mathscr{M}$ can be extended to a connection $i_*\nabla$ on $i_*\mathscr{M}$ and a connection $j_*\nabla$ on $j_*\mathscr{M}$. The global (geometric) Fourier transform of the D-module $i_*\mathscr{M}$ on $\mathbb{A}^1_k$ is defined to be
$$\mathscr{F}(i_*\mathscr{M}) = p_{2+}(p_1^* i_*\mathscr{M} \otimes_{\mathscr{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}} \mathscr{L}_{tt'})[1],$$
where $\otimes$ and p_{2+} are derived functors of D-modules. This definition is analogous to
$$\widehat{f}(t') = \int f(t) e^{itt'}\, dt.$$
More precisely, we have
$$\begin{aligned}
\mathscr{F}(i_*\mathscr{M}) &\cong R^1 p_{2*}\big(p_1^* i_*\mathscr{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathscr{O}_{\mathbb{A}^1_k}} i_*\mathscr{M})\big) \\
&\cong \alpha^* \alpha_* R^1 p_{2*}\big(p_1^* i_*\mathscr{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathscr{O}_{\mathbb{A}^1_k}} i_*\mathscr{M})\big) \\
&\cong \alpha^* R^1 \bar{p}_{2*} \mu_*\big(p_1^* i_*\mathscr{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathscr{O}_{\mathbb{A}^1_k}} i_*\mathscr{M})\big) \\
&\cong \alpha^* R^1 \bar{p}_{2*}\big(\bar{p}_1^* j_*\mathscr{M} \otimes \mu_*\mathscr{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k} \otimes j_*\mathscr{M}) \otimes \mu_*\mathscr{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}\big).
\end{aligned}$$
Consider the complex
$$(*) \qquad \big(\bar{p}_1^* j_*\mathscr{M} \otimes \mu_*\mathscr{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k} \otimes j_*\mathscr{M}) \otimes \mu_*\mathscr{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}\big).$$
We have
$$\mathscr{F}(i_*\mathscr{M})|_{\eta_{\infty'}} = R^1 \bar{p}_{2*}(*)|_{\eta_{\infty'}}.$$

To study $\mathscr{F}(i_*\mathcal{M})|_{\eta_{\infty'}}$, one needs to study $R^1\bar{p}_{2*}(*)|_{\mathrm{Spf}\,k[[z']]}$. The complex $(*)$ involves quasicoherent sheaves that may not be coherent sheaves. To study the localization of $(*)$ on $\mathrm{Spf}\,k[[z']]$, we need to transform them into coherent sheaves. For this reason, Bloch and Esnault [2004] rewrote $(*)$ in terms of the cohomology of a complex of coherent modules. They found a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of the connection $j_*\mathcal{M}$ such that $(\bar{p}_1^* j_*\nabla + t'dt)(\bar{p}_1^*\mathcal{V}) \subset \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k}(T)\otimes\mathcal{W})$ and the inclusion of complexes

$$\big(\bar{p}_1^{*}\mathcal{V} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k}(T)\otimes\mathcal{W})\big) \subset (*)$$

is a quasi-isomorphism. Here $T = \mathbb{P}^1_k - U$. However, for any good lattice pair $\mathcal{V}$, $\mathcal{W}$ of the connection $j_*\mathcal{M}$, the conditions above do not hold, because the differential form $t'dt$ is singular on $\mathbb{P}^1_k \times \{\infty\} \cup \{\infty\} \times \mathbb{P}^1_k$. We only have

$$(\bar{p}_1^* j_*\nabla + t'dt)(\bar{p}_1^*\mathcal{V}) \subset \bar{p}_1^*\big(\Omega^1_{\mathbb{P}^1_k}(T)\otimes(\mathcal{W}+\mathcal{V}(\{\infty\}))\big)(\mathbb{P}^1_k\times\{\infty\})$$

and a subcomplex

$$\big(\bar{p}_1^{*}\mathcal{V} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*\big(\Omega^1_{\mathbb{P}^1_k}(T)\otimes(\mathcal{W}+\mathcal{V}(\{\infty\}))\big)(\mathbb{P}^1_k\times\{\infty\})\big) \tag{1-1}$$

of $(*)$. This inclusion of complexes (1-1) $\subset$ $(*)$ is still not a quasi-isomorphism. Using Deligne's construction of good lattice pairs, we find a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of $j_*\mathcal{M}$ in Lemma 2.3 such that $(1\text{-}1)|_{\mathbb{P}^1_k\otimes_k k(t')} \subset (*)|_{\mathbb{P}^1_k\otimes_k k(t')}$ is a quasi-isomorphism. From this, we get the following stationary phase formula.

Theorem 1.1. *Let $\mathcal{M}$ be a vector bundle with a connection ∇ on a nonempty open subscheme U of $\mathbb{A}^1_k$, and let $i : U \hookrightarrow \mathbb{A}^1$ be the inclusion. Suppose all points in $\mathbb{A}^1_k - U$ are k-rational. Then the natural map*

$$\begin{aligned}\mathscr{F}(i_*\mathcal{M})|_{\eta'_\infty} \to \bigoplus_{x\in\mathbb{A}^1_k-U} &\mathrm{coker}\big((\mathcal{M}|_{\eta_x})((z')) \xrightarrow{z'\partial_{t_x}+1} (\mathcal{M}|_{\eta_x})((z'))\big)\\ &\oplus \mathrm{coker}\big((\mathcal{M}|_{\eta_\infty})((z')) \xrightarrow{z'\partial_z - \frac{1}{z^2}} (\mathcal{M}|_{\eta_\infty})((z'))\big)\end{aligned} \tag{1-2}$$

is an isomorphism of formal connections on $k((z'))$.

The direct summands on the right side of (1-2) induce the definition of local Fourier transforms for formal connections.

The paper is organized as follows. In Section 2, we discuss the good lattice pairs of connections on a smooth curve. Passing to the stalks, we discuss the good lattice pairs of connections on a discrete valuation field. In Section 3, we prove the stationary phase formula using proper base change theorem between formal schemes.

2. Good lattice pairs

Let X be a smooth algebraic curve over k and $j : X \hookrightarrow \overline{X}$ the smooth compactification. Let $\mathcal{F}$ be a vector bundle on X with a connection ∇. Set $\Sigma = \overline{X} - X$. A *pair of good lattices* $\mathcal{V}, \mathcal{W}$ *of* $j_*\mathcal{F}$ is a pair of vector bundles on $\overline{X}$ which extends $\mathcal{F}$ and satisfies the following conditions:

(1) $\mathcal{V} \subset \mathcal{W} \subset j_*\mathcal{F}$.

(2) $\nabla(\mathcal{V}) \subset \Omega^1_{\overline{X}}(\Sigma) \otimes \mathcal{W}$.

(3) For any effective divisor D supported on Σ, the inclusion of complexes

$$\big(\mathcal{V} \xrightarrow{\nabla} \Omega^1_{\overline{X}}(\Sigma) \otimes \mathcal{W}\big) \to \big(\mathcal{V}(D) \xrightarrow{\nabla} \Omega^1_{\overline{X}}(\Sigma) \otimes \mathcal{W}(D)\big)$$

is a quasi-isomorphism. Taking the direct limit with respect to D, we get a quasi-isomorphism:

$$\big(\mathcal{V} \xrightarrow{\nabla} \Omega^1_{\overline{X}}(\Sigma) \otimes \mathcal{W})\big) \to \big(j_*\mathcal{F} \xrightarrow{\nabla} \Omega^1_{\overline{X}} \otimes j_*\mathcal{F}\big).$$

The existence of good lattice pairs can be passed to the stalks. So we only need to consider the local case: good lattice pairs of connections on a discrete valuation field.

Let K be a discrete valuation field with the valuation v. Let A be the valuation ring, t a uniformizer, and ∂ a continuous derivation on K such that $\partial(t) = 1$ and $\partial(A) \subseteq A$.

Definition 2.1. A *connection* on K (of rank k, where k is finite) is a k-dimensional vector space M over K with an additive map $\partial : M \to M$ satisfying $\partial(fm) = f\partial(m) + \partial(f)m$ for any $f \in K$ and $m \in M$.

Let r be the rank of the connection M. Set $\tau = t\partial$. There exists a cyclic element $v \in M$, in the sense that the elements $\tau^i v$, for $0 \le i \le r-1$, form a basis of M over K. We have

$$\tau^r v = \sum_{0 \le i \le r-1} a_i \tau^i v$$

for some $a_i \in K$. The *Newton polygon* $N(M)$ of M is the convex hull of

$$\{(u, v) \mid u \le i, v \ge v(a_i)\}$$

in the plane $\mathbb{R}^2$. The *slopes* of M are the slopes of nonvertical edges of $N(M)$, and we eliminate the slope 0 if the horizontal edge is situated in $u \le 0$. The slopes are independent of the choice of the cyclic elements. The sum of all the slopes of M is called the *irregularity* of M, and is denoted by $i(M)$. Then

$$i(M) = \max_{0 \le i \le r} (0, -v(a_i)).$$

A *lattice* of M is a finitely generated A-submodule V of M that spans M. For any artinian A-module V, the length of V is denoted by $\ell(V)$.

Definition 2.2. A pair of lattices V, W of (M, ∂) is called *good* if the following conditions are satisfied

(1) $V \subset W \subset M$.

(2) $\partial V \subset (1/t)W$.

(3) For any $i \in \mathbb{N}$, the natural inclusion of complexes

$$\Big(V \xrightarrow{\partial} \frac{1}{t}W\Big) \to \Big(\frac{1}{t^i}V \xrightarrow{\partial} \frac{1}{t^{i+1}}W\Big)$$

is a quasi-isomorphism.

Note that if V, W is a good lattice pair, so is $(1/t^i)W$, $(1/t^i)W$ for any $i \in \mathbb{N}$.

Condition (3) above is equivalent to the following:

(3′) For any $i \in \mathbb{N}$, the map

$$\frac{1}{t^i}V \Big/ \frac{1}{t^{i-1}}V \xrightarrow{\mathrm{gr}_i \partial} \frac{1}{t^{i+1}}W \Big/ \frac{1}{t^i}W$$

induced by ∂ is an isomorphism.

One can show that $i(M) = \ell(W/V)$.

Lemma 2.3. *Let $k \hookrightarrow k'$ be an extension of fields of characteristic 0. Let ∂_t be the natural derivation on $k(t)$ and on $k'(t)$. The variable t defines a discrete valuation v on $k(t)$ and $k'(t)$. Let A and A' be their discrete valuation rings, respectively. Suppose c is an element in k' which is not algebraic over k. Let M be a connection on $k(t)$, and let M_c be the connection on $k'(t)$ whose underlying space is the $k'(t)$-vector space $M \otimes_{k(t)} k'(t)$, and with the operation ∂_t defined by*

$$\partial_t(m \otimes f) = \partial_t(m) \otimes f + m \otimes \partial_t(f) - m \otimes \frac{c}{t^2}$$

for any $m \in M$ and any $f \in k'(t)$. Then there exists a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of M, such that $\mathcal{V} \otimes_A A'$, $(\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A'$ is also a good lattice pair of the connection M_c on $k'(t)$.

Proof. Set $r = \mathrm{rk} M$. Choose a cyclic element v of M. Let ε be the basis $\{\tau^i v \mid 0 \le i \le r-1\}$ of M over $k(t')$. We have $\tau^r v = \sum_{0 \le i < r} a_i \tau^i v$ for some $a_i \in K$. The irregularity $i(M)$ of M is $\max_{0 \le i < r}(0, -v(a_i))$. Consider the Newton polygon of the differential operator $\tau^r - \sum_{0 \le i \le r-1} a_i \tau^i$. Let j be the integer such that $(j, v(a_j))$ is a vertex of this Newton polygon, and such that the slopes of this Newton polygon on the right side (respectively left side) of $(j, v(a_j))$ is > 1 (respectively ≤ 1). Set $a_r = 1$. Then we have

$$\begin{aligned} v(a_{j+i}) - v(a_j) &> i \text{ for any } 1 \leq i \leq r - j, \\ v(a_{j-i}) - v(a_j) &\geq -i \text{ for any } 0 \leq i \leq j. \end{aligned}$$

Then

$$v(a_j) - j = \min_{0 \leq i \leq r}(v(a_i) - i). \tag{2-1}$$

The matrix of the differential operator τ with respect to the basis ε is

$$\Gamma = \begin{pmatrix} 0 & & & a_0 \\ 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{r-1} \end{pmatrix}.$$

The characteristic polynomial of Γ is $\lambda^r - \sum_{0 \leq i \leq r-1} a_i \lambda^i$. Let

$$\Lambda = \operatorname{diag}\{1, \ldots, 1, t, \ldots, t^{r-j+i(M)+v(a_j)}\},$$

and let $e = \varepsilon\Lambda = \{e_i \mid 0 \leq i < r\}$. Set $l = j - v(a_j) - i(M) \geq 0$. Then the matrix of the differential operator τ with respect to the basis e is

$$\Gamma' = \begin{pmatrix} 0 & & & & & & t^{r-l}a_0 \\ 1 & & & & & & t^{r-l}a_1 \\ & \ddots & & & & & \vdots \\ & & 1 & & & & t^{r-l}a_{l-1} \\ & & & \frac{1}{t} & & & t^{r-l-1}a_l \\ & & & & \ddots & & \vdots \\ & & & & & \frac{1}{t} & a_{r-1} \end{pmatrix} + \operatorname{diag}\{0, \cdots, 0, 1, \cdots, r - l\}.$$

Let $P(\lambda) = \lambda^r - \sum_{0 \leq i \leq r-1} a'_i \lambda^i$ be the characteristic polynomial of Γ'. Since

$$\Gamma' = \Lambda^{-1}\Gamma\Lambda + \operatorname{diag}\{0, \ldots, 0, 1, \ldots, r - l\},$$

we have

$$a'_i - a_i \in \sum_{i<j<r} \mathbb{Z}a_j + \mathbb{Z} \subset K.$$

So

$$\max\{0, -v(a'_i) \mid 0 \leq i < r\} = \max\{0, -v(a_i) \mid 0 \leq i < r\} = i(M).$$

Write $P(\lambda) = t^{-i(M)} \sum_i b_i \lambda^i$, $b_i \in K$. Then $b_i \in A$ and $v(b_i) = 0$ for at least one i. The residue polynomial $\sum_i \bar{b}_i \lambda^i$ of $\sum_i b_i \lambda^i$ is nonzero. For almost all $n \in \mathbb{Z}$,

$\sum_i \bar{b}_i(-n)^i \neq 0$. In this case, we have

$$-v(\det(n+\Gamma')) = -v((-1)^r P(-n)) = -v\Big(t^{-i(M)}\Big(\sum_i b_i(-n)^i\Big)\Big) = i(M).$$

Then, for almost all $n \in \mathbb{Z}$,

$$i(M) = -v(\det(n+\Gamma')). \tag{2-2}$$

Let V be the lattice of M generated by e. Define

$$[(n+\Gamma')V : V] = \ell((n+\Gamma')V + V/V) - \ell((n+\Gamma')V + V/(n+\Gamma')V). \tag{2-3}$$

By [Deligne 1970, p. 48, Proposition 2], we have

$$[(n+\Gamma')V : V] = -v(\det(\mathrm{n}+\Gamma')). \tag{2-4}$$

Let W be the lattice of M generated by

$$e_0, \dots, e_{l-1}, \frac{1}{t}e_l, \dots, \frac{1}{t}e_{r-1}.$$

Then $\ell(W/V) = r-l$. Since $((n+\Gamma')V+V)/W$ is an artinian A-module generated by the single element

$$x = \sum_{0\le i\le l-1} a_i t^{r-l} e_i + \sum_{l\le i\le r-1} a_i t^{r-1-i} e_i = \sum_{0\le i\le l-1} a_i t^{r-l} e_i + \sum_{l\le i\le r-1} a_i t^{r-i}\frac{1}{t} e_i.$$

For any i, we have $i(M) \ge -v(a_i)$ and $v(a_j) - j \le v(a_i) - i$. Then

$$v(t^{i(M)+l-r} a_i t^{r-l}) \ge 0 \quad \text{and} \quad v(t^{i(M)+l-r} a_i t^{r-i}) \ge v(t^{i(M)+l-r} a_j t^{r-j}) = 0.$$

Then the annihilator of x in $((n+\Gamma')V+V)/W$ is $t^{i(M)+l-r}$. So

$$\ell((n+\Gamma')V + V/W) = i(M) + l - r.$$

Then

$$\ell((n+\Gamma')V + V/V) = \ell(W/V) + \ell((n+\Gamma')V + V/W) = i(M). \tag{2-5}$$

Comparing this equality with (2-2), (2-3), and (2-4), we get

$$\ell((n+\Gamma')V + V/(n+\Gamma')V) = 0$$

for almost $n \in \mathbb{Z}$, that is, $(n+\Gamma')V \supset V$ for almost all $n \in \mathbb{Z}$.

The A-module

$$(n+\Gamma')V + \frac{1}{t}V \Big/ \frac{1}{t}V$$

is artinian and is generated by one element x whose annihilator is

$$t^{i(M)+l-r} = t^{j-v(a_j)-r}.$$

Then

$$\ell\Big((n+\Gamma')V+\frac{1}{t}V\Big/V\Big)=\ell\Big((n+\Gamma')V+\frac{1}{t}V\Big/\frac{1}{t}V\Big)+\ell\frac{1}{t}V\Big/V \tag{2-6}$$
$$=j-v(a_j)=\sum_{\lambda:\text{slope of }M}\max(\lambda,1).$$

The matrix of the differential operator τ with respect to the basis ε of M_c is $\Gamma-c/t$. The characteristic polynomial $P'(\lambda)$ of $\Gamma-c/t$ is

$$P'(\lambda)=\Big(\lambda+\frac{c}{t}\Big)^r-\sum_{0\le i<r}a_i\Big(\lambda+\frac{c}{t}\Big)^i.$$

Write $P'(\lambda)=\lambda^r+\sum_{0\le i<r}b_i\lambda^i$ for some $b_i\in k'(t)$. Then

$$b_0=\Big(\frac{c}{t}\Big)^r-\sum_{0\le i<r}a_i\Big(\frac{c}{t}\Big)^i=\frac{a_j}{t^j}\bigg(\frac{1}{a_jt^{r-j}}c^r-\sum_{0\le i<r}\frac{a_i}{a_jt^{i-j}}c^i\bigg).$$

By (2-1), we have

$$\frac{1}{a_jt^{r-j}}c^r-\sum_{0\le i<r}\frac{a_i}{a_jt^{i-j}}c^i\in A[c],$$

and its residue in k' is a nonzero polynomial over k of c. Since c is not algebraic over k, this residue is nonzero. Then we have

$$v(b_0)=v\Big(\frac{a_j}{t^j}\Big)=v(a_j)-j.$$

Also by (2-1), we have $v(b_i)\ge v(b_0)$ for any $0\le i<r$. So

$$\max_{0\le i<r}(0,-v(b_i))=j-v(a_j)=i(M_c).$$

The matrix of the differential operator τ with respect to the basis e of M_c is $\Gamma''=\Gamma'-c/t$. Write the characteristic polynomial of Γ'' as $\lambda^r+\sum_{0\le i<r}b'_i\lambda^i$ for some $b'_i\in k'(t)$. By a similar proof as above, we have

$$\max_{0\le i<r}(0,-v(b'_i))=\max_{0\le i<r}(0,-v(b_i))=i(M_c).$$

For almost $n\in\mathbb{Z}$, we have

$$-v(\det(n+\Gamma''))=i(M_c).$$

Let $V'=V\otimes_A A'$. We have $(n+\Gamma'')V'+V'\subseteq\frac{1}{t}V'+\Gamma'V'$; therefore So

$$\ell((n+\Gamma'')V'+V'/V')\le\ell\Big(\frac{1}{t}V'+\Gamma'V'\Big/V'\Big). \tag{2-7}$$

Since $A\to A'$ is flat and $k\otimes_A A'=k'$, for any artinian A-module M, one can prove $\ell(M)=\ell(M\otimes_A A')$. Since $(1/t)V+\Gamma'V/V$ is an artinian A-module, by (2-6), we have

$$\ell\Big(\frac{1}{t}V'+\Gamma'V'\Big/V'\Big)=\ell\Big(\frac{1}{t}V+\Gamma'V\Big/V\Big)=j-v(a_j). \tag{2-8}$$

By (2-4), we have, for almost $n\in\mathbb{Z}$,

$$\begin{aligned}\ell((n{+}\Gamma'')V'+V'/V')&\geq \ell((n{+}\Gamma'')V'+V'/V')-\ell((n{+}\Gamma'')V'+V'/(n{+}\Gamma'')V')\\ &=-v(\det(n+\Gamma''))=j-v(a_j).\end{aligned}$$

Comparing this inequality with (2-7) and (2-8), we have for almost $n\in\mathbb{Z}$,

$$\ell((n+\Gamma'')V'+V'/V')=j-v(a_j);$$

$$\ell((n+\Gamma'')V'+V'/(n+\Gamma'')V')=0;$$

$$(n+\Gamma'')V'+V'=\frac{1}{t}V'+\Gamma'V'=\Big(\frac{1}{t}V+\Gamma'V\Big)\otimes_A A'. \tag{2-9}$$

So for almost $n\in\mathbb{Z}$, $(n{+}\Gamma'')V'\supseteq V'$. Let $e'=(1/t^N)e$. The matrix of τ with respect to the basis e' of M (respectively M_c') is $\Gamma_1:=\Gamma'-N$ (respectively $\Gamma_2:=\Gamma''-N$). Let $\mathcal{V}=(1/t^N)V$ and let $\mathcal{V}'=(1/t^N)V'$. Choose N large enough so that for any $n\leq 0$, we have

$$(n+\Gamma_1)\mathcal{V}\supset\mathcal{V}\quad\text{and}\quad(n+\Gamma_2)\mathcal{V}'\supseteq\mathcal{V}'.$$

Let $\mathcal{W}=\Gamma_1\mathcal{V}$. By (2-9), we have $\Gamma_2\mathcal{V}'=(\mathcal{W}+(1/t)\mathcal{V})\otimes_A A'$. Let's prove $\mathcal{V}$, $\mathcal{W}$ is a good lattice of M now. We only need to verify condition (3$'$) for any $i\in\mathbb{N}$. Conjugating by $1/t^i$, the A-linear map

$$\mathrm{gr}_i\tau:\frac{1}{t^i}\mathcal{V}\Big/\frac{1}{t^{i-1}}\mathcal{V}\to\frac{1}{t^i}\mathcal{W}\Big/\frac{1}{t^{i-1}}\mathcal{W}$$

can be identified with the A-linear map

$$\mathrm{gr}_0\tau-i=\Gamma_1-i:\mathcal{V}/t\mathcal{V}\to\mathcal{W}/t\mathcal{W}.$$

Since $(\Gamma_1-i)\mathcal{V}\supset\mathcal{V}$, we have

$$(\Gamma_1-i)\mathcal{V}=(\Gamma_1-i)\mathcal{V}+\mathcal{V}\supset\Gamma_1\mathcal{V}=\mathcal{W}.$$

So $\Gamma_1-i:\mathcal{V}/t\mathcal{V}\to\mathcal{W}/t\mathcal{W}$ is surjective. But the domain and the range of $\mathrm{gr}_i\tau$ are artinian A-modules of the same length r, so $\mathrm{gr}_0\tau-i$ is an isomorphism and so is $\mathrm{gr}_i\tau$. This proves $\mathcal{V}$, $\mathcal{W}$ is a good lattice pair of M. Repeating the proof, we conclude that $\mathcal{V}\otimes_A A'$, $(\mathcal{W}+(1/t)\mathcal{V})\otimes_A A'$ is a good lattice pair of M_c. □

Remark 2.4. Lemma 2.3 is the main technical lemma for the proof of the stationary phase principle in the next section. Lemma 2.3 also allows us to choose a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of M such that

$$\dim_k\Big(\mathcal{W}+\frac{1}{t}\mathcal{V}\Big/\mathcal{V}\Big)=\sum_{\lambda:\text{slope of }M}\max(\lambda,1). \tag{2-10}$$

Formula (2-10) is easily seen to give a new proof of the following result:

Lemma 2.5 [Bloch and Esnault 2004, Lemma 3.3]. *Let M be a connection on K. The slopes of M are all ≤ 1 (respectively ≥ 1) if and only if there exists a good lattice pair $\mathcal{V}$, $\mathcal{W}$ such that $\mathcal{W} \subseteq (1/t)\mathcal{V}$ (respectively $\mathcal{W} \supseteq (1/t)\mathcal{V}$).*

(Note that the original proof by Bloch and Esnault needs the assumption that K is complete.)

3. Stationary phase principle

Let $K = k(t')$. For any scheme X over k and any $\mathcal{O}_X$-modules $\mathcal{F}$, let $X_K = X \otimes_k K$ and $\mathcal{F}_K = \mathcal{F}|_{X_K}$. For any k-morphism $f : X \to Y$, let $f_K : X_K \to Y_K$ be the base change of f.

We keep the notation used in Section 1. In this section we prove Theorem 1.1.

For any $x \in T_K = T$, $(\mathcal{V}_K)_x$, $(\mathcal{W}_K)_x$ is a good lattice pair of the connection $(j_{K*}\mathcal{M}_K)_x$ on $K(t_x)$. Since t' is not algebraic over k, by Lemma 2.3, we may assume that

$$\mathcal{V}_\infty \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty}, \ \Big(\mathcal{W}_\infty + \frac{1}{z}\mathcal{V}_\infty\Big) \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty}$$

is a good lattice pair of the connection

$$\partial_z - \frac{t'}{z^2} : (j_{K*}\mathcal{M}_K)_\infty \to (j_{K*}\mathcal{M}_K)_\infty.$$

Lemma 3.1. *The inclusion of complexes* (1-1) $\subset$ $(*)$ *induces a quasi-isomorphism*

$$(1\text{-}1)|_{\mathbb{P}^1_K} \simeq (*)|_{\mathbb{P}^1_K}.$$

Proof. We have

$$(1\text{-}1)|_{\mathbb{P}^1_K} = \big(\mathcal{V}_K \xrightarrow{j_{K*}\nabla_K + t'dt} \Omega^1_{\mathbb{P}^1_K}(T_K) \otimes (\mathcal{W}_K + \mathcal{V}_K(\{\infty\}))\big),$$
$$(*)|_{\mathbb{P}^1_K} = \big(j_{K*}\mathcal{M}_K \xrightarrow{j_{K*}\nabla_K + t'dt} \Omega^1_{\mathbb{P}^1_K} \otimes j_{K*}\mathcal{M}_K\big).$$

First we have $(1\text{-}1)|_{U_K} = (*)|_{U_K}$. For any $x \in S_K$, let's prove $(1\text{-}1)|_{\mathbb{P}^1_K} \subset (*)|_{\mathbb{P}^1_K}$ induces a quasi-isomorphism on the stalks at x. It suffices to show that

$$\left(\frac{1}{t_x^i}(\mathcal{V}_K)_x \Big/ \frac{1}{t_x^{i-1}}(\mathcal{V}_K)_x\right) \xrightarrow{\mathrm{gr}_i(\partial_{t_x} + t')} \left(\frac{1}{t_x^{i+1}}(\mathcal{W}_K)_x \Big/ \frac{1}{t_x^i}(\mathcal{W}_K)_x\right)$$

is an isomorphism for any $i \geq 1$. As $(\mathcal{V}_K)_x \subset (\mathcal{W}_K)_x$, the map $\mathrm{gr}_i(\partial_{t_x} + t')$ is equal to $\mathrm{gr}_i(\partial_{t_x})$, which is an isomorphism by the definition of good lattices. The inclusion

$$\big((1\text{-}1)|_{\mathbb{P}^1_K}\big)_\infty \to \big((*)|_{\mathbb{P}^1_K}\big)_\infty$$

can be written as

$$\Big(\mathcal{V}_\infty \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty} \xrightarrow{\partial_z - \frac{t'}{z^2}} \frac{1}{z}\Big(\mathcal{W}_\infty + \frac{1}{z}\mathcal{V}_\infty\Big) \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty}\Big)$$
$$\subset \Big((j_{K*}\mathcal{M}_K)_\infty \xrightarrow{\partial_z - \frac{t'}{z^2}} (j_{K*}\mathcal{M}_K)_\infty\Big).$$

It is a quasi-isomorphism by the assumption on $\mathcal{V}_\infty$ and $\mathcal{W}_\infty$. □

Lemma 3.2. $$R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'} \cong R^1\bar{p}_{2*}(*)|_{\eta'}.$$

Proof. Consider the Cartesian diagram

$$\begin{tikzcd} \mathbb{P}^1_K \arrow[r] \arrow[d] & \eta' = \operatorname{Spec} K \arrow[d] \\ \mathbb{P}^1_k \times \mathbb{P}^1_k \arrow[r, "\bar{p}_2"] & \mathbb{P}^1_k. \end{tikzcd} \tag{3-1}$$

By Lemma 3.1, we have

$$R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'} \cong H^1(\mathbb{P}^1_K,\ (1\text{-}1)|_{\mathbb{P}^1_K}) \cong H^1(\mathbb{P}^1_K,\ (*)|_{\mathbb{P}^1_K}) \cong R^1\bar{p}_{2*}(*)|_{\eta'}. \qquad \square$$

Corollary 3.3. $$\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} = R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'_\infty}.$$

Denote by $\mathbb{P}^1_k[[z']]$ the formal completion of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ along its closed subscheme $\mathbb{P}^1_k \times \{\infty\}$. For any coherent sheaf $\mathcal{K}$ on $\mathbb{P}^1_k$, let $\mathcal{K}[[z']] = \mathcal{K}|_{\mathbb{P}^1_k[[z']]}$.

Lemma 3.4 [Bloch and Esnault 2004, Corollary 2.2].

$$R^1\bar{p}_{2*}(1\text{-}1) \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} k[[z']] \cong H^1\Big(\mathbb{P}^1[[z']], \mathcal{V}[[z']] \xrightarrow{z'\nabla + dt} \Omega^1_{\mathbb{P}^1}(T) \otimes \mathcal{W}[[z']]\Big).$$

Lemma 3.5 [Bloch and Esnault 2004, Lemma 2.4 and Corollary 2.5]. *Let $\mathcal{H}$ be the complex*

$$\mathcal{V}[[z']] \xrightarrow{z'\nabla + dt} \big(\Omega^1_{\mathbb{P}^1_k}(T) \otimes (\mathcal{W} + \mathcal{V}(\{\infty\}))\big)[[z']].$$

Then $\mathcal{H}^0$ equals (0) *and $\mathcal{H}^1$ is supported on $T \subset \mathbb{P}^1_k = \mathbb{P}^1_k[[z']]$. For any $x \in T$, let $\widehat{\mathcal{V}_x} = \mathcal{V}_x \otimes_{\mathcal{O}_{\mathbb{P}^1_{k,x}}} k[[t_x]]$ and $\widehat{\mathcal{W}_x} = \mathcal{W}_x \otimes_{\mathcal{O}_{\mathbb{P}^1_{k,x}}} k[[t_x]]$. We have*

$$\mathcal{H}^1_x = \operatorname{coker}\Big(\widehat{\mathcal{V}_x}[[z']] \xrightarrow{z'\nabla + dt} \Omega^1_{\mathbb{P}^1_k}(T) \otimes (\widehat{\mathcal{W}_x} + \widehat{\mathcal{V}(\{\infty\})_x})[[z']]\Big). \qquad \square$$

Corollary 3.6.

$$H^1(\mathbb{P}^1_k[[z']], \mathcal{H}) = \bigoplus_{x \in S} \operatorname{coker}\Big(\widehat{\mathcal{V}_x}[[z']] \xrightarrow{z'\partial_{t_x} + 1} \frac{1}{t_x}\widehat{\mathcal{W}_x}[[z']]\Big)$$
$$\oplus \operatorname{coker}\Big(\widehat{\mathcal{V}_\infty}[[z']] \xrightarrow{z'\partial_z - \frac{1}{z^2}} \frac{1}{z}\Big(\widehat{\mathcal{W}_\infty} + \frac{1}{z}\widehat{\mathcal{V}_\infty}\Big)[[z']]\Big).$$

Combining Corollary 3.3, Lemma 3.4 and Corollary 3.6, we have

$$\begin{aligned}\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} &= R^1\bar{p}_{2*}(1\text{-}1)\otimes_{\mathbb{O}_{\mathbb{P}^1_k}} k[[z']]\otimes_{k[[z']]} k((z'))\\ &= \bigoplus_{x\in S}\operatorname{coker}\big(\widehat{\mathcal{V}_x}((z'))\xrightarrow{z'\partial_{t_x}+1}\frac{1}{t_x}\widehat{\mathcal{W}_x}((z'))\big)\\ &\qquad\oplus\operatorname{coker}\Big(\widehat{\mathcal{V}_\infty lpz'})\xrightarrow{z'\partial_z-\frac{1}{z^2}}\frac{1}{z}(\widehat{\mathcal{W}_\infty}+\frac{1}{z}\widehat{\mathcal{V}_\infty})((z'))\Big).\end{aligned}$$

The left side of this equality is independent of the choice $\mathcal{V}$ and $\mathcal{W}$. For any $i\in\mathbb{N}$, $\mathcal{V}(iT)$ and $\mathcal{W}(iT)$ still satisfy the condition of Lemma 3.1. Then the above equality holds if we replace $\mathcal{V}$ and $\mathcal{W}$ by $\mathcal{V}(iT)$ and $\mathcal{W}(iT)$, respectively. Taking the direct limit on i, we have

$$\begin{aligned}\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} &= \bigoplus_{x\in S}\operatorname{coker}\big((\mathcal{M}|_{\eta_x})((z'))\xrightarrow{z'\partial_{t_x}+1}(\mathcal{M}|_{\eta_x})((z'))\big)\\ &\qquad\oplus\operatorname{coker}\Big((\mathcal{M}|_{\eta_\infty})((z'))\xrightarrow{z'\partial_z-\frac{1}{z^2}}(\mathcal{M}|_{\eta_\infty})((z'))\Big).\end{aligned}$$

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MONOTONICITY AND UNIQUENESS OF A 3D TRANSONIC SHOCK SOLUTION IN A CONIC NOZZLE WITH VARIABLE END PRESSURE

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We focus on the uniqueness problem of a 3D transonic shock solution in a conic nozzle when the variable end pressure in the diverging part of the nozzle lies in an appropriate scope. By establishing the monotonicity of the position of shock surface relative to the end pressure, we remove the nonphysical assumptions on the transonic shock past a fixed point made in previous studies and further obtain uniqueness.

1. Introduction and the main results

We study the uniqueness of a 3D transonic shock in a conic nozzle when the variable end pressure of the diverging part lies in an appropriate scope. The transonic shock problem in a nozzle is a fundamental one in fluid dynamics and has been extensively studied by many authors under various assumptions, for example, that either the transonic flow is quasi-one-dimensional or that the transonic shock goes through some fixed point in advance; see [Liu 1982; Embid et al. 1984; Chen et al. 2007; Chen 2008; Chen and Yuan 2008; Xin and Yin 2008a; 2008b; Xin et al. 2009] and so on. However, Courant and Friedrichs [1948, p. 386] indicated that transonic shock in a nozzle can be formulated as follows: Given appropriately large end pressure $p_e(x)$, if the upstream flow is still supersonic behind the throat of the three-dimensional de Laval nozzle, then at a certain place in the diverging part of the nozzle, a shock front intervenes and the gas is compressed and slowed down to subsonic speed. The position and the strength of the shock front are automatically adjusted so that the end pressure at the exit becomes $p_e(x)$. This statement indicates that the position of the transonic shock should be completely

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free. More importantly, the assumption of shock going through some fixed point in advance will lead in general to the transonic shock problem not being well-posed [Xin and Yin 2008a; Xin et al. 2009]. On the other hand, Courant and Friedrichs [1948, pp. 372, 375] pointed out that it is a question of great importance to know under what circumstances a steady flow involving shocks is uniquely determined and stable by the boundary conditions and by the conditions at the entrance, and when further conditions at the exit are appropriate. Motivated by these two basic problems, in this paper, we will establish the uniqueness result on a 3D transonic shock solution for the 3D Euler system when the variable end pressure $p_e(x)$ of the conic part of the nozzle lies in an appropriate scope without the assumption that the shock goes through a fixed point in advance. The existence of a 3D transonic shock solution under suitable restrictions on the end pressures was given in [Li et al. 2010].

We will consider only the isentropic gas for simplicity. By a slight modification, our discussions also apply to the nonisentropic case. The steady isentropic Euler system in three-dimensional spaces is

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \end{cases} \tag{1-1}$$

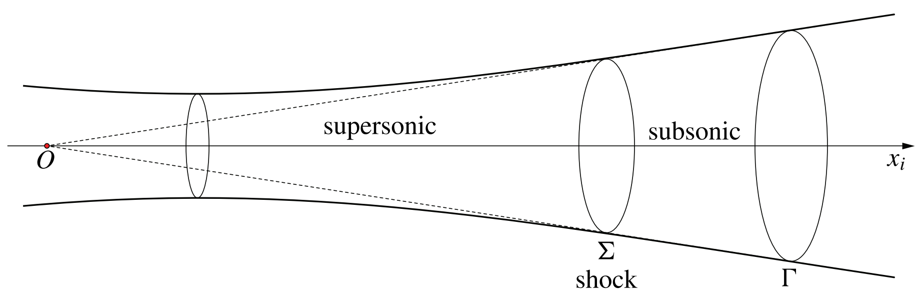
where $u = (u_1, u_2, u_3)$, ρ and P are the velocity, density and pressure, respectively. Moreover, the pressure function $P = P(\rho)$ is smooth with $P'(\rho) > 0$ for $\rho > 0$, and $c(\rho) = \sqrt{P'(\rho)}$ is called the local sound speed.

For ideal polytropic gases, the equation of state is given by

$$P = A\rho^{\gamma},$$

where A and γ are positive constants and $1 < \gamma < 3$.

It will be assumed that the nozzle wall Γ is $C^{4,\alpha}$-regular for $X_0 - 1 \le r = \sqrt{x_1^2 + x_2^2 + x_3^2} \le X_0 + 1$, where $X_0 > 1$ is a fixed constant and $\alpha \in (0, 1)$, and the wall Γ consists of two curved surfaces Π_1 and Π_2, where Π_1 includes the converging part of the nozzle and Π_2 is the conic diverging part of the nozzle (see figure). More precisely, the equation of Π_2 is represented by $x_2^2 + x_3^2 = x_1^2 \tan^2 \theta_0$ with $x_1 > 0$ and $X_0 < r < X_0 + 1$, where $0 < \theta_0 < \pi/2$ is sufficiently small. For



simplicity, we suppose that the $C^{4,\alpha}$-smooth supersonic incoming flow

$$(\rho_0^-(x), u_{1,0}^-(x), u_{2,0}^-(x), u_{3,0}^-(x))$$

is symmetric near $r = X_0$, where

$$\rho_0^-(x) = \rho_0^-(r) \quad \text{and} \quad u_{i,0}^-(x) = \frac{U_0^-(r)x_i}{r} \quad \text{for } i = 1, 2, 3$$

(this assumption can be easily realized by the hyperbolicity of the supersonic incoming flow and the symmetry of the nozzle wall for $X_0 < r < X_0 + 1$).

Denote the equation of the possible multidimensional shock Σ in the nozzle by $x_1 = \eta(x_2, x_3)$ and the flow field behind the shock by

$$(\rho^+(x), u_1^+(x), u_2^+(x), u_3^+(x)).$$

Then the Rankine–Hugoniot conditions on Σ are

$$\text{(1-2)} \qquad \begin{cases} [\rho u_1] - \partial_2\eta(x_2, x_3)[\rho u_2] - \partial_3\eta(x_2, x_3)[\rho u_3] = 0, \\ [P + \rho u_1^2] - \partial_2\eta(x_2, x_3)[\rho u_1 u_2] - \partial_3\eta(x_2, x_3)[\rho u_1 u_3] = 0, \\ [\rho u_1 u_2] - \partial_2\eta(x_2, x_3)[P + \rho u_2^2] - \partial_3\eta(x_2, x_3)[\rho u_2 u_3] = 0, \\ [\rho u_1 u_3] - \partial_2\eta(x_2, x_3)[\rho u_2 u_3] - \partial_3\eta(x_2, x_3)[P + \rho u_3^2] = 0. \end{cases}$$

In addition, $P^+(x)$ should satisfy the physical entropy condition (see [Courant and Friedrichs 1948])

$$\text{(1-3)} \qquad P^+(x) > P^-(x) \quad \text{on } x_1 = \eta(x_2, x_3).$$

On the exit of the nozzle, we place the end pressure condition

$$\text{(1-4)} \qquad P^+(x) = P_e + \varepsilon P_0(x_2, x_3) \quad \text{on } r = X_0 + 1,$$

where $\varepsilon > 0$ is sufficiently small and

$$P_0(x_2, x_3) \in C^{3,\alpha}\{(x_2, x_3) : x_2^2 + x_3^2 \le (X_0 + 1)^2 \sin^2\theta_0\}.$$

The positive constant P_e stands for the end pressure when a symmetric shock lies at the position $r = r_0$ with $r_0 \in (X_0, X_0 + 1)$ and the supersonic incoming flow admits the state $(\rho_0^-(r), U_0^-(r))$. For detailed information on P_e, see Theorem A.1 in Appendix A.

The flow is assumed to be tangent to the nozzle wall Γ, thus,

$$\text{(1-5)} \qquad x_1 u_1^+ \tau^2 - x_2 u_2^+ - x_3 u_3^+ = 0 \quad \text{on } x_2^2 + x_3^2 = x_1^2 \tan^2\theta_0.$$

Finally, X_0 and θ_0 are assumed to satisfy

$$\text{(1-6)} \qquad X_0\theta_0 = 1 \quad \text{and} \quad \frac{\eta_0}{2} < \theta_0 < \eta_0,$$

where $\eta_0 > 0$ is a suitably small constant. This assumption means that the nozzle wall Γ is close to the cylindrical surface $x_2^2 + x_3^2 = 1$ for $X_0 \le r \le X_0 + 1$.

Theorem 1.1 (uniqueness). *Under the assumptions above and*

$$M_0^-(X_0) \equiv \frac{U_0^-(X_0)}{c(\rho_0^-(X_0))} > \sqrt{\frac{2^{\gamma+1}-2}{\gamma}},$$

then for large X_0 and $0 < \varepsilon < 1/X_0^2$, Equation (1-1) *with the boundary conditions* (1-2)–(1-5) *has no more than one solution*

$$(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x); \eta(x_2, x_3))$$

with the following estimates:

(i) $\eta(x_2, x_3) \in C^{4,\alpha}(\bar{S})$, *where $S = \{(x_2, x_3) : (\eta(x_2, x_3), x_2, x_3) \in \Sigma\}$ is the projection of the shock surface Σ on the x_2x_3-plane. Moreover, there exists a constant $C_0 > 0$ (depending only on α and the supersonic incoming flow) such that*

$$\|\eta(x_2, x_3) - \sqrt{r_0^2 - x_2^2 - x_3^2}\|_{L^\infty(\bar{S})} \le C_0 X_0 \varepsilon,$$
$$\|\nabla_{x_2,x_3}(\eta(x_2, x_3) - \sqrt{r_0^2 - x_2^2 - x_3^2})\|_{C^{3,\alpha}(\bar{S})} \le C_0 \varepsilon.$$

(ii) *Let*

$$\Omega_+ = \{(x_1, x_2, x_3) : \eta(x_2, x_3) < x_1 < \sqrt{(X_0+1)^2 - x_2^2 - x_3^2}, x_2^2 + x_3^2 \le x_1^2 \tan^2 \theta_0\}.$$

The solution $(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x)) \in C^{3,\alpha}(\bar{\Omega}_+)$ satisfies

$$\|(P^+(x), u_1^+(x), u_2^+(x), u_3^+(x)) - (\hat{P}_0^+(r), \hat{u}_{1,0}^+(x), \hat{u}_{2,0}^+(x), \hat{u}_{3,0}^+(x))\|_{C^{3,\alpha}(\bar{\Omega}_+)} \le C_0 \varepsilon,$$

where

$$\hat{u}_{i,0}^+(x) = \hat{U}_0^+(r)\frac{x_i}{r} \quad \text{for } i = 1, 2, 3$$

and $(\hat{P}_0^+(r), \hat{U}_0^+(r))$ is the extension of the subsonic part of the background solution $(P_0^+(r), U_0^+(r))$ in Ω_+ (given in more detail in Theorem A.1 and Remark A.2).

Remark 1.1. The solution is required to have $C^{3,\alpha}$ regularity in Theorem 1.1. This is plausible, as in to [Li et al. 2009], since such a $C^{3,\alpha}$ smooth solution can be obtained as in [Li et al. 2010] under suitable assumptions on the compatibility conditions of the variable end pressure. It will be also shown that the position of the shock depends on the given end pressure monotonically. This will be given more precisely in Proposition 2.2. In addition, the order $X_0\varepsilon$ in the bound on

$$\|\eta(x_2, x_3) - \sqrt{r_0^2 - x_2^2 - x_3^2}\|_{L^\infty(\bar{S})}$$

comes essentially from the relation between the shock position and the end pressure (see (4-8)). As pointed out in [Li et al. 2009], this actually means that the shock position will move with order $X_0 O(\varepsilon)$ when the end pressure changes in order $O(\varepsilon)$ in (1-4).

Remark 1.2. The uniqueness result in [Xin and Yin 2008b] needs the key assumption that the transonic shock goes through a fixed point which is determined by the resulting ordinary differential equation in the case of the symmetric solutions. Using a completely different method, we remove this assumption.

Remark 1.3. If the transonic shock lies in a converging part of the symmetric nozzle, then a similar result to Theorem 1.1 still holds true. However, as shown in [Xin and Yin 2008b], an unsteady symmetric transonic shock is structurally unstable in a global-in-time sense when it lies in the symmetric converging part of the nozzle.

Remark 1.4. In Theorem 1.1, we assume that the regularity of the transonic shock surface is higher than that of the transonic shock solution $(\rho^+, u_1^+, u_2^+, u_3^+)$. The necessity of this assumption is plausible, in view of the existence result in [Li et al. 2010] under the condition of axisymmetric exit pressure. The assumption is also natural, as it comes up in the existence and stability theory of multidimensional shocks in [Majda 1983a; 1983b].

The steady transonic problem has been studied in great detail; see [Courant and Friedrichs 1948; Liu 1982; Gilbarg and Trudinger 1983; Embid et al. 1984; Morawetz 1994; Čanić et al. 2000; Kuz'min 2002; Zheng 2003; 2006; Chen et al. 2007; Chen 2008; Chen and Yuan 2008; Xin and Yin 2008a; 2008b; Xin et al. 2009; Li et al. 2010] and the references therein. However, most known results deal with 2D problems or 3D problems with special symmetries, or make additional a priori assumptions on shock positions. In this paper, we consider the uniqueness problem for general exit pressure and without stringent conditions on shock locations.

Next we comment on the proofs of the main results. Compared with previous studies, one of the main difficulties is the uncertainty of the shock position. As in the 2-dimensional case [Li et al. 2009], we overcome this difficulty by deriving the monotonic dependence of the shock position on the end pressure along the nozzle wall. Although the strategy here is somewhat similar to [Li et al. 2009], much more delicate and technical a priori estimates are needed to overcome some essential difficulties occurring in the 3-dimensional case. In particular, more complicated and careful analysis is needed for the estimates on the difference of two possible pressures P^+, $\tilde{P}^+$ and the suitable regularity arguments of the difference of two possible velocities (u_1^+, u_2^+, u_3^+), $(\tilde{u}_1^+, \tilde{u}_2^+, \tilde{u}_3^+)$ in the x_2 and x_3 directions. The pressure difference solves a second-order elliptic equation, while the velocity differences satisfy hyperbolic equations. Thus it would be plausible that the regularities of the velocity difference are lower than that of the pressure difference. This leads to the difficulty in deriving the $C^{3,\alpha}$-regularity of the difference of the shock surfaces. Our key observation to overcome this difficulty is that the difference $(u_i^+ - \tilde{u}_i^+)$ for $i = 2, 3$ satisfies a first-order elliptic system with respect to the variables x_2 and x_3 in the interior of subsonic domain Ω_+. Combining this with the transport equations for the velocity differences, we can obtain the $C^{2,\alpha}$-estimate of the velocity difference in the full variable x in Ω_+. This will yield the same regularities of the differences of the pressure and velocity simultaneously.

The rest of the paper is organized as follows. In Section 2, we reformulate the problem (1-1) with the boundary conditions (1-2)–(1-5) by suitable decompositions. To this end, first we transform the nozzle wall Π_2 into a cylindrical surface $y_2^2 + y_3^2 = 1$ and give a suitable decomposition on the velocity $u^+ = (u_1^+, u_2^+, u_3^+)$. Then we decompose the resulting 4×4 three-dimensional Euler system (1-1) into a second-order elliptic equation on the density ρ^+ with mixed boundary conditions and three first-order equations on the velocity components U_1^+, U_2^+ and U_3^+ by making use of Bernoulli's law. Furthermore, by an analysis of the R-H conditions (1-2) and the first equation in (1-1), we can show that (U_2^+, U_3^+) is governed by the Cauchy–Riemann system on the shock surface (see (2-9)–(2-10)). In Section 3, by use of the decomposition techniques in Section 2, we can establish some a priori estimates on the derivatives of the difference $(Y_1, Y_2, Y_3, Y_4, Y_5)$ of two possible solutions $(U_1^+, U_2^+, U_3^+, \rho^+, \xi_1)$ and $(V_1^+, V_2^+, V_3^+, q^+, \xi_2)$. In this process, we especially observe that Y_2 and Y_3 also satisfy a first-order elliptic system with a parameter y_1 in the interior of the nozzle so that one can obtain the same regularity of (Y_2, Y_3) as the pressure difference Y_4 and the suitable $C^{2,\alpha}$-estimates (see Lemma 3.5). With Bernoulli's law, this gives the analogous estimate on the gradients of Y_1 in Lemma 3.6. In Section 4, based on the estimates given in Section 3, we can determine the position of the shock surface and complete the proof of the uniqueness result in Theorem 1.1. Finally, for the reader's convenience, descriptions of the background solution illustrated in [Xin and Yin 2008b] are given in Appendix A. Some useful computations and estimates are given in Appendix B.

In the remainder of the paper, we will use the following conventions: $O(\varepsilon)$ and $O(1)$ mean that there exists a constant $C_1 > 0$, independent of X_0 and ε, such that

$$\|O(\varepsilon)\|_{C^{1,\alpha}} \le C_1\varepsilon \quad \text{and} \quad \|O(1)\|_{C^{1,\alpha}} \le C_1,$$

respectively. $O(1/X_0^m)$ for $m > 0$ means that there exists a generic constant $C_2 > 0$ independent of X_0 and ε such that

$$\|O(1/X_0^m)\|_{C^{1,\alpha}} \le C_2/X_0^m.$$

Also we set $\tau = \tan\theta_0$.

2. Reformulation in terms of radial and angular velocities

In this section, we first decompose the velocity $u=(u_1^+, u_2^+, u_3^+)$ as (U_1^+, U_2^+, U_3^+), where U_1^+ is the radial velocity and U_2^+ and U_3^+ are the angular velocities. Then we reformulate the nonlinear problem (1-1) with (1-2)–(1-5) to obtain a second-order elliptic equation on ρ^+ and a coupled system on U_2^+, U_3^+ and the first-order equation on U_1^+. The relations between (ρ^+, U_1^+) and (U_2^+, U_3^+) on the shock Σ will also be derived.

Due to the symmetry of the nozzle in the diverging part, it is convenient to introduce a coordinate transformation where $\tau = \tan\theta_0$.

$$\begin{cases} y_1 = \sqrt{x_1^2+x_2^2+x_3^2}, & \\ y_i = \dfrac{x_i}{x_1\tau}, & i=2,3, \end{cases} \tag{2-1}$$

and a decomposition of (u_1^+, u_2^+, u_3^+)

$$\begin{cases} u_1^+ = \dfrac{U_1^+ - y_2\tau U_2^+ - y_3\tau U_3^+}{1+(y_2^2+y_3^2)\tau^2}, \\[2ex] u_2^+ = \dfrac{y_2\tau U_1^+ + (1+y_3^2\tau^2)U_2^+ - y_2y_3\tau^2U_3^+}{1+(y_2^2+y_3^2)\tau^2}, \\[2ex] u_3^+ = \dfrac{y_3\tau U_1^+ - y_2y_3\tau^2U_2^+ + (1+y_2^2\tau^2)U_3^+}{1+(y_2^2+y_3^2)\tau^2}. \end{cases} \tag{2-2}$$

The transformation (2-1) changes the domain

$$\Omega = \{(x_1,x_2,x_3) : X_0 \le \sqrt{x_1^2+x_2^2+x_3^2} \le X_0+1,\ x_2^2+x_3^2 \le x_1^2\tau^2\}$$

and

$$\Omega_+ = \{(x_1,x_2,x_3) : \eta(x_2,x_3) < x_1 < \sqrt{(X_0+1)^2 - x_2^2 - x_3^2},\ x_2^2+x_3^2 \le x_1^2\tau^2$$

into the domains

$$\omega = \{(y_1,y_2,y_3) : X_0 \le y_1 \le X_0+1,\ y_2^2+y_3^2 \le 1\}$$

and

$$\omega_+ = \{(y_1,y_2,y_3) : \xi(y_2,y_3) \le y_1 \le X_0+1,\ y_2^2+y_3^2 \le 1\},$$

respectively. Here $y_1 = \xi(y_2,y_3)$ stands for the equation of the shock surface Σ in the new coordinates $y=(y_1,y_2,y_3)$.

To simplify notation, set

$$
\begin{cases}
D_0 = \dfrac{1}{y_1\sqrt{1+(y_2^2+y_3^2)\tau^2}}, \\[2ex]
D_1 = \dfrac{1}{\sqrt{1+(y_2^2+y_3^2)\tau^2}}\partial_{y_1}, \\[2ex]
D_i = \dfrac{\sqrt{1+(y_2^2+y_3^2)\tau^2}}{y_1\tau}\partial_{y_i}, \quad i = 2, 3.
\end{cases}
\tag{2-3}
$$

Then for any C^1 solution, a direct but tedious computation yields that (1-1) takes the form

$$
\begin{cases}
U_1^+ D_1\rho^+ + U_2^+ D_2\rho^+ + U_3^+ D_3\rho^+ \\
\qquad\qquad + \rho^+(D_1U_1^+ + D_2U_2^+ + D_3U_3^+) = f_1, \\[1ex]
\rho^+U_1^+ D_1U_1^+ + \rho^+U_2^+ D_2U_1^+ + \rho^+U_3^+ D_3U_1^+ \\
\qquad\qquad + \big(1+(y_2^2+y_3^2)\tau^2\big)c^2(\rho^+)D_1\rho^+ = f_2, \\[1ex]
\rho^+U_1^+ D_1U_2^+ + \rho^+U_2^+ D_2U_2^+ + \rho^+U_3^+ D_3U_2^+ \\
\qquad + (1+y_2^2\tau^2)c^2(\rho^+)D_2\rho^+ + y_2y_3\tau^2c^2(\rho^+)D_3\rho^+ = f_3, \\[1ex]
\rho^+U_1^+ D_1U_3^+ + \rho^+U_2^+ D_2U_3^+ + \rho^+U_3^+ D_3U_3^+ \\
\qquad + y_2y_3\tau^2c^2(\rho^+)D_2\rho^+ + (1+y_3^2\tau^2)c^2(\rho^+)D_3\rho^+ = f_4,
\end{cases}
\tag{2-4}
$$

and on the shock position $y_1 = \xi(y_2, y_3)$, Equation (1-2) becomes

$$
\begin{cases}
\dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1] - \partial_{y_2}\xi[\rho U_2] - \partial_{y_3}\xi[\rho U_3] = 0, \\[2ex]
\dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1^2 + (1+(y_2^2+y_3^2)\tau^2)P] \\
\qquad\qquad - \partial_{y_2}\xi[\rho U_1U_2] - \partial_{y_3}\xi[\rho U_1U_3] = 0, \\[2ex]
\dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1U_2] - \partial_{y_2}\xi[\rho U_2^2 + (1+y_2^2\tau^2)P] \\
\qquad\qquad - \partial_{y_3}\xi[\rho U_2U_3 + y_2y_3\tau^2P] = 0, \\[2ex]
\dfrac{y_1\tau}{1+(y_2^2+y_3^2)\tau^2}[\rho U_1U_3] - \partial_{y_2}\xi[\rho U_2U_3 + y_2y_3\tau^2P] \\
\qquad\qquad - \partial_{y_3}\xi[\rho U_3^2 + (1+y_3^2\tau^2)P] = 0,
\end{cases}
\tag{2-5}
$$

where

$$\begin{cases} f_1 = -2\rho^+ D_0\big(U_1^+ - y_2\tau U_2^+ - y_3\tau U_3^+\big), \\ f_2 = \rho^+ D_0\big(U_1^+(y_2\tau U_2^+ + y_3\tau U_3^+) + (U_2^+)^2 + (U_3^+)^2 \\ \qquad\qquad\qquad\qquad\qquad + (y_3\tau U_2^+ - y_2\tau U_3^+)^2\big), \\ f_3 = -\rho^+ D_0\big(U_1^+U_2^+ - y_2\tau(U_2^+)^2 - y_3\tau U_2^+U_3^+\big), \\ f_4 = -\rho^+ D_0\big(U_1^+U_3^+ - y_2\tau U_2^+U_3^+ - y_3\tau^2(U_3^+)^2\big). \end{cases} \tag{2-6}$$

Meanwhile, (1-5) is changed into

$$y_2U_2^+ + y_3U_3^+ = 0 \quad \text{on } y_2^2 + y_3^2 = 1. \tag{2-7}$$

Since the transformation (2-1) between the coordinate systems (x_1, x_2, x_3) and (y_1, y_2, y_3) preserves the $C^{4,\alpha}$ norm, from now on, we will use (y_1, y_2, y_3) to discuss our problem instead of (x_1, x_2, x_3). In addition, we will neglect the "+" superscripts for notational simplification.

The third and the fourth equalities in (2-5) give

$$\partial_{y_2}\xi(y_2, y_3) = \frac{\Delta_2}{\Delta_1}, \quad \partial_{y_3}\xi(y_2, y_3) = \frac{\Delta_3}{\Delta_1}, \tag{2-8}$$

where

$$\Delta_1 = \rho\big((1 + y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2U_2U_3 + (1 + y_2^2\tau^2)U_3^2\big) + [P](1 + (y_2^2 + y_3^2)\tau^2),$$

$$\Delta_2 = \frac{\xi(y_2, y_3)\tau\rho U_1}{1 + (y_2^2 + y_3^2)\tau^2}(U_2 + y_3^2\tau^2U_2 - y_2y_3\tau^2U_3),$$

$$\Delta_3 = \frac{\xi(y_2, y_3)\tau\rho U_1}{1 + (y_2^2 + y_3^2)\tau^2}(-y_2y_3\tau^2U_2 + U_3 + y_2^2\tau^2U_3).$$

It follows from the compatibility condition

$$\partial_{y_3}(\partial_{y_2}\xi) = \partial_{y_2}(\partial_{y_3}\xi)$$

that

$$\begin{aligned} (\partial_{y_3}\xi\,\partial_{y_1} + \partial_{y_3})U_2 - (\partial_{y_2}\xi\,\partial_{y_1} + \partial_{y_2})U_3 \\ = H_0(y_2, y_3, \rho, U_2, U_3, \xi, \nabla_{y_2,y_3}\rho, \nabla_{y_2,y_3}U_2, \nabla_{y_2,y_3}U_3, \nabla_{y_2,y_3}\xi) \\ \text{on } y_1 = \xi(y_2, y_3), \end{aligned} \tag{2-9}$$

where for large X_0,

$$\begin{aligned} H_0 = {} & O(|U_2|^2 + |U_3|^2) + O(|\nabla_{y_2,y_3}\rho|^2) \\ & + O(|\nabla_{y_2,y_3}U_2|^2) + O(|\nabla_{y_2,y_3}U_3|^2) + O(|\nabla_{y_2,y_3}\xi|^2) \\ & + O(1/X_0)(|U_2| + |U_3| + |\nabla_{y_2,y_3}\rho| + |\nabla_{y_2,y_3}U_2| + |\nabla_{y_2,y_3}U_3| + |\nabla_{y_2,y_3}\xi|). \end{aligned}$$

The concrete expression of H_0 is given in Lemma B.1 in Appendix B.

In addition, the first equation in (2-4) can be rewritten as

$$D_2U_2 + D_3U_3 = \frac{1}{\rho}(f_1 - \rho D_1U_1 - U_1D_1\rho - U_2D_2\rho - U_3D_3\rho). \tag{2-10}$$

It is clear that for small $|\nabla_{y_2,y_3}\xi|$, Equations (2-9) and (2-10) consist of a first-order elliptic system for (U_2, U_3) on the shock surface $y_1 = \xi(y_2, y_3)$.

Next we determine the equations of U_2, U_3 in ω_+ and their boundary conditions. By the third and fourth equations of (2-4) and (2-9), (U_2, U_3) satisfies

$$\begin{cases} \rho U_1D_1U_2 + \rho U_2D_2U_2 + \rho U_3D_3U_2 \\ \qquad + (1+y_2^2\tau^2)c^2(\rho)D_2\rho + y_2y_3\tau^2c^2(\rho)D_3\rho = f_3, \\ \rho U_1D_1U_3 + \rho U_2D_2U_3 + \rho U_3D_3U_3 \\ \qquad + y_2y_3\tau^2c^2(\rho)D_2\rho + (1+y_3^2\tau^2)c^2(\rho)D_3\rho = f_4, \\ (\partial_{y_3}\xi\partial_{y_1} + \partial_{y_3})U_2 - (\partial_{y_2}\xi\partial_{y_1} + \partial_{y_2})U_3 = H_0 \quad \text{on } y_1 = \xi(y_2, y_3), \\ y_2U_2 + y_3U_3 = 0 \quad \text{on } y_2^2 + y_3^2 = 1. \end{cases} \tag{2-11}$$

Next, U_1 can be obtained from the equation

$$(\rho U_1D_1 + \rho U_2D_2 + \rho U_3D_3) \times \left(\frac{U_1^2 + U_2^2 + U_3^2 + (y_3\tau U_2 - y_2\tau U_3)^2}{2\big(1 + (y_2^2 + y_3^2)\tau^2\big)} + h(\rho)\right) = 0 \tag{2-12}$$

with

$$h'(\rho) = \frac{c^2(\rho)}{\rho}.$$

Finally, we determine the equation and the boundary conditions for the density ρ. By (2-7) and the third and the fourth equations in (2-4), the corresponding boundary condition of ρ on $y_2^2 + y_3^2 = 1$ is

$$y_2\partial_{y_2}\rho + y_3\partial_{y_3}\rho = \frac{\rho(U_2^2 + U_3^2)}{(1+\tau^2)c^2(\rho)} \quad \text{on } y_2^2 + y_3^2 = 1. \tag{2-13}$$

We now derive a Dirichlet boundary condition for ρ on the shock Σ. Substituting the expression (2-8) into the first two equations of (2-5) yields on Σ

$$\begin{cases} G_1(\rho, U) \equiv [\rho U_1]\tilde{\Delta}_1 - [\rho U_2]\tilde{\Delta}_2 - [\rho U_3]\tilde{\Delta}_3 = 0, \\ G_2(\rho, U) \equiv [P + \rho U_1^2]\tilde{\Delta}_1 - [\rho U_1U_2]\tilde{\Delta}_2 - [\rho U_1U_3]\tilde{\Delta}_3 = 0, \end{cases} \tag{2-14}$$

with

$$\begin{cases} \tilde{\Delta}_1 = \Delta_1, \\ \tilde{\Delta}_2 = \rho U_1 (U_2 + y_3^2 \tau^2 U_2 - y_2 y_3 \tau^2 U_3), \\ \tilde{\Delta}_3 = \rho U_1 (-y_2 y_3 \tau^2 U_2 + U_3 + y_2^2 \tau^2 U_3). \end{cases}$$

In terms of (2-1), the background solution

$$(P_0^{\pm}(x), u_{1,0}^{\pm}(x), u_{2,0}^{\pm}(x), u_{3,0}^{\pm}(x))$$

in Appendix A is changed into

$$(\overline{P}_0^{\pm}(y_1), \overline{U}_{1,0}^{\pm}(y), \overline{U}_{2,0}^{\pm}(y), \overline{U}_{3,0}^{\pm}(y)) = (P_0^{\pm}(y_1), \sqrt{1 + (y_2^2 + y_3^2)\tau^2}\, U_0^{\pm}(y_1), 0, 0). \tag{2-15}$$

Then by Remarks A.1 and A.2 of Appendix A and a direct computation, there exists a constant $C > 0$ such that

$$\left| \frac{d^k \overline{P}_0^{\pm}(y_1)}{dy_1} \right| + \left| \partial_{y_1}^k \overline{U}_{1,0}^{\pm}(y) \right| \le \frac{C}{X_0^k}, \quad k = 1, 2, 3, 4, \tag{2-16}$$

$$\left| \partial_{y_2}^k \overline{U}_{1,0}^{\pm}(y) \right| + \left| \partial_{y_3}^k \overline{U}_{1,0}^{\pm}(y) \right| \le \frac{C}{X_0^2}. \tag{2-17}$$

Therefore, due to (2-16), (2-14) and the implicit function theorem, a direct computation yields on Σ

$$(U_1 - \overline{U}_{1,0}^{+}(r_0), \rho - \bar{\rho}_0^{+}(r_0)) = (\tilde{g}_1, \tilde{g}_2)(U_2^2, U_3^2, \overline{P}_0^{-} - \overline{P}_0^{-}(r_0), \overline{U}_{1,0}^{-} - \overline{U}_{1,0}^{-}(r_0)), \tag{2-18}$$

where $\tilde{g}_i$ satisfies

$$\tilde{g}_i = \big(O(\varepsilon) + O(1/X_0)\big)\big(O(U_2) + O(U_3) + O(\xi - r_0)\big). \tag{2-19}$$

Equation (2-19) implies that on the shock surface, the influence of U_2 and U_3 on $U_1 - \overline{U}_{1,0}^{+}(r_0)$ and $\rho^{+} - \bar{\rho}_0^{+}(r_0)$ can be almost "neglected".

Additionally, as in [Xin and Yin 2008b, Section 5], one can combined equations (2-4) in the form

$$\begin{aligned} &D_1(\text{the second equation}) + D_2(\text{the third equation}) + D_3(\text{the fourth equation}) \\ &\qquad - D_1(U_1 \times \text{the first equation}) - D_2(U_2 \times \text{the first equation}) \\ &\qquad\qquad - D_3(U_3 \times \text{the first equation}) + (D_1 U_1 + D_2 U_2 + D_3 U_3) f_1, \end{aligned}$$

obtaining a second-order equation on ρ with mixed boundary value conditions (by (2-18), (2-13) and (1-4)) as follows:

$$
\begin{cases}
D_1\Big(\big(c^2(\rho)-U_1^2+(y_2^2+y_3^2)\tau^2c^2(\rho)\big)D_1\rho \\
\qquad -U_1U_2D_2\rho-U_1U_3D_3\rho\Big) \\
+D_2\Big(-U_1U_2D_1\rho+\big(c^2(\rho)-U_2^2+y_2\tau^2c^2(\rho)\big)D_2\rho \\
\qquad +(y_2y_3\tau^2c^2(\rho)-U_2U_3)D_3\rho\Big) \\
+D_3\Big(-U_1U_3D_1\rho+\big(y_2y_3\tau^2c^2(\rho)-U_2U_3\big)D_2\rho \\
\qquad +\big(c^2(\rho)-U_3^2+y_3^2\tau^2c^2(\rho)\big)D_3\rho\Big) \\
\qquad\qquad =H_1(y_2,y_3,\rho,U,\nabla\rho,\nabla U) & \text{in } \omega_+, \\
\rho-\bar{\rho}_0^+(r_0)=\tilde{g}_2 & \text{on } y_1=\xi(y_2,y_3), \\
y_2\partial_{y_2}\rho+y_3\partial_{y_3}\rho=\dfrac{\rho(U_2^2+U_3^2)}{(1+\tau^2)c^2(\rho)} & \text{on } y_2^2+y_3^2=1, \\
P(\rho)=P_e+\varepsilon\tilde{P}_0(y_2,y_3) & \text{on } y_1=X_0+1,
\end{cases}
\tag{2-20}
$$

where $\tilde{P}_0(y_2,y_3)$ is the function $P_0(x_2,x_3)$ under the transformation (2-1) and

$$
\begin{aligned}
&H_1(y_2,y_3,\rho,U,\nabla\rho,\nabla U) \\
&\quad =D_1(\rho U_1)D_2U_2+D_1(\rho U_1)D_3U_3-D_1(\rho U_2)D_2U_1-D_1(\rho U_3)D_3U_1 \\
&\qquad +D_2(\rho U_2)D_1U_1+D_2(\rho U_2)D_3U_3-D_2(\rho U_1)D_1U_2-D_2(\rho U_3)D_3U_2 \\
&\qquad +D_3(\rho U_3)D_1U_1+D_3(\rho U_3)D_2U_2-D_3(\rho U_1)D_1U_3-D_3(\rho U_2)D_2U_3 \\
&\qquad +\rho U_1([D_1,D_2]U_2+[D_1,D_3]U_3)+\rho U_2([D_2,D_1]U_1+[D_2,D_3]U_3) \\
&\qquad\qquad +\rho U_3([D_3,D_1]U_1+[D_3,D_2]U_2) \\
&\qquad +D_1\Big(\rho D_0\big(U_1(y_2\tau U_2+y_3\tau U_3)+(1+y_3^2\tau^2)U_2^2-2y_2y_3\tau^2U_2U_3+(1+y_2^2\tau^2)U_3^2 \\
&\qquad\qquad +2U_1(U_1-y_2\tau U_2-y_3\tau U_3)\big)\Big) \\
&\qquad +D_2\big(\rho D_0(U_1U_2-y_2\tau U_2^2-y_3\tau U_2U_3)\big)+D_3\big(\rho D_0(U_1U_3-y_2\tau U_2U_3-y_3\tau U_3^2)\big),
\end{aligned}
$$

where $[D_i,D_j]=D_iD_j-D_jD_i$.

Therefore, we only need to prove the next result to show Theorem 1.1.

Theorem 2.1. *Let the assumptions of Theorem 1.1 hold. Then the problem* (2-9)–(2-12), (2-18) *and* (2-20) *has no more than one solution*

$$(P(y),U_1(y),U_2(y),U_3(y);\xi(y_2,y_3))$$

with the following estimates.

(1) $\xi(y_2, y_3) \in C^{4,\alpha}(\overline{B_1(0)})$ *with* $B_1(0)$ *a unit circle centered at* $(0, 0)$, *and there exists a constant* $C > 0$ (*depending on* α *and the supersonic incoming flow*) *such that*

$$\|\xi(y_2, y_3) - r_0\|_{L^\infty(\overline{B_1(0)})} \le C X_0 \varepsilon, \quad \|\nabla_{y_2,y_3}(\xi(y_2, y_3) - r_0)\|_{C^{3,\alpha}(\overline{B_1(0)})} \le C\varepsilon.$$

(2) *If* $\omega_+ = \{(y_1, y_2, y_3) : \xi(y_2, y_3) < y_1 < X_0 + 1,\ y_2^2 + y_3^2 < 1\}$, *then*

$$(P(y), U_1(y), U_2(y), U_3(y)) \in C^{3,\alpha}(\overline{\omega_+})$$

satisfies

$$\|(P(y), U_1(y), U_2(y), U_3(y)) - (\bar{P}_0^+(y_1), \bar{U}_{1,0}^+(y), 0, 0)\|_{C^{3,\alpha}(\overline{\omega_+})} \le C\varepsilon.$$

To prove Theorem 2.1, as in [Xin and Yin 2008b], we first reduce the free boundary problem (2-9)–(2-12), (2-18) and (2-20) into a fixed boundary problem by the transformation

$$\begin{cases} z_1 = \dfrac{y_1 - \xi(y_2, y_3)}{X_0 + 1 - \xi(y_2, y_3)}, \\ z_i = y_i \qquad\qquad\qquad i = 2, 3. \end{cases} \tag{2-21}$$

Under (2-21), the region ω_+ is changed into

$$E_+ = \{(z_1, z_2, z_3) : 0 < z_1 < 1,\ z_2^2 + z_3^2 < 1\}. \tag{2-22}$$

Correspondingly,

$$\begin{cases} D_0 = \dfrac{1}{\big(\xi(z_2, z_3) + z_1(X_0 + 1 - \xi(z_2, z_3))\big)\sqrt{1 + (z_2^2 + z_3^2)\tau^2}}, \\ D_1 = \dfrac{1}{\sqrt{1 + (z_2^2 + z_3^2)\tau^2}} \dfrac{1}{X_0 + 1 - \xi(z_2, z_3)} \partial_{z_1}, \\ D_i = \dfrac{\sqrt{1 + (z_2^2 + z_3^2)\tau^2}}{\big(\xi(z_2, z_3) + z_1(X_0 + 1 - \xi(z_2, z_3))\big)\tau} \\ \qquad\qquad \times \left(\dfrac{(z_1 - 1)\partial z_i \xi}{X_0 + 1 - \xi(z_2, z_3)} \partial_{z_1} + \partial_{z_2} \right), \quad i = 2, 3. \end{cases} \tag{2-23}$$

In next section, we will establish some basic estimates on the problem (2-9)–(2-12), (2-18) and (2-20) in the coordinate $z = (z_1, z_2, z_3)$, which are crucial in the proof of Theorem 2.1.

A further by-product of the analysis for Theorems 1.1 and 2.1 is estimates on the location of the shock and its monotonic dependence on the end pressure.

Proposition 2.2. *Let the assumptions of Theorem 1.1 hold. Suppose the problem* (2-4) *with* (2-5), (2-7) *has two* $C^{3,\alpha}$ *solutions*

$$(\rho, U_1, U_2, U_3; \xi_1(y_2, y_3)) \quad \textit{and} \quad (q, V_1, V_2, V_3; \xi_2(y_2, y_3))$$

which satisfy the exit pressure conditions

$$P_e + \varepsilon(P_0(x_2, x_3) + C_{0,1}) \quad \textit{and} \quad P_e + \varepsilon(P_0(x_2, x_3) + C_{0,2})$$

at $r = X_0 + 1$, *respectively, and which admit the estimates in Theorem 2.1*, *with the two constants satisfying* $C_{0,1} < C_{0,2}$. *Then*

$$\xi_1(y_2, y_3) > \xi_2(y_2, y_3). \tag{2-24}$$

3. A priori estimates

In this section, we will derive some elementary estimates on the difference of two possible solutions to the problem (2-9)–(2-12), (2-18) and (2-20). Based on these estimates, we can show the monotonicity of the end pressure on the position of the shock along the nozzle wall. Assume that the problem (2-9)–(2-12), (2-18) and (2-20) has two solutions $(\rho, U_1, U_2, U_3; \xi_1(z_2, z_3))$ and $(q, V_1, V_2, V_3; \xi_2(z_2, z_3))$, which satisfy the assumptions in Theorem 2.1. Denote by $Q = P(q)$ the pressure for the density q. In addition, (D_0, D_1, D_2, D_3) and $(\widetilde{D_0}, \widetilde{D_1}, \widetilde{D_2}, \widetilde{D_3})$ satisfy (2-23) with $(q, V_1, V_2, V_3; \xi_2(z_2, z_3))$ instead of $(\rho, U_1, U_2, U_3; \xi(z_2, z_3))$ in the $(\widetilde{D_0}, \widetilde{D_1}, \widetilde{D_2}, \widetilde{D_3})$ case.

Set

$$\begin{cases} (Y_i, Y_4)(z_1, z_2, z_3) \\ \qquad = (U_i, \rho)(\xi_1(z_2, z_3) + z_1(X_0 + 1 - \xi_1(z_2, z_3)), z_2, z_3) \\ \qquad\quad - (V_i, q)(\xi_2(z_2, z_3) + z_1(X_0 + 1 - \xi_2(z_2, z_3)), z_2, z_3), \quad i = 1, 2, 3, \\ Y_5(z_2, z_3) = \xi_1(z_2, z_3) - \xi_2(z_2, z_3). \end{cases}$$

We estimate the derivatives of Y_i for $i = 1, 2, 3, 4, 5$ in a series of lemmas.

Lemma 3.1. *Under the assumptions of Theorem 2.1*, *the following estimates hold*:

$$\begin{cases} D_0 - \widetilde{D_0} = O(1/X_0^2) Y_5, \\ D_1 - \widetilde{D_1} = O(1) Y_5 \partial_{z_1}, \\ D_i - \widetilde{D_i} = O(\varepsilon) Y_5 \partial_{z_1} + O(1) \partial_{z_2} Y_5 \partial_{z_1} + O(1/X_0) Y_5 \partial_{z_2}, \quad i = 2, 3. \end{cases} \tag{3-1}$$

Proof. We estimate $D_1 - \widetilde{D_1}$ only since the other terms can be treated analogously.

By (2-23), one has

$$D_1 - \widetilde{D_1} = \frac{Y_5}{(X_0+1-\xi_1(z_2, z_3))(X_0+1-\xi_2(z_2, z_3))\sqrt{1+(z_2^2+z_3^2)\tau^2}} \partial_{z_1},$$

where

$$\left\| \frac{1}{(X_0+1-\xi_1(z_2,z_3))(X_0+1-\xi_2(z_2,z_3))\sqrt{1+(z_2^2+z_3^2)\tau^2}} \right\|_{C^{1,\alpha}} \leq C.$$

This immediately implies $D_1 - \widetilde{D_1} = O(1)Y_5\partial_{z_1}$. □

Lemma 3.2 (estimates of $\nabla_{z_2,z_3}Y_5$). *Under the assumptions of Theorem 2.1, we have*

$$\begin{aligned}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}} &\leq C\varepsilon\|(Y_1, (\varepsilon X_0^2)^{-1}Y_2, (\varepsilon X_0^2)^{-1}Y_3, Y_4, Y_5)\|_{C^{1,\alpha}} \\ &\quad + \frac{C}{X_0^2}\|\nabla_{z_2,z_3}(\varepsilon Y_1, \varepsilon X_0^2 Y_4)\|_{C^{1,\alpha}} \\ &\quad + C\|(\partial_{z_2}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}} + \frac{C}{X_0^2}\|(\partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned} \tag{3-2}$$

Remark 3.1. It follows from (3-2) that the term $\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}$ is controlled mainly by $\|\partial_{z_2}Y_2\|_{C^{1,\alpha}} + \|\partial_{z_3}Y_3\|_{C^{1,\alpha}}$.

Proof of Lemma 3.2. Equation (2-8) yields

$$\begin{cases} \partial_{z_2}\xi_1(z_2,z_3) = \dfrac{\Delta_2}{\Delta_1}, & \partial_{z_3}\xi_1(z_2,z_3) = \dfrac{\Delta_3}{\Delta_1}, \\ \partial_{z_2}\xi_2(z_2,z_3) = \dfrac{\widetilde{\Delta_2}}{\widetilde{\Delta_1}}, & \partial_{z_3}\xi_2(z_2,z_3) = \dfrac{\widetilde{\Delta_3}}{\widetilde{\Delta_1}}, \\ z_2\partial_{z_2}Y_5 + z_3\partial_{z_3}Y_5 = 0 \quad \text{on } l, \end{cases}$$

where $\tilde{\Delta}_i$ for $i = 1, 2, 3$ has a similar expression to Δ_i with $(q, V_1, V_2, V_3; \xi_2(z_2, z_3))$ instead of $(\rho, U_1, U_2, U_3; \xi(z_2, z_3))$, and l denotes the circle $\{z : z_1 = 0,\ z_2^2+z_3^2 = 1\}$.

This shows that on $z_1 = 0$,

$$\begin{cases} \partial_{z_2}Y_5 = O(\varepsilon)\cdot(Y_1, Y_4, X_0^{-1}Y_5) + O(1)Y_2 + O(1/X_0^2)Y_3, \\ \partial_{z_3}Y_5 = O(\varepsilon)\cdot(Y_1, Y_4, X_0^{-1}Y_5) + O(1/X_0^2)Y_2 + O(1)Y_3, \\ z_2\partial_{z_2}Y_5 + z_3\partial_{z_3}Y_5 = 0 \quad \text{on } l, \end{cases} \tag{3-3}$$

From this, one can obtain a first-order elliptic system on $(\partial_{z_2}Y_5, \partial_{z_3}Y_5)$ as

$$\begin{cases} \partial_{z_2}(\partial_{z_2}Y_5) + \partial_{z_3}(\partial_{z_3}Y_5) = F_1 & \text{on } z_1 = 0, \\ \partial_{z_3}(\partial_{z_2}Y_5) - \partial_{z_2}(\partial_{z_3}Y_5) = 0 & \text{on } z_1 = 0, \\ z_2\partial_{z_2}Y_5 + z_3\partial_{z_3}Y_5 = 0 & \text{on } l, \end{cases} \tag{3-4}$$

with

$$\begin{aligned} F_1 &= O(\varepsilon)\cdot(Y_1, Y_4, X_0^{-1}Y_5) \\ &\quad + O(1/X_0^2)\cdot(Y_2, Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3) + O(1)\partial_{z_2}Y_2 + O(1)\partial_{z_3}Y_3 \\ &\quad + O(\varepsilon)\cdot(\partial_{z_2}Y_1, \partial_{z_2}Y_4, X_0^{-1}\partial_{z_2}Y_5, \partial_{z_3}Y_1, \partial_{z_3}Y_4, X_0^{-1}\partial_{z_3}Y_5). \end{aligned}$$

It follows from the Hilbert problem for first-order elliptic systems with index -2 that (see [Bers 1950; 1951; Vekua 1952])

$$\|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{2,\alpha}} \le C\|F_1\|_{C^{1,\alpha}}. \tag{3-5}$$

This yields (3-2). □

Lemma 3.3 (estimates of $\partial_{z_1} Y_i$ for $i = 1, 2, 3, 4$). *Under the assumptions of Theorem 2.1, we have the following estimates*:

$$\begin{aligned}
\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} &\\
\le \frac{C}{X_0}&\|(Y_1, X_0^{-1} Y_2, X_0^{-1} Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\
&+ C\varepsilon\|(\partial_{z_2} Y_1, \partial_{z_2} Y_3, \partial_{z_2} Y_4, \partial_{z_2} Y_5, \partial_{z_3} Y_1, \partial_{z_3} Y_2, \partial_{z_3} Y_4, \partial_{z_3} Y_5)\|_{C^{1,\alpha}}\\
&+ C\|(\partial_{z_2} Y_2, \partial_{z_3} Y_3)\|_{C^{1,\alpha}},
\end{aligned} \tag{3-6}$$

$$\begin{aligned}
\|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}} &\\
\le C\varepsilon&\|(Y_1, (\varepsilon X_0)^{-1} Y_2, (\varepsilon X_0)^{-1} Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\
&+ C\varepsilon\|(\partial_{z_2} Y_1, \partial_{z_2} Y_2, \partial_{z_2} Y_3, \partial_{z_3} Y_1, \partial_{z_3} Y_2, \partial_{z_3} Y_3)\|_{C^{1,\alpha}}\\
&+ \frac{C}{X_0}\|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{1,\alpha}} + C\|(\partial_{z_2} Y_4, \partial_{z_3} Y_4)\|_{C^{1,\alpha}},
\end{aligned} \tag{3-7}$$

$$\begin{aligned}
\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^{\alpha}} &\\
\le \frac{C}{X_0^2}&\|(Y_1, X_0 Y_2, X_0 Y_3, Y_4, Y_5)\|_{C^{1,\alpha}}\\
&+ C\varepsilon\|(\partial_{z_2} Y_1, \partial_{z_2} Y_3, \partial_{z_3} Y_1, \partial_{z_3} Y_2)\|_{C^{1,\alpha}}\\
&+ \frac{C}{X_0}\|(\partial_{z_2} Y_5, \partial_{z_3} Y_5, \partial_{z_2} Y_2, \partial_{z_3} Y_3)\|_{C^{1,\alpha}} + C\|(\partial_{z_2} Y_4, \partial_{z_3} Y_4)\|_{C^{1,\alpha}}.
\end{aligned} \tag{3-8}$$

Remark 3.2. Equations (3-6) and (3-7) imply the terms $\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}}$ and $\|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}}$ are controlled mainly by

$$\frac{C}{X_0}\|Y_5\|_{C^{1,\alpha}} + C(\|\partial_{z_2} Y_2\|_{C^{1,\alpha}} + \|\partial_{z_3} Y_3\|_{C^{1,\alpha}}) \quad \text{and} \quad C(\|\partial_{z_2} Y_4\|_{C^{1,\alpha}} + \|\partial_{z_3} Y_4\|_{C^{1,\alpha}}),$$

respectively. In fact, $(C/X_0)\|Y_5\|_{C^{1,\alpha}}$ is not a "good" term (see Remark 4.1). To overcome this difficulty and for more applications (see Remark 3.4), we must treat the term $\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^\alpha}$ instead of $\|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}}$. Fortunately, the term $\|(\partial_{z_1}^2 Y_1, \partial_{z_1}^2 Y_4)\|_{C^\alpha}$ can be controlled mainly by

$$\frac{C}{X_0^2}\|Y_5\|_{C^{1,\alpha}}, \quad \frac{C}{X_0}\|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}} \quad \text{and} \quad C\|(\partial_{z_2} Y_4, \partial_{z_3} Y_4)\|_{C^{1,\alpha}},$$

which are all "good" (roughly speaking, a "good" term can be directly absorbed by the left hand side in the related a priori estimates).

Proof of Lemma 3.3. It follows from (2-4), Lemma 3.1 and the assumptions in Theorem 2.1 that $\partial_{z_1} Y_i$ for $i = 1, 2, 3, 4$ satisfy

$$
\text{(3-9)} \quad \begin{cases}
\rho\partial_{z_1}Y_1 + U_1\partial_{z_1}Y_4 \\
\qquad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) + O(1)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3) \\
\qquad\quad + O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, Y_5, \varepsilon\partial_{z_1}Y_4), \\
\rho U_1\partial_{z_1}Y_1 + (1+(z_2^2+z_3^2)\tau^2)c^2(\rho)\partial_{z_1}Y_4 \\
\qquad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_1, \partial_{z_2}Y_1, \partial_{z_3}Y_1, X_0^{-1}\partial_{z_2}Y_5, X_0^{-1}\partial_{z_3}Y_5), \\
\partial_{z_1}Y_2 = O(\varepsilon)\cdot(Y_1, (\varepsilon X_0)^{-1}Y_2, Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_2, \partial_{z_1}Y_4, \partial_{z_2}Y_2, \partial_{z_3}Y_2, (\varepsilon X_0^2)^{-1}\partial_{z_3}Y_4) \\
\qquad\quad + O(1/X_0)(\partial_{z_2}Y_5, X_0^{-2}\partial_{z_3}Y_5) + O(1)\partial_{z_2}Y_4, \\
\partial_{z_1}Y_3 = O(\varepsilon)\cdot(Y_1, Y_2, (\varepsilon X_0)^{-1}Y_3, Y_4, Y_5) \\
\qquad\quad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_3, (\varepsilon X_0^2)^{-1}\partial_{z_2}Y_4, \partial_{z_3}Y_3) \\
\qquad\quad + O(1/X_0)\cdot(X_0^{-2}\partial_{z_2}Y_5, \partial_{z_3}Y_5) + O(1)\partial_{z_3}Y_4.
\end{cases}
$$

So a direct computation yields (3-6) and (3-7).

From the expressions of $\partial_{z_1}Y_1$ and $\partial_{z_1}Y_4$ obtained by solving the first and second equations in (3-9), one has again for $i = 1, 4$,

$$
\begin{aligned}
\text{(3-10)} \quad \partial_{z_1}^2 Y_i &= O(1/X_0^2)\cdot(Y_1, Y_2, Y_3, Y_4) \\
&\quad + O(1/X_0)\cdot(\partial_{z_1}Y_1, X_0^{-1}\partial_{z_1}Y_2, X_0^{-1}\partial_{z_1}Y_3, \partial_{z_1}Y_4, Y_5) \\
&\quad + \partial_{z_1}\big(O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_1, \partial_{z_3}Y_4, \partial_{z_3}Y_5)\big) \\
&\quad + O(1/X_0^2)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3) + O(1)\cdot(\partial_{z_1z_2}^2 Y_2, \partial_{z_1z_3}^2 Y_3).
\end{aligned}
$$

Equation (3-8) follows from (3-10) and a direct computation. □

Next, we estimate $\nabla_{z_2,z_3}Y_2$ and $\nabla_{z_2,z_3}Y_3$.

Lemma 3.4 (estimates of $Y_2(0, z_2, z_3)$ and $Y_3(0, z_2, z_3)$). *Under the assumptions of Theorem 2.1, we have*

$$
\begin{aligned}
\text{(3-11)} \quad &\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B}B_1(0))} \\
&\le \frac{C}{X_0}\|(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}} \\
&\quad + C\varepsilon\|(\partial_{z_1}Y_1, \partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_1}Y_4)\|_{C^{1,\alpha}} \\
&\quad + C\|(\partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_3}Y_1, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{1,\alpha}}.
\end{aligned}
$$

Remark 3.3. It follows from (3-11) that $\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B}B_1(0))}$ is controlled mainly by $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$ and $C\|(\partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_3}Y_1, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}$.

Proof of Lemma 3.4. From (2-9)–(2-10), the assumptions in Theorem 2.1, and a direct computation, it follows that on $z_1 = 0$,

$$\begin{cases} \partial_{z_3} Y_2 - \partial_{z_2} Y_3 = F_2, \\ \partial_{z_2} Y_2 + \partial_{z_3} Y_3 = F_3, \\ \ z_2 Y_2 + z_3 Y_3 = 0 \quad \text{on } z_2^2 + z_3^2 = 1, \end{cases} \tag{3-12}$$

with

$$\begin{aligned} F_2 = O(\varepsilon) \cdot (Y_1, Y_4, X_0^{-1} Y_5) + O(1/X_0^2) \cdot (Y_2, Y_3) \\ + O(\varepsilon)(\partial_{z_2} Y_1, (\varepsilon X_0^2)^{-1} \partial_{z_2} Y_2, \varepsilon \partial_{z_2} Y_3, \partial_{z_2} Y_4, X_0^{-1} \partial_{z_2} Y_5) \\ + O(\varepsilon) \cdot (\partial_{z_3} Y_1, (\varepsilon X_0^2)^{-1} \partial_{z_3} Y_2, (\varepsilon X_0^2)^{-1} \partial_{z_3} Y_3, \partial_{z_3} Y_4, \partial_{z_3} Y_5) \\ + O(\varepsilon)(\varepsilon \partial_{z_1} Y_1, \partial_{z_1} Y_2, X_0^{-2} \partial_{z_1} Y_3, \varepsilon \partial_{z_1} Y_4), \\ F_3 = O(1/X_0) \cdot (Y_1, X_0^{-1} Y_2, X_0^{-1} Y_3, Y_4, Y_5) \\ + O(\varepsilon) \cdot (\partial_{z_1} Y_2, \partial_{z_1} Y_3, \partial_{z_2} Y_4, \partial_{z_2} Y_5, \partial_{z_3} Y_4, \partial_{z_3} Y_5) \\ + O(1) \cdot (\partial_{z_1} Y_1, \partial_{z_1} Y_4), \end{aligned}$$

where F_3 is given in Lemma B.2 of Appendix B.

As in (3-5), one can obtain from (3-12) that

$$\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B} B_1(0))} \le C \|(F_2, F_3)\|_{C^{1,\alpha}(\bar{B} B_1(0))}. \tag{3-13}$$

On the other hand, due to the second equation and the boundary condition in (3-12),

$$\int_{B_1(0)} F_3 \, ds = \int_{B_1(0)} (\partial_{z_2} Y_2 + \partial_{z_3} Y_3) \, ds = \int_{\partial B_1(0)} (z_2 Y_2 + z_3 Y_3) \, dl = 0 \quad \text{on } z_1 = 0.$$

Since $F_3 \in C^{1,\alpha}(\Omega)$, it follows from the integral mean value theorem that there exists a point (z_2*, z_3*) such that

$$F_3(0, z_2*, z_3*) = 0.$$

This implies

$$\|F_3(0, z_2, z_3)\|_{C^{1,\alpha}} \le C \|\nabla_{z_2, z_3} F_3(0, z_2, z_3)\|_{C^\alpha}.$$

Combining this with (3-13) and a direct computation yields

$$\begin{aligned} &\|(Y_2(0, z_2, z_3), Y_3(0, z_2, z_3))\|_{C^{2,\alpha}(\bar{B} B_1(0))} \\ &\le \frac{C}{X_0} \Big\| \Big(Y_1, X_0^{-1} Y_2, X_0^{-1} Y_3, Y_4, \frac{1}{X_0} Y_5 \Big) \Big\|_{C^{1,\alpha}} \\ &\qquad + C\varepsilon \|(\partial_{z_1} Y_1, \partial_{z_1} Y_2, \partial_{z_1} Y_3, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} + C \|(\partial_{z_2} Y_1, \partial_{z_2} Y_4, \partial_{z_3} Y_1, \partial_{z_3} Y_4)\|_{C^{1,\alpha}} \\ &\qquad + \frac{C}{X_0} \|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{1,\alpha}}, \end{aligned}$$

which completes the proof of Lemma 3.4. □

Using Lemmas 3.3–3.4 and Lemma B.3 in Appendix B, we can estimate $\nabla_{z_2,z_3} Y_2$ and $\nabla_{z_2,z_3} Y_3$ as follows:

Lemma 3.5 (estimates of $\partial_{z_2} Y_2$, $\partial_{z_3} Y_2$ and $\partial_{z_2} Y_3$, $\partial_{z_3} Y_3$). *Under the assumptions of Theorem 2.1*, $\partial_{z_2} Y_2$, $\partial_{z_3} Y_2$ *and* $\partial_{z_2} Y_3$, $\partial_{z_3} Y_3$ *satisfy*

$$
\begin{aligned}
(3\text{-}14)\quad & \|(\partial_{z_2} Y_2, \partial_{z_3} Y_2, \partial_{z_2} Y_3, \partial_{z_3} Y_3)\|_{C^{1,\alpha}} \\
&\leq \frac{C}{X_0}\big(\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1} Y_5)\|_{C^{1,\alpha}} + \|(\partial_{z_2} Y_5, \partial_{z_3} Y_5)\|_{C^{2,\alpha}}\big) \\
&\qquad + C\|(\partial_{z_2} Y_1, \partial_{z_2} Y_4, \partial_{z_3} Y_1, \partial_{z_3} Y_4)\|_{C^{1,\alpha}}.
\end{aligned}
$$

Remark 3.4. Thanks to (3-8), the right hand side of (3-14) can be controlled by the "good" term $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$. This can be seen in (3-16) and (3-17) below.

Proof of Lemma 3.5. This lemma is proved by the characteristic method.

Under the coordinate $z = (z_1, z_2, z_3)$, the characteristics curves

$$(z_2^1(s; z), z_3^1(s; z)) \quad \text{and} \quad (z_2^2(s; z), z_3^2(s; z))$$

of the first-order differential operators

$$U_1 D_1 + U_2 D_2 + U_3 D_3 \quad \text{and} \quad V_1 \widetilde{D_1} + V_2 \widetilde{D_2} + V_3 \widetilde{D_3},$$

respectively, through the point $z = (z_1, z_2, z_3)$, can be defined as

$$
\begin{cases}
\dfrac{dz_i^1(s; z)}{ds} = \dfrac{U_i\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1)), z_2^1, z_3^1\big)}{\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1))\big) A_1 \tau}, \\
z_i^1(z_1; z) = z_i, \quad i = 2, 3,
\end{cases}
$$

$$
\begin{cases}
\dfrac{dz_i^2(s; z)}{ds} = \dfrac{V_i\big(\xi_2(z_2^2, z_3^2) + s(X_0 + 1 - \xi_2(z_2^2, z_3^2)), z_2^2, z_3^2\big)}{\big(\xi_2(z_2^2, z_3^2) + s(X_0 + 1 - \xi_2(z_2^2, z_3^2))\big) A_2 \tau}, \\
z_i^2(z_1; z) = z_i, \quad i = 2, 3,
\end{cases}
$$

where

$$
\begin{aligned}
A_1 = \frac{1}{X_0 + 1 - \xi_1(z_2^1, z_3^1)} \Bigg(& \frac{U_1}{1 + (z_2^1)^2 \tau^2 + (z_3^1)^2 \tau^2} \\
& + \frac{(s-1)\partial_{z_2}\xi_1(z_2^1, z_3^1) U_2 + (s-1)\partial_{z_3}\xi_1(z_2^1, z_3^1) U_3}{\big(\xi_1(z_2^1, z_3^1) + s(X_0 + 1 - \xi_1(z_2^1, z_3^1))\big)\tau} \Bigg),
\end{aligned}
$$

and A_2 can be defined similarly by replacing (ξ_1, U_1, U_2, U_3) with (ξ_2, V_1, V_2, V_3).

Denote by $z_2^1(0;z)=\beta_1$, $z_3^1(0;z)=\beta_2$ and $z_2^2(0;z)=\widetilde{\beta}_1$, $z_3^2(0;z)=\widetilde{\beta}_2$. Then for $i=2,3$,

$$z_i^1(s;z)=\int_0^s \frac{U_i\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1)),z_2^1,z_3^1\big)}{\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1))\big)A_1\tau}\,dt+\beta_{i-1},$$
$$z_i=\int_0^{z_1} \frac{U_i\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1)),z_2^1,z_3^1\big)}{\big(\xi_1(z_2^1,z_3^1)+t(X_0+1-\xi_1(z_2^1,z_3^1))\big)A_1\tau}\,dt+\beta_{i-1}.$$

Similarly, $z_i^2(s,z)$ and z_i have the same expressions with (β_{i-1},ξ_1,V_i) replaced by $(\tilde{\beta}_{i-1},\xi_2,V_i)$.

From this, we can obtain immediately that for $i=2,3$,

$$\|\beta_{i-1}-z_i\|_{C^{2,\alpha}}\le C\|U_i\|_{C^{2,\alpha}},\quad \|\widetilde{\beta_{i-1}}-z_i\|_{C^{2,\alpha}}\le C\|V_i\|_{C^{2,\alpha}}.$$

Next define $l^1(s;z)=(z_2^1-z_2^2)(s;z)$ and $l^2(s;z)=(z_3^1-z_3^2)(s;z)$. Then by direct computation,

$$\begin{cases} \dfrac{dl^1(s;z)}{ds}=O(\varepsilon)\cdot(l^1,l^2)(s;z) \\ \qquad\qquad +O(\varepsilon)\cdot(Y_1,Y_3,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)(s,z_2^1,z_3^1) \\ \qquad\qquad +O(1)Y_2(s,z_2^1,z_3^1), \\ l^1(0;z)=\beta_1-\widetilde{\beta}_1,\quad l^1(z_1;z)=0, \end{cases}$$

and similarly for $l^2(s;z)$.

Therefore

$$\begin{cases} \|l^1\|_{C^{2,\alpha}}+\|\beta_1-\widetilde{\beta}_1\|_{C^{2,\alpha}} \\ \qquad\qquad \le C\|Y_2\|_{C^{2,\alpha}}+C\varepsilon\|(Y_1,Y_3,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)\|_{C^{2,\alpha}}, \\ \|l^2\|_{C^{2,\alpha}}+\|\beta_2-\widetilde{\beta}_2\|_{C^{2,\alpha}} \\ \qquad\qquad \le C\|Y_3\|_{C^{2,\alpha}}+C\varepsilon\|(Y_1,Y_2,Y_5,\varepsilon\partial_{z_2}Y_5,\varepsilon\partial_{z_3}Y_5)\|_{C^{2,\alpha}}. \end{cases}\tag{3-15}$$

By Lemma B.2 in Appendix B, (Y_2,Y_3) satisfies

$$\begin{cases} \partial_{z_2}Y_2+\partial_{z_3}Y_3=F_3 & \text{in } E_+, \\ \partial_{z_3}Y_2-\partial_{z_2}Y_3=F_4 & \text{in } E_+, \\ z_2Y_2+z_3Y_3=0 & \text{on } z_2^2+z_3^2=1, \end{cases}\tag{3-16}$$

where F_3 and F_4 are given in Lemma B.2.

A direct computation yields

$$\begin{cases} \partial_{z_1}F_3=O(1)(\partial_{z_1}^2Y_1,\partial_{z_1}^2Y_4)+\text{some “good” terms}, \\ \nabla_{z_2,z_3}F_3 \text{ consists of “good” terms}. \end{cases}\tag{3-17}$$

Therefore, it follows from Lemma B.3 of Appendix B and Lemmas 3.3–3.4 that

$$\begin{aligned}
&\|(\partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\|_{C^{1,\alpha}} \\
&\quad\le C\Big(\sum_{i=2}^{3}\|\partial_{z_1}Y_i\|_{C^{1,\alpha}} + \|\nabla F_3\|_{C^{1,\alpha}} + \|F_4\|_{C^{1,\alpha}}\Big) \\
&\quad\le \frac{C}{X_0}\big(\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}} + \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big) \\
&\qquad\qquad + C\|(\partial_{z_2}Y_1, \partial_{z_2}Y_4, \partial_{z_3}Y_1, \partial_{z_3}Y_4)\|_{C^{1,\alpha}},
\end{aligned}$$

which completes the proof of Lemma 3.5. □

Lemma 3.6 (estimates of $\partial_{z_2}Y_1,\ \partial_{z_3}Y_1$). *Under the assumptions of Theorem 2.1, Y_1 satisfies*

$$\begin{aligned}
&\|(\partial_{z_2}Y_1, \partial_{z_3}Y_1)\|_{C^{1,\alpha}} \\
&\quad\le \frac{C}{X_0^2}\|(\varepsilon Y_1, Y_2, Y_3, Y_4, Y_5, \partial_{z_1}Y_4, X_0\partial_{z_2}Y_5, X_0\partial_{z_3}Y_5)\|_{C^{2,\alpha}} \\
&\qquad\qquad + C\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}.
\end{aligned}\tag{3-18}$$

Proof. Applying the characteristic method to (2-12) as in the proof of Lemma 3.5, we arrive at

$$\begin{aligned}
Y_1 &= O(1/X_0^2)\cdot(l^1, l^2) + O(\varepsilon)\cdot(Y_2, Y_3) \\
&\qquad + O(1)Y_4 + O(1)\cdot(Y_1, \varepsilon Y_2, \varepsilon Y_3, Y_4)(0, \beta_1(z), \beta_2(z)).
\end{aligned}$$

It follows from (2-18) that on $z_1 = 0$,

$$Y_i = O(\varepsilon)\cdot(Y_2, Y_3) + O(1/X_0)Y_5, \quad i = 1, 4. \tag{3-19}$$

By the assumptions of Theorem 2.1 and Equations (2-16)–(2-17), a direct computation yields

$$\begin{aligned}
&\partial_{z_i}Y_1 \\
&\quad= \partial_{z_i}\big(O(1/X_0^2)\cdot(l^1, l^2) + O(\varepsilon)\cdot(Y_2, Y_3) + O(\varepsilon)\cdot(Y_2, Y_3)(0, \beta_1(z), \beta_2(z))\big) \\
&\qquad + O(1/X_0^2)Y_4 + O(1/X_0^2)\cdot(Y_1, Y_4)(0, \beta_1(z), \beta_2(z)) + O(1)\partial_{z_i}Y_4 \\
&\qquad + O(1)\cdot(\partial_{z_i}Y_1, \partial_{z_i}Y_4)(0, \beta_1(z), \beta_2(z)), \quad i = 2, 3,
\end{aligned}\tag{3-20}$$

and on $z_1 = 0$,

$$\begin{aligned}
&\partial_{z_i}Y_j \\
&\quad= \partial_{z_i}(O(\varepsilon)\cdot(Y_2, Y_3)) + o(1/X_0^2)Y_5 + O(1/X_0)\partial_{z_i}Y_5, \quad i = 2, 3,\ j = 1, 4.
\end{aligned}\tag{3-21}$$

So, combining (3-20) and (3-21) with (3-14) and (3-15) yields (3-18). □

Lemmas 3.2–3.6 essentially convert the estimates on $\|\nabla_{z_2,z_3}Y_5\|_{C^{2,\alpha}}$, $\|\nabla_z Y_1\|_{C^{1,\alpha}}$, $\|\nabla_z(Y_2, Y_3)\|_{C^{1,\alpha}}$ and $\|\partial_{z_1}Y_4\|_{C^{1,\alpha}}$ into an estimate on $\|\nabla_{z_2,z_3}Y_4\|_{C^{1,\alpha}}$, so we now focus on of $\|\nabla_{z_2,z_3}Y_4\|_{C^{1,\alpha}}$. First, we derive from (2-20) some second-order elliptic equations with corresponding boundary conditions for $z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4$ and $z_3\partial_{z_2}Y_4 - z_2\partial_{z_3}Y_4$. This will enable one to obtain their $C^{1,\alpha}$ boundary estimates on the nozzle wall by the theory of second-order elliptic equations with mixed boundary conditions (in this process, one cannot obtain the global $C^{1,\alpha}$ estimates directly in the whole domain due to the appearance of a singularity in the equation for $z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4$; see (3-24)). This and a simple computation yield the $C^{1,\alpha}$ estimates of $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ on the boundary $z_2^2 + z_3^3 = 1$. Subsequently, we can use the second-order elliptic equations and the corresponding boundary conditions for $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ to obtain $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty}$ and further $C^{1,\alpha}$ estimates.

Lemma 3.7 (estimates of $\partial_{z_2}Y_4$, $\partial_{z_3}Y_4$). *Under the assumptions of Theorem 2.1, $\partial_{z_2}Y_4$, and $\partial_{z_3}Y_4$ satisfy*

$$
\begin{aligned}
\text{(3-22)}\quad \|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} \le \frac{C}{X_0}&\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\
&+\frac{C}{X_0}\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\\
&+C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.
\end{aligned}
$$

Remark 3.5. By (3-22), the norm $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}$ has been controlled by "good" terms, in particular, $(C/X_0^2)\|Y_5\|_{C^{1,\alpha}}$.

Proof of Lemma 3.7. It follows from (2-20), (3-19), Lemma 3.1 and a direct computation that

$$
\text{(3-23)}\quad
\begin{cases}
\widetilde{D}_1\big((c^2(\rho) - U_1^2)\widetilde{D}_1Y_4 + c^2(\rho)(z_2^2\tau^2 + z_3^2\tau^2)\widetilde{D}_1Y_4\\
\qquad - U_1U_2\widetilde{D}_2Y_4 - U_1U_3\widetilde{D}_3Y_4\big)\\
+\widetilde{D}_2\big(-U_1U_2\widetilde{D}_1Y_4 + (c^2(\rho) - U_2^2)\widetilde{D}_2Y_4 + z_2^2\tau^2c^2(\rho)\widetilde{D}_2Y_4\\
\qquad - U_2U_3\widetilde{D}_3Y_4 + z_2z_3\tau^2c^2(\rho)\widetilde{D}_3Y_4\big)\\
+\widetilde{D}_3\big(-U_1U_3\widetilde{D}_1Y_4 - U_2U_3\widetilde{D}_2Y_4 + z_2z_3\tau^2c^2(\rho)\widetilde{D}_2Y_4\\
\qquad + (c^2(\rho) - U_3^2)\widetilde{D}_3Y_4 + z_3^2\tau^2c^2(\rho)\widetilde{D}_3Y_4\big)\\
\quad = H_2(Y, \nabla Y) & \text{in } E_+,\\
Y_4 = O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(1/X_0)Y_5 & \text{on } z_1 = 0,\\
Y_4 = 0 & \text{on } z_1 = 1,\\
z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4 = O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(\varepsilon^2)Y_4 & \text{on } z_2^2 + z_3^2 = 1,
\end{cases}
$$

with

$$
\begin{aligned}
&H_2(Y, \nabla Y)\\
&\quad = \widetilde{D_1}\big(O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4)\big)\\
&\qquad + \widetilde{D_2}\big(O(\varepsilon/X_0)\cdot(Y_1, \varepsilon^{-1}Y_2, X_0^{-1}Y_3, X_0Y_4, Y_5, \varepsilon^{-1}\partial_{z_2}Y_5, (\varepsilon X_0^2)^{-1}\partial_{z_3}Y_5)\big)\\
&\qquad + \widetilde{D_3}\big(O(\varepsilon/X_0)\cdot(Y_1, X_0^{-1}Y_2, \varepsilon^{-1}Y_3, X_0Y_4, Y_5, (\varepsilon X_0^2)^{-1}\partial_{z_2}Y_5, \varepsilon^{-1}\partial_{z_3}Y_5)\big)\\
&\qquad\qquad + O(1/X_0)\cdot\big(\varepsilon Y_1, X_0^{-2}Y_2, X_0^{-2}Y_3, \varepsilon Y_4, X_0^{-1}Y_5\big)\\
&\qquad\qquad\quad + O(1/X_0^2)\cdot\big(\varepsilon X_0^2\partial_{z_1}Y_1, \partial_{z_1}Y_2, \partial_{z_1}Y_3, \varepsilon X_0^2\partial_{z_1}Y_4\big)\\
&\qquad\qquad\qquad + O(\varepsilon)\cdot\big(\partial_{z_2}Y_1, (\varepsilon X_0)^{-1}\partial_{z_2}Y_2, \partial_{z_2}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5\big)\\
&\qquad\qquad\qquad\quad + O(\varepsilon)\cdot\big(\partial_{z_3}Y_1, \partial_{z_3}Y_2, (\varepsilon X_0^{-1})\partial_{z_3}Y_3, \partial_{z_3}Y_4, \partial_{z_3}Y_5\big),
\end{aligned}
$$

where we use the formula of H_1 on page 140 and the assumptions in Theorem 2.1.

Next, define

$$
M_1 = z_2\partial_{z_2}Y_4 + z_3\partial_{z_3}Y_4 \quad \text{and} \quad M_2 = z_3\partial_{z_2}Y_4 - z_2\partial_{z_3}Y_4
$$

Applying $z_2\partial_{z_2} + z_3\partial_{z_3}$ to the first three equalities of (3-23) yields

$$
\text{(3-24)}\quad
\begin{cases}
\widetilde{D_1}\big((c^2(\rho) - U_1^2)\widetilde{D_1}M_1 + c^2(\rho)(z_2^2\tau^2 + z_3^2\tau^2)\widetilde{D_1}M_1\\
\qquad - U_1U_2\widetilde{D_2}M_1 - U_1U_3\widetilde{D_3}M_1\big)\\
\quad + \widetilde{D_2}\Big(- U_1U_2\widetilde{D_1}M_1 + (c^2(\rho) - U_2^2)\widetilde{D_2}M_1\\
\qquad + z_2^2\tau^2c^2(\rho)\widetilde{D_2}M_1 - U_2U_3\widetilde{D_3}M_1 + z_2z_3\tau^2c^2(\rho)\widetilde{D_3}M_1\\
\qquad + O(1)\dfrac{z_2M_1 + z_3M_2}{z_2^2 + z_3^2} + O(1)\dfrac{z_3M_1 - z_2M_2}{z_2^2 + z_3^2}\Big)\\
\quad + \widetilde{D_3}\Big(- U_1U_3\widetilde{D_1}M_1 - U_2U_3\widetilde{D_2}M_1\\
\qquad + z_2z_3\tau^2c^2(\rho)\widetilde{D_2}M_1 + (c^2(\rho) - U_3^2)\widetilde{D_3}M_1 + z_3^2\tau^2c^2(\rho)\widetilde{D_3}M_1\\
\qquad + O(1)\dfrac{z_2M_1 + z_3M_2}{z_2^2 + z_3^2} + O(1)\dfrac{z_3M_1 - z_2M_2}{z_2^2 + z_3^2}\Big)\\
\quad = (z_2\partial_{z_2} + z_3\partial_{z_3})H_2(Y, \nabla Y) + H_3(Y, \nabla Y) & \text{in } E_+,\\
M_1 = O(\varepsilon)\cdot(Y_2, Y_3, \partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\\
\qquad + O(1/X_0)(X_0^{-1}Y_5, \partial_{z_2}Y_5, \partial_{z_3}Y_5) & \text{on } z_1 = 0,\\
M_1 = 0 & \text{on } z_1 = 1,\\
M_1 = O(\varepsilon)\cdot(Y_2, Y_3, \varepsilon Y_4) & \text{on } z_2^2 + z_3^2 = 1,
\end{cases}
$$

where

$$
\begin{aligned}
&H_3(Y, \nabla Y)\\
&= O(1/X_0^2)\cdot(Y_5,\, \partial_{z_2}Y_5,\, \partial_{z_3}Y_5) + O(1/X_0^2)\partial_{z_1}\big(O(1)\partial_{z_1}Y_4 + O(\varepsilon)\partial_{z_2}Y_4 + O(\varepsilon)\partial_{z_3}Y_4\big)\\
&\quad + \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_2}Y_4 + O(1/X_0^2)\partial_{z_3}Y_4\big)\\
&\quad + \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_2}Y_4 + O(1)\partial_{z_3}Y_4\big)\\
&\quad + O(1)\partial_{z_1}\big(O(1/X_0^2)\partial_{z_1}Y_4 + O(\varepsilon)\partial_{z_2}Y_4 + O(\varepsilon)\partial_{z_3}Y_4\big)\\
&\quad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_2}Y_4 + O(1/X_0^2)\partial_{z_3}Y_4\big)\\
&\quad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_2}Y_4 + O(1/X_0^2)\partial_{z_3}Y_4\big),
\end{aligned}
$$

and the singular terms

$$
O(1)\frac{z_2M_1 + z_3M_2}{z_2^2 + z_3^2} \quad \text{and} \quad O(1)\frac{z_3M_1 - z_2M_2}{z_2^2 + z_3^2}
$$

in (3-24) arise essentially from the computation

$$
\begin{aligned}
&(z_2\partial_{z_2} + z_3\partial_{z_3})\big(\widetilde{D_2}(c^2(\rho)\widetilde{D_2}Y_4) + \widetilde{D_3}(c^2(\rho)\widetilde{D_3}Y_4)\big)\\
&\quad = \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_2}Y_4\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_3}Y_4\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_2}Y_4)\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_3}Y_4)\big)\\
&\qquad + \widetilde{D_2}\big(c^2(\rho)\widetilde{D_2}M_1 - 2c^2(\rho)\partial_{z_2}Y_4\big) + \widetilde{D_3}\big(c^2(\rho)\widetilde{D_3}M_1 - 2c^2(\rho)\partial_{z_3}Y_4\big)\\
&\quad = \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_2}Y_4\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_3}Y_4\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_2}Y_4)\big)\\
&\qquad + \big(O(\varepsilon)\partial_{z_1} + O(1)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1/X_0^2)\partial_{z_3}Y_4)\big)\\
&\qquad + \widetilde{D_2}\left(c^2(\rho)\widetilde{D_2}M_1 + O(1)\frac{z_2M_1 + z_3M_2}{z_2^2 + z_3^2}\right)\\
&\qquad + \widetilde{D_3}\left(c^2(\rho)\widetilde{D_3}M_1 + O(1)\frac{z_3M_1 - z_2M_2}{z_2^2 + z_3^2}\right).
\end{aligned}
$$

The factors

$$\frac{z_2}{z_2^2+z_3^2} \quad \text{and} \quad \frac{z_3}{z_2^2+z_3^2}$$

in the second-order elliptic Equation (3-24) have a strong singularity on $z_2^2+z_3^2=0$. Thus it is difficult to use the standard theory on second-order elliptic equations to derive directly the global $C^{1,\alpha}$ estimate on M_1 in E_+. To overcome this difficulty, we first establish the boundary $C^{1,\alpha}$ estimate of M_1. In fact, the compatibility conditions on the intersection curve between the shock surface Σ and the nozzle wall Π_2 (see [Xin and Yin 2008b, Appendix B]) as well as the natural compatibility conditions on the intersection curve between the end $r = X_0 + 1$ and Π_2 due to the $C^{3,\alpha}$ regularity assumption of the solution have the following implication: From the estimates on the boundary of the second-order elliptic equations with the divergence form and the Dirichlet boundary values on the cornered domain (see [Azzam 1980; 1981; Lieberman 1986; 1988]), we have

$$\begin{aligned}
(3\text{-}25)\quad & \|M_1\|_{C^{1,\alpha}(\bar{B}E_+^0)} \\
&\le C\big(\|M_1\|_{L^\infty} + \|M_2\|_{C^\alpha} + \|H_2\|_{C^\alpha} + \|H_3\|_{C^\alpha} \\
&\qquad\qquad + \|M_1|_{z_1=0}\|_{C^{1,\alpha}} + \|M_1|_{z_2^2+z_3^2=1}\|_{C^{1,\alpha}}\big) \\
&\le C(\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty} + \|M_2\|_{C^{1,\alpha}}) + C\varepsilon\|(\partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}} \\
&\quad + \frac{C}{X_0}\|(Y_1, Y_2, Y_3, X_0^{-1}Y_5, \partial_{z_1}Y_1, X_0^{-1}\partial_{z_1}Y_2, X_0^{-1}\partial_{z_1}Y_3, \partial_{z_2}Y_2, \partial_{z_3}Y_3)\|_{C^{1,\alpha}} \\
&\quad + \frac{C}{X_0}\|(Y_4, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} + \frac{C}{X_0}\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{1,\alpha}},
\end{aligned}$$

where the subdomain E_+^0 of E_+ contains the nozzle wall $\{z : 0 < z_1 < 1,\ z_2^2+z_3^2=1\}$.

Similar analysis gives a second-order elliptic equation for M_2 with suitable boundary conditions. In fact, by the fourth equality in (3-23), one has

$$\begin{aligned}
&(z_2\partial_{z_2} + z_3\partial_{z_3})M_2 \\
&\qquad = O(\varepsilon)\cdot(Y_2, Y_3, \varepsilon Y_4, \partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3, \varepsilon M_2) \quad \text{on } z_2^2+z_3^2=1.
\end{aligned}$$

Note that

$$\begin{aligned}
&(z_3\partial_{z_2} - z_2\partial_{z_3})\big(\widetilde{D}_2(c^2(\rho)\widetilde{D}_2Y_4) + \widetilde{D}_3(c^2(\rho)\widetilde{D}_3Y_4)\big) \\
&\qquad = \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_2}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_2}Y_4\big) \\
&\qquad\quad + \big(O(\varepsilon)\partial_{z_1} + O(1/X_0^2)\partial_{z_3}\big)\big(O(\varepsilon)\partial_{z_1}Y_4 + O(1)\partial_{z_3}Y_4\big) \\
&\qquad\quad + \big(O(1)\partial_{z_2} + O(1)\partial_{z_3}\big)\big(O(\varepsilon)\cdot(\partial_{z_1}Y_4, \partial_{z_2}Y_4)\big) \\
&\qquad\quad + \widetilde{D}_2(c^2(\rho)\widetilde{D}_2M_2) + \widetilde{D}_3(c^2(\rho)\widetilde{D}_3M_2).
\end{aligned}$$

Then we can show that M_2 solves

$$\left\{\begin{aligned}
&\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}M_2+c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}M_2\\
&\qquad -U_1U_2\widetilde{D_2}M_2-U_1U_3\widetilde{D_3}M_2\big)\\
&+\widetilde{D_2}\big(-U_1U_2\widetilde{D_1}M_2+(c^2(\rho)-U_2^2)\widetilde{D_2}M_2\\
&\qquad +z_2^2\tau^2c^2(\rho)\widetilde{D_2}M_2-U_2U_3\widetilde{D_3}M_2\\
&\qquad +z_2z_3\tau^2c^2(\rho)\widetilde{D_3}M_2\big)\\
&+\widetilde{D_3}\big(-U_1U_3\widetilde{D_1}M_2-U_2U_3\widetilde{D_2}M_2\\
&\qquad +z_2z_3\tau^2c^2(\rho)\widetilde{D_2}M_2+(c^2(\rho)-U_3^2)\widetilde{D_3}M_2\\
&\qquad +z_3^2\tau^2c^2(\rho)\widetilde{D_3}M_2\big)\\
&\quad =(z_3\partial_{z_2}-z_2\partial_{z_3})H_2(Y,\nabla Y)+\widetilde{H_3}(Y,\nabla Y) \qquad \text{in } E_+,\\
&M_2=O(\varepsilon)\cdot\big(Y_2,Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3\big)\\
&\qquad +O(1/X_0)\cdot\big(X_0^{-1}Y_5,\partial_{z_2}Y_5,\partial_{z_3}Y_5\big) \qquad \text{on } z_1=0,\\
&M_2=0 \qquad \text{on } z_1=1,\\
&(z_2\partial_{z_2}+z_3\partial_{z_3})M_2\\
&\quad =O(\varepsilon)\\
&\qquad \cdot\big(Y_2,Y_3,\varepsilon Y_4,\partial_{z_2}Y_2,\partial_{z_3}Y_2,\partial_{z_2}Y_3,\partial_{z_3}Y_3,\varepsilon M_2\big) \quad \text{on } z_2^2+z_3^2=1,
\end{aligned}\right. \tag{3-26}$$

where $\widetilde{H_3}(Y,\nabla Y)$ has the same property as $H_3(Y,\nabla Y)$ in (3-24).

Since the equation in (3-26) has no singular terms, a global $C^{1,\alpha}$ estimate of M_2 in E_+ can easily be given as

$$\begin{aligned}
&\|M_2\|_{C^{1,\alpha}}\\
&\le C\big(\|H_2\|_{C^\alpha}+\|\tilde{H}_3\|_{C^\alpha}+\|M_2|_{z_1=0}\|_{C^{1,\alpha}}\\
&\qquad\qquad +\|(z_2\partial_{z_2}+z_3\partial_{z_3})M_2|_{z_2^2+z_3^2=1}\|_{C^\alpha}\big)\\
&\le \frac{C}{X_0}\|(Y_1,Y_2,Y_3,Y_4,X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\
&\quad +\frac{C}{X_0}\|(\partial_{z_1}Y_1,X_0^{-1}\partial_{z_1}Y_2,X_0^{-1}\partial_{z_1}Y_3,\partial_{z_2}Y_2,\partial_{z_3}Y_3,\partial_{z_1}Y_4,\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}}\\
&\qquad +C\varepsilon\|(\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}+\frac{C}{X_0}\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}.
\end{aligned}\tag{3-27}$$

Next, we treat the bounds of $\|\partial_{z_2}Y_4\|_{L^\infty}$ and $\|\partial_{z_3}Y_4\|_{L^\infty}$ in (3-25).

As with (3-24), the first three equations of (3-23) imply that $\partial_{z_2} Y_4$ satisfies

$$
\begin{cases}
\widetilde{D_1}\big((c^2(\rho)-U_1^2)\widetilde{D_1}(\partial_{z_2}Y_4)+c^2(\rho)(z_2^2\tau^2+z_3^2\tau^2)\widetilde{D_1}(\partial_{z_2}Y_4) \\
\qquad -U_1U_2\widetilde{D_2}(\partial_{z_2}Y_4)-U_1U_3\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
+\widetilde{D_2}\big(-U_1U_2\widetilde{D_1}(\partial_{z_2}Y_4)+(c^2(\rho)-U_2^2)\widetilde{D_2}(\partial_{z_2}Y_4)+z_2^2\tau^2c^2(\rho)\widetilde{D_2}(\partial_{z_2}Y_4) \\
\qquad -U_2U_3\widetilde{D_3}(\partial_{z_2}Y_4)+z_2z_3\tau^2c^2(\rho)\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
+\widetilde{D_3}\big(-U_1U_3\widetilde{D_1}(\partial_{z_2}Y_4)-U_2U_3\widetilde{D_2}(\partial_{z_2}Y_4)+z_2z_3\tau^2c^2(\rho)\widetilde{D_2}(\partial_{z_2}Y_4) \\
\qquad +(c^2(\rho)-U_3^2)\widetilde{D_3}(\partial_{z_2}Y_4)+z_3^2\tau^2c^2(\rho)\widetilde{D_3}(\partial_{z_2}Y_4)\big) \\
\qquad = \partial_{z_2}H_2(Y,\nabla Y)+\hat{H}_3(Y,\nabla Y) & \text{in } E_+, \\
\partial_{z_2}Y_4 = O(\varepsilon)\cdot(Y_2,Y_3,\partial_{z_2}Y_2,\partial_{z_2}Y_3)+O(1/X_0)\cdot(X_0^{-1}Y_5,\partial_{z_2}Y_5) & \text{on } z_1=0, \\
\partial_{z_2}Y_4 = 0 & \text{on } z_1=1,
\end{cases}
\tag{3-28}
$$

where $\hat{H}_3(Y,\nabla Y)$ has the same property as $H_3(Y,\nabla Y)$ in (3-24).

By the maximum principle for second-order elliptic equations of divergence form with the Dirichlet boundary condition [Gilbarg and Trudinger 1983, Theorem 8.16], we have

$$
\begin{aligned}
&\|\partial_{z_2}Y_4\|_{L^\infty} \\
&\quad \le C\big(\|\partial_{z_2}Y_4|_{z_1=0}\|_{L^\infty}+\|\partial_{z_2}Y_4|_{z_1=1}\|_{L^\infty}+\|\partial_{z_2}Y_4|_{z_2^2+z_3^2=1}\|_{L^\infty} \\
&\qquad\qquad +\|H_2\|_{C^\alpha}+\|\hat{H}_3\|_{C^\alpha}\big).
\end{aligned}
\tag{3-29}
$$

Since $M_1 = O(\varepsilon)\cdot(Y_2, Y_3, \varepsilon Y_4)$ on $z_2^2+z_3^2=1$, a simple computation yields

$$
\begin{aligned}
\|\partial_{z_2}Y_4\|_{L^\infty} &\le \|M_1|_{z_2^2+z_3^2=1}\|_{L^\infty}+\|M_2|_{z_2^2+z_3^2=1}\|_{L^\infty} \\
&\le C\varepsilon\|(Y_2,Y_3,\varepsilon Y_4)\|_{L^\infty}+C\|M_2\|_{C^{1,\alpha}}.
\end{aligned}
\tag{3-30}
$$

Substituting (3-30), (3-25), (3-27) and the boundary value conditions of (3-28) into (3-29) gives

$$
\begin{aligned}
\|\partial_{z_2}Y_4\|_{L^\infty} \le{}& \frac{C}{X_0}\|(Y_1,Y_2,Y_3,Y_4,X_0^{-1}Y_5)\|_{C^{1,\alpha}} \\
&+\frac{C}{X_0}\big(\|(\partial_{z_1}Y_1,\partial_{z_2}Y_2,\partial_{z_3}Y_3,\partial_{z_1}Y_4,\partial_{z_2}Y_4,\partial_{z_3}Y_4)\|_{C^{1,\alpha}} \\
&\qquad +\|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big) \\
&+C\varepsilon\|(\partial_{z_1}Y_2,\partial_{z_1}Y_3,\partial_{z_3}Y_2,\partial_{z_2}Y_3)\|_{C^{1,\alpha}}.
\end{aligned}
\tag{3-31}
$$

Similarly,

$$\begin{aligned}(3\text{-}32)\quad \|\partial_{z_3}Y_4\|_{L^\infty} \le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}} &\\ +\frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} &\\ +\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big) &\\ +C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.&\end{aligned}$$

So far, we have shown that the "large" term $\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty} + \|M_2\|_{C^{1,\alpha}}$ in the right hand side of (3-25) can be controlled by the "good" terms in (3-27) and (3-31)–(3-32). This means that $\|M_1\|_{C^{1,\alpha}(\bar{B}E_+^0)}$ has the same estimate as in (3-31)–(3-32). Namely,

$$\begin{aligned}(3\text{-}33)\quad &\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}(\bar{B}E_+^0)}\\ &\le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\ &\quad +\frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}\\ &\quad +\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big)\\ &\quad +C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

From this and the equations on $\partial_{z_2}Y_4$ and $\partial_{z_3}Y_4$ (see (3-28)), one has

$$\begin{aligned}&\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}}\\ &\le C\big(\|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{L^\infty} + \|(\partial_{z_2}Y_4, \partial_{z_3}Y_4)|_{\partial E^+}\|_{C^{1,\alpha}} + \|H_2\|_{C^\alpha} + \|\hat{H}_3\|_{C^\alpha}\big)\\ &\le \frac{C}{X_0}\|(Y_1, Y_2, Y_3, Y_4, X_0^{-1}Y_5)\|_{C^{1,\alpha}}\\ &\quad +\frac{C}{X_0}\big(\|(\partial_{z_1}Y_1, \partial_{z_2}Y_2, \partial_{z_3}Y_3, \partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_3}Y_4)\|_{C^{1,\alpha}} + \|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}\big)\\ &\quad +C\varepsilon\|(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_3}Y_2, \partial_{z_2}Y_3)\|_{C^{1,\alpha}}.\end{aligned}$$

This completes the proof of Lemma 3.7. □

Remark 3.6. We now explain the importance of deriving the $C^{2,\alpha}$-regularity estimates on Y_4 and (Y_1, Y_2, Y_3) simultaneously. The crucial estimate in (3-14) which bounds $\|(\partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\|_{C^{1,\alpha}}$ in terms of $\|(\nabla Y_1, \nabla Y_4)\|_{C^{1,\alpha}}$ and $\|(\partial_{z_2}Y_5, \partial_{z_3}Y_5)\|_{C^{2,\alpha}}$ follows from the key observation that though the system (2-11) is hyperbolic, the lower-dimensional first-order system (3-16) is elliptic. Indeed, without (3-16), the standard characteristic method for (2-11) gives only that (Y_2, Y_3) has the same $C^{1,\alpha}$ regularity as $(\partial_{z_2}Y_4, \partial_{z_3}Y_4) \in C^{1,\alpha}$. In this case, one can estimate $\|(\partial_{z_2}Y_2, \partial_{z_3}Y_2, \partial_{z_2}Y_3, \partial_{z_3}Y_3)\|_{C^\alpha}$ in terms of the right hand side of (3-14) by the proof of Lemma 3.5. Then, from the proof of (3-6), one can estimate $\|(\partial_{z_1}Y_1, \partial_{z_1}Y_4)\|_{C^\alpha}$ which gives an estimate of $\|(Y_2, Y_3)\|_{C^{1,\alpha}}$ on $z_1 = 0$ using the proof of (3-11). Together with boundary condition on $z_1 = 0$ in (3-28), this yields the desired estimate on $\|(\partial_{z_2} Y_4, \partial_{z_3} Y_4)\|_{C^\alpha}$. However, neither $C^{1,\alpha}$ estimates on $(\nabla Y_1, \nabla Y_2, \nabla Y_3, \nabla Y_4)$ nor $C^{2,\alpha}$ estimates on $\nabla_{z_2,z_3} Y_5$ can be obtained in this way.

Remark 3.7. We have established a priori estimates for the gradients of solutions instead of solutions themselves. Trying to derive a priori estimates on a solution directly would give from (3-9) that

$$\|\partial_{z_1} Y_4\|_{C^{1,\alpha}} \le C_1 \|(\partial_{z_2} Y_2, \partial_{z_3} Y_3)\|_{C^{1,\alpha}} + \text{positive terms with “good” coefficients,}$$

while (3-12) yields

$$\|(\partial_{z_2} Y_2, \partial_{z_2} Y_3)\|_{C^{1,\alpha}} \le C_2 \|\partial_{z_1} Y_4\|_{C^{1,\alpha}} + \text{positive terms with “good” coefficients.}$$

However, it seems extremely difficult to get precise estimates on C_1 and C_2 so that $C_1 \cdot C_2 < 1$. Thus the direct estimate cannot yield useful information on $\partial_{z_1} Y_4$, $\partial_{z_2} Y_2$ and $\partial_{z_3} Y_3$.

4. Proofs of Theorem 1.1 and Proposition 2.2

Due to the equivalence between Theorem 1.1 and Theorem 2.1, it suffices to prove Theorem 2.1 only.

To this end, we first show that $\xi_1(0, 1) = \xi_2(0, 1)$ by contradiction. Without loss of generality, assume that

$$\xi_1(0, 1) < \xi_2(0, 1). \tag{4-1}$$

We will show the corresponding end pressures are different, contradicting (1-4).

Lemma 4.1. *For $\varepsilon_0 < 1/X_0^2$ in Theorem 2.1, one has*

$$\begin{cases} \|(\partial_{z_1} Y_1, \partial_{z_1} Y_4)\|_{C^{1,\alpha}} \le C|Y_4(0, 0, 1)|, \\ \|(\partial_{z_1} Y_2, \partial_{z_1} Y_3)\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}|Y_4(0, 0, 1)|, \\ \displaystyle\sum_{i=1}^{5}\sum_{j=2}^{3} \|\partial_{z_j} Y_i\|_{C^{1,\alpha}} \le \dfrac{C}{X_0}|Y_4(0, 0, 1)|. \end{cases} \tag{4-2}$$

Remark 4.1. Thanks to the appearance of the term $(1/X_0^2)\|Y_5\|_{C^{1,\alpha}}$ in the right hand sides of (3-11), (3-14), (3-18) and (3-22), we can obtain the desired estimates (4-2), which will be the key in deriving the monotonicity of shock position on the end pressure and further obtaining the uniqueness result. Indeed, if the dominant term on the right hand sides of (3-11), (3-14), (3-18) and (3-22) is $(1/X_0)\|Y_5\|_{C^{1,\alpha}}$, then Lemma B.4 implies that $Y_5(0, 1) \sim X_0 Y_4(0, 0, 1)$ and the third estimate in (4-7) becomes

$$\|(\partial_{z_1}Y_2,\partial_{z_1}Y_3)\|_{C^{1,\alpha}}+\sum_{i=2}^{3}\sum_{j=1}^{5}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}}\le \frac{C}{X_0}|Y_5(0,1)|.$$

In this case, by Equation (4-11) below, one can only show that $\partial_{z_1}Y_4=O(1/X_0)Y_5$. Thus, Equation (4-13) becomes $\partial_{z_1}Y_4=O(1)Y_4$, which yields no useful information on Y_4. It is then unclear how to proceed to obtain the monotonic dependence of the shock position on the end pressure.

Proof of Lemma 4.1. By the estimates in Lemmas 3.2–3.7 and a direct computation,

$$\begin{cases}\|(\partial_{z_1}Y_1,\partial_{z_1}Y_4)\|_{C^{1,\alpha}}\le \dfrac{C}{X_0}\displaystyle\sum_{i=1}^{5}\|Y_i\|_{C^{1,\alpha}},\\ \|(\partial_{z_1}Y_2,\partial_{z_1}Y_3)\|_{C^{1,\alpha}}\\ \quad+\displaystyle\sum_{i=2}^{3}\sum_{j=1}^{4}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}}\le \dfrac{C}{X_0}\Big(\displaystyle\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}}+X_0^{-1}\|Y_5\|_{C^{1,\alpha}}\Big),\\ \|(\partial_{z_2}Y_5,\partial_{z_3}Y_5)\|_{C^{2,\alpha}}\le \dfrac{C}{X_0}\Big(\displaystyle\sum_{i=1}^{4}\|Y_i\|_{C^{1,\alpha}}+X_0^{-1}\|Y_5\|_{C^{1,\alpha}}\Big).\end{cases}\tag{4-3}$$

Note that

$$\begin{cases}\|(Y_1,Y_4)\|_{C^{1,\alpha}}\le C(|(Y_1,Y_4)(0,0,1)|+\|\nabla(Y_1,Y_4)\|_{C^{1,\alpha}}),\\ \|Y_5\|_{C^{1,\alpha}}\le C(|Y_5(0,1)|+\|\nabla Y_5\|_{C^{2,\alpha}}).\end{cases}\tag{4-4}$$

The nonslip condition (2-7) implies that $z_2Y_2+z_3Y_3=0$ on $z_2^2+z_3^2=1$ and further $Y_2(z_1,1,0)=Y_3(z_1,0,1)=0$, so

$$\|(Y_2,Y_3)\|_{C^{1,\alpha}}\le C\|\nabla(Y_2,Y_3)\|_{C^{1,\alpha}}.\tag{4-5}$$

In addition, at the point (0, 0, 1), Equation (3-19) implies

$$|Y_1(0,0,1)|+|Y_4(0,0,1)|\le\frac{C}{X_0}|Y_5(0,1)|+C\varepsilon(\|Y_2\|_{L^\infty}+\|Y_3\|_{L^\infty}).\tag{4-6}$$

Substituting (4-4)–(4-6) into (4-3) yields

$$\begin{cases}|Y_1(0,0,1)|+|Y_4(0,0,1)|+X_0|Y_2(0,0,1)|\le\dfrac{C}{X_0}|Y_5(0,1)|,\\ \|\partial_{z_1}Y_1\|_{C^{1,\alpha}}+\|\partial_{z_1}Y_4\|_{C^{1,\alpha}}\le\dfrac{C}{X_0}|Y_5(0,1)|,\\ \|(\partial_{z_1}Y_2,\partial_{z_1}Y_3)\|_{C^{1,\alpha}}+\displaystyle\sum_{i=2}^{3}\sum_{j=1}^{5}\|\partial_{z_i}Y_j\|_{C^{1,\alpha}}\le\dfrac{C}{X_0^2}|Y_5(0,1)|.\end{cases}\tag{4-7}$$

In addition, by Lemma B.4,

$$|Y_5(0,1)| \le C X_0 |Y_4(0,0,1)|. \tag{4-8}$$

Combining (4-8) with (4-7) yields Lemma 4.1. □

Lemma 4.2. *Suppose that* (4-1) *and the assumptions in Theorem 2.1 hold. If* $\rho_0^+(r_0) > 2\rho_0^-(r_0)$, *then*

$$Y_4(0,0,1) > 0. \tag{4-9}$$

Proof. Lemma B.4 implies that $Y_4(0,0,1)$ and $Y_5(0,1)$ satisfy

$$Y_4 = a_0 Y_5 + O(1/X_0^2) Y_5,$$

where $a_0 < 0$ and $a_0 = O(1/X_0)$.

Thus by (4-1), we have $Y_4(0,0,1) > 0$. □

Remark 4.2. If $M_0^-(X_0) > \sqrt{(2^{\gamma+1}-1)/\gamma}$, then by [Li et al. 2009, Lemma 5.1], we can show that $\rho_0^+(r_0) > 2\rho_0^-(r_0)$ in Lemma 4.2.

Based on Lemmas 4.1 and 4.2, we can now prove Theorem 2.1.

Proof of Theorem 2.1. It follows from (2-4) and a direct computation that

$$\begin{cases} U_1 \widetilde{D_1} Y_4 + \rho \widetilde{D_1} Y_1 \\ \quad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4) + a_1 Y_5 \\ \qquad + O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \varepsilon\partial_{z_1}Y_4, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, \partial_{z_3}Y_5) \\ \qquad + O(1)\cdot(\partial_{z_2}Y_2, \partial_{z_3}Y_3), \\ \rho U_1 \widetilde{D_1} Y_1 + c^2(\rho)\widetilde{D_1} Y_4 \\ \quad = O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, \varepsilon X_0 Y_3, Y_4) + a_2 Y_5 \\ \qquad + O(\varepsilon)\cdot(\varepsilon\partial_{z_1}Y_1, (\varepsilon X_0^2)^{-1}\partial_{z_1}Y_4, \partial_{z_2}Y_1, X_0^{-1}\partial_{z_2}Y_5, \partial_{z_3}Y_1, X_0^{-1}\partial_{z_3}Y_5), \end{cases} \tag{4-10}$$

where, abbreviating $\xi_1(z_2,z_3)$ by ξ_1 and $\xi_2(z_2,z_3)$ by ξ_2,

$$\begin{aligned} a_1 = &-\frac{\partial_{z_1}(\rho U_1)}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(X_0+1-\xi_1)(X_0+1-\xi_2)} \\ &+\frac{2(1-z_1)\rho U_1}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(\xi_1+z_1(X_0+1-\xi_1))(\xi_2+z_1(X_0+1-\xi_2))} \\ &+O(\varepsilon/X_0), \\ a_2 = &-\frac{c^2(\rho)\partial_{z_1}\rho+\rho U_1\partial_{z_1}U_1}{\sqrt{1+(z_2^2+z_3^2\tau^2)}(X_0+1-\xi_1)(X_0+1-\xi_2)} \\ &+O(1/X_0^3), \end{aligned}$$

It follows from (4-10) that

$$\begin{aligned}
\text{(4-11)}\quad \partial_{z_1} Y_4 = a(z) Y_5 + O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4) + O(1)(\partial_{z_2} Y_2, \partial_{z_3} Y_3)\\
+ O(\varepsilon)\cdot(\varepsilon\partial_{z_1} Y_1, \partial_{z_1} Y_2, \partial_{z_1} Y_3, (\varepsilon X_0^2)^{-1}\\
\times \partial_{z_1} Y_4, \partial_{z_2} Y_1, \partial_{z_2} Y_4, \partial_{z_2} Y_5, \partial_{z_3} Y_1, \partial_{z_3} Y_4, \partial_{z_3} Y_5),
\end{aligned}$$

where, again abbreviating $\xi_1(z_2, z_3)$ by ξ_1 and $\xi_2(z_2, z_3)$ by ξ_2,

$$\begin{aligned}
a(z) &= \frac{(X_0+1-\xi_2)\sqrt{1+(z_2^2+z_3^2)\tau^2}}{c^2(\rho)-U_1^2}(a_2 - a_1 U_1)\\
\text{(4-12)}\quad &= -\frac{\partial_{z_1}\rho}{X_0+1-\xi_1}\\
&\quad - \frac{2(X_0+1-\xi_2)(1-z_1)\rho U_1^2}{(c^2(\rho)-U_1^2)(\xi_1+z_1(X_0+1-\xi_1))(\xi_2+z_1(X_0+1-\xi_2))}\\
&\quad + O(1/X_0^3),
\end{aligned}$$

It should be pointed out here that the "good" coefficient $O(1/X_0^2)$ in the term of $\partial_{z_1} Y_4$ on the right hand side of (4-11) can be derived from (2-17), the assumptions on the solutions, and $\varepsilon < 1/X_0^2$ in Theorem 2.1.

In addition, under the assumptions of Theorem 2.1, one has

$$\begin{cases}
\partial_{z_1}\rho = \partial_r \rho_0^+(r_0) + O(\varepsilon),\\
c^2(\rho) - U_1^2 = c^2(\rho_0^+(r_0)) - (U_0^+(r_0))^2 + O(1/X_0^2),
\end{cases}$$

which yields

$$\partial_{z_1}\rho > 0, \quad c^2(\rho) - U_1^2 > 0.$$

Hence, it follows from (4-12) that $a(z)$ is a negative function in subsonic domain. In addition, (4-1) implies $Y_5(0, 1) < 0$. So $a(z)Y_5(0, 1)$ is always nonnegative along the line $(z_1, 0, 1)$. Thus along the line $(z_1, 0, 1)$, by Lemma 4.1, (4-11) can be reduced into

$$\text{(4-13)}\quad \begin{cases}
\partial_{z_1} Y_4 \geq b(z) Y_4(0, 0, 1),\\
Y_4(0, 0, 1) > 0,
\end{cases}$$

where $\|b(z)\|_{L^\infty} \leq O(1/X_0)$. This yields

$$\text{(4-14)}\quad Y_4(z_1, 0, 1) > C_1 Y_4(0, 0, 1) > 0$$

for some constant $C_1 > 0$, which contradicts the end pressure condition (1-4), so contradicts (4-1). Thus $Y_5(0, 0, 1) = 0$.

So by Lemma 4.1,

$$Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = 0.$$

This completes the proof of Theorem 2.1 and thus of Theorem 1.1. □

Proof of Proposition 2.2. It follows from the assumptions in Proposition 2.2 that $C_{0,1} < C_{0,2}$ and $Y_4(1, z_2, z_3) < 0$.

We claim that

$$Y_5(0, 1) > 0. \tag{4-15}$$

Otherwise, if $Y_5(0, 1) < 0$, then (4-13)–(4-14) imply $C_{0,1} > C_{0,2}$. If $Y_5(0, 1) = 0$, then $Y_4(0, 0, 1) = 0$ by Lemma B.4 and further $Y_4 \equiv 0$ by Lemma 4.1, hence $C_{0,1} = C_{0,2}$. Both cases contradict that $C_{0,1} < C_{0,2}$.

Since $Y_5 = Y_5(0, 1) + O(1)\partial_{z_2} Y_5 + O(1)\partial_{z_3} Y_5$, the third equality in (4-7) gives

$$Y_5(z_2, z_3) = Y_5(0, 1) + O(1/X_0^2)Y_5(0, 1). \tag{4-16}$$

Combining (4-16) and (4-15) yields $Y_5(z_2, z_3) > 0$ which implies $\xi_1(y_2, y_3) > \xi_2(y_2, y_3)$. □

Appendix A: Analysis of the background solution

Under the assumptions given in Section 1, we describe the transonic solution of the problem (1-1) with (1-2)–(1-5) when the end pressure is a given suitable constant P_e. Such a solution is called the background solution and can be obtained by solving the related ordinary differential equations. In fact, the analysis of this background solution was given in [Courant and Friedrichs 1948, Section 147]; see also [Xin and Yin 2008b, Section 2]. For the reader's convenience and the requirements of our computations in this paper, we state the main facts here.

Theorem A.1 (existence of a transonic shock for the constant end pressure). *For the 3D nozzle and the supersonic incoming flow given in Section 1, there exist two constant pressures P_1 and P_2 with $P_1 < P_2$, determined by the incoming flow and the nozzle, such that if the end pressure $P_e \in (P_1, P_2)$, then the system* (1-1) *has a symmetric transonic shock solution,*

$$(P, u_1, u_2, u_3) = \begin{cases} (P_0^-(r), u_{1,0}^-(x), u_{2,0}^-(x), u_{3,0}^-(x)) & \text{for } r < r_0, \\ (P_0^+(r), u_{1,0}^+(x), u_{2,0}^+(x), u_{3,0}^+(x)) & \text{for } r > r_0, \end{cases}$$

where $u_{i,0}^\pm = U_0^\pm x_i/r$ for $i = 1, 2, 3$ and $(P_0^\pm(r), U_0^\pm(r))$ is $C^{4,\alpha}$-smooth. Moreover, the position $r = r_0$ with $X_0 < r_0 < X_0 + 1$ and the strength of the shock are determined by P_e.

Proof. See [Xin and Yin 2008b, Section 2]. □

Remark A.1. By (1-6) and the analysis of [Xin and Yin 2008b, Theorem A, Section 2], there exists a constant $C > 0$ independent of X_0 such that for $r_0 \le r \le X_0+1$,

$$\left|\frac{d^k U_0^+(r)}{dr^k}\right| + \left|\frac{d^k P_0^+(r)}{dr^k}\right| \le \frac{C}{X_0^k}, \quad k = 1, 2, 3.$$

Remark A.2. It follows from (2-1) that we can obtain an extension $(\hat{\rho}_0^+(r), \hat{U}_0^+(r))$ of $(\rho_0^+(r), U_0^+(r))$ for $r \in (X_0, X_0+1)$ and large X_0.

Appendix B

We first give a detailed computation for H_0 in (2-9), and then derive a first-order elliptic system on (U_2, U_3) in the interior of the nozzle. Next, we discuss the regularity problem of solutions to a class of first-order elliptic system which includes a parameter. Finally, we derive a relation between $Y_4(0, 0, 1)$ and $Y_5(0, 1)$ used in Lemmas 4.1 and 4.2.

Lemma B.1. *In* (2-9), *the function H_0 admits the estimate*

$$\begin{aligned} H_0 &= O(|U_2|^2+|U_3|^2)+O(|\nabla_{y_2,y_3}\rho|^2) \\ &\quad + O(|\nabla_{y_2,y_3}U_2|^2)+O(|\nabla_{y_2,y_3}U_3|^2)+O(|\nabla_{y_2,y_3}\xi|^2) \\ &\quad\ + O(1/X_0)(|U_2|+|U_3|+|\nabla_{y_2,y_3}\rho|+|\nabla_{y_2,y_3}U_2|+|\nabla_{y_2,y_3}U_3|+|\nabla_{y_2,y_3}\xi|). \end{aligned}$$

Proof. It follows from

$$\partial_{y_3}\Big(\frac{\Delta_2}{\Delta_1}(\xi(y_2, y_3), y_2, y_3)\Big) = \partial_{y_2}\Big(\frac{\Delta_3}{\Delta_1}(\xi(y_2, y_3), y_2, y_3)\Big)$$

that

$$\partial_{y_3}\Delta_2 - \partial_{y_2}\Delta_3 = \frac{\Delta_2\partial_{y_3}\Delta_1 - \Delta_3\partial_{y_2}\Delta_1}{\Delta_1}. \tag{B-1}$$

Since

$$\begin{aligned} \partial_{y_3}\Delta_2 = \frac{y_1\tau\rho U_1}{1+(y_2^2+y_3^2)\tau^2}\Big(&(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 + y_3^2\tau^2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 \\ &- y_2y_3\tau^2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3 + 2y_3\tau^2U_2 - y_2\tau^2U_3\Big) \\ &+ \frac{\partial_{y_3}\xi\tau\rho U_1 + \xi\tau(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})(\rho U_1)}{1+(y_2^2+y_3^2)\tau^2}(U_2+y_3^2\tau^2U_2-y_2y_3\tau^2U_3) \\ &- \frac{2y_1y_3\tau^3\rho U_1}{(1+(y_2^2+y_3^2)\tau^2)^2}(U_2+y_3^2\tau^2U_2-y_2y_3\tau^2U_3), \end{aligned}$$

$$\begin{aligned} \partial_{y_2}\Delta_3 = \frac{y_1\tau\rho U_1}{1+(y_2^2+y_3^2)\tau^2}\Big(&(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 + y_2^2\tau^2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 \\ &- y_2y_3\tau^2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 + 2y_2\tau^2U_3 - y_3\tau^2U_2\Big) \\ &+ \frac{\partial_{y_2}\xi\tau\rho U_1 + \xi\tau(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})(\rho U_1)}{1+(y_2^2+y_3^2)\tau^2}(U_3+y_2^2\tau^2U_3-y_2y_3\tau^2U_2) \\ &- \frac{2y_1y_2\tau^3\rho U_1}{(1+(y_2^2+y_3^2)\tau^2)^2}(U_3+y_2^2\tau^2U_3-y_2y_3\tau^2U_2), \end{aligned}$$

$$\begin{aligned}
\partial_{y_2}\Delta_1 = \rho\Big(&2(1+y_3^2\tau^2)U_2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 - 2y_2y_3\tau^2U_3(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_2 \\
&- 2y_2y_3\tau^2U_2(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 + 2(1+y_2^2\tau^2)U_3(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3\Big) \\
&+ (1+(y_2^2+y_3^2)\tau^2)[(\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})P] + 2y_2\tau^2[P] \\
&+ (\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})\rho((1+y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2U_2U_3 + (1+y_2^2\tau^2)U_3^2) \\
&+ \rho(2y_2\tau^2U_3^2 - 2y_3\tau^2U_2U_3), \\
\partial_{y_3}\Delta_1 = \rho\Big(&2(1+y_3^2\tau^2)U_2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 - 2y_2y_3\tau^2U_3(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 \\
&- 2y_2y_3\tau^2U_2(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3 + 2(1+y_2^2\tau^2)U_3(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_3\Big) \\
&+ (1+(y_2^2+y_3^2)\tau^2)[(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})P] + 2y_3\tau^2[P] \\
&+ (\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})\rho((1+y_3^2\tau^2)U_2^2 - 2y_2y_3\tau^2U_2U_3 + (1+y_2^2\tau^2)U_3^2) \\
&+ \rho(2y_3\tau^2U_2^2 - 2y_2\tau^2U_2U_3),
\end{aligned}$$

substituting these expressions into (B-1) yields

$$\begin{aligned}
&(\partial_{y_3}\xi\partial_{y_1}+\partial_{y_3})U_2 - (\partial_{y_2}\xi\partial_{y_1}+\partial_{y_2})U_3 \\
&\qquad = H_0(y_2, y_3, \rho, U_2, U_3, \xi, \nabla_{y_2,y_3}\rho, \nabla_{y_2,y_3}U_2, \nabla_{y_2,y_3}U_3, \nabla_{y_2,y_3}\xi),
\end{aligned}$$

where

$$\begin{aligned}
H_0 = &\,O(|U_2|^2+|U_3|^2) + O(|\nabla_{y_2,y_3}\rho|^2) + O(|\nabla_{y_2,y_3}U_2|^2) + O(|\nabla_{y_2,y_3}U_3|^2) + O(|\nabla_{y_2,y_3}\xi|^2) \\
&+ O(1/X_0)\big(|U_2| + |U_3| + |\nabla_{y_2,y_3}\rho| + |\nabla_{y_2,y_3}U_2| + |\nabla_{y_2,y_3}U_3| + |\nabla_{y_2,y_3}\xi|\big).
\end{aligned}$$

This completes the proof of Lemma B.1. □

Lemma B.2. *Under the assumptions of Theorem 2.1, we have*

$$\begin{cases}
\partial_{z_2}Y_2 + \partial_{z_3}Y_3 = F_3 & \text{in } E_+, \\
\partial_{z_3}Y_2 - \partial_{z_2}Y_3 = F_4 & \text{in } E_+, \\
z_2Y_2 + z_3Y_3 = 0 & \text{on } z_2^2+z_3^2 = 1,
\end{cases}$$

with

$$\begin{aligned}
F_3 = &\,O(1/X_0)\cdot(Y_1, X_0^{-1}Y_2, X_0^{-1}Y_3, Y_4, Y_5) \\
&+ O(\varepsilon)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_4, \partial_{z_2}Y_5, \partial_{z_3}Y_4, \partial_{z_3}Y_5) \\
&+ O(1)\cdot(\partial_{z_1}Y_1, \partial_{z_1}Y_4), \\
F_4 = &\,O(\varepsilon)\cdot(l^1, l^2) + O(1)\cdot(\partial_{z_3}Y_2, \partial_{z_2}Y_3)(0, \beta_1(z), \beta_2(z)) \\
&+ O(1/X_0)\cdot(\varepsilon Y_1, X_0^{-2}Y_2, X_0^{-2}Y_3, \varepsilon Y_4, X_0^{-2}Y_5) \\
&+ O(\varepsilon)\cdot(\partial_{z_1}Y_1, \partial_{z_1}Y_4, \partial_{z_2}Y_1, \partial_{z_3}Y_1) \\
&+ O(1/X_0^2)\cdot(\partial_{z_1}Y_2, \partial_{z_1}Y_3, \partial_{z_2}Y_2, X_0\partial_{z_2}Y_3, \partial_{z_2}Y_4, X_0^{-1}\partial_{z_2}Y_5, \\
&\qquad X_0\partial_{z_3}Y_2, \partial_{z_3}Y_3, \partial_{z_3}Y_4, \partial_{z_3}Y_5),
\end{aligned}$$

where l^i and β_i for $i = 1, 2$ are defined as in Lemma 3.5.

Proof. By the first and the second equations in (2-11) we obtain

$$
\begin{aligned}
\text{(B-2)}\quad & c^2(\rho)\big((1+z_2^2\tau^2)(1+z_3^2\tau^2)-z_2^2z_3^2\tau^4\big)D_2\rho \\
&= (1+z_3^2\tau^2)(\rho U_1D_1U_2+\rho U_2D_2U_2+\rho U_3D_3U_2) \\
&\quad - z_2z_3\tau^2(\rho U_1D_1U_3+\rho U_2D_2U_3+\rho U_3D_3U_3) \\
&\quad + \rho D_0((1+z_3^2\tau^2)U_2-z_2z_3\tau^2U_3)(U_1-z_2\tau U_2-z_2\tau U_3)
\end{aligned}
$$

$$
\begin{aligned}
\text{(B-3)}\quad & c^2(\rho)\big((1+z_2^2\tau^2)(1+z_3^2\tau^2)-z_2^2z_3^2\tau^4\big)D_3\rho \\
&= (1+z_2^2\tau^2)(\rho U_1D_1U_3+\rho U_2D_2U_3+\rho U_3D_3U_3) \\
&\quad - z_2z_3\tau^2(\rho U_1D_1U_2+\rho U_2D_2U_2+\rho U_3D_3U_2) \\
&\quad + \rho D_0((1+z_2^2\tau^2)U_3-z_2z_3\tau^2U_2)(U_1-z_2\tau U_2-z_2\tau U_3).
\end{aligned}
$$

Applying ∂_{y_3} to (B-2) and ∂_{y_2} to (B-3), and then subtracting them results in

$$
\begin{aligned}
\text{(B-4)}\quad & (\rho U_1D_1+\rho U_2D_2+\rho U_3D_3)\big(\partial_{z_3}U_2-\partial_{z_2}U_3+O(\varepsilon)\partial_{z_1}U_2+O(\varepsilon)\partial_{z_1}U_3\big) \\
&+(\rho U_1D_1+\rho U_2D_2+\rho U_3D_3)\big(z_2z_3\tau^2\partial_{z_2}U_2-z_2^2\tau^2\partial_{z_2}U_3 \\
&\qquad +z_3^2\tau^2\partial_{z_3}U_2-z_2z_3\tau^2\partial_{z_3}U_3\big) \\
&= H_4(z,U,\rho,\nabla U,\nabla\rho),
\end{aligned}
$$

where

$$
\begin{aligned}
H_4(z,\rho,U,\nabla\rho,\nabla U) &= O(|U_2|^2+|U_3|^2)+O(|\nabla U|^2)+O(|\nabla\rho|^2) \\
&\quad + O(1/X_0+\varepsilon)(|U_2|+|U_3|+|\nabla\rho|+|\nabla U|).
\end{aligned}
$$

Finally, due to the first equation in (2-4) and (B-4), a direct computation implies

$$
\begin{cases}
\partial_{z_2}Y_2+\partial_{z_3}Y_3=F_3 & \text{in } E_+,\\
\partial_{z_3}Y_2-\partial_{z_2}Y_3=F_4 & \text{in } E_+,\\
z_2Y_2+z_3Y_3=0 & \text{on } z_2^2+z_3^2=1,
\end{cases}
$$

and F_i for $i=3,4$ has the same properties as stated in Lemma B.2. □

Lemma B.3. *Assume that the problem*

$$
\text{(B-5)}\quad
\begin{cases}
\partial_2u_1+\partial_3u_2=f_1(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\},\\
\partial_3u_1-\partial_2u_2=f_2(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\},\\
\partial_1u_1=f_3(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\},\\
\partial_1u_2=f_4(x_1,x_2,x_3) & \textit{in } \Omega=\{(x_1,x_2,x_3):[0,1]\times B_1(0)\},\\
x_2u_1+x_3u_2=0 & \textit{on } \Gamma=\{(x_1,x_2,x_3):[0,1]\times \partial B_1(0)\}
\end{cases}
$$

has a $C^{2,\alpha}(\bar{\Omega})$ *solution* (u_1, u_2), *where* $f_i \in C^{1,\alpha}(\bar{\Omega})$ *for* $i = 1, 2, 3, 4$. *Then*

$$\sum_{i=2}^{3}\sum_{j=1}^{2}\|\partial_{x_i}u_j\|_{C^{1,\alpha}(\bar{\Omega})} \le C\bigg(\|\nabla f_1\|_{C^{\alpha}(\bar{\Omega})} + \sum_{i=2}^{4}\|f_i\|_{C^{1,\alpha}(\bar{\Omega})}\bigg). \tag{B-6}$$

Proof. Set $\Sigma_1 = \{(0, x_2, x_3) : x_2^2 + x_3^2 \le 1\}$ and $\Sigma_2 = \{(1, x_2, x_3) : x_2^2 + x_3^2 \le 1\}$.

First, we assert

$$\sum_{i=1}^{2}\|u_j\|_{C^{2,\alpha}(\Sigma_i)} + \|u_j\|_{C^{2,\alpha}(\Gamma)} \le C\sum_{k=1}^{4}\|f_k\|_{C^{1,\alpha}(\bar{\Omega})}, \quad j = 1, 2. \tag{B-7}$$

Indeed, it follows from (B-5) that on Σ_i for $i = 1, 2$,

$$\begin{cases} \partial_2 u_1 + \partial_3 u_2 = f_1(i-1, x_2, x_3) & \text{in } B_1(0),\\ \partial_3 u_1 - \partial_2 u_2 = f_2(i-1, x_2, x_3) & \text{in } B_1(0),\\ x_2 u_1 + x_2 u_2 = 0 & \text{on } \Sigma_i. \end{cases}$$

Thus, by the solution of the index -2 Hilbert problem in [Bers 1950; 1951; Vekua 1952],

$$\|u_1\|_{C^{2,\alpha}(\Sigma_i)} + \|u_2\|_{C^{2,\alpha}(\Sigma_i)} \le C(\|f_1\|_{C^{1,\alpha}(\Sigma_i)} + \|f_2\|_{C^{1,\alpha}(\Sigma_i)}), \quad i = 1, 2. \tag{B-8}$$

For notational convenience, set $w_1 = x_2 u_1 + x_3 u_2$ and $w_2 = x_3 u_2 - x_2 u_1$.

Equation (B-5) implies that w_1 and w_2 satisfy the following the second-order elliptic equations, respectively:

$$\begin{cases} (\partial_1^2 + \partial_2^2 + \partial_3^2)w_1 = \partial_1(x_2 f_2 + x_3 f_4) & \\ \qquad\qquad + \partial_2(x_2 f_1 - x_3 f_2) + \partial_3(x_2 f_2 + x_3 f_3) & \text{in } \Omega,\\ w_1 = 0 & \text{on } \Gamma, \end{cases} \tag{B-9}$$

$$\begin{cases} (\partial_1^2 + \partial_2^2 + \partial_3^2)w_2 = \partial_1(x_3 f_3 - x_2 f_4) & \\ \qquad\qquad + \partial_2(x_2 f_2 + x_3 f_1) - \partial_3(x_2 f_1 - x_3 f_2) & \text{in } \Omega,\\ (x_2\partial_2 + x_3\partial_3)w_2 = f_2 & \text{on } \Gamma. \end{cases} \tag{B-10}$$

For the problem (B-9), it follows from [Gilbarg and Trudinger 1983, Theorem 3.7, Theorem 6.6] that

$$\|w_1\|_{C^{2,\alpha}(\bar{\Omega})} \le C\bigg(\sum_{i=1}^{2}\|w_1\|_{C^{2,\alpha}(\Sigma_i)} + \sum_{j=1}^{4}\|f_j\|_{C^{1,\alpha}(\bar{\Omega})}\bigg). \tag{B-11}$$

For (B-10), the compatibility conditions at corners and $C^{2,\alpha}$ estimates of solutions to second-order elliptic equations with mixed boundary conditions in [Xin et al.

2009, Lemma A] imply that

$$\|w_2\|_{C^{2,\alpha}(\bar\Omega)} \le C\biggl(\sum_{i=1}^{2}\|w_2\|_{C^{2,\alpha}(\Sigma_i)} + \sum_{j=1}^{4}\|f_j\|_{C^{1,\alpha}(\bar\Omega)}\biggr). \tag{B-12}$$

Transforming w_1 and w_2 back to u_1 and u_2 via

$$u_1 = \frac{x_2w_1 + x_3w_2}{x_2^2+x_3^2} \quad \text{and} \quad u_2 = \frac{x_3w_1 - x_2w_2}{x_2^2+x_3^2}$$

gives

$$\begin{aligned}\|u_1\|_{C^{2,\alpha}(\Gamma)} + \|u_2\|_{C^{2,\alpha}(\Gamma)} &\le C(\|w_1\|_{C^{2,\alpha}(\bar\Omega)} + \|w_2\|_{C^{2,\alpha}(\bar\Omega)})\\ &\le C\biggl(\sum_{i=1}^{2}(\|w_1\|_{C^{2,\alpha}(\Sigma_i)} + \|w_2\|_{C^{2,\alpha}(\Sigma_i)}) + \sum_{j=1}^{4}\|f_j\|_{C^{1,\alpha}(\bar\Omega)}\biggr).\end{aligned} \tag{B-13}$$

This, together with (B-5), yields (B-7).

Next, we derive the second-order elliptic equations on u_1 and u_2. By (B-5),

$$\begin{cases}(\partial_1^2+\partial_2^2+\partial_3^2)u_1 = \partial_1 f_3 + \partial_2 f_1 + \partial_3 f_2 & \text{in } \Omega,\\ (\partial_1^2+\partial_2^2+\partial_3^2)u_2 = \partial_1 f_4 - \partial_2 f_2 + \partial_3 f_1 & \text{in } \Omega.\end{cases}$$

Thus,

$$\begin{aligned}&\|u_1\|_{C^{2,\alpha}(\bar\Omega)} + \|u_2\|_{C^{2,\alpha}(\bar\Omega)}\\ &\qquad\le C\biggl(\sum_{i=1}^{2}\|u_j\|_{C^{2,\alpha}(\Sigma_i)} + \|u_j\|_{C^{2,\alpha}(\Gamma)} + \sum_{i=1}^{4}\|f_i\|_{C^{1,\alpha}(\bar\Omega)}\biggr).\end{aligned} \tag{B-14}$$

Substituting (B-7) into (B-14) yields

$$\|u_1\|_{C^{2,\alpha}(\bar\Omega)} + \|u_2\|_{C^{2,\alpha}(\bar\Omega)} \le C\sum_{i=1}^{4}\|f_i\|_{C^{1,\alpha}(\bar\Omega)}. \tag{B-15}$$

For each $x_1 \in [0, 1]$, from the first and the fifth equations in (B-5) it follows that

$$\int_{B_1(0)} f_1(x_1, x_2, x_3)\,dx_2\,dx_3 = \int_{\partial B_1(0)} (x_2u_1 + x_3u_2)\,dl = 0,$$

so by $f_1 \in C^{1,\alpha}(\Omega)$ and the integral mean value theorem, there exists some point $(x_2^*(x_1), x_3^*(x_1)) \in B_1(0)$ such that

$$f_1(x_1, x_2^*(x_1), x_3^*(x_1)) = 0.$$

This implies

$$\|f_1\|_{C^{1,\alpha}(\bar\Omega)} \le C\|\nabla f_1\|_{C^{\alpha}(\bar\Omega)}. \tag{B-16}$$

Substituting (B-16) into (B-15) yields (B-6). □

Lemma B.4. *Under the assumptions of Lemma 4.2, at the point* (0, 0, 1),

$$Y_4 = a_0 Y_5 + O(1/X_0^2)Y_5, \quad Y_5 = O(X_0)Y_4,$$

where $a_0 < 0$ *and* $a_0 = O(1/X_0)$.

Proof. In the coordinate (y_1, y_2, y_3), the background solution $(\rho_0^{\pm}(y_1), U_0^{\pm}(y_1))$ satisfies (see Appendix A),

$$\begin{cases} \dfrac{d\rho_0^{\pm}(y_1)}{dy_1} = \dfrac{2(M_0^{\pm}(y_1))^2\rho_0^{\pm}(y_1)}{y_1(1-(M_0^{\pm}(y_1))^2)}, \\ \dfrac{dU_0^{\pm}(y_1)}{dy_1} = -\dfrac{2U_0^{\pm}(y_1)}{y_1(1-(M_0^{\pm}(y_1))^2)}, \\ \dfrac{dM_0^{\pm}(y_1)}{dy_1} = -\dfrac{M_0^{\pm}(y_1)(2+(\gamma-1)M_0^{\pm}(y_1))}{y_1(1-(M_0^{\pm}(y_1))^2)}, \end{cases} \tag{B-17}$$

where

$$M_0^{\pm}(y_1) = \frac{U_0^{\pm}(y_1)}{c(\rho_0^{\pm}(y_1))}$$

denote the Mach numbers of supersonic coming flow and subsonic flow, respectively.

By (B-17) and (2-16)–(2-17),

$$\begin{cases} M_0^{-}(y_1) = M_0^{-}(X_0) + O(1/X_0), \\ \rho_0^{-}(y_1) = \rho_0^{-}(X_0) + O(1/X_0), \\ U_{1,0}^{-}(y_1) = U_0^{-}(X_0) + O(1/X_0). \end{cases} \tag{B-18}$$

In addition, it follows from (2-5) that at the point $z = (0, 0, 1)$,

$$\begin{cases} \rho Y_1 + V_1 Y_4 \\ \quad = \big(\rho_0^{-}(\xi_1(0,1))\bar U_0^{-}(\xi_1(0,1),0,1) - \rho_0^{-}(\xi_2(0,1))\bar U_0^{-}(\xi_2(0,1),0,1)\big) \\ \qquad + O(\varepsilon^2)Y_1 + O(\varepsilon)Y_2 + O(\varepsilon)Y_3 + O(\varepsilon^2)Y_4 + O(\varepsilon^2)Y_5, \\ \rho(U_1+V_1)Y_1 + V_1^2 Y_4 + (1+\tau^2)c^2(\tilde\rho)Y_4 \\ \quad = \big((\rho_0^{-}(\bar U_0^{-})^2)(\xi_1(0,1),0,1) - (\rho_0^{-}(\bar U_0^{-})^2)(\xi_2(0,1),0,1)\big) \\ \qquad + (1+\tau^2)\big(P_0^{-}(\xi_1(0,1)) - P_0^{-}(\xi_2(0,1))\big) + O(\varepsilon^2)Y_1 + O(\varepsilon)Y_2 + O(\varepsilon)Y_3 \\ \qquad + O(\varepsilon^2)Y_4 + O(\varepsilon^2)Y_5. \end{cases}$$

Using this and a direct computation gives

$$
\begin{aligned}
\text{(B-19)}\quad &\big((1+\tau^2)c^2(\tilde\rho)-U_1(\xi_1(0,1),0,1)V_1(\xi_2(0,1),0,1)\big)Y_4\\
&=(1+\tau^2)\big(P_0^-(\xi_1(0,1))-P_0^-(\xi_2(0,1))\big)\\
&\quad+(\rho_0^-(U_0^-)^2)(\xi_1(0,1),0,1)-(\rho_0^-(U_0^-)^2)(\xi_2(0,1),0,1)\\
&\quad-\Big(\big((\rho_0^-U_0^-)(\xi_1(0,1),0,1)-(\rho_0^-U_0^-)(\xi_2(0,1),0,1)\big)\\
&\qquad\times\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)\Big)\\
&\quad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon^2)Y_4+O(\varepsilon^2)Y_5.
\end{aligned}
$$

Since

$$
\begin{cases}
\dfrac{d(\rho_0^-(r)U_0^-(r))}{dr}=-\dfrac{2\rho_0^-(r)U_0^-(r)}{r},\\[2mm]
\dfrac{d\big(\rho_0^-(r)(U_0^-(r))^2+P_0^-(r)\big)}{dr}=-\dfrac{2\rho_0^-(r)(U_0^-(r))^2}{r},
\end{cases}
$$

we have

$$
\begin{aligned}
\text{(B-20)}\quad &(1+\tau^2)\big(P_0^-(\xi_1(0,1))-P_0^-(\xi_2(0,1))\big)\\
&\quad+(\rho_0^-(\overline{U}_0^-)^2)(\xi_1(0,1),0,1)-(\rho_0^-(\overline{U}_0^-)^2)(\xi_2(0,1),0,1)\\
&\quad-\Big(\big((\rho_0^-\overline{U}_0^-)(\xi_1(0,1),0,1)-(\rho_0^-\overline{U}_0^-)(\xi_2(0,1),0,1)\big)\\
&\qquad\times\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)\Big)\\
&=-\frac{2\rho_0^-(\tilde\xi)(\overline{U}_0^-(\tilde\xi))^2}{\tilde\xi}Y_5(0,1)\\
&\quad+\frac{2\rho_0^-(\tilde\xi)\overline{U}_0^-(\tilde\xi)}{\tilde\xi}\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big)Y_5(0,1)
\end{aligned}
$$

$$
\begin{aligned}
\text{(B-21)}\quad &\big((1+\tau^2)c^2(\tilde\rho)-U_1V_1\big)Y_4\\
&=-\frac{2(\rho_0^-U_0^-)(\tilde\xi)}{\tilde\xi}\Big(U_0^-(\tilde\xi)-\big(U_1(\xi_1(0,1))+V_1(\xi_2(0,1))\big)\Big)Y_5\\
&\quad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon^2)Y_4+O(\varepsilon^2)Y_5,
\end{aligned}
$$

where $\tilde\rho$ and $\tilde\xi$ are the values derived by the mean value theorem on the functions $P(\rho)-P(q)$ and $G(\xi_1(0,1))-G(\xi_2(0,1))$ with

$$
\begin{aligned}
G(y_1)&=(1+\tau^2)P_0^-(y_1)+(\rho_0^-(U_0^-)^2)(y_1,0,1)\\
&\quad-(\rho_0^-U_0^-)(y_1,0,1)\big(U_1(\xi_1(0,1),0,1)+V_1(\xi_2(0,1),0,1)\big),
\end{aligned}
$$

respectively.

Substituting (B-19)–(B-20) into (B-18) yields

$$\begin{aligned}\text{(B-22)}\quad &\big((1+\tau^2)c^2(\tilde{\rho})-U_1V_1\big)Y_4\\ &=-\frac{2(\rho_0^-U_0^-)(\tilde{\xi})}{\tilde{\xi}}\bigg(U_0^-(\tilde{\xi})-\bigg(\frac{(\rho_0^-U_0^-)(\xi_1(0,1))}{\rho(\xi_1(0,1))}\\ &\qquad\qquad+\frac{(\rho_0^-U_0^-)(\xi_2(0,1))}{q(\xi_2(0,1))}\bigg)\bigg)Y_5\\ &\qquad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon)Y_4+O(\varepsilon^2)Y_5.\end{aligned}$$

Due to the assumptions in Theorem 2.1, we have

$$\begin{aligned}\rho(\xi_1(0,1))&=\widehat{\rho_0}^+(r_0)+O(\varepsilon),\\ q(\xi_2(0,1))&=\widehat{\rho_0}^+(r_0)+O(\varepsilon),\\ \rho_0^-(\xi_i(0,1))&=\rho_0^-(r_0)+O(\varepsilon),\quad i=1,2.\end{aligned}$$

Then for $\rho_0^+(r_0)>2\rho_0^-(r_0)$ and small ε,

$$\text{(B-23)}\qquad \begin{cases}\rho(\xi_1(0,1))>2\rho_0^-(\xi_1(0,1)),\\ q(\xi_2(0,1))>2\rho_0^-(\xi_2(0,1)).\end{cases}$$

Moreover,

$$\begin{aligned}U_0^-(\tilde{\xi})&=U_0^-(\xi_1(0,1))+O(1/X_0)(\xi_1(0,1)-\tilde{\xi}),\\ U_0^-(\tilde{\xi})&=U_0^-(\xi_2(0,1))+O(1/X_0)(\xi_2(0,1)-\tilde{\xi}),\\ \tilde{\rho}&=\rho(\xi_1(0,1))+O(1)Y_4,\\ V_1&=U_1+O(1)Y_1.\end{aligned}$$

So (B-22) becomes

$$\begin{aligned}\text{(B-24)}\quad &\big((1+\tau^2)c^2(\rho(\xi_1(0,1))-U_1^2\big)Y_4\\ &=-\frac{(\rho_0^-U_0^-)(\tilde{\xi})}{\tilde{\xi}}\bigg(U_0^-(\xi_1(0,1))+U_0^-(\xi_2(0,1))\\ &\qquad-\bigg(\frac{2(\rho_0^-U_0^-)(\xi_1(0,1))}{\rho(\xi_1(0,1))}+\frac{2(\rho_0^-U_0^-)(\xi_2(0,1))}{q(\xi_2(0,1))}\bigg)\bigg)Y_5\\ &\qquad+O(\varepsilon^2)Y_1+O(\varepsilon)Y_2+O(\varepsilon)Y_3+O(\varepsilon)Y_4+O(1/X_0^2)Y_5.\end{aligned}$$

By (4-4), (4-6) and (B-23)–(B-24), we obtain that at the point (0, 0, 1)

$$Y_4=a_0Y_5+O(1/X_0^2)Y_5\quad\text{and}\quad Y_5=O(X_0)Y_4,$$

where $a_0<0$ and $a_0=O(1/X_0)$, which completes the proof of Lemma B.4. □

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REFINED OPEN NONCOMMUTATIVE DONALDSON–THOMAS INVARIANTS FOR SMALL CREPANT RESOLUTIONS

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We study analogs of noncommutative Donaldson–Thomas invariants corresponding to the refined topological vertex for small crepant resolutions of toric Calabi–Yau 3-folds. We give three definitions of the invariants which are equivalent to each others and provide "wall-crossing" formulas for the invariants. In particular, we get normalized generating functions which are unchanged under wall-crossing.

Introduction

Donaldson–Thomas theory [Thomas 2000] is intersection theory on the moduli spaces of ideal sheaves on a smooth variety, which is conjecturally equivalent to Gromov–Witten theory [Maulik et al. 2006]. For a Calabi–Yau 3-fold, the virtual dimension of the moduli space is zero and hence Donaldson–Thomas invariants are said to be counting invariants of ideal sheaves. It is known that they coincide with the weighted Euler characteristics of the moduli spaces weighted by the Behrend functions [2009]. Recently, the Donaldson–Thomas invariants of Calabi–Yau 3-folds have been studied using categorical methods; see, for example, [Joyce 2008; 2007; Toda 2009; 2010; Kontsevich and Soibelman 2008; Joyce and Song 2010].

On the other hand, a smooth variety Y sometimes has a noncommutative associative algebra A such that the derived category of coherent sheaves on Y is equivalent to the derived category of A-modules. Derived McKay correspondence [Kapranov and Vasserot 2000; Bridgeland et al. 2001] and Van den Bergh's noncommutative crepant resolutions [2004] are typical examples. In such cases, B. Szendrői proposed to study counting invariants of A-modules (*noncommutative Donaldson–Thomas invariants*) and relations with the original Donaldson–Thomas invariants on Y [Szendrői 2008]. In [Nagao and Nakajima 2011; Nagao 2011a], we provided wall-crossing formulas which relate generating functions of the Donaldson–Thomas and noncommutative Donaldson–Thomas invariants for small crepant resolutions of toric Calabi–Yau 3-folds. (We say a resolution of a 3-fold is *small* if the dimension of each fiber is less than or equal to 1.)

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The aim of this paper is to propose new invariants generalizing noncommutative Donaldson–Thomas invariants and to provide "wall-crossing formulas" for small crepant resolutions of toric Calabi–Yau 3-folds. We have two directions of generalizations:

- "open" version:[1] corresponding to counting invariants of sheaves on Y with noncompact supports,[2]
- refined version: corresponding to refined topological vertex [Iqbal et al. 2009].[3]

Let $Y \to X$ be a projective small crepant resolution of an affine toric Calabi–Yau 3-fold. Recall that giving an affine toric Calabi–Yau 3-fold is equivalent to giving a convex lattice polygon. Existence of a small crepant resolution is equivalent to absence of interior lattice points in the polygon. It is easy to classify such polygons and X is one of the following:

- $X = X_{L^+,L^-} := \{\mathrm{xy} = \mathrm{z}^{L^+}\mathrm{w}^{L^-}\} \subset \mathbb{C}^4$ for $L^+ > 0$ and $L^- \geq 0$, or
- $X = X_{(\mathbb{Z}/2\mathbb{Z})^2} := \mathbb{C}^3/(\mathbb{Z}/2\mathbb{Z})^2$ where $(\mathbb{Z}/2\mathbb{Z})^2$ acts on $\mathbb{C}^3$ with weights $(1, 0)$, $(0, 1)$ and $(1, 1)$.

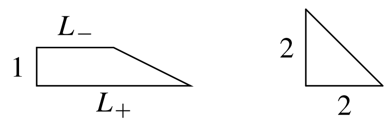


Figure 1. Polygons for X_{L^+,L^-} and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$.

In this paper, we study the first case. We put $L := L^+ + L^-$. Note that $X_{1,1}$ is called the *conifold* and $X_{L,0}$ is isomorphic to $\mathbb{C} \times \mathbb{C}^2/(\mathbb{Z}/L\mathbb{Z})$.

Given a pair of Young diagrams $\nu = (\nu_+, \nu_-)$ and an L-tuple of Young diagrams

$$\lambda = \bigl(\lambda^{(1/2)}, \dots, \lambda^{(L-1/2)}\bigr),$$

the generating function of *refined open noncommutative Donaldson–Thomas invariants* (roncDT, in short)

$$\mathfrak{X}^{Y}_{\lambda,\nu}(\vec{q}) = \mathfrak{X}^{Y}_{\lambda,\nu}(q_+, q_-, q_1 \dots, q_{L-1}),$$

which is denoted by $\mathfrak{X}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}$ in the body of this paper, is defined by counting the number of the following data:

- an $(L-1)$-tuple of Young diagrams $\vec{\nu} = (\nu^{(1)}, \dots, \nu^{(L-1)})$, and

[1] The word "open" stems from such terminologies as "open topological string theory". According to [Aganagic et al. 2005], open topological string partition function is given by summing up the generating functions of these invariants over Young diagrams.

[2] As far as we know, there is no definition of "open" invariants for general Calabi–Yau 3-folds.

[3] See [Behrend et al. 2009] for a geometric definition of refined invariants.

- an L-tuple of 3-dimensional Young diagrams $\vec{\Lambda} = (\Lambda^{(1/2)}, \ldots, \Lambda^{(L-1/2)})$ such that $\Lambda^{(j)}$ is of type $(\lambda^{(j)}, \nu^{(j+1/2)}, {}^{\mathrm{t}}\nu^{(j-1/2)})$ or $(\lambda^{(j)}, {}^{\mathrm{t}}\nu^{(j-1/2)}, \nu^{(j+1/2)})$ (see Section 5.3 for details).

Such data parametrize torus fixed ideal sheaves on the small crepant resolution Y. In particular,

$$\mathcal{Z}^{Y}_{\varnothing,\varnothing}(\vec{q})\big|_{q_+=q_-}$$

coincides with the generating function of Euler characteristic versions of the Donaldson–Thomas invariants of Y.[4]

Let A be a noncommutative crepant resolution of X. Let $\mathbb{Z}_{\mathrm{h}}$ denote the set of half integers and let $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ be a bijection such that $\theta(h+L) = \theta(h) + L$ and such that

$$\theta(1/2) + \cdots + \theta(L-1/2) = 1/2 + \cdots + (L-1/2).$$

We will define generating functions $\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q})$, which are denoted by $\mathcal{Z}_{\sigma,\lambda,\nu,\theta}(\vec{q})$ in the body of this paper (see Section 3.4), satisfying these properties:

- $\mathcal{Z}^{A}_{\varnothing,\varnothing,\mathrm{id}}(\vec{q})\big|_{q_+=q_-=q_0^{1/2}}$ coincides with the generating function $\mathcal{Z}^{A}_{\mathrm{NCDT,eu}}$ of Euler characteristic versions[5] of noncommutative Donaldson–Thomas invariants for the noncommutative crepant resolution A; see [Mozgovoy and Reineke 2010] and the remark on page 184.
- "$\lim\limits_{\theta\to\infty}$" $\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q}) = \mathcal{Z}^{Y}_{\lambda,\nu}(\vec{q})$; see Theorem 5.4.8. (The limit in this equation is, in fact, equivalent to a limit in the space of stability conditions for the category of finite-dimensional A-modules.)[6]

Moreover, for $i \in I := \mathbb{Z}/L\mathbb{Z}$ we can define the new bijection $\mu_i(\theta)\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ (see §1.2.1) and

- $\mathcal{Z}^{A}_{\lambda,\nu,\mu_i(\theta)}(\vec{q})/\mathcal{Z}^{A}_{\lambda,\nu,\theta}(\vec{q})$ is given explicitly (Theorem 4.2.2 and 4.4.2).

In [Nagao and Nakajima 2011; Nagao 2011a], we realized the $\mathcal{Z}^{A}_{\varnothing,\varnothing,\theta}(\vec{q})\big|_{q_+=q_-}$ as generating functions of virtual counting of certain moduli spaces and these moduli spaces are constructed using geometric invariant theory. In this story, θ determines a chamber in the space of stability parameters and the chamber corresponding to θ is adjacent to the chamber corresponding to $\mu_i(\theta)$ by a single wall. This is the reason we call Theorem 4.2.2 and 4.4.2 as wall-crossing formulas, even though our

[4] The Euler characteristic version of the Donaldson–Thomas invariant coincides with the Donaldson–Thomas invariant up to sign [Maulik et al. 2006].

[5] The Euler characteristic version of the noncommutative Donaldson–Thomas invariant coincides with the noncommutative Donaldson–Thomas invariant up to sign [Nagao 2011a; Mozgovoy and Reineke 2010].

[6] A moduli space of stable A-modules with the specific numerical data gives a crepant resolution of X [Ishii and Ueda 2008]. The direction in which we take limit in the space of stability conditions determines a stability parameter in the construction of a crepant resolution.

definition of the invariants and the proof of the formula are given in combinatorial ways. In fact, in the subsequent paper [Nagao 2011b] we provide an alternative geometric definition, in which θ determines a chamber in the space of Bridgeland's stability conditions for the category of finite-dimensional A-modules.

As consequences of the wall-crossing formula, we get

- Corollaries 4.5.2 and 5.5.2: $\mathcal{Z}^A_{\lambda,\nu,\theta}/\mathcal{Z}^A_{\lambda,\varnothing,\theta} = \mathcal{Z}^Y_{\lambda,\nu}/\mathcal{Z}^Y_{\lambda,\varnothing}$ for any θ, λ and ν.
- Corollaries 4.5.4 and 5.5.4: $\left(\mathcal{Z}^A_{\lambda,\nu,\theta}/\mathcal{Z}^A_{\varnothing,\varnothing,\theta}\right)\big|_{q_+=q_-} = \left(\mathcal{Z}^Y_{\lambda,\nu}/\mathcal{Z}^Y_{\varnothing,\varnothing}\right)\big|_{q_+=q_-}$ for any θ, λ and ν such that $c_\lambda[j]=0$ for any j (see §1.3.1 for notation).

By the results in [Nagao and Nakajima 2011; Nagao 2011a], these formulas should be interpreted as stability of the normalized generating functions under wall crossing. We can find such stability of normalized generating functions in other contexts such as flop invariance and DT-PT correspondence. Categorical interpretations of such normalized generating functions and their stability are expected.

Now, we summarize the prior study on noncommutative Donaldson–Thomas invariants:

- Szendrői's formula on the generating function of noncommutative Donaldson–Thomas invariants of the conifold was shown by B. Young [2009] in a purely combinatorial way. The main tool is an operation called *dimer shuffling*.
- J. Brian and Young [2010] generalized the Szendrői–Young formula for $X_{L,0}$ and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$. The method is different from the one used in [Young 2009]: they use vertex operator method.
- In [Nagao and Nakajima 2011], we gave an interpretation of Szendrői–Young formula as a consequence of the wall-crossing formula. From our point of view, the argument there can be translated into combinatorial language by localization, yielding the argument in [Young 2009]. In particular, dimer shuffling is nothing but "mutation" in the categorical language.
- In [Nagao 2011a], we generalized the results in [Nagao and Nakajima 2011] for arbitrary small crepant resolutions of toric Calabi–Yau 3-folds.
- In [Joyce and Song 2010], the authors study noncommutative Donaldson–Thomas invariants of small crepant resolutions of toric Calabi–Yau 3-folds as examples of their theory of generalized Donaldson–Thomas invariants.
- T. Dimofte and S. Gukov [2010] provided a refined version of Szendrői–Young formula for the conifold.
- See [Jafferis and Moore 2008; Chuang and Jafferis 2009; Aganagic et al. 2011; Chuang and Pan 2010; Aganagic and Yamazaki 2010; Dimofte et al. 2011] for developments in physics.

In this paper, we define the roncDT invariants using a dimer model (Section 2), which is purely combinatorial.

In Section 3, we give an interpretation of the dimer model as a crystal melting model.[7] We construct an A-module $M^{\max}_{\sigma,\lambda,\nu,\theta}$ such that giving a dimer configuration is equivalent to giving a finite-dimensional torus invariant quotient module of $M^{\max}_{\sigma,\lambda,\nu,\theta}$. Hence the roncDT invariant coincides with the Euler characteristic of the moduli space of finite-dimensional quotient modules of $M^{\max}_{\sigma,\lambda,\nu,\theta}$; see [Nagao 2011b].[8]

In Section 4, we introduce the notion of dimer shuffling to prove the first main result of this paper: the wall-crossing formula (Theorems 4.2.2 and 4.4.2).

Finally we study the limit behavior of the dimer model in Section 5. The second main result is that the generating function given by the refined topological vertex for Y appears as the limit (Theorem 5.4.8).

While preparing the papers, the author was informed from J. Bryan that he and his collaborators C. Cadman and B. Young provided an explicit formula of $\mathfrak{Z}^{A}_{\lambda,\nu,\mathrm{id}}|_{q_+=q_-}$ for $X_{L,0}$ and $X_{(\mathbb{Z}/2\mathbb{Z})^2}$ using vertex operator methods [Bryan et al. 2012; ≥ 2011]. In a subsequent paper [Nagao 2011b], we provide an explicit formula of $\mathfrak{Z}^{A}_{\lambda,\nu,\theta}$ for X_{L_+,L_-} using vertex operator methods.

A physicist may refer to [Nagao and Yamazaki 2010], in which we explain the result of this paper in a physical context.

We conclude this introduction by definition some notation.

Indices. Let $\mathbb{Z}_{\mathrm{h}}$ denote the set of half integers and L be a positive integer. We set $I := \mathbb{Z}/L\mathbb{Z}$ and $I_{\mathrm{h}} := \mathbb{Z}_{\mathrm{h}}/L\mathbb{Z}$. The two natural projections $\mathbb{Z} \to I$ and $\mathbb{Z}_{\mathrm{h}} \to I_{\mathrm{h}}$ are denoted by the same symbol π. We sometimes identify I and I_{h} with $\{0, \dots, L-1\}$ and $\{1/2, \dots, L-1/2\}$ respectively.

The symbols n, h, i and j are used for elements in $\mathbb{Z}$, $\mathbb{Z}_{\mathrm{h}}$, I and I_{h} respectively.

For $n \in \mathbb{Z}$ and $h \in \mathbb{Z}_{\mathrm{h}}$, we define $c(n), c(h) \in \mathbb{Z}$ by

$$n = c(n) \cdot L + \pi(n), \quad h = c(h) \cdot L + \pi(h).$$

Young diagrams. A *Young diagram* ν is a map $\nu\colon \mathbb{Z} \to \mathbb{Z}$ such that $\nu(n) = |n|$ if $|n| \gg 0$ and $\nu(n) - \nu(n-1) = \pm 1$ for any $n \in \mathbb{Z}$. The map $\mathbb{Z}_{\mathrm{h}} \to \{\pm 1\}$ given by $j \mapsto \nu(j+1/2) - \nu(j-1/2)$ is also denoted by ν.

By an abuse of notation, we sometimes identify $+$ and $-$ with 1 and -1.

[7] From the geometric point of view, the crystal melting model is more natural. But in this paper we adapt the definition using the dimer model since it is more convenient when we prove some technical lemmas, which we also use in [Nagao 2011b].

[8] In the case when $\nu_+ = \nu_- = \varnothing$, the moduli spaces have symmetric obstruction theory and the invariant in this paper coincides with the weighted Euler characteristic up to sign.

A Young diagram can be represented by a nonincreasing sequence of positive integers. We fix the notation as in Figure 2.

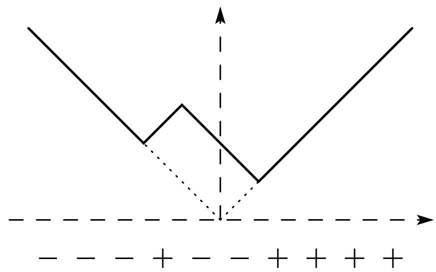


Figure 2. $\nu = (1, 1)$, ${}^{\mathrm{t}}\nu = (2)$.

Formal variables. Let q_+, q_- and $q_0, \dots, q_{L-1}$ be formal variables. We use q_+, q_- and $q_1, \dots, q_{L-1}$ for generating functions of refined invariants. Substituting $q_+ = q_- = (q_0)^{1/2}$, we get generating functions of nonrefined invariants.

Let $P := \mathbb{Z} \cdot I$ be the lattice with the basis $\{\alpha_i \mid i \in I\}$. For an element $\alpha = \sum \alpha^i \cdot \alpha_i \in P \quad (\alpha^i \in \mathbb{Z})$, we put $q^\alpha := \prod (q_i)^{\alpha^i}$.

For $\alpha, \alpha' \in P$, we say $\alpha < \alpha'$ or $q^\alpha < q^{\alpha'}$ if $\alpha' - \alpha \in P^+ := \mathbb{Z}_{\geq 0} \cdot I$.

1. Preliminaries

1.1. *Affine root system.*

1.1.1. For $h, h' \in \mathbb{Z}_{\mathrm{h}}$, we define $\alpha_{[h,h']} \in P$ by

$$\alpha_{[h,h']} := \sum_{n=h+1/2}^{h'-1/2} \alpha_{\pi(n)}$$

if $h < h'$, $\alpha_{[h,h']} = 1$ if $h = h'$ and $\alpha_{[h,h']} = -\alpha_{[h',h]}$ if $h > h'$. We set

$$\Lambda := \{\alpha_{[h,h']} \in P \mid h \neq h'\},$$
$$\Lambda^{\mathrm{re},+} := \{\alpha_{[h,h']} \in \Lambda \mid h < h',\ h \not\equiv h' \ (\mathrm{mod}\, L)\}.$$

An element in Λ (resp. $\Lambda^{\mathrm{re},+}$) is called a *root* (resp. *positive real root*) of the affine root system of type A_{L-1}.

1.1.2. The element $\delta := \alpha_0 + \cdots \alpha_{L-1} \in P$ is called the *minimal imaginary root*. We set

$$\Lambda^{\mathrm{fin},+} := \left\{\alpha_{[j,j']} \in \Lambda \mid 1/2 \leq j < j' \leq L - 1/2\right\}$$

and

$$\Lambda_+^{\mathrm{re},+} := \left\{\alpha_{[j,j']} + N\delta \mid \alpha_{[j,j']} \in \Lambda^{\mathrm{fin},+},\ N \geq 0\right\}. \tag{1-1}$$

Example 1.1.3. In the case of $L=4$, we have

$$\Lambda^{\mathrm{fin},+} := \{\alpha_1,\,\alpha_2,\,\alpha_3,\,\alpha_2+\alpha_3,\,\alpha_1+\alpha_2,\,\alpha_1+\alpha_2+\alpha_3\}.$$

1.1.4. For a root $\alpha \in \Lambda$, we take h and h' such that $\alpha = \alpha_{[h,h']}$ and set

$$j_-(\alpha) := \pi(h), \text{ and } j_+(\alpha) := \pi(h').$$

We also put

$$B^{\alpha} := \{(h,h') \in (\mathbb{Z}_{\mathrm{h}})^2 \mid \alpha_{[h,h']} = \alpha\}.$$

1.1.5. Let Θ denote the set of bijections $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ such that

- $\theta(h+L) = \theta(h)+L$ for any $h \in \mathbb{Z}_{\mathrm{h}}$, and
- $\displaystyle\sum_{h=1/2}^{L-1/2} \theta(h) = \sum_{h=1/2}^{L-1/2} h.$

Example 1.1.6. In the case of $L=4$, the correspondence

$$\tfrac{1}{2} \mapsto -\tfrac{1}{2}, \quad \tfrac{3}{2} \mapsto \tfrac{3}{2}, \quad \tfrac{5}{2} \mapsto \tfrac{5}{2}, \quad \tfrac{7}{2} \mapsto \tfrac{9}{2}$$

gives an elements in Θ. Let $\mu_0(\mathrm{id})$ denote this map (see §1.2.1 for notation).

1.1.7. For $\theta \in \Theta$ and $i \in I$, we define $\alpha(\theta,i) := \alpha_{[\theta(n-1/2),\theta(n+1/2)]}$ $(n \in \pi^{-1}(i))$.

Example 1.1.8.

$$\begin{aligned}
\alpha(\mathrm{id},0) &= \alpha_0, & \alpha(\mu_0(\mathrm{id}),0) &= -\alpha_0,\\
\alpha(\mathrm{id},1) &= \alpha_1, & \alpha(\mu_0(\mathrm{id}),1) &= \alpha_0+\alpha_1,\\
\alpha(\mathrm{id},2) &= \alpha_2, & \alpha(\mu_0(\mathrm{id}),2) &= \alpha_2,\\
\alpha(\mathrm{id},3) &= \alpha_3, & \alpha(\mu_0(\mathrm{id}),3) &= \alpha_0+\alpha_3.
\end{aligned}$$

1.1.9. If $\alpha = \alpha_{[h,h']}$ is a positive real root, we write $\theta(\alpha) > 0$ if $\theta^{-1}(h) > \theta^{-1}(h')$, and we write $\theta(\alpha) < 0$ if $\theta^{-1}(h) < \theta^{-1}(h')$. We set

$$\Lambda_{\theta}^{\mathrm{re},+} := \{\alpha \in \Lambda^{\mathrm{re},+} \mid \theta(\alpha) > 0\}. \tag{1-2}$$

Example 1.1.10. We have $\Lambda_{\mathrm{id}}^{\mathrm{re},+} = \varnothing$ and $\Lambda_{\mu_0(\mathrm{id})}^{\mathrm{re},+} = \{\alpha_0\}$.

Remark. As we mentioned in the introduction, we studied moduli spaces of representations of a noncommutative crepant resolution of X_{L^+,L^-} in [Nagao 2011a]. In this theory, the space of stability conditions can be canonically identified with $P^* \otimes \mathbb{R}$ and the walls are classified as follows:

- the walls $W_\alpha := (\mathbb{R}\cdot\alpha)^{\perp} \subset P^* \otimes \mathbb{R}$ $\quad(\alpha \in \Lambda^{\mathrm{re},+})$, and
- the wall $W_\delta := (\mathbb{R}\cdot\delta)^{\perp}$, which separates the Donaldson–Thomas and Pandharipande–Thomas chambers.

The maps $\theta\colon \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ as above parametrize the chambers on one side of the wall W_δ. The notation $\theta(\alpha) \gtrless 0$ respects this parametrization.

1.2. ***Wall-crossing.***

1.2.1. For $i \in I$, let $\mu_i : \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ be the map given by

$$\mu_i(h) = \begin{cases} h-1 & \text{if } \pi(h-1/2) = i, \\ h+1 & \text{if } \pi(h+1/2) = i, \\ h & \text{otherwise.} \end{cases}$$

For $\theta \in \Theta$, we put $\mu_i(\theta) := \theta \circ \mu_i$.

Remark. The chambers corresponding to θ and $\mu_i(\theta)$ are separated by the wall $W_{\alpha(\theta,i)}$, which is the reason for the title of this subsection. From the viewpoint of the affine root system, wall crossing corresponds to simple reflection; from the viewpoint of noncommutative crepant resolutions, it corresponds to mutation; and from the viewpoint of dimer models, to dimer shuffling.

1.2.2. Let $\boldsymbol{i} = (i_1, i_2, \dots) \in I^{\mathbb{Z}_{>0}}$ be a sequence of elements in I. For $b > 0$, we define

$$\theta_{\boldsymbol{i},b} := \mu_{i_{b-1}}(\cdots(\mu_{i_1}(\mathrm{id}))\cdots) \in \Theta, \qquad \alpha_{\boldsymbol{i},b} := \alpha(\theta_{\boldsymbol{i},b}, i_b).$$

We say $\boldsymbol{i} \in I^{\mathbb{Z}_{>0}}$ is a *minimal expression* if $\theta_{\boldsymbol{i},b}(\alpha_{\boldsymbol{i},b}) < 0$ for any $b > 0$. For a minimal expression $\boldsymbol{i}$, we have

$$\Lambda^{\mathrm{re},+}_{\theta_{\boldsymbol{i},b}} = \{\alpha_{\boldsymbol{i},1}, \dots, \alpha_{\boldsymbol{i},b-1}\}.$$

1.3. ***Core and quotient of a Young diagram.***

1.3.1. Let $\sigma : I_{\mathrm{h}} \to \{\pm\}$ and $\lambda : \mathbb{Z}_{\mathrm{h}} \to \{\pm\}$ be maps such that $\lambda(h) = \pm\sigma(\pi(h))$ if $\pm h \gg 0$. We define integers $c_\lambda[j]$ and Young diagrams $\lambda^{[j]}$ for $j \in I_{\mathrm{h}}$ by

$$\lambda(h) = \lambda^{[\pi(h)]}\big(\sigma(j(h)) \cdot (c(h) - c_\lambda[\pi(h)] + 1/2)\big).$$

Remark. In the case $\sigma \equiv +$ and $\sum c_\lambda[j] = 0$, the sequence $\big(c_\lambda[j]\big)$ of integers and the sequence $\big(\lambda^{[j]}\big)$ of Young diagrams are called the L-core and the L-quotient of the Young diagram λ.

1.3.2. We put

$$B^{\alpha,\pm}_{\sigma,\lambda} := \{(h, h') \in B^\alpha \mid -\lambda(h)\sigma(h) = \lambda(h')\sigma(h') = \pm\}. \tag{1-3}$$

Lemma 1.3.3.

$$\big|B^{\alpha,+}_{\sigma,\lambda}\big| - \big|B^{\alpha,-}_{\sigma,\lambda}\big| = \alpha^0 + c_\lambda[j_-(\alpha)] - c_\lambda[j_+(\alpha)].$$

Proof. We write simply $j_\pm$ for $j_\pm(\alpha)$. Note that we have

$$B^\alpha = \big\{\big(cL + j_-, (c + \alpha^0)L + j_+\big) \,\big|\, c \in \mathbb{Z}\big\}.$$

For an integer N, we set

$$B^\alpha_N := \big\{\big(cL + j_-, (c + \alpha^0)L + j_+\big) \,\big|\, c \in [-N, N-1]\big\}.$$

Take a sufficiently large integer N. Then

$$B^{\alpha,+}_{\sigma,\lambda},\, B^{\alpha,-}_{\sigma,\lambda} \subset B^{\alpha}_{N}$$

and so

$$\begin{aligned}
&\left|B^{\alpha,+}_{\sigma,\lambda}\right| - \left|B^{\alpha,-}_{\sigma,\lambda}\right| \\
&\quad = -\sharp\bigl\{(h,h') \in B^{\alpha}_{N} \bigm| \lambda(h)\sigma(h) = +\bigr\} + \sharp\bigl\{(h,h') \in B^{\alpha}_{N} \bigm| \lambda(h')\sigma(h') = +\bigr\} \\
&\quad = -\sharp\bigl\{c \in [-N, N-1] \bigm| \lambda^{[j_-]}\bigl(\sigma(j_-)\cdot(c - c_\lambda[j_-] + 1/2)\bigr) = \sigma(j_-)\bigr\} \\
&\qquad\quad + \sharp\bigl\{c \in [-N, N-1] \bigm| \lambda^{[j_+]}\bigl(\sigma(j_+)\cdot(c + \alpha^0 - c_\lambda[j_+] + 1/2)\bigr) = \sigma(j_+)\bigr\} \\
&\quad = -(N - c_\lambda[j_-] - 1/2) + (N + \alpha^0 - c_\lambda[j_+] - 1/2) \\
&\quad = \alpha^0 + c_\lambda[j_-] - c_\lambda[j_+]. \qquad \square
\end{aligned}$$

For σ, λ, θ and i, we put

$$B^{i,\pm}_{\sigma,\lambda,\theta} := \bigl\{n \in \pi^{-1}(i) \mid (\theta(n-1/2), \theta(n+1/2)) \in B^{\alpha(\theta,i),\pm}_{\sigma,\lambda}\bigr\}. \tag{1-4}$$

2. Dimer model

2.1. *Dimer configurations.*

2.1.1. We fix the following data:

- a map $\sigma : I_{\mathrm{h}} \to \{\pm\}$,
- a map $\lambda : \mathbb{Z}_{\mathrm{h}} \to \{\pm\}$ such that $\lambda(h) = \pm\sigma(\pi(h))$ for $\pm h \gg 0$,
- a pair of Young diagrams $\nu = (\nu_+, \nu_-)$,
- a bijection $\theta : \mathbb{Z}_{\mathrm{h}} \to \mathbb{Z}_{\mathrm{h}}$ in Θ.

We put $\tilde{\sigma} := \sigma \circ \pi \circ \theta$, $\tilde{\lambda} := \lambda \circ \theta$ and $L_\pm := |\sigma^{-1}(\pm)|$.

2.1.2. We consider the following graph in the (x, y)-plane. First, we set

$$H(\sigma,\theta) := \bigl\{n \in \mathbb{Z} \mid \tilde{\sigma}(n-1/2) = \tilde{\sigma}(n+1/2)\bigr\}, \quad I_H(\sigma,\theta) := \pi(H(\sigma,\theta)), \tag{2-1}$$

$$S(\sigma,\theta) := \bigl\{n \in \mathbb{Z} \mid \tilde{\sigma}(n-1/2) \neq \tilde{\sigma}(n+1/2)\bigr\}, \quad I_S(\sigma,\theta) := \pi(S(\sigma,\theta)) \tag{2-2}$$

and for $n \in H(\sigma,\theta)$ we put $\tilde{\sigma}(n) := \tilde{\sigma}(n \pm 1/2)$.

The set of the vertices is given by

$$\begin{aligned}
\mathcal{V} := &\{(n,m) \mid n \in S(\sigma,\theta),\ n-m\text{: odd}\} \\
&\sqcup \{(n-1/2, m) \mid n \in H(\sigma,\theta),\ n-m\text{: odd}\} \\
&\sqcup \{(n+1/2, m) \mid n \in H(\sigma,\theta),\ n-m\text{: odd}\},
\end{aligned}$$

which are denoted by $v(n,m)$, $v_{\mathrm{l}}(n-1/2, m)$ and $v_{\mathrm{r}}(n+1/2, m)$ respectively.

The set of the edges is given by

$$\mathscr{E} := \{e_{\mathrm{h}}(n,m) \mid n \in H(\sigma,\theta),\ n-m : \mathrm{odd}\} \sqcup \{e_{\mathrm{s}}(h,k) \mid h,k \in \mathbb{Z}_{\mathrm{h}}\},$$

where

- $e_{\mathrm{h}}(n,m)$ connects $v_{\mathrm{l}}(n-1/2,m)$ and $v_{\mathrm{r}}(n+1/2,m)$,
- $e_{\mathrm{s}}(h,k)$ connects $v(h-1/2,k+1/2)$ or $v_{\mathrm{r}}(h,k+1/2)$ and $v(h+1/2,k-1/2)$ or $v_{\mathrm{l}}(h,k-1/2)$ if $h-k$ is even, and
- $e_{\mathrm{s}}(h,k)$ connects $v(h-1/2,k-1/2)$ or $v_{\mathrm{r}}(h,k-1/2)$ and $v(h+1/2,k+1/2)$ or $v_{\mathrm{l}}(h,k+1/2)$ if $h-k$ is odd.

We put

$$\mathscr{F} := \{(n,m) \in \mathbb{Z}^2 \mid n+m : \mathrm{even}\}, \quad \mathscr{F}_i := \{(n,m) \in \mathscr{F} \mid n \in \pi^{-1}(i)\} \tag{2-3}$$

for $i \in I$. Note that $\mathscr{E}$ divides the plain into disjoint hexagons and quadrilaterals. The hexagons are parametrized by the set

$$\mathscr{F}_{\mathrm{H}} := \{(n,m) \in \mathscr{F} \mid n \in H(\sigma,\theta)\}$$

and the quadrilaterals are parametrized by the set

$$\mathscr{F}_{\mathrm{S}} := \{(n,m) \in \mathscr{F} \mid n \in S(\sigma,\theta)\}.$$

For $(n,m) \in \mathscr{F}$, let $f(n,m)$ denote the corresponding hexagon or quadrilateral.

Example 2.1.3. In Figure 3, we show the graph associated with $L=3$, σ given by

$$\sigma(1/2) = +, \quad \sigma(3/2) = -, \quad \sigma(5/2) = -,$$

and $\theta = \mathrm{id}$ ($L_+ = 1$, $L_- = 2$).

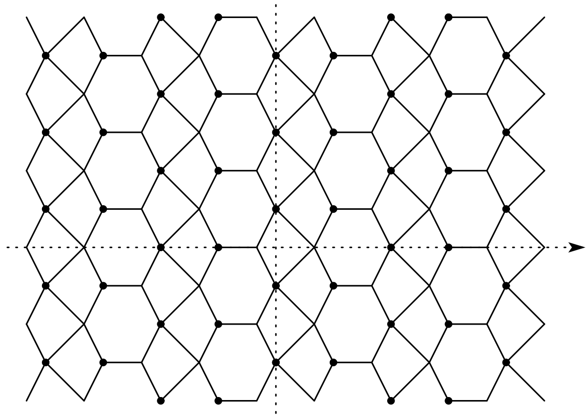


Figure 3. Graph and $\mathscr{V}_+$ for Example 2.1.3.

2.1.4. We set

$$\mathcal{V}_\pm := \{v(n,m) \mid \tilde{\sigma}(n+1/2) = \pm\} \\ \sqcup \{v_{\mathrm{l}}(n-1/2,m) \mid \tilde{\sigma}(n) = \mp\} \sqcup \{v_{\mathrm{r}}(n+1/2,m) \mid \tilde{\sigma}(n) = \pm\}.$$

Note that $\mathcal{V} = \mathcal{V}_+ \sqcup \mathcal{V}_-$ and each element in $\mathcal{E}$ connects an element in $\mathcal{V}_+$ and an element in $\mathcal{V}_-$ (see Figure 3 for example).

A *perfect matching* is a subset of $\mathcal{E}$ giving a bijection between $\mathcal{V}_+$ and $\mathcal{V}_-$.

2.1.5. We define the map $F_{\sigma,\lambda,\theta} \colon \mathbb{Z} \to \mathbb{Z}$ by $F_{\sigma,\lambda,\theta}(0) = 0$ and

$$F_{\sigma,\lambda,\theta}(n) = F_{\sigma,\lambda,\theta}(n-1) - \tilde{\lambda}(n-1/2). \tag{2-4}$$

For $k \in \mathbb{Z}_{\mathrm{h}}$, we set

$$\mathcal{P}^{k,\pm}_{\sigma,\lambda,\theta} := \bigl\{e_{\mathrm{h}}\bigl(n,\, F_{\sigma,\lambda,\theta}(n)+2k\bigr) \bigm| n \in \mathbb{Z},\ \tilde{\sigma}(n) = \mp\bigr\} \\ \sqcup \bigl\{e_{\mathrm{s}}\bigl(h,\, \tfrac{1}{2}(F_{\sigma,\lambda,\theta}(h-1/2) + F_{\sigma,\lambda,\theta}(h+1/2)) + 2k\bigr) \bigm| h \in \mathbb{Z}_{\mathrm{h}},\ \tilde{\sigma}(h) = \pm\bigr\}.$$

For a Young diagram η, define the perfect matching

$$\mathcal{P}^{\eta}_{\sigma,\lambda,\theta} := \bigsqcup_{k\in\mathbb{Z}_{\mathrm{h}}} \mathcal{P}^{k,\eta(k)}_{\sigma,\lambda,\theta}.$$

Example 2.1.6. In Figure 4, we show the perfect matching associated with σ as in Example 2.1.3, $\theta = \mathrm{id}$, $\eta = \varnothing$, and λ given by

$$\lambda(h) = \begin{cases} + & \text{if } h = -5/2, \\ - & \text{if } h = 1/2, \\ \mathrm{sgn}(h)\sigma(h) & \text{otherwise.} \end{cases}$$

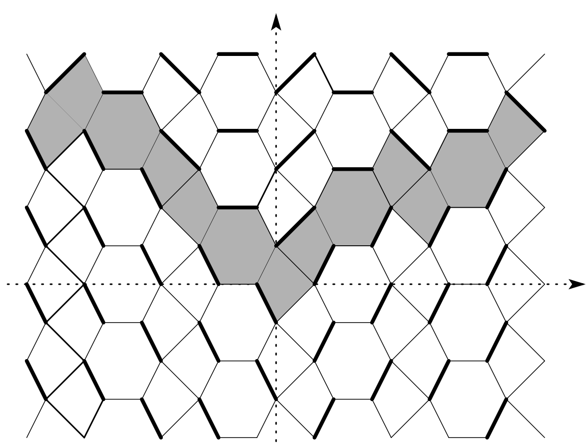


Figure 4. Example 2.1.6: $\{f(n,\, F_{\sigma,\lambda,\mathrm{id}}(n)) \mid n \in \mathbb{Z}\}$ and $\mathcal{P}^{\varnothing}_{\sigma,\lambda,\mathrm{id}}$.

2.1.7. Define the perfect matching

$$\mathcal{P}^{\pm}_{\sigma,\lambda,\theta} := \bigl\{e_{\mathrm{h}}(n,m) \mid \tilde{\sigma}(n) = \mp\bigr\} \sqcup \bigl\{e_{\mathrm{s}}(h,k) \mid \tilde{\sigma}(h) = \pm,\ h \cdot \tilde{\lambda}(h) - k : \text{even}\bigr\}.$$

Definition 2.1.8. A perfect matching D is said to be *a dimer configuration of type* $(\sigma, \lambda, \nu, \theta)$ if D coincides with $\mathcal{P}^{\nu_\pm}_{\sigma,\lambda,\theta}$ in the area $\{\pm x > m\}$ and $\mathcal{P}^{\pm}_{\sigma,\lambda,\theta}$ in the area $\{\pm y > m\}$ for $m \gg 0$.

Remark. A dimer configuration of type $(\sigma, \vec{\varnothing}, \vec{\varnothing}, \mathrm{id})$ is "a perfect matching congruent to the canonical perfect matching" in the terminology of [Mozgovoy and Reineke 2010].

2.1.9. For $f \in \mathcal{F}$, let $\partial f \subset \mathcal{E}$ denote the set of edges surrounding the face f. By moving f around clockwise, we can determine an orientation for each element in ∂f. Let $\partial^{\pm} f \subset \partial f$ denote the subset of edges starting from elements in $\mathcal{V}_{\pm}$.

For an edge $e \in \mathcal{E}$, let $f^{\pm}(e)$ denote the unique face such that $e \in \partial^{\pm} f^{\pm}(e)$.

2.2. *Weights.*

2.2.1. For $h \in \mathbb{Z}_{\mathrm{h}}$, we define the monomials $w_{\sigma,\lambda}(h)$ by the conditions

$$w_{\sigma,\lambda}(h) = \begin{cases} \bigl(Q_{\sigma(h)}\bigr)^{c(h)-c_\lambda[j(h)]} q^{(j(h))}_{\sigma(h)} & \text{if } h \gg 0, \\ \bigl(Q_{-\sigma(h)}\bigr)^{c(h)-c_\lambda[j(h)]} q^{(j(h))}_{-\sigma(h)} & \text{if } h \ll 0, \end{cases}$$

and

$$w_{\sigma,\lambda}(h)/w_{\sigma,\lambda}(h-L) = q_{\lambda(h)} \cdot q_{\lambda(h-L)} \cdot q_1 \cdot \cdots \cdot q_{L-1},$$

where

$$Q_{\pm} := (q_{\pm})^2 \cdot q_1 \cdot \cdots \cdot q_{L-1},\ \ q^{(j)}_{\pm} := q_{\pm} \cdot q_1 \cdot \cdots \cdot q_{j-1/2}.$$

Note that for $h \neq h'$ we have

$$w_\lambda(h')/w_\lambda(h)\big|_{q_+=q_-=(q_0)^{1/2}} = q^{\alpha_{[h,h']}}. \tag{2-5}$$

Example 2.2.2. Figure 5 shows the weight $w_{\sigma,\lambda}$ for σ and λ as in Example 2.1.6.

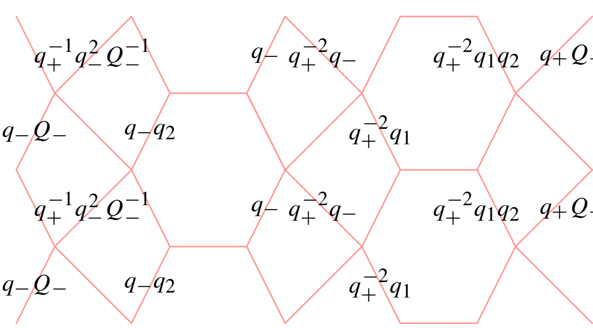


Figure 5. The weight $w_{\sigma,\lambda}$.

2.2.3. To an edge $e \in \mathscr{E}$ we associate the weight $w_{\sigma,\lambda,\theta}(e)$ by

$$w_{\sigma,\lambda,\theta}(e_{\mathrm{s}}(h,k)) := \begin{cases} w_{\sigma,\lambda}(\theta(h))^{\tilde{\sigma}(h)\cdot\tilde{\lambda}(h)} & \text{if } h\cdot\tilde{\lambda}(h)-k \text{ is odd,} \\ 1 & \text{if } h\cdot\tilde{\lambda}(h)-k \text{ is even,} \end{cases} \tag{2-6}$$

$$w_{\lambda,\sigma,\theta}(e_{\mathrm{h}}(n,m)) := 1. \tag{2-7}$$

2.2.4. Fix σ and λ. Then the set $\bigsqcup_{\alpha\in\Lambda^{\mathrm{re},+}} B^{\alpha,-}_{\sigma,\lambda}$ is finite. We define

$$F^{\alpha}_{\sigma,\lambda} := \prod_{(h,h')\in B^{\alpha,-}_{\sigma,\lambda}} \frac{w_{\sigma,\lambda}(h')}{w_{\sigma,\lambda}(h)}, \qquad F^{\theta}_{\sigma,\lambda} := \prod_{\substack{\alpha\in\Lambda^{\mathrm{re},+};\ \theta(\alpha)<0,\\ \sigma(j^-(\alpha))\neq\sigma(j^+(\alpha)).}} F^{\alpha}_{\sigma,\lambda}. \tag{2-8}$$

2.2.5. Note that for a dimer configuration D of type $(\sigma,\lambda,\nu,\theta)$ we have only a finite number of $e\in D$ such that $w_{\sigma,\lambda,\theta}(e)\neq 1$.

Definition 2.2.6. For a dimer configuration D of type $(\sigma,\lambda,\nu,\theta)$, we define the weight $w_{\sigma,\lambda,\theta}(D)$ by

$$w_{\sigma,\lambda,\theta}(D) := F^{\theta}_{\sigma,\lambda}\cdot\prod_{e\in D} w_{\sigma,\lambda,\theta}(e). \tag{2-9}$$

(See (2-6)–(2-8) for notation.)

Remark. We will define the generating function $\mathscr{Z}_{\sigma,\lambda,\nu,\theta}$ by the sum of weighs of all dimer configurations of type $(\sigma,\lambda,\nu,\theta)$.[9]

2.2.7. For a finite subset $\mathscr{E}'\subset\mathscr{E}$, we put

$$w_{\sigma,\lambda,\theta}(\mathscr{E}') := \prod_{e\in\mathscr{E}'} w_{\sigma,\lambda,\theta}(e)$$

and for a face $f\in\mathscr{F}$ we put

$$w_{\sigma,\lambda,\theta}(f) := \frac{w_{\sigma,\lambda,\theta}(\partial^- f)}{w_{\sigma,\lambda,\theta}(\partial^+ f)}. \tag{2-10}$$

For an integer n we set

$$w_{\sigma,\lambda,\theta}(n) := \frac{w_{\sigma,\lambda}(\theta(n+1/2))}{w_{\sigma,\lambda}(\theta(n-1/2))};$$

then

$$w_{\sigma,\lambda,\theta}(f(n,m)) = w_{\sigma,\lambda,\theta}(n)$$

for any $(n,m)\in\mathscr{F}$. By (2-5), we have

$$w_{\sigma,\lambda,\theta}(n)\big|_{q_+=q_-=(q_0)^{1/2}} = q^{\alpha(\theta,i)}.$$

[9] We will leave the definition of the generating function until Section 3.4 since we will use Proposition 3.3.9 to prove that the number of dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ is finite.

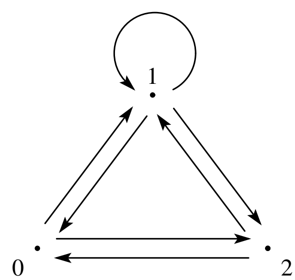
3. The viewpoint of noncommutative crepant resolutions

3.1. ***Noncommutative crepant resolutions.*** Let Γ be a lattice in the (x, y)-plane generated by $(L, 0)$ and $(0, 2)$. The graph given in §2.1.2 is invariant under the action of Γ and so gives a graph on the torus $\mathbb{R}^2/\,\Gamma$. This gives a quiver with a potential $A=(Q_{\sigma,\theta}, w_{\sigma,\theta})$ as in [Nagao 2011a]. The vertices of $Q_{\sigma,\theta}$ are parametrized by I and the arrows are given by

$$\Bigl(\bigsqcup_{j\in I_{\mathrm{h}}} h_j^+\Bigr)\sqcup\Bigl(\bigsqcup_{j\in I_{\mathrm{h}}} h_j^-\Bigr)\sqcup\Bigl(\bigsqcup_{i\in I_H(\sigma,\theta)} r_i\Bigr)$$

(see (2-1) for notation). Here h_j^+ (resp. h_j^-) is an edge from $j-1/2$ to $j+1/2$ (resp. from $j+1/2$ to $j-1/2$) and r_i is an edge from i to itself. See [Nagao 2011a, §1.2] for the definition of the potential $w_{\sigma,\theta}$.

Example 3.1.1. Here is the quiver $Q_{\sigma,\mathrm{id}}$ for σ as in Example 2.1.6:



Remarks.
- The center of A is isomorphic to $R:=\mathbb{C}[\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{w}]/(\mathrm{xy}=\mathrm{z}^{L_+}\mathrm{w}^{L_-})$. In [Nagao 2011a, Theorem 1.14 and 1.19], we showed that A is a noncommutative crepant resolution of $X=\operatorname{Spec} R$.
- The affine 3-fold X is toric. In fact,

$$T=\operatorname{Spec}\tilde{R}:=\operatorname{Spec}\mathbb{C}[\mathrm{x}^{\pm}, \mathrm{y}^{\pm}, \mathrm{z}^{\pm}, \mathrm{w}^{\pm}]/(\mathrm{xy}=\mathrm{z}^{L_+}\mathrm{w}^{L_-})\subset X$$

  is a 3-dimensional torus.

3.2. ***Dimer model and noncommutative crepant resolution.***

3.2.1. We will construct an A-module $M(D)$ for a dimer configuration D. Let $V_i=V_i(D)$ $(i\in I)$ be vector space with the basis

$$\bigl\{b[D; x, y, z] \mid (x, y)\in\mathcal{F}_i,\ z\in\mathbb{Z}_{\geq 0}\bigr\}$$

(see (2-3) for notation). We define the map $h_j^{\pm}\colon V_{j\mp 1/2}\to V_{j\pm 1/2}$ by setting

$$h_j^{\pm}(b[D; x, y, z])=\begin{cases} b[D; x\pm 1, y-\tilde{\sigma}(j), z] & \text{if } e_{\mathrm{s}}\bigl(x\pm\frac{1}{2}, y-\frac{1}{2}\tilde{\sigma}(j)\bigr)\notin D,\\ b[D; x\pm 1, y-\tilde{\sigma}(j), z+1] & \text{if } e_{\mathrm{s}}\bigl(x\pm\frac{1}{2}, y-\frac{1}{2}\tilde{\sigma}(j)\bigr)\in D,\end{cases}$$

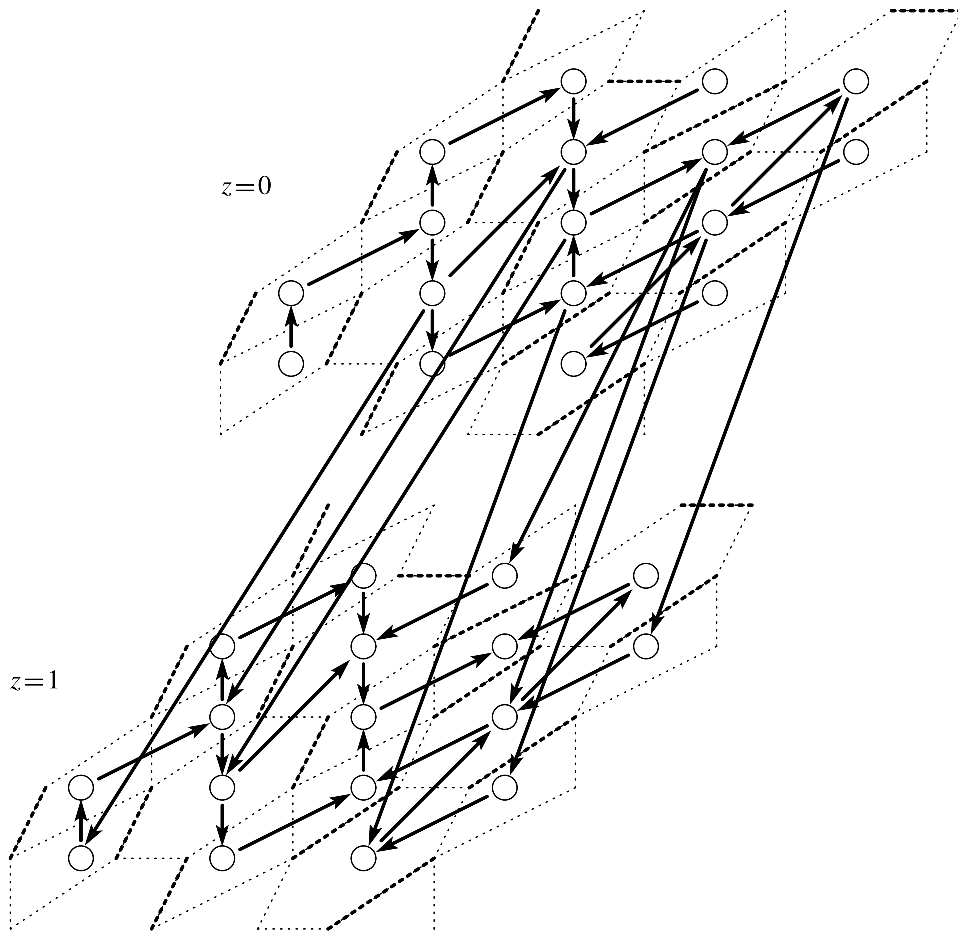


Figure 6. An example of $M(D)$.

and $r_i : V_i \to V_i$ by

$$r_i(b[D; x, y, z]) = \begin{cases} b[D; x, y+\tilde{\sigma}(j), z] & \text{if } e_{\mathrm{h}}(x, y+\tilde{\sigma}(j)/2) \notin D, \\ b[D; x, y+\tilde{\sigma}(j), z+1] & \text{if } e_{\mathrm{h}}(x, y+\tilde{\sigma}(j)/2) \in D. \end{cases}$$

3.2.2. Let $\mathscr{C} \subset \mathscr{E}$ be a subset which gives a closed zigzag curve without self-intersection. By moving along the zigzag curve clockwisely, we can determine an orientation for each element in $\mathscr{C}$. Let $\mathscr{C}^{\pm} \subset \mathscr{C}$ denote the subset of edges starting from elements in $\mathscr{V}_{\pm}$.

Let D be a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$. A subset $\mathscr{C}$ as above is said to be a *positive cycle* with respect to D if $\mathscr{C} \cap D = \mathscr{C}^{+}$, and it is said to be a *negative cycle* with respect to D if $\mathscr{C}^{-}$.

3.2.3. Given a dimer configuration D and a positive cycle $\mathscr{C}$ with respect to D, let $D_{\mathscr{C}}$ be the dimer configuration given by

$$D_{\mathscr{C}} = (D \backslash \mathscr{C}^+) \cup \mathscr{C}^-.$$

Then we can check the following lemma:

Lemma 3.2.4. *The surjection $M(D) \to M(D_{\mathscr{C}})$ given by*

$$b[D; x, y, z] \mapsto \begin{cases} 0 & \text{if } (x, y) \in \mathscr{C}^\circ \text{ and } z = 0, \\ b[D_{\mathscr{C}}; x, y, z-1] & \text{if } (x, y) \in \mathscr{C}^\circ \text{ and } z \geq 1, \\ b[D_{\mathscr{C}}; x, y, z] & \text{if } (x, y) \notin \mathscr{C}^\circ, \end{cases}$$

is a homomorphism of A-modules, where $\mathscr{C}^\circ$ is the interior of the closed zigzag curve. Moreover,

$$w_{\sigma,\lambda,\theta}(D_{\mathscr{C}}) = w_{\sigma,\lambda,\theta}(D) \cdot \prod_{f \in \mathscr{C}^\circ} w_{\sigma,\lambda,\theta}(f).$$

3.3. *Crystal melting interpretation.* In this subsection, we show that a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$ corresponds to a (torus invariant) quotient A-module of the A-module $M^{\max} = M^{\max}_{\sigma,\lambda,\nu,\theta}$. In the physicists' terminology, studying such quotient modules is called *the crystal melting model* (see [Ooguri and Yamazaki 2009]) and $M^{\max}$ is called the *grand state* of the model.

3.3.1. We define a Young diagram $G_{\sigma,\lambda,\theta} \colon \mathbb{Z} \to \mathbb{Z}$ by the following conditions:

- $G_{\sigma,\lambda,\theta}(n) = |n|$ if $|n| \gg 0$, and
- $G_{\sigma,\lambda,\theta}(n) = G_{\sigma,\lambda,\theta}(n-1) + \tilde{\sigma}(n-1/2)\tilde{\lambda}(n-1/2)$ for any n.

We define a map $G_{\sigma,\lambda,\theta} \colon \mathscr{F} \to \mathbb{Z}$ by

$$G_{\sigma,\lambda,\theta}(n, m) := G(n)_{\sigma,\lambda,\theta} + 2 \cdot |m - F_{\sigma,\lambda,\theta}(n)|, \tag{3-1}$$

where $F_{\sigma,\lambda,\theta}(n)$ is given in (2-4).

Example 3.3.2. In the case of Example 2.1.6, we have

$$(G_{\sigma,\lambda,\mathrm{id}}(n))_{n \in \mathbb{Z}} = (\ldots, 6, 5, 4, 3, 4, 3, 2, 1, 2, 3, 4, 5, 6, \ldots)$$

and $G_{\sigma,\lambda,\mathrm{id}}(n, m)$ is given in Figure 7.

3.3.3. We define two maps $F^{\pm}_{\sigma,\lambda,\theta} \colon \mathbb{Z} \to \mathbb{Z}$ by the following conditions:

- $F^{\pm}_{\sigma,\lambda,\theta}(n) = F_{\sigma,\lambda,\theta}(n)$ if $\pm n \gg 0$.
- $F^{\pm}_{\sigma,\lambda,\theta}(n) = F^{\pm}_{\sigma,\lambda,\theta}(n-1) \mp \tilde{\sigma}(n-1/2)$ for any n.

Then we define two maps $G^{\nu_\pm,\pm}_{\sigma,\lambda,\theta} \colon \mathscr{F} \to \mathbb{Z}$ by

$$G^{\nu_\pm,\pm}_{\sigma,\lambda,\theta}(n, m) := \nu_\pm(m - F^{\pm}_{\sigma,\lambda,\theta}(n)) \pm n. \tag{3-2}$$

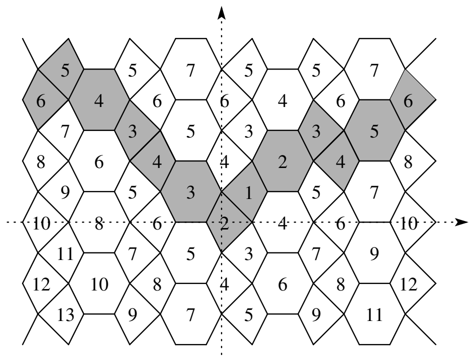
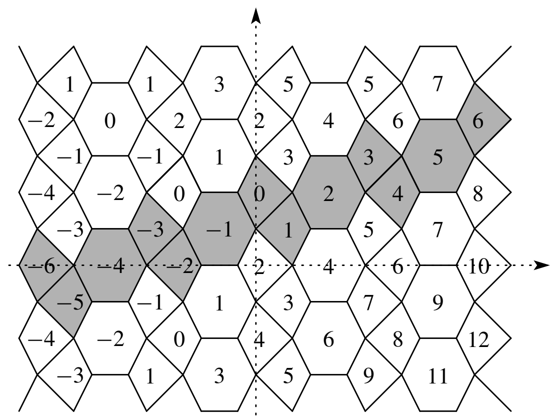


Figure 7. $G_{\sigma,\lambda,\mathrm{id}}(n, m)$.

Example 3.3.4. Figure 8 shows $G^{\varnothing,+}_{\sigma,\lambda,\mathrm{id}}$ and $G^{\square,-}_{\sigma,\lambda,\mathrm{id}}$ for σ and λ as in Example 2.1.6.



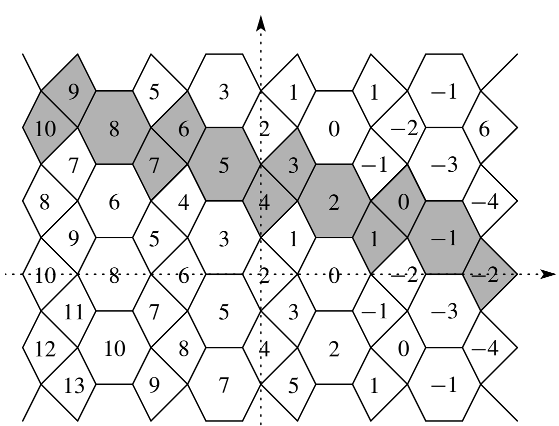


Figure 8. $G^{\varnothing,+}_{\sigma,\lambda,\theta}$ (top) and $G^{\square,-}_{\sigma,\lambda,\theta}$ (bottom).

3.3.5. We define a map $G^{\nu}_{\sigma,\lambda,\theta}\colon \mathcal{F} \to \mathbb{Z}$ by

$$G^{\nu}_{\sigma,\lambda,\theta}(n,m) := \max\bigl(G_{\sigma,\lambda,\theta}(n,m),\, G^{\nu_+,+}_{\sigma,\lambda,\theta}(n,m),\, G^{\mu_-,-}_{\sigma,\lambda,\theta}(n,m)\bigr).$$

We can verify that

$$G^{\nu}_{\sigma,\lambda,\theta}(f^+(e)) = G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) + 1 \text{ or } G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) - 3.$$

for an edge $e \in \mathcal{E}$ (see §2.1.9 for notation). We define a perfect matching $D^{\max} = D^{\max}_{\sigma,\lambda,\nu,\theta}$ by

$$e \in D^{\max} \iff G^{\nu}_{\sigma,\lambda,\theta}(f^+(e)) = G^{\nu}_{\sigma,\lambda,\theta}(f^-(e)) - 3.$$

Let $M^{\max} = M^{\max}_{\sigma,\lambda,\nu,\theta} := M(D^{\max}_{\sigma,\lambda,\nu,\theta})$ denote the corresponding A-module.

Example 3.3.6. In Figure 9, we show $G^{\vec{\varnothing}}_{\sigma,\lambda,\mathrm{id}}$ and $D^{\max}_{\sigma,\lambda,\vec{\varnothing},\mathrm{id}}$ for σ and λ as in Example 2.1.6.

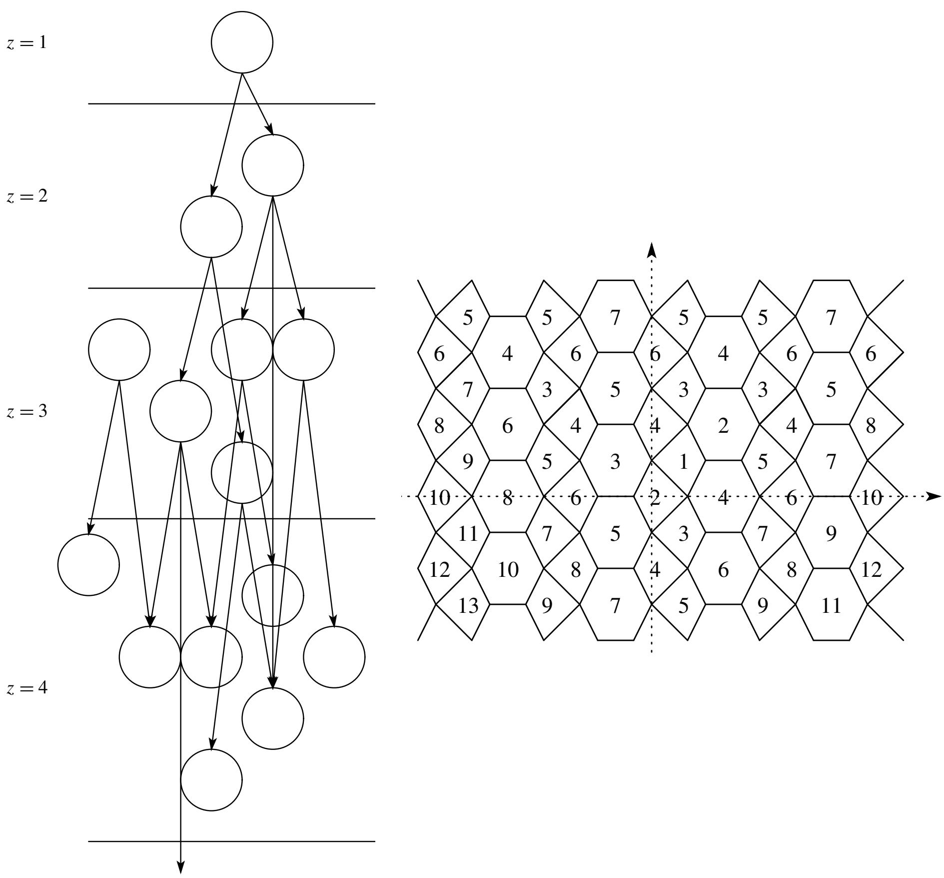


Figure 9. $G^{\vec{\varnothing}}_{\sigma,\lambda,\mathrm{id}}$ and $D^{\max}_{\sigma,\lambda,\vec{\varnothing},\mathrm{id}}$.

Remark. The graph of the map $m \mapsto G^{\nu}_{\sigma,\lambda,\theta}(n,m)$ determines a Young diagram. This is what we denote by $\mathcal{V}_{\min}(n)$ in [Nagao 2011b, §3.1].

Lemma 3.3.7. *There is no positive cycle with respect to $D^{\max}$.*

Proof. Assume that we have a positive cycle $\mathscr{C}$. For an edge $e \in \partial\mathscr{C}$, let $f_{\text{in}}(e)$ (resp. $f_{\text{out}}(e)$) be the unique face such that $e \in \partial f_{\text{in}}(e)$ and $f_{\text{in}}(e) \in \mathscr{C}^\circ$ (resp. $e \in \partial f_{\text{out}}(e)$ and $f_{\text{out}}(e) \notin \mathscr{C}^\circ$). Then we have

$$G^{\nu}_{\sigma,\lambda,\theta}(f_{\text{in}}(e)) > G^{\nu}_{\sigma,\lambda,\theta}(f_{\text{out}}(e)). \tag{3-3}$$

Take a face $(n,m) \in \mathscr{C}^\circ$. If $G^{\nu}_{\sigma,\lambda,\theta}(n,m) = G^{\nu,\pm}_{\sigma,\lambda,\theta}(n,m)$, then

$$(n \pm n',\, F^{\pm}_{\sigma,\lambda,\theta}(n \pm n') - F^{\pm}_{\sigma,\lambda,\theta}(n) + m) \in \mathscr{C}^\circ$$

for any $n' \geq 0$ by (3-2) and (3-3), and this is a contradiction. On the other hands, if $G^{\mu}_{\sigma,\lambda,\theta}(n,m) = G_{\sigma,\lambda,\theta}(n,m)$ and $\pm m \mp F_{\sigma,\lambda,\theta}(n) \geq 0$, then $(n, m \pm m') \in \mathscr{C}^\circ$ for any $m' \geq 0$ by (3-1) and (3-3), and this is also a contradiction. Hence the claim follows. □

3.3.8. For a map $H : \mathscr{F} \to \mathbb{Z}_{\geq 0}$, let $V_i^H \subset V_i(D^{\max})$ $(i \in I)$ be the subspace spanned by the elements

$$\big\{ b[D^{\max}; x, y, z] \mid (x,y) \in \mathscr{F}_i,\ z \geq H(x,y) \big\}.$$

The following proposition gives a one-to-one correspondence between dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ and finite-dimensional quotient modules of $M^{\max}_{\sigma,\lambda,\nu,\theta}$.

Proposition 3.3.9. *Given a monomial $\boldsymbol{q}$, we have a natural bijection between*

- *the set of dimer configurations of type $(\sigma,\lambda,\nu,\theta)$ with weight $\boldsymbol{q}$, and*
- *the set of maps $H : \mathscr{F} \to \mathbb{Z}_{\geq 0}$ satisfying the following conditions*:
  - *$H(f) = 0$ except for only a finite number of $f \in \mathscr{F}$,*
  - *$(V_i^H)_{i \in I}$ is stable under the action of A, and*
  - *$w_{\sigma,\lambda,\theta}(D^{\max}) \cdot \prod_f w_{\sigma,\lambda,\theta}(f)^{H(f)} = \boldsymbol{q}$.*

Proof. Let D be a dimer configuration of type $(\sigma,\lambda,\nu,\theta)$. By Lemma 3.3.7, $(D \cup D^{\max}) \backslash (D \cap D^{\max})$ is a disjoint union $\sqcup \mathscr{C}_\gamma$ of a finite number of positive cycles. We define a map $H_D : \mathscr{F} \to \mathbb{Z}_{\geq 0}$ by

$$H_D(f) := \sharp\{\mathscr{C}_\gamma \mid f \in \mathscr{C}_\gamma^\circ\}.$$

Then we can verify the claim using Lemma 3.2.4. □

Remark. The graph of the map $m \mapsto G^{\nu}_{\sigma,\lambda,\theta}(n,m) + 2H(n,m)$ determines a Young diagram. This is what we denote by $\mathcal{V}(n)$ in [Nagao 2011b, §3.1].

3.4. ***Generating function.*** From the description given by Proposition 3.3.9, we can verify that, fixing a monomial $\boldsymbol{q}$, we have only a finite number of dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ with weight $\boldsymbol{q}$.

Definition 3.4.1. We define the generating function by

$$\mathcal{Z}_{\sigma,\lambda,\nu,\theta} = \mathcal{Z}_{\sigma,\lambda,\nu,\theta}(\vec{q}) := \sum_{D} w_{\sigma,\lambda,\theta}(D),$$

where the sum is taken over all dimer configurations of type $(\sigma, \lambda, \nu, \theta)$. In particular, we put

$$\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu} := \mathcal{Z}_{\sigma,\lambda,\nu,\mathrm{id}_{\mathbb{Z}_{\mathrm{h}}}}.$$

Remark. Note that $\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu} \cdot w_{\sigma,\lambda,\theta}\big(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\big)^{-1}$ is a formal power series in q_+, q_- and $q_1, \dots, q_{L-1}$.

4. Dimer shuffling and wall-crossing formula

4.1. ***Dimer shuffling at a hexagon.*** In this and next subsections, we study the relation between dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ and of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ for $i \in I_H(\sigma, \theta)$.

4.1.1. For $(n, m) \in \mathcal{F}$ and $M \in \mathbb{Z}_{>0} \sqcup \{\infty\}$. we put

$$f(n, m; \pm, M) := \bigcup_{m'=0}^{M-1} f(n, m \pm m')$$

We define $\partial f(n, m; \pm, M)$ and $\partial^{\pm} f(n, m; \pm, M)$ in the same way as in §2.1.9 and §3.2.2.

4.1.2. For a dimer configuration D and $n \in B^{i,\pm}_{\sigma,\lambda,\theta}$, let $m(D, n)$ denote the unique integer such that

$$\partial f(n, m(D, n); \sigma(i), \infty) \cap D = \partial^{\pm} f(n, m(D, n); \sigma(i), \infty).$$

4.1.3. For a dimer configuration D and $i \in I$, we consider the following conditions:

(4-1) $\partial f \cap D \neq \partial^- f$ for any $f \in \mathcal{F}_i$,

(4-2) $\partial f \cap D \neq \partial^+ f$ if $f \in \mathcal{F}_i \backslash \{f(n, m(D, n)) \mid n \in B^{i,\pm}_{\sigma,\lambda,\theta}\}$,

(4-3) $\partial f(n, m(D, n) - 2\sigma(i)) \cap D \neq \partial^- f(n, m(D, n) - 2\sigma(i))$ for $n \in B^{i,\pm}_{\sigma,\lambda,\theta}$.

4.1.4. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1), we set

$$E_i(D^\circ) := \big\{(n, m) \in \mathcal{F}_i \mid \partial f(n, m) \cap D^\circ = \partial^+ f(n, m)\big\},$$

and define the map $M^i_{D^\circ}\colon E_i(D^\circ) \to \mathbb{Z}_{>0} \sqcup \{\infty\}$ by

$$M^i_{D^\circ}(n,m) := \max\{M \mid \partial f(n,m;\sigma(i),M) \cap D^\circ = \partial^+ f(n,m;\sigma(i),M)\}.$$

Note that

$$\bigl(M^i_{D^\circ}\bigr)^{-1}(\infty) = \bigl\{(n,m_n) \mid n \in B^{i,+}_{\sigma,\lambda,\theta}\bigr\}.$$

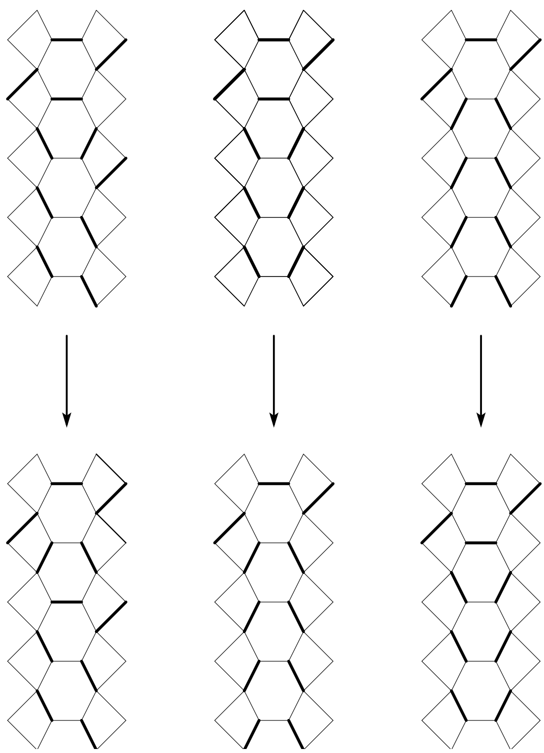
We put $E_i^{\mathrm{fin}}(D^\circ) := E_i(D^\circ)\backslash\bigl(M^i_{D^\circ}\bigr)^{-1}(\infty)$.

Definition 4.1.5. For a dimer configuration D° of type $(\sigma,\lambda,\nu,\theta)$ satisfying the condition (4-1), let $\mu_i(D^\circ)$ be the a dimer configuration of type $(\sigma,\lambda,\nu,\mu_i(\theta))$ given by

$$\Biggl(D^\circ\Bigg\backslash\Biggl(\bigcup_{(n,m)\in E_i(D^\circ)} \partial^+ f\bigl(n,m;\sigma(i),M^i_{D^\circ}(n,m)\bigr) \cup \bigcup_{n\in B^{i,-}_{\sigma,\lambda,\theta}} \partial^- f(n,m;\sigma(i),\infty)\Biggr)\Biggr)$$
$$\sqcup\Biggl(\bigcup_{(n,m)\in E_i(D^\circ)} \partial^- f\bigl(n,m;\sigma(i),M^i_{D^\circ}(n,m)\bigr) \cup \bigcup_{n\in B^{i,-}_{\sigma,\lambda,\theta}} \partial^+ f(n,m;\sigma(i),\infty)\Biggr).$$

Note that $\mu_i(D^\circ)$ satisfies the condition (4-2) and (4-3).

Example 4.1.6. Here are some examples of dimer shuffling at hexagons.



Lemma 4.1.7. $$w_{\sigma,\lambda,\mu_i(\theta)}(\mu_i(D^\circ)) = w_{\sigma,\lambda,\theta}(D^\circ).$$

Proof. For $n \in \pi^{-1}(i)$ and $m \in \mathbb{Z}$ such that $n+m$ is odd, we put

$$D^\circ(n,m) := \{e_{\mathrm{s}}(n+\varepsilon_1, m+\varepsilon_2)\ (\varepsilon_1, \varepsilon_2 = \pm 1/2)\} \cap D^\circ.$$

Assume that

$$(n,m-1),\,(n,m+1) \notin \bigcup_{(n,m)\in E_i(D^\circ)} f\big(n,m;\sigma(i), M^i_{D^\circ}(n,m)\big). \tag{4-4}$$

Then $D^\circ(n,m)$ is one of the following:

$$\varnothing, \quad \{e_{\mathrm{s}}(n\pm 1/2, m\pm 1/2)\}, \quad \{e_{\mathrm{s}}(n\pm 1/2, m\mp 1/2)\}.$$

In particular, we have

$$w_{\sigma,\lambda,\theta}(D^\circ(n,m)) = w_{\sigma,\lambda,\mu_i(\theta)}(D^\circ(n,m)).$$

Hence

$$w_{\sigma,\lambda,\theta}\big(D^\circ \cap \mu_i(D^\circ)\big) = w_{\sigma,\lambda,\mu_i(\theta)}\big(D^\circ \cap \mu_i(D^\circ)\big).$$

The claim follows from this and the fact that

$$w_{\sigma,\lambda,\theta}\big(\partial^{\pm} f(n,m,M)\big) = w_{\sigma,\lambda,\mu_i(\theta)}\big(\partial^{\mp} f(n,m,M)\big)$$

for $n \in \pi^{-1}(i)$. □

4.2. *Wall-crossing formula at a hexagon.*

Lemma 4.2.1.

$$\mathfrak{Z}_{\sigma,\lambda,\nu,\theta} = \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}} \frac{1}{1+w_{\sigma,\lambda,\theta}(n)} \prod_{(n,m)\in E^{\mathrm{fin}}_i(D^\circ)} \frac{1+w_{\sigma,\lambda,\theta}(n)^{M^i_{D^\circ}(n,m)+1}}{1+w_{\sigma,\lambda,\theta}(n)},$$

where the sum is taken over all dimer configurations D° of type $(\sigma,\lambda,\nu,\theta)$ satisfying the condition (4-1).

Proof. For a map $s\colon E_i(D^\circ) \to \mathbb{Z}_{\geq 0}$ such that $s(n,m) \leq M^i_{D^\circ}(n,m)$, we define the dimer configuration

$$D^\circ_s := \Bigg(D^\circ \Big\backslash \bigcup_{(n,m)\in E_i(D^\circ)} \partial^+ f(n,m;\sigma(i), s(n,m))\Bigg) \sqcup \bigcup_{(n,m)\in E_i(D^\circ)} \partial^- f(n,m;\sigma(i), s(n,m)).$$

Then

$$w_{\sigma,\lambda,\theta}(D^\circ_s) = w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in E_i(D^\circ)} w_{\sigma,\lambda,\theta}(n)^{s(n,m)}.$$

Note that any dimer configuration D is uniquely realized as $D^\circ(s)$ by some D° and s. Hence we have

$$\begin{aligned}\mathfrak{X}_{\sigma,\lambda,\nu,\theta} &= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ)\cdot\biggl(\sum_{s}\prod_{(n,m)\in E_i(D^\circ)} w_{\sigma,\lambda,\theta}(n)^{s(n,m)}\biggr)\\ &= \sum_{D^\circ} w_{\sigma,\lambda,\nu,\theta}(D^\circ)\prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}}\frac{1}{1-w_{\sigma,\lambda,\theta}(n)}\\ &\qquad\qquad\times\prod_{(n,m)\in E_i^{\mathrm{fin}}(D^\circ)}\frac{1+w_{\sigma,\lambda,\theta}(n)^{M^i_{D^\circ}(n,m)+1}}{1+w_{\sigma,\lambda,\theta}(n)}. \qquad \square\end{aligned}$$

Theorem 4.2.2.

$$\mathfrak{X}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \mathfrak{X}_{\sigma,\lambda,\nu,\theta}\prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}}\bigl(1-w_{\sigma,\lambda,\theta}(n)\bigr)\prod_{n\in B^{i,-}_{\sigma,\lambda,\theta}}\frac{1}{1-w_{\sigma,\lambda,\theta}(n)}.$$

Proof. As Lemma 4.2.1, we get

$$\begin{aligned}\mathfrak{X}_{\sigma,\lambda,\nu,\mu_i(\theta)} &= \sum_{D^\bullet} w_{\sigma,\lambda,\mu_i(\theta)}(D^\bullet)\prod_{n\in B^{i,+}_{\sigma,\lambda,\mu_i(\theta)}}\frac{1}{1-w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}}\\ &\qquad\qquad\times\prod_{(n,m)\in \check{E}_i(D^\bullet)}\frac{1+w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-\check{M}^i_{D^\bullet}(n,m)-1}}{1+w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}},\end{aligned}$$

where the sum is taken over all dimer configurations $D^\bullet$ of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying (4-2), (4-3), and

$$\begin{aligned}\check{E}_i(D^\bullet) &:= \bigl\{(n,m)\in\mathcal{F}_i \bigm| \partial f(n,m)\cap D^\bullet = \partial^- f(n,m)\bigr\},\\ \check{M}^i_{D^\bullet}(n,m) &:= \max\bigl\{M \bigm| \partial f(n,m;\sigma(i),M)\cap D^\bullet = \partial^- f(n,m;-\sigma(i),M)\bigr\}.\end{aligned}$$

Note that μ_i gives a one-to-one correspondence between dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ satisfying (4-1) and those of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying (4-2) and (4-3). Hence the claim follows from

- $B^{i,\pm}_{\sigma,\lambda,\mu_i(\theta)} = B^{i,\mp}_{\sigma,\lambda,\theta}$,
- $w_{\sigma,\lambda,\mu_i(\theta)}(n) = w_{\sigma,\lambda,\theta}(n)^{-1}$ for $n\in\pi^{-1}(i)$,
- $(n,m)\mapsto(n, m+\sigma(i)\cdot(M^i_{D^\circ}(n,m)-1))$ gives a bijection between $E_i^{\mathrm{fin}}(D^\circ)$ and $\check{E}_i(\mu_i(D^\circ))$ which respects $M^i_{D^\circ}$ and $\check{M}^i_{\mu_i(D^\circ)}$,

and Lemma 4.2.1. $\square$

4.3. ***Dimer shuffling at a quadrilateral.*** In this subsection, we study the relation between dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ and of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ for $i\in I_S(\sigma,\theta)$.

4.3.1. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1) and $n \in \pi^{-1}(i)$, we define

$$E^1_n(D^\circ) := \{(n, m) \in \mathcal{F} \mid \partial f(n, m) \cap D^\circ = \partial^+ f(n, m)\},$$
$$E^2_n(D^\circ) := \{(n, m) \in \mathcal{F} \mid \partial f(n, m) \cap D^\circ = \varnothing\}.$$

Lemma 4.3.2. $|E^1_n(D^\circ)| - |E^2_n(D^\circ)| = \begin{cases} \mp 1 & \text{if } n \in B^{i,\pm}_{\sigma,\lambda,\theta}, \\ 0 & \text{otherwise}. \end{cases}$

(See (1-4) for notation.)

Proof. For $n, m \in \mathbb{Z}$ such that $n + m$ is odd, we define $\varepsilon_{D^\circ}(n, m)$ by

$$\varepsilon_{D^\circ}(n, m) := \begin{cases} + & \text{if } e_{\mathrm{s}}(n + 1/2, m + 1/2), e_{\mathrm{s}}(n - 1/2, m + 1/2) \notin D, \\ - & \text{if } e_{\mathrm{s}}(n + 1/2, m - 1/2), e_{\mathrm{s}}(n - 1/2, m - 1/2) \notin D. \end{cases}$$

Then for $(n, m) \in \mathcal{F}$, we have

$$(n, m) \in E^1_n(D^\circ) \iff \varepsilon_{D^\circ}(n, m \pm 1) = \pm,$$
$$(n, m) \in E^2_n(D^\circ) \iff \varepsilon_{D^\circ}(n, m \pm 1) = \mp,$$

and $\varepsilon_{D^\circ}(n, m) = \mp\tilde{\lambda}(n \pm 1/2)$ if $\tilde{\sigma}(n \pm 1/2) \cdot m \gg 0$. The claim follows. $\square$

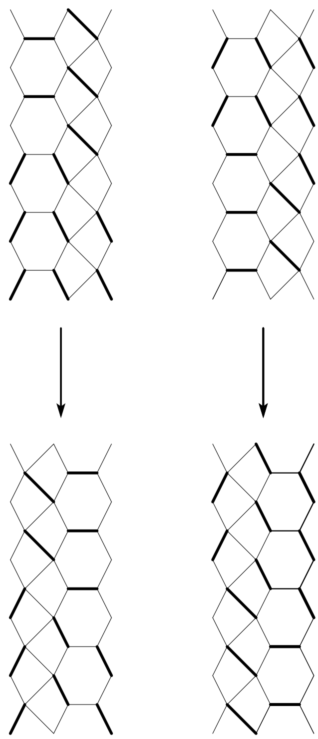
4.3.3. For a dimer configuration D° of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1), we define a dimer configuration $\mu_i(D^\circ)$ of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ as follows:

- If $\pi(h) \neq i \pm 1/2$, we have
$$e_{\mathrm{s}}(h, k) \in D^\circ \iff e_{\mathrm{s}}(h, k) \in \mu_i(D^\circ),$$
- If $n \in I_H(\sigma, \theta)$ and $\pi(n) \neq i \pm 1$, we have
$$e_{\mathrm{h}}(n, m) \in D^\circ \iff e_{\mathrm{h}}(n, m) \in \mu_i(D^\circ),$$
- For $(n, m) \in \mathcal{F}_i$ we have
$$D^\circ(f(n, m)) = \varnothing \iff \mu_i(D^\circ)(f(n, m)) = \partial^-_{\sigma,\mu_i(\theta)}(f(n, m)),$$
$$D^\circ(f(n, m)) = \partial^+_{\sigma,\theta}(f(n, m)) \iff \mu_i(D^\circ)(f(n, m)) = \varnothing,$$
(Here we use notation such as $\partial^\pm_{\sigma,\theta}(f(n, m))$ in order to emphasize that the notions like $\partial^\pm(f(n, m))$ given in §2.1.9 depend on σ and θ.)
- If $D^\circ(f(n, m)) \neq \varnothing, \partial^+_{\sigma,\theta}(f(n, m))$ for $(n, m) \in \mathcal{F}_i$, we have
$$e_{\mathrm{s}}(n + \varepsilon_1, m + \varepsilon_2) \in D^\circ \iff e_{\mathrm{s}}(n - \varepsilon_1, m - \varepsilon_2) \in \mu_i(D^\circ) \quad (\varepsilon_1, \varepsilon_2 = \pm 1/2),$$
- If $\sigma(i \pm 3/2) \neq \sigma(i \pm 1/2)$, we have
$$e_{\mathrm{s}}(n \pm 1/2, m - 1), e_{\mathrm{s}}(n \pm 1/2, m + 1) \notin D^\circ \iff e_{\mathrm{h}}(n \pm 1, m) \in \mu_i(D^\circ).$$

Note that $\mu_i(D^\circ)$ satisfies the condition

$$D(f) \neq \partial^+ f \text{ for any } f \in \mathcal{F}_i. \tag{4-5}$$

Example 4.3.4. Here are some examples of dimer shuffling at squares.



Lemma 4.3.5. $$w_{\sigma,\lambda,\mu_i(\theta)}(\mu_i(D^\circ)) = w_{\sigma,\lambda,\theta}(D^\circ).$$

Proof. We have $w_{\sigma,\lambda,\theta}(\partial^+_{\sigma,\theta} f) = w_{\sigma,\lambda,\mu_i(\theta)}(\partial^-_{\sigma,\mu_i(\theta)} f)$ for $f \in \mathcal{F}_i$, and

$$w_{\sigma,\lambda,\theta}(\partial^+_{\sigma,\theta} f) = \begin{cases} 1 & \text{if } n \in B^{i,+}_{\sigma,\lambda,\theta}, \\ w_{\sigma,\lambda,\theta}(n)^{-1} & \text{if } n \in B^{i,-}_{\sigma,\lambda,\theta}. \end{cases}$$

Thus, the claim follows from Lemma 4.3.2 and (2-9). □

4.4. *Wall-crossing formula at a quadrilateral.*

Lemma 4.4.1. $\mathfrak{X}_{\sigma,\lambda,\nu,\theta} = \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \cdot \prod_{n\in\pi^{-1}(i)} \bigl(1 + w_{\sigma,\lambda,\theta}(n)\bigr)^{|E^1_n(D^\circ)|}$.

Proof. We set

$$E^1_i(D^\circ) := \bigcup_{n\in\pi^{-1}(i)} E^1_n(D^\circ), \quad E^2_i(D^\circ) := \bigcup_{n\in\pi^{-1}(i)} E^2_n(D^\circ).$$

Given a subset $S \subset E_i^1(D^\circ)$, we get a dimer configuration D_S° of type $(\sigma, \lambda, \nu, \theta)$ such that

$$D_S^\circ := \big(D \backslash \bigcup \partial^+ f\big) \cup \bigcup \partial^+ f,$$

and we have

$$w_{\sigma,\lambda,\theta}(D_S^\circ) = w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in S} w_{\sigma,\lambda,\theta}(n).$$

Note that any dimer configuration D is uniquely realized as D_S° by some D° and S. Hence we have

$$\begin{aligned}
\mathscr{Z}_{\sigma,\lambda,\nu,\theta} &= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ)\Bigg(\sum_S \prod_{(n,m)\in S} w_{\sigma,\lambda,\theta}(n)\Bigg)\\
&= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{(n,m)\in E_i^1(D^\circ)} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)\\
&= \sum_{D^\circ} w_{\sigma,\lambda,\theta}(D^\circ) \prod_{n\in\pi^{-1}(i)} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)^{|E_n^1(D^\circ)|}. \qquad \square
\end{aligned}$$

Theorem 4.4.2.

$$\mathscr{Z}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \mathscr{Z}_{\sigma,\lambda,\nu,\theta} \prod_{n\in B^{i,+}_{\sigma,\lambda,\theta}} \big(1 + w_{\sigma,\lambda,\theta}(n)\big)^{-1} \prod_{n\in B^{i,-}_{\sigma,\lambda,\theta}} \big(1 + w_{\sigma,\lambda,\theta}(n)\big).$$

Proof. Let $D^\bullet$ be a dimer configuration of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying (4-5). We put

$$\tilde{E}_n^1(D^\bullet) := \big\{(n, m) \in \mathscr{F} \;\big|\; \partial_{\sigma,\mu_i(\theta)} f(n, m) \cap D^\bullet = \partial^-_{\sigma,\mu_i(\theta)} f(n, m)\big\}.$$

Then, as Lemma 4.4.1, we get

$$\mathscr{Z}_{\sigma,\lambda,\nu,\mu_i(\theta)} = \sum_{D^\bullet} w_{\sigma,\lambda,\mu_i(\theta)}(D^\bullet) \prod_{n\in\pi^{-1}(i)} \big(1 + w_{\sigma,\lambda,\mu_i(\theta)}(n)^{-1}\big)^{|\tilde{E}_n^1(D^\bullet)|},$$

where the sum is taken over all dimer configurations $D^\bullet$ of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying the condition (4-5). Note that μ_i gives a one-to-one correspondence of dimer configurations of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition (4-1) and ones of type $(\sigma, \lambda, \nu, \mu_i(\theta))$ satisfying the condition (4-5). Hence the claim follows from the equalities $\tilde{E}_n^1(\mu_i(D^\circ)) = E_n^2(D^\circ)$ and $w_{\sigma,\lambda,\mu_i(\theta)}(n) = w_{\sigma,\lambda,\theta}(n)^{-1}$, both valid for $n \in \pi^{-1}(i)$, together with Lemma 4.4.1. $\square$

4.5. *Conclusion.* For σ and $\alpha \in \Lambda^{\mathrm{re},+}$, we put

$$\sigma(\alpha) := \sigma(j^-(\alpha)) \cdot \sigma(j^+(\alpha)). \tag{4-6}$$

Combining Theorem 4.2.2 and 4.4.2, we get:

Theorem 4.5.1. *$\mathcal{Z}_{\sigma,\lambda,\nu,\theta}$ has the value*

$$\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu} \prod_{\alpha\in\Lambda^{\mathrm{re},+}_{\theta}} \left(\prod_{(h,h')\in B^{\alpha,+}_{\sigma,\lambda}} \Bigl(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Bigr)^{\sigma(\alpha)} \prod_{(h,h')\in B^{\alpha,-}_{\sigma,\lambda}} \Bigl(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\Bigr)^{-\sigma(\alpha)} \right).$$

(See (1-2) and (1-3) for notation.)

Since the second term in this expression does not depend on ν, we have:

Corollary 4.5.2.
$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu,\theta}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing},\theta}} = \frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\vec{\varnothing}}}.$$

Lemma 1.3.3 and Theorem 4.5.1 yield:

Theorem 4.5.3. (See (1-2) for notation.)

$$\begin{aligned}\mathcal{Z}_{\sigma,\lambda,\nu,\theta}\Big|_{q_+=q_-=(q_0)^{1/2}} \\ = \mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}} \prod_{\alpha\in\Lambda^{\mathrm{re},+}_{\theta}} \bigl(1-\sigma(\alpha)\cdot q^{\alpha}\bigr)^{\sigma(\alpha)[\alpha^0+c_\lambda(j_-(\alpha))-c_\lambda(j_+(\alpha))]}.\end{aligned}$$

Since the second term on the right depends only on the $c_\lambda[j]$ and not on λ and ν, we have:

Corollary 4.5.4. *If $c_\lambda[j]=0$ for any j, we have*

$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu,\theta}}{\mathcal{Z}_{\sigma,\vec{\varnothing},\vec{\varnothing},\theta}\Big|_{q_+=q_-}} = \frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}}.$$

5. Refined topological vertex via dimer model

5.1. *Refined topological vertex for $\mathbb{C}^3$.*

5.1.1. A Young diagram can be regarded as a subset of $(\mathbb{Z}_{\geq 0})^2$. For a Young diagram λ, let

$$\begin{aligned}\Lambda^{\mathrm{x}}(\lambda) &= \bigl\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3 \mid (y,z)\in\lambda\bigr\},\\ \Lambda^{\mathrm{y}}(\lambda) &= \bigl\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3 \mid (z,x)\in\lambda\bigr\},\\ \Lambda^{\mathrm{z}}(\lambda) &= \bigl\{(x,y,z)\in(\mathbb{Z}_{\geq 0})^3 \mid (x,y)\in\lambda\bigr\}.\end{aligned}$$

5.1.2. Given a triple $(\lambda_x,\lambda_y,\lambda_z)$ of Young diagrams, define

$$\Lambda^{\min} := \Lambda^{\mathrm{x}}(\lambda_x)\cup\Lambda^{\mathrm{y}}(\lambda_y)\cup\Lambda^{\mathrm{z}}(\lambda_z)\subset(\mathbb{Z}_{\geq 0})^3.$$

5.1.3. A subset Λ of $(\mathbb{Z}_{\geq 0})^3$ is said to be a 3-*dimensional Young diagram of type* $(\lambda_x, \lambda_y, \lambda_z)$ if the following conditions are satisfied:

- If $(x, y, z) \notin \Lambda$, then $(x+1, y, z),\ (x, y+1, z),\ (x, y, z+1) \notin \Lambda$.
- $\Lambda \supset \Lambda^{\min}$.
- $|\Lambda \backslash \Lambda^{\min}| < \infty$.

5.1.4. For a Young diagram λ, we define a monomial $w_\lambda(m)$ for each $m \in \mathbb{Z}$ by

$$w_\lambda(m) = q_{\lambda(m-1/2)} \cdot q_{\lambda(m+1/2)} \cdot q_1 \cdot \dots \cdot q_{L-1}. \tag{5-1}$$

For a finite subset S of $(\mathbb{Z}_{\geq 0})^3$ we define the weight $w(S)$ by

$$w(S) := \prod_{(x,y,z)\in S} w_{\lambda_x}(y-z).$$

For a positive integer N, we set $C_N := [0, N]^3$. Given a 3-dimensional Young diagram Λ of type $(\lambda_x, \lambda_y, \lambda_z)$, we take a sufficiently large N such that $\Lambda \backslash \Lambda^{\min} \subset C_N$ and define the weight $w(\Lambda)$ of Λ by

$$w(\Lambda) := \frac{w(\Lambda \cap C_N)}{w(\Lambda^{\mathrm{x}}(\lambda_x) \cap C_N) w(\Lambda^{\mathrm{y}}(\lambda_y) \cap C_N)\, w(\Lambda^{\mathrm{z}}(\lambda_z) \cap C_N)}.$$

Note that this is well-defined.

Remarks. • In the definition of $w(\Lambda)$, the three axes do not play the same role. The x-axis is called the *preferred axis* for the refined topological vertex.

- If we replace the definition (5-1) with

  $$(q_{\lambda(m-1/2)})^2 \cdot q_1 \cdot \dots \cdot q_{L-1},$$

  then the weight coincides with the one in [Iqbal et al. 2009]. Our weight coincides with the one in [Dimofte and Gukov 2010].

We define the generating function

$$G_{\lambda_x,\lambda_y,\lambda_z}(\vec{q}\,) := \sum w(\Lambda),$$

where the sum is taken over all 3-dimensional Young diagrams of type $(\lambda_x, \lambda_y, \lambda_z)$.

5.2. ***Dimer model for $L = 1$.*** In the case $L = 1$, the graph in §2.1.2 gives a hexagon lattice. As we have only two choices of σ, we put $\sigma(1/2) = +$. We take id as θ. We omit σ and id from the notation in this subsection. Note that λ is a single 2-dimensional Young diagram.

It is well-known that giving a dimer configuration of type (λ, ν) is equivalent to giving a 3-dimensional Young diagram of type $(\lambda, \nu_+, {}^{\mathrm{t}}\nu_-)$. Let $D(\Lambda)$ be the dimer configuration corresponding to a 3-dimensional Young diagram Λ.

For a Young diagram $\eta = (\eta_{(1)}, \eta_{(2)}, \dots)$ and a monomial p, we put

$$w(\eta;\, p, Q) := \prod \big(pQ^{i-1}\big)^{\eta_{(i)}}.$$

Then we can verify the following:

$$w_\lambda(D(\Lambda)) = w(\nu_-;\, q_+, Q_+)\, w(\nu_+;\, q_-, Q_-)\, w(\Lambda). \tag{5-2}$$

Example 5.2.1. As we show in Figure 10, we have

$$\begin{aligned} w_\varnothing(\Lambda^{\min}_{\varnothing,(1),\varnothing}) &= w((1);\, q_-, Q_-) &&= q_-, \\ w_\varnothing(\Lambda^{\min}_{\varnothing,(2),\varnothing}) &= w((2);\, q_-, Q_-) &&= q_-^2, \\ w_\varnothing(\Lambda^{\min}_{\varnothing,(1,2),\varnothing}) &= w((2,1);\, q_-, Q_-) &&= q_-^3 Q_-. \end{aligned}$$

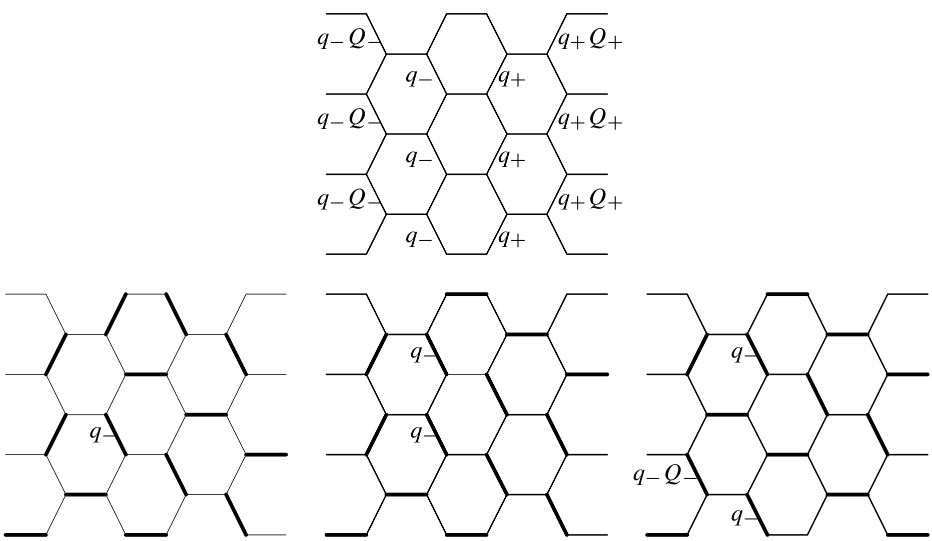


Figure 10. $D(\Lambda^{\min}_{\varnothing,(1),\varnothing})$, $D(\Lambda^{\min}_{\varnothing,(2),\varnothing})$ and $D(\Lambda^{\min}_{\varnothing,(1,2),\varnothing})$.

In particular, we have

$$\mathcal{Z}_{\lambda,\nu} = w(\nu_-;\, q_+, Q_+) \cdot w(\nu_+;\, q_-, Q_-) \cdot G_{\lambda,\nu_+,{}^{\mathrm{t}}\nu_-},$$

where $\mathcal{Z}_{\lambda,\nu}$ is the generating function given in Definition 3.4.1.

5.3. ***Refined topological vertex for a small resolution.*** We will define generating functions $\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}(\vec{q}\,)$. First, we consider the following data: let $\vec{\nu} = (\nu^{(1)}, \dots, \nu^{(L-1)})$ be an $(L-1)$-tuple of Young diagrams and $\vec{\Lambda} = (\Lambda^{(1/2)}, \dots, \Lambda^{(L-1/2)})$ be an L-tuple of 3-dimensional Young diagrams such that $\Lambda^{(j)}$ is

- of type $(\lambda^{(j)}, \nu^{(j+1/2)}, {}^{\mathrm{t}}\nu^{(j-1/2)})$ if $\sigma(j) = +$,
- of type $(\lambda^{(j)}, {}^{\mathrm{t}}\nu^{(j-1/2)}, \nu^{(j+1/2)})$ if $\sigma(j) = -$,

where we put $\nu^{(0)} := \nu_-$ and $\nu^{(L)} := \nu_+$. We say that the data $(\vec{\Lambda}, \vec{\nu})$ is of type (σ, λ, ν). We define the weight $w(\vec{\Lambda}, \vec{\nu})$ of the data $(\vec{\Lambda}, \vec{\nu})$ by

$$w_\sigma(\vec{\Lambda}, \vec{\nu}) := w(\nu_+; q_-, Q_-) \cdot w(\nu_-; q_+, Q_+) \Bigg(\prod_{j=1/2}^{L-1/2} w(\Lambda^{(j)}) \Bigg) \Bigg(\prod_{i=1}^{L-1} w_\sigma^i(\mu^{(i)}) \Bigg),$$

where $w_\sigma^i(\mu^{(i)})$ is given by

$$w_\sigma^i(\mu^{(i)}) := \prod_{(\alpha,\beta)\in\mu^i} \begin{cases} q_i \cdot Q^{2\alpha+1} & \text{if } \sigma(i-\frac{1}{2}) = \sigma(i+\frac{1}{2}) = +, \\ q_i \cdot Q^{2\beta+1} & \text{if } \sigma(i-\frac{1}{2}) = \sigma(i+\frac{1}{2}) = -, \\ q_i \cdot Q \cdot Q_+^\alpha \cdot Q_-^\beta & \text{if } \sigma(i-\frac{1}{2}) = +,\ \ \sigma(i+\frac{1}{2}) = -, \\ q_i \cdot Q \cdot Q_-^\alpha \cdot Q_+^\beta & \text{if } \sigma(i-\frac{1}{2}) = -,\ \ \sigma(i+\frac{1}{2}) = +. \end{cases} \tag{5-3}$$

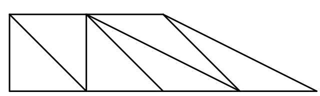
We consider the generating function

$$\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\mu}(\vec{q}\,) := \sum w_\sigma(\vec{\Lambda}, \vec{\nu})$$

where the sum is taken over all the data as above.

Remark. This is the generating function of the refined topological vertex associated to Y_σ, where $Y_\sigma \to X$ is the crepant resolution constructed from σ (see [Nagao 2011a, §1.1] for the construction of Y_σ). Here is the polygon corresponding to Y_σ, for σ given by

$$(\sigma(1/2), \ldots, \sigma(11/2)) = (+, -, +, +, -, +):$$



5.4. *Limit behavior of the dimer model.*

5.4.1. Let $\boldsymbol{i} \in I^{\mathbb{Z}_{>0}}$ be a minimal expression such that for any $N \in \mathbb{Z}_{\geq 0}$ we have $b(N) \in \mathbb{Z}_{>0}$ such that $\alpha_{\boldsymbol{i},b} > N\delta$ for any $b > b(N)$.

Lemma 5.4.2. *Given σ, λ and a monomial $\boldsymbol{q}$, there exists an integer B_1 such that the following condition holds: for any $b \geq B_1$,*

- *any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$ satisfies* (4-1),
- *any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b+1})$ with weight $\boldsymbol{q}$ satisfies* (4-2), *and*
- *μ_{i_b} gives a one-to-one correspondence between dimer configurations of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b+1})$ with weight $\boldsymbol{q}$.*

Proof. Take N_2 such that

$$q^{N_2\delta} > \boldsymbol{q} \cdot w_{\sigma,\lambda,\theta}\bigl(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\bigr)^{-1}.$$

By Theorem 4.5.3 and the remark just before Section 4,

$$\mathscr{Z}_{\sigma,\lambda,\nu,\theta} \cdot w_{\sigma,\lambda,\theta}\bigl(D^{\max}_{\sigma,\lambda,\nu.\mathrm{id}}\bigr)^{-1}\big|_{q_+=q_-=(q_0)^{1/2}}$$

is a polynomial in $q_0, \ldots, q_{L-1}$. Thus, there does not exist any dimer configuration with weight $\boldsymbol{q}-\alpha(\boldsymbol{i}, b)$ for any $b > b(N_2) =: B_1$, where $b(N_2)$ is taken as in §5.4.1.

Assume that we have a dimer configuration type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$ and $f \in \mathscr{F}$ such that $D(f) = \partial^-(f)$. Then we get a dimer configuration $D \cup \partial^+(f)\backslash\partial^-(f)$ with weight $\boldsymbol{q}-\alpha(\theta, i)$, which is a contradiction. We can check the second claim similarly and the third claim immediately follows from the first and second ones. □

5.4.3. Given σ, λ, we can take an integer N_2 such that

- $\tilde{\sigma}(h) = \pm\tilde{\lambda}(h)$ for any $h \in \mathbb{Z}_{\mathrm{h}}$ such that $\pm h > N_2 L$,
- $e^{\mathrm{s}}(h, k) \notin D^{\max}_{\sigma,\lambda,\theta_{i,B_1}}$ for any h and k such that $h < N_2 L$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and
- $e^{\mathrm{s}}(h, k) \notin D^{\max}_{\sigma,\lambda,\theta_{i,B_1}}$ for any h and k such that $h > N_2 L$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Take a monomial $\boldsymbol{q}$. Since we have only a finite number of dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},B_1})$ with weight $\boldsymbol{q}$ and each dimer configuration has only finite difference with $D^{\max}_{\sigma,\lambda,\nu,\theta_{i,B_1}}$, we can take an integer N_4 such that

- $\tilde{\sigma}(h) = \pm\tilde{\lambda}(h)$ for any $h \in \mathbb{Z}_{\mathrm{h}}$ such that $\pm h > L N_4$,
- $e^{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < L N_4$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and
- $e^{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h > L N_4$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Lemma 5.4.4. *Let D be a dimer configuration of type $(\sigma, \lambda, \nu, \theta)$ satisfying the condition* (4-1). *Take $h \in \pi^{-1}(i + 1/2)$ such that $\tilde{\sigma}(h) = \tilde{\lambda}(h)$ and assume that $e_{\mathrm{s}}(h, k) \notin D$ for any $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is odd. Then $e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h))$ is not in $\mu_i(D)$.*

Similarly, take $h \in \pi^{-1}(i + 1/2)$ such that $\tilde{\sigma}(h) = -\tilde{\lambda}(h)$ and assume that $e_{\mathrm{s}}(h, k) \notin D$ for any $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is even. Then $e_{\mathrm{s}}(h+1, k+\tilde{\sigma}(h))$ is not in $\mu_i(D)$.

Proof. In the case $i \in I_S$, for any $h, k \in \mathbb{Z}_{\mathrm{h}}$ such that $\tilde{\sigma}(h) = \tilde{\lambda}(h)$ and $h\tilde{\sigma}(h) - k$ is odd, we can verify

$$e_{\mathrm{s}}(h, k) \notin D \Longrightarrow e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h)) \notin \mu_i(D)$$

from the definition of $\mu_i(D)$ in §4.3.3.

In the case $i \in I_S$, assume we have $k \in \mathbb{Z}_{\mathrm{h}}$ such that $h\tilde{\sigma}(h) - k$ is odd and $e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h)) \in \mu_i(D)$. From Definition 4.1.5, we have $e_{\mathrm{s}}(h-1, k-\tilde{\sigma}(h)) \in D$. Since $e_{\mathrm{s}}(h, k-2\tilde{\sigma}(h)) \notin D$, we have $e_{\mathrm{s}}(h, k-\tilde{\sigma}(h)) \in D$. Then, since $\tilde{\sigma}(h) = \tilde{\lambda}(h)$, there exists m such that $\sigma(i)(m-k) > 0$ and $\partial f(h-1/2, m) \cap D = \partial^- f(h-1/2, m)$, which is a contradiction. $\square$

5.4.5. Given σ, λ and a monomial $\boldsymbol{q}$, take B_1 and N_4 as in Lemma 5.4.2 and §5.4.3. By the definition of N_4 and Lemma 5.4.4, we have the following lemma:

Lemma 5.4.6. *For any $b \geq B_1$ and any dimer configuration of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weight $\boldsymbol{q}$, we have*

- *$e_{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < \theta_{\boldsymbol{i},b}^{-1}(\pi(h)) - 2LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is even, and*
- *$e_{\mathrm{s}}(h, k) \notin D$ for any h and k such that $h < \theta_{\boldsymbol{i},b}^{-1}(\pi(h)) + 2LN_4$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.*

5.4.7. We assume that

$$\theta_{\boldsymbol{i},b}^{-1}(1/2) < \theta_{\boldsymbol{i},b}^{-1}(3/2) < \cdots < \theta_{\boldsymbol{i},b}^{-1}(L - 1/2)$$

for any $b > 0$.

Given σ, λ and a monomial $\boldsymbol{q}$, take B_5 such that $B_5 > b(2N_4)$ and $B_5 > B_1$. The following theorem is the main result of this section:

Theorem 5.4.8. *For any $b > B_5$, we have a bijection between*

- *the set of dimer configurations of type $(\sigma, \lambda, \nu, \theta_{\boldsymbol{i},b})$ with weights $\boldsymbol{q}$, and*
- *the set of data $(\vec{\Lambda}, \vec{\nu})$ as in Section 5.3 of type (σ, λ, ν) with weights $\boldsymbol{q}$.*

Proof. First, we divide the (x, y)-plane into the following $2L + 1$ areas:

$$C_j := \{\theta^{-1}(j) - 2LN_4 < x < \theta^{-1}(j) + 2LN_4\} \quad (j \in I_{\mathrm{h}}),$$
$$C_0 := \{x < \theta^{-1}(1/2) - 2LN_4\},$$
$$C_i := \{\theta^{-1}(i - 1/2) + 2LN_4 < x < \theta^{-1}(i + 1/2) - 2LN_4\} \quad (1 \leq i \leq L - 1),$$
$$C_L := \{\theta^{-1}(L - 1/2) + 2LN_4 < x\}.$$

By Lemma 5.4.6, in the area C_j we have

- $e^{\mathrm{s}}[h, k] \notin D$ for any h and k such that $\pi(h) > j$ and $h \cdot \tilde{\sigma}(h) - k$ is even;
- $e^{\mathrm{s}}[h, k] \notin D$ for any h and k such that $\pi(h) < j$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Removing these edges, we get a new graph. A face of the new graph is a union of L-tuple of elements in $\mathcal{F}$. If we regard such a union as a hexagon, the dimer configuration D gives a dimer configuration for the hexagon lattice — in other words, a three-dimensional diagram. Let $\Lambda^{(j)}$ denote this three-dimensional diagram. (See Example 5.4.9.)

Similarly, in the area C_j we have

- $e^{\mathrm{s}}[h,k] \notin D$ for any h and k such that $\pi(h) > i$ and $h \cdot \tilde{\sigma}(h) - k$ is even;
- $e^{\mathrm{s}}[h,k] \notin D$ for any h and k such that $\pi(h) < i$ and $h \cdot \tilde{\sigma}(h) - k$ is odd.

Removing these edges, we get a new graph, which is an infinite disjoint union of zigzag paths. For each zigzag path, we have two choices of perfect matching and so the dimer configuration D gives a Young diagram $\nu^{(i)}$. We can verify that the datum $(\vec{\Lambda}, \vec{\nu})$ satisfies the conditions in Section 5.3. Note that the reverse construction also works.

We have to check the correspondence above respects the weights. Note that all edges of in the area C_i have weights $= 1$. By (5-2), the contribution of the part in the area C_j is given by

$$w\big(\,\nu^{(j-1/2)}; q_+^{(s_i(j))}, Q_+\big)\, w\big(\,\nu^{(j+1/2)}; \big(q_+^{(s_i(j))}\big)^{-1} Q, Q_-\big)\, w(\Lambda^{(j)}) \text{if } \sigma(j) = +,$$
$$w\big({}^{\mathrm{t}}\nu^{(j-1/2)}; q_+^{(s_i(j))}, Q_+\big)\, w\big({}^{\mathrm{t}}\nu^{(j+1/2)}; \big(q_+^{(s_i(j))}\big)^{-1} Q, Q_-\big)\, w(\Lambda^{(j)}) \text{if } \sigma(j) = -.$$

Combining these contributions, we get the claim. □

Example 5.4.9. We take σ as in Example 2.1.3 and $\lambda = \varnothing$. Assume that $\theta(1/2) = N + 1/2$ and $\theta(5/2) = -N + 5/2$ for $N \gg 0$. In Figure 11, we show the weight (after putting $q_+ = q_- = q_0^{1/2}$) of edges in the area $C_{1/2}$. We can idenfity the graph in the area $C_{1/2}$ with a hexagon lattice as shown in Figure 12.

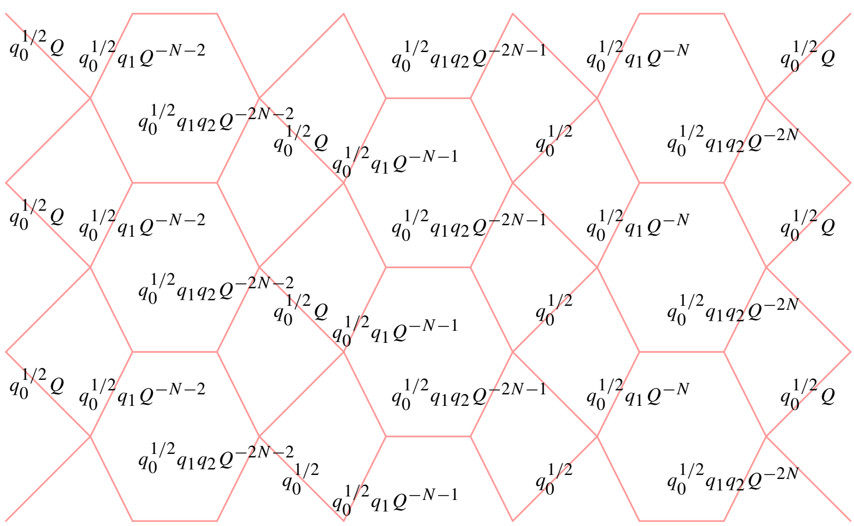


Figure 11. The graph in the area $C_{1/2}$.

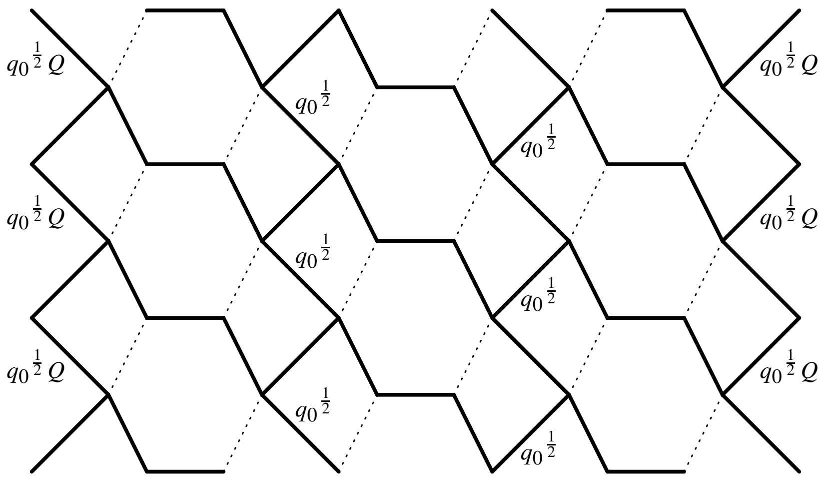


Figure 12. Identification with a hexagon lattice.

Remark. In general, we have the permutation $s_i \in \mathfrak{S}_{I_\mathrm{h}}$ of the set I_h satisfying the following condition: for sufficiently large b we have

$$\theta_{i,b}^{-1}(s_i(1/2)) < \theta_{i,b}^{-1}(s_i(3/2)) < \cdots < \theta_{i,b}^{-1}(s_i(L-1/2)).$$

The permutation s_i determines the direnction in which we take limit in the space of stability conditions. It is the refine topological vertex associated to $Y_{\sigma\circ s_i}$ what we get in the limit.

5.5. *Conclusion.* Note that

$$\bigcup_{b=1}^{\infty} \Lambda_{\theta_{i,b}}^{\mathrm{re},+} = \Lambda_{+}^{\mathrm{re},+}.$$

Combining Theorem 4.5.1 and Theorem 5.4.8, we have:

Theorem 5.5.1. *$\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{RTV}}$ has the value*

$$\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{NCDT}} \prod_{\alpha\in\Lambda_{+}^{\mathrm{re},+}} \left(\prod_{(h,h')\in B_{\sigma,\lambda}^{\alpha,+}} \left(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\right)^{\sigma(\alpha)} \prod_{(h,h')\in B_{\sigma,\lambda}^{\alpha,-}} \left(1-\sigma(\alpha)\frac{w_\lambda(h')}{w_\lambda(h)}\right)^{-\sigma(\alpha)} \right).$$

(See (1-1), (1-3) and (4-6) for notation.)

Since the second term in this expression does not depend on ν, we have:

Corollary 5.5.2.
$$\frac{\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{RTV}}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing}}^{\mathrm{RTV}}} = \frac{\mathcal{Z}_{\sigma,\lambda,\nu}^{\mathrm{NCDT}}}{\mathcal{Z}_{\sigma,\lambda,\vec{\varnothing}}^{\mathrm{NCDT}}}.$$

Combining Theorem 4.5.3 and Theorem 5.4.8, we have:

Theorem 5.5.3.

$$\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}} = \mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}\Big|_{q_+=q_-=(q_0)^{1/2}} \prod_{\alpha\in\Lambda_+^{\mathrm{re},+}} \big(1-\sigma(\alpha)\cdot q^{\alpha}\big)^{\sigma(\alpha)\cdot[\alpha^0+c_\lambda(j_-(\alpha))-c_\lambda(j_+(\alpha))]}.$$

(See (1-1), (1-3) and (4-6) for notation.)

Since the second term in the right-hand side depend only on $c_\lambda[j]$'s but not on λ and ν, we have the following:

Corollary 5.5.4. *If $c_\lambda[j]=0$ for any j, we have*

$$\frac{\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{RTV}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}} = \frac{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\lambda,\nu}}{\mathcal{Z}^{\mathrm{NCDT}}_{\sigma,\vec{\varnothing},\vec{\varnothing}}\Big|_{q_+=q_-}}.$$

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THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS ON COMPACT SETS

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The main goal of this paper is to study the Dirichlet problem on a compact set $K \subset \mathbb{R}^n$. Initially we consider the space $H(K)$ of functions on K that can be uniformly approximated by functions harmonic in a neighborhood of K as possible solutions. As in the classical theory, we show $C(\partial_f K) \cong H(K)$ for compact sets with $\partial_f K$ closed, where $\partial_f K$ is the fine boundary of K. However, in general, a continuous solution cannot be expected, even for continuous data on $\partial_f K$. Consequently, we show that for any bounded continuous boundary data on $\partial_f K$, the solution can be found in a class of finely harmonic functions. Also, in complete analogy with the classical situation, this class is isometrically isomorphic to the set of bounded continuous functions on $\partial_f K$ for all compact sets K.

1. Introduction

The Dirichlet problem for harmonic functions on domains in $\mathbb{R}^n$ is important not only for its own sake but also because of its influence on potential theory. Many now-standard notions — regular points, fine topology, etc. — first appeared in the study of this problem. The main goal of this paper is to extend the classic theory to compact sets $K \subset \mathbb{R}^n$.

One possible extension can be found in the abstract theory of balayage spaces [Bliedtner and Hansen 1986; Hansen 1985]. However, we feel that the gain in transparency resulting from a direct geometric approach more than justifies the use of new techniques.

The Dirichlet problem can be thought of as having two components: the data set and the data itself. One uses an initial function defined on the data set to construct a solution (a harmonic function) on the rest of the domain, which must have a prescribed regularity as it approaches the data set. Classically, the data set is taken to be the topological boundary of the domain. One of the main goals of this paper is to establish that the natural choice for the data set on compact sets is the fine

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boundary $\partial_f K$ of K, which is shown in Lemma 5.1 to be the Choquet boundary of K with respect to subharmonic functions on K. We limit ourselves to initial functions that are continuous and bounded on $\partial_f K$, as in the classical case.

In Section 3, we introduce Jensen measures as our main tool and extend potential theory to compact sets $K \subset \mathbb{R}^n$ by defining harmonic functions and subharmonic functions on K. We devote Section 4 to the construction and study of harmonic measures on compact sets. The harmonic measure on K is shown to be a maximal Jensen measure. This is used to see that harmonic measures are concentrated on the fine boundary (Corollary 5.3). In Section 6 we study the Dirichlet problem for compact sets. As in the classical theory, our Theorem 6.1 shows $C(\partial_f K) \cong H(K)$ for a class of compact sets. However, in general, a continuous solution cannot be expected, even for continuous data on $\partial_f K$, as we illustrate in Example 6.2. Therefore we show that the solution can be found in the class of finely harmonic functions introduced in that section. By Theorem 6.5, in complete analogy with the classical situation, this class is isometrically isomorphic to the set of bounded continuous functions on $\partial_f K$, denoted $C_b(\partial_f K)$, for all compact sets K.

2. Basic facts

Let $\mathcal{M}(\Omega)$ denote the space of finite signed Radon measures on $\Omega \subset \mathbb{R}^n$, and $C_0(\mathbb{R}^n)$ the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity. We often use $\mu(f)$ to denote $\int f\,d\mu$.

Classical potential theory. Let D be an open set in $\mathbb{R}^n$, with $n \geq 2$. For any $f \in C(\partial D)$, the *Dirichlet problem* on D is to find a unique function h that is harmonic on D and continuous on $\overline{D}$ such that $h|_{\partial D} = f$. The function f is commonly referred to as the *boundary data*, and the corresponding h is said to be the *solution* of the Dirichlet problem on D with boundary data f. The punctured disk in $\mathbb{R}^2$ is a fundamental example that shows that the Dirichlet problem cannot be solved for any continuous boundary data. However, for a bounded open set U, the method of Perron allows one to assign a function that is harmonic on U to any continuous (or simply measurable) boundary data. The concept of a regular domain was developed to establish the continuity of the Perron solution to the boundary. A bounded connected open set $D \subset \mathbb{R}^n$ is a *regular* domain if the Dirichlet problem is solvable on D for any continuous boundary data. Therefore, on a regular domain, $C(\partial D)$ is isometrically isomorphic to $H(D)$, the space of continuous functions on $\overline{D}$ that are harmonic on D. For any $f \in C(\partial D)$, let $h_f \in H(D)$ denote the solution of the Dirichlet problem on D with boundary data f. Put $z \in D$. The point evaluation

$$H_z : f \mapsto h_f(z)$$

is a positive bounded linear functional on $C(\partial D)$. By the Riesz representation theorem, there is a Radon measure $\omega_D(z, \cdot)$ on ∂D that represents H_z; that is,

$$h_f(z) = \int_{\partial D} f(\zeta)\, d\omega_D(z, \zeta)$$

for all $f \in C(\partial D)$. The measure $\omega_D(z, \cdot)$ is called the *harmonic measure* of D with barycenter at z. See [Armitage and Gardiner 2001] for more details on potential theory.

Jensen measures. If D is an open set in $\mathbb{R}^n$, we call μ a *Jensen measure* on D with barycenter $z \in D$ if μ is a probability measure (a positive Radon measure of unit mass) whose support is compactly contained in D and if for every subharmonic function f on D the *subaveraging inequality* $f(z) \le \mu(f)$ holds. The set of Jensen measures on D with barycenter $z \in D$ we denote by $\mathcal{J}_z(D)$.

One could define the set of Jensen measures $\mathcal{J}_z^c(D)$ with respect to the continuous subharmonic functions on D. However, the following theorem shows that the set of Jensen measures would not be changed.

Theorem 2.1. *Let D be a bounded open subset of $\mathbb{R}^n$. For every $z \in D$, the sets $\mathcal{J}_z(D)$ and $\mathcal{J}_z^c(D)$ are equal.*

Proof. Since clearly $\mathcal{J}_z(D) \subseteq \mathcal{J}_z^c(D)$ for all $z \in D$, we show the reverse inclusion.

Pick some $z_0 \in D$ and let $\mu \in \mathcal{J}_{z_0}^c(D)$. Then we must show $f(z_0) \le \mu(f)$ for every function f that is subharmonic on D. The support of μ is compactly contained in D.

Because f is subharmonic on D, we can find a decreasing sequence $\{f_n\}$ of continuous subharmonic functions that converge to f. Since $\mu \in \mathcal{J}_{z_0}^c(D)$, we have $f(z_0) \le \mu(f_n)$ for every f_n. By the Lebesgue monotone convergence theorem, it follows that $f(z_0) \le \mu(f)$. Thus $\mu \in \mathcal{J}_{z_0}(D)$. □

Since $\mathcal{J}_z(D) = \mathcal{J}_z^c(D)$ for all $z \in D$, to check that $\mu \in \mathcal{J}_z(D)$ it suffices to check that μ has the subaveraging property for every continuous subharmonic function.

Examples of Jensen measures with barycenter at $z \in D$ include the Dirac measure at z, that is, δ_z, the harmonic measure with barycenter at z for any regular domain that is compactly contained in D, and the average over any ball (or sphere) centered at z that is contained in D. The following proposition demonstrates some basic properties of sets of Jensen measures.

Proposition 2.2 [Cole and Ransford 2001, Proposition 2.1]. *Let D_1 and D_2 be open subsets of $\mathbb{R}^n$ with $D_1 \subset D_2$. Let $z \in D_1$.*

(i) *If $\mu \in \mathcal{J}_z(D_1)$, then also $\mu \in \mathcal{J}_z(D_2)$.*

(ii) *If $\mu \in \mathcal{J}_z(D_2)$ and* $\operatorname{supp}(\mu) \subset D_1$, *and if each bounded component of $\mathbb{R}^n \setminus D_1$ meets $\mathbb{R}^n \setminus D_2$, then $\mu \in \mathcal{J}_z(D_2)$.*

Jensen measures and subharmonic functions are, in a sense, dual to each other. This duality is illustrated by the following theorem.

Theorem 2.3 [Cole and Ransford 1997, Corollary 1.7]. *Let D be an open subset of $\mathbb{R}^n$ that possesses a Green's function. Let*

$$\phi : D \to [-\infty, \infty)$$

be a Borel measurable function that is locally bounded above. Then

$$\sup\{v(z) : v \in S(D), v \le \phi\} = \inf\{\mu(\phi) : \mu \in \mathcal{J}_z(D)\}$$

for each $z \in D$, where $S(D)$ denotes the set of subharmonic functions on D.

Fine topology. The two books [Brelot 1971; Fuglede 1972] are classical references on the fine topology, and many books on potential theory contain chapters on the topic, for example [Armitage and Gardiner 2001, Chapter 7].

The *fine topology* on $\mathbb{R}^n$ is the coarsest topology on $\mathbb{R}^n$ such that all subharmonic functions are continuous in the extended sense of taking values in $[-\infty, \infty]$.

When referring to a topological concept in the fine topology, we follow the standard policy of either using the words "fine" or "finely" prior to the topological concept, or attaching the letter f to the associated symbol. For example, the fine boundary of K, $\partial_f K$, is the boundary of K in the fine topology. The fine topology is strictly finer than the Euclidean topology.

Many of the key concepts of classical potential theory have analogous definitions in relation to the fine topology. We recall a few of them. Relative to a finely open set V in $\mathbb{R}^n$, the *harmonic measure* $\delta_x^{V^c}$ is defined as the swept-out of the Dirac measure δ_x on the complement of V. A function u is said to be *finely hypoharmonic* on a finely open set U if it is upper finite, finely upper semicontinuous, and if

$$u(x) \le \delta_x^{V^c}(u) < \infty$$

for all $x \in V$ and all relatively compact finely open sets V with fine closure contained in U. A function h is said to be *finely harmonic* if h and $-h$ are finely hypoharmonic. Also, the *fine Dirichlet problem on U* for a finely continuous function f defined on the fine boundary of a bounded finely open set U consists of finding a finely harmonic extension of f to U. The development of the fine Dirichlet problem is quite similar to that of the classical. Fuglede [1972, Theorem 14.6] establishes a Perron solution for the fine Dirichlet problem, showing that there exists a Perron solution H_f^U that is finely harmonic on U for any numerical function f on $\partial_f U$ that is $\delta_x^{\partial_f U}$ integrable for every $x \in U$. The same theorem also provides us with the desired continuity at the boundary, showing that the fine limit of $H_f^U(x)$ tends to $f(y)$ as $x \in U$ goes to y for every finely "regular" boundary point $y \in \partial_f U$ at which f is finely continuous.

3. Harmonic and subharmonic functions on compact sets

We now begin our study of potential theory on compact sets. For compact sets that are not connected, the Hausdorff property allows us to reduce Dirichlet-type problems on the compact set to solving separate problems on each connected component. Therefore, in what follows, we work on compact sets K in $\mathbb{R}^n$ that need not be connected, with the understanding that we can always separate the problem by working on the connected components of K individually.

There are currently three equivalent ways to define harmonic and subharmonic functions on compact sets.

Definition 3.1 (exterior). Let $H(K)$ (resp. $S(K)$) be the uniform closures of all functions in $C(K)$ that are restrictions of harmonic (resp. subharmonic) functions on a neighborhood of K.

Definition 3.2 (interior). One can define $H(K)$ and $S(K)$ as the subspaces of $C(K)$ consisting of functions that are finely harmonic and finely subharmonic, respectively, on the fine interior of K.

The equivalence of these definitions of $H(K)$ was shown in [Debiard and Gaveau 1974], and of $S(K)$ in [Bliedtner and Hansen 1975; 1978].

For the third definition of $H(K)$, we extend the notion of Jensen measures to compact sets.

Definition 3.3. We define the set of Jensen measures on K with barycenter at $z \in K$ as the intersection of all sets $\mathcal{J}_z(U)$, that is,

$$\mathcal{J}_z(K) = \bigcap_{K \subset U} \mathcal{J}_z(U),$$

where U is any open set containing K.

Another definition of $H(K)$ was introduced in [Poletsky 1997] using the notion of Jensen measures.

Definition 3.4 (via Jensen measures). The set $H(K)$ is the subspace of $C(K)$ consisting of functions h such that $h(x) = \mu(h)$ for all $\mu \in \mathcal{J}_x(K)$ and $x \in K$.

It was shown in [Poletsky 1997] that this definition is equivalent to the exterior definition above.

The next lemma shows that this last construction extends to subharmonic functions in the ideal way.

Lemma 3.5. *A function is in $S(K)$ if and only if it is continuous and satisfies the subaveraging property with respect to every Jensen measure on K; that is,*

$$S(K) = \{f \in C(K) : f(z) \le \mu(f), \textit{ for all } \mu \in \mathcal{J}_z(K) \textit{ and every } z \in K\}.$$

Proof. We use the exterior definition of $S(K)$ to show "$\subseteq$". Take $f \in C(K)$ and let $\{f_j\}$ be a sequence of subharmonic functions defined in a neighborhood of K such that $\{f_j\}$ is converging uniformly to f. Then $f_j(z) \leq \mu(f_j)$ for any $\mu \in \mathcal{J}_z(K)$. Because the convergence is uniform, we have $f(z) \leq \mu(f)$.

Now suppose that f is in the set on the right. The subaveraging condition implies that f is finely subharmonic on the fine interior of K, and by assumption, f is continuous. Therefore f satisfies the interior definition of $S(K)$. □

Recall the exterior definition of $S(K)$ as the uniform limits of continuous functions subharmonic in neighborhoods of K. Proposition 3.6 shows that the defining sequence for any function in $S(K)$ may be taken to be increasing. This is a simple consequence of a duality theorem of Edwards [Gamelin 1978, Theorem 1.2; Cole and Ransford 1997].

Proposition 3.6. *Every function in $S(K)$ is the limit of an increasing sequence of continuous subharmonic functions defined on neighborhoods of K.*

Proof. Edwards's theorem states that if p is a continuous function on K, then for all $z \in K$ we have

$$Ep(z) := \sup\{f(z) : f \in S(K),\ f \leq p\} = \inf\{\mu(p) : \mu \in \mathcal{J}_z(K)\}.$$

From the proof of this theorem, it follows that Ep is lower semicontinuous and is the limit of an increasing sequence of continuous subharmonic functions on neighborhoods of K. The result follows by observing that $p = Ep$ whenever $p \in S(K)$. □

4. Harmonic measure on a compact set

To use the exterior definition of $H(K)$, we typically want to approximate K by a decreasing sequence of regular domains. A decreasing sequence of regular domains $\{U_j\}$ is said to be *converging* to K if for every $\varepsilon > 0$, there is a j_0 such that U_j contains K and lies in the ε-neighborhood K_ε of K when $j \geq j_0$. Furthermore, we require that U_{j+1} be compactly contained in U_j; that is, $\overline{U}_{j+1} \subset U_j$ for all j. The existence of such a sequence is provided by [Hervé 1962, Proposition 7.1].

Theorem 4.1 allows us to define a harmonic measure on K. For a decreasing sequence of regular domains $\{U_j\}$, we let $\omega_{U_j}(z, \cdot)$ denote the harmonic measure on U_j with barycenter at $z \in U_j$.

Theorem 4.1. *If $\{U_j\}$ is a sequence of regular domains converging to a compact set $K \subset \mathbb{R}^n$, then for every $z \in K$, the harmonic measures $\omega_{U_j}(z, \cdot)$ converge weak*. Also, this limit does not depend on the choice of the sequence of domains $\{U_j\}$.*

Proof. Since ω_{U_j} are measures of unit mass supported on a compact set in $\mathbb{R}^n$, by Alaoglu's theorem they must have a limit point. To show that this point is unique, it suffices to show that for every $z \in K$, the limit

$$\lim_{j\to\infty} \int_{\partial U_j} u(\zeta)\, d\omega_{U_j}(z,\zeta) \tag{1}$$

exists for every $u \in C(\overline{U}_1)$.

First, we show that the limit in (1) exists when u is continuous and subharmonic in a neighborhood of K. The solution u_j of the Dirichlet problem on U_j with boundary value u is equal to

$$u_j(z) = \int_{\partial U_j} u(\zeta)\, d\omega_{U_j}(z,\zeta).$$

Since u is subharmonic, we have $u_j \geq u$ on U_j. Then, since $u_{j+1} = u$ on ∂U_{j+1} and $u_j \geq u = u_{j+1}$ on ∂U_{j+1}, the maximum principle for harmonic functions implies that $u_j \geq u_{j+1}$ on U_{j+1}. Thus $\{u_j\}$ is a decreasing sequence on K, and we see that for every $z \in K$, the limit in (1) exists.

If $u \in C^2(\overline{U}_1)$, then we may represent u as a difference of two $C^2(\overline{U}_1)$ functions that are subharmonic on U_1. By the argument above, the limit in (1) exists.

Because $C^2(\overline{U}_1)$ is dense in $C(\overline{U}_1)$, we see that the limit in (1) always exists. $\square$

Definition 4.2. We define the harmonic measure $\omega_K(z,\cdot)$ on a compact set K with barycenter $z \in K$ as the weak* limit of $\omega_{U_j}(z,\cdot)$, chosen as above.

To use this definition for the Dirichlet problem, we must check that the support of $\omega_K(z,\cdot)$ lies on the boundary of K. Actually, in Section 5, we are able to give more specific information about $\omega_K(z,\cdot)$; see Corollary 5.3.

Lemma 4.3. *The support of $\omega_K(z,\cdot)$ is contained in ∂K.*

Proof. Let W be a neighborhood of ∂K. Let $\{U_j\}$ be a sequence of domains converging to K, and take a sequence $z_j \in \partial U_j$. Then there exists a subsequence $\{z_{j_k}\}$ that must be converging to some $z_0 \in K$. Because $z_j \in \partial U_j$, we know z_j is not in K. Therefore, the limit of z_{j_k} cannot be in the interior of K. Thus z_0 is in $\partial K \subset W$. Consequently, there is j_0 such that $\partial U_j \subset W$ for each $j \geq j_0$.

Let $x \in \mathbb{R}^n \setminus \partial K$, and take W to be a neighborhood of ∂K such that $x \notin \overline{W}$. There is an $r > 0$ such that $\overline{B(x,r)} \cap \overline{W} = \varnothing$. Since $\omega_{U_j}(z,\cdot)$ has support on ∂U_j, which is contained in $\overline{W}$ for large j, we have $\omega_{U_j}(z, B(x,r)) = 0$. since $b(x,r)$ is open, the portmanteau theorem shows that

$$\liminf_{j\to\infty} \omega_{U_j}(z, B(x,r)) \geq \omega_K(z, B(x,r)).$$

Hence, $\omega_K(z, B(x,r)) = 0$, and x is not in the support of $\omega_K(z,\cdot)$. $\square$

The following theorem brings our study back to the topic of Jensen measures.

Theorem 4.4. *The harmonic measure on K is a Jensen measure on K.*

Proof. Because $\omega_K(z,\cdot)$ is defined as the weak* limit of probability measures, $\omega_K(z,\cdot)$ is a probability measure.

Recall that for $z \in K$, we defined $\mathcal{J}_z(K) = \cap \mathcal{J}_z(U)$, where $K \subset U$. However, it suffices to see that $\mathcal{J}_z(K) = \cap \mathcal{J}_z(U_j)$, where $\{U_j\}$ is any sequence of domains converging to K. We show $\omega_K(z,\cdot) \in \mathcal{J}_z(U_j)$ for all j.

Pick some j. Then let f be a continuous subharmonic function on U_j. Then

$$f(z) \le \int_{\partial U_l} f(\zeta)\, d\omega_{U_l}(z,\zeta)$$

for all $l > j$. Then, taking the weak* limit, we have

$$f(z) \le \int_{\partial K} f(\zeta)\, d\omega_K(z,\zeta).$$

Therefore, $\omega_K(z,\cdot)$ satisfies the subaveraging inequality for every continuous subharmonic function on U_j, and $\omega_K(z,\cdot)$ is a probability measure with support contained in U_j. Thus $\omega_K(z,\cdot)$ must be in $\mathcal{J}_z^c(U_j)$, which is equal to $\mathcal{J}_z(U_j)$ by Theorem 2.1. Thus $\omega_K(z,\cdot) \in \mathcal{J}_z(K)$. □

Following [Gamelin 1978, p. 16], a partial ordering on the set of Jensen measures is defined below. The notation $\mathcal{J}(K)$ is used to stand for the union of all Jensen measures on K; that is,

$$\mathcal{J}(K) = \bigcup_{z \in K} \mathcal{J}_z(K).$$

Definition 4.5. For $\mu, \nu \in \mathcal{J}(K)$, we say that $\mu \succeq \nu$ if for every $\phi \in S(K)$ we have $\mu(\phi) \ge \nu(\phi)$. Furthermore, a Jensen measure μ is maximal if there is no $\nu \succeq \mu$ with $\nu \ne \mu$ where $\nu \in \mathcal{J}(K)$.

Lemma 4.6. *If $\mu \in \mathcal{J}_{z_1}(K)$ and $\nu \in \mathcal{J}_{z_2}(K)$ with $z_1 \ne z_2$, then μ and ν are not comparable.*

Proof. Recall that the coordinate functions π_i are harmonic. Because $z_1 \ne z_2$, they must differ in at least one coordinate, say, the i-th. Assume without loss of generality that $\pi_i(z_1) > \pi_i(z_2)$. Then $\mu(\pi_i) > \nu(\pi_i)$. However, $-\pi_i$ is also harmonic, and so $\nu(-\pi_i) > \mu(-\pi_i)$. Therefore, μ and ν are not comparable, and if $\mu \succeq \nu$, then they have a common barycenter. □

We now demonstrate that the harmonic measure is maximal with respect to this ordering. The maximality of the harmonic measure proved below is the Littlewood subordination principle [Duren 1970, Theorem 1.7] when K is the closed unit ball in the plane.

Theorem 4.7. *For all $z \in K$, the measure $\omega_K(z,\cdot)$ is maximal in $\mathcal{J}(K)$.*

Proof. By Lemma 4.6, it suffices to show that $\omega_K(z,\cdot)$ is maximal in $\mathcal{J}_z(K)$ for any $z \in K$.

Pick any $z_0 \in K$. We show that $\omega_K(z_0,\cdot)$ majorizes every measure $\mu \in \mathcal{J}_{z_0}(K)$. Consider a decreasing sequence of regular domains $\{U_j\}$ converging to K. Take any $\phi \in S^c(K)$. By Proposition 3.6, we may find a sequence $\phi_j \in S^c(U_j)$ increasing to ϕ. Furthermore, we extend ϕ as $\tilde{\phi} \in C_0(\mathbb{R}^n)$, while keeping $\tilde{\phi} \geq \phi_j$ for all j. Define harmonic functions Φ_j on U_j by

$$\Phi_j(x) = \int_{\partial U_{j+1}} \phi_j(\zeta)\, d\omega_{U_{j+1}}(x,\zeta).$$

Therefore, since ϕ_j is subharmonic, $\Phi_j \geq \phi_j$ on U_{j+1}, so

$$\int_{\partial U_{j+1}} \phi_j(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) = \Phi_j(z_0) = \mu(\Phi_j) \geq \mu(\phi_j).$$

Because $\tilde{\phi} \geq \phi_j$, we have

$$\int_{\partial U_{j+1}} \tilde{\phi}(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) \geq \mu(\phi_j) \tag{2}$$

for all j. By taking weak* limits, we have that

$$\lim_{j\to\infty} \int_{\partial U_{j+1}} \tilde{\phi}(\zeta)\, d\omega_{U_{j+1}}(z_0,\zeta) = \int_{\partial K} \phi(\zeta)\, d\omega_K(z_0,\zeta).$$

The Lebesgue monotone convergence theorem gives

$$\lim_{j\to\infty} \mu(\phi_j) = \mu(\phi).$$

Therefore, by taking the limit by j of (2), we see

$$\int_{\partial K} \phi(\zeta)\, d\omega_K(z_0,\zeta) \geq \mu(\phi).$$

We now have $\omega_K(z_0,\cdot) \succeq \mu$. If any $\nu \in \mathcal{J}_{z_0}(K)$ has the property $\nu \succeq \omega_K(z_0,\cdot)$, by the antisymmetry property of partial orderings, we have $\nu = \omega_K(z_0,\cdot)$. Thus the measure $\omega_K(z_0,\cdot)$ is maximal in $\mathcal{J}_{z_0}(K)$. □

The maximality of harmonic measures implies that they are trivial at the points $z \in K$ such that $\mathcal{J}_z(K) = \{\delta_z\}$, which, by Lemma 5.1, are precisely the fine boundary points.

Corollary 4.8. *The harmonic measure $\omega_K(z_0,\cdot)$ is equal to δ_{z_0} if and only if $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$.*

Proof. Suppose $\omega_K(z_0, \cdot\,) = \delta_{z_0}$. Consider the function $\rho(z) = \|z - z_0\|^2 \in S^c(K)$. Then for any $\mu \in \mathcal{J}_{z_0}$, by the maximality of $\omega_K(z_0, \cdot\,)$, we have

$$0 = \rho(z_0) \leq \mu(\rho) \leq \int_{\partial K} \rho(\zeta)\, d\omega_K(z_0, \zeta) = \rho(z_0) = 0.$$

Because $\rho(z) > 0$ for all $z \neq 0$, and μ is a probability measure, we see that $\mu = \delta_{z_0}$. Thus $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$.

For the reverse implication, we have $\omega_K(z_0, \cdot\,) \in \mathcal{J}_{z_0}(K)$ from Theorem 4.4. $\square$

5. The boundary

Gamelin [1978] introduces a version of Choquet theory for cones of functions on compact sets. (Actually, it applies to sets of functions that are slightly more general than the cones we define.)

Following his guidance, we consider a set $\mathcal{R}$ of functions mapping a compact set $K \subset \mathbb{R}^n$ to $[-\infty, \infty)$, with the following properties:

(i) $\mathcal{R}$ includes the constant functions;

(ii) if $c \in \mathbb{R}^+$ and $f \in \mathcal{R}$, then $cf \in \mathcal{R}$;

(iii) if $f, g \in \mathcal{R}$, then $f + g \in \mathcal{R}$; and

(iv) $\mathcal{R}$ separates the points of K.

One then considers a set of $\mathcal{R}$-measures for $z \in K$ defined as the set of probability measures μ on K such that

$$f(z) \leq \mu(f)$$

for all $f \in \mathcal{R}$.

Naturally, our model for $\mathcal{R}$ will be $S(K)$. It then follows that when $\mathcal{R} = S(K)$, the $\mathcal{R}$-measures for $z \in K$ are precisely $\mathcal{J}_z(K)$. We now state some classic results from [Gamelin 1978] that we need in the following sections.

One can define the Choquet boundary of K with respect to $S(K)$ as

$$\mathrm{Ch}_{S(K)}\, K = \{z \in K : \mathcal{J}_z(K) = \{\delta_z\}\}.$$

Many nice properties of the Choquet boundary are known. In particular, we need the following characterization; see also, for example, [Bliedtner and Hansen 1986, VI.4.1; Hansen 1985].

Lemma 5.1. *The Choquet boundary of K with respect to $S(K)$ is the fine boundary of K; that is,*

$$\mathrm{Ch}_{S(K)}\, K = \partial_f K.$$

Proof. Since the fine topology is strictly finer than the Euclidean topology, any point in the interior of K will also be in the fine interior of K, and any point of $\mathbb{R}^n \setminus K$ can be separated from K by a Euclidean (and therefore fine) open set. Thus the fine boundary of K is contained in ∂K. The result follows immediately from [Poletsky 1997, Theorem 3.3] or [Bliedtner and Hansen 1986, Proposition 3.1], which states that $\mathcal{J}_z(K) = \{\delta_z\}$ if and only if the complement of K is non-thin at z, that is, z is a fine boundary point of K. □

The set $\partial_f K$ is also called the stable boundary of K. In fact, Lemma 5.1 shows that $\mathrm{Ch}_{S(K)}\, K$ is the finely regular boundary of the fine interior of K. For details on finely regular boundary points and related concepts, see [Bliedtner and Hansen 1986, VII.5–7; Hansen 1985].

With this association, the result of Brelot [1971, p. 89] about the stable boundary points of K shows that $\mathrm{Ch}_{S(K)}\, K$ is dense in ∂K.

Theorem 5.2. *The fine boundary of K (and therefore the Choquet boundary of K with respect to $S(K)$) is dense in the topological boundary of K.*

In general, the fine boundary is not closed, as Example 6.2 of Section 6 shows. So we cannot claim that it is the support of measures. Also, as Theorem 5.2 shows, the closure of $\mathbb{O}_k$ is the boundary of K. In particular, it may coincide with K for porous Swiss cheeses [Gamelin 1969, pp. 25–26].

Recall that a measure $\mu \in \mathcal{M}(K)$ is concentrated on a set E if for every set $F \subset K \setminus E$, we have $\mu(F) = 0$. A probability measure μ is concentrated on a set E if and only if $\mu(E) = 1$. From [Gamelin 1978, p. 19], we know that all maximal measures are concentrated on $\mathrm{Ch}_{S(K)}\, K = \partial_f K$. With this observation, the next corollary immediately follows from Theorem 4.7 (which states that the harmonic measure is maximal).

Corollary 5.3. *For every z in K, the harmonic measure with barycenter at z is concentrated on $\partial_f K$.*

6. The Dirichlet problem on compact sets

In the classical setting, we know that any continuous function in the boundary of a domain $D \subset \mathbb{R}^n$ extends harmonically to D and continuously to $\overline{D}$ if and only if every point of the boundary is regular. For general compact sets in $\mathbb{R}^n$, we have:

Theorem 6.1. *If K is a compact set in $\mathbb{R}^n$, then any function $\phi \in C(\partial_f K)$ extends to a function in $H(K)$ if and only if the set $\partial_f K$ is closed. Also, the solution is given by*

$$\Phi(z) = \int_{\partial_f K} \phi(\zeta)\, d\omega_K(z, \zeta), \quad z \in K,$$

and $H(K)$ is isometrically isomorphic to $C(\partial_f K)$.

From this it also follows that the swept-out point mass at z onto the complement of K is just $\omega_K(z, \cdot)$.

Proof. Suppose the set $\partial_f K$ is closed. Consider a continuous function ϕ on $\partial_f K$. Assume that

$$\Phi(z) = \int_{\partial_f K} \phi(\zeta)\, d\omega_K(z, \zeta), \quad z \in K.$$

Because $\partial_f K$ is closed, by Theorem 5.2, we have $\partial_f K = \partial K$. Also, because $\omega_K(z, \cdot) = \delta_z$ for every $z \in \partial_f K$, we see that $\Phi = \phi$ on $\partial_f K$.

Let z_j be a sequence in K converging to $z_0 \in \partial_f K$. Because z_0 is in $\partial_f K = \mathrm{Ch}_{S(K)} K$, we have $\mathcal{J}_{z_0}(K) = \{\delta_{z_0}\}$. Because $\mathcal{J}(K)$ is weak* compact [Gamelin 1978, p. 3], any sequence of measures $\mu_j \in \mathcal{J}_{z_j}(K)$ must converge weak* to δ_{z_0}. In particular, $\omega_{U_j}(z_j, \cdot)$ is weak* converging to δ_{z_0}. Hence, $\Phi(z_j)$ is converging to $\Phi(z_0) = \phi(z_0)$, and Φ is continuous at the boundary of K.

Because $\partial_f K$ is closed, $\phi \in C(\partial_f K) = C(\partial K)$. We extend ϕ continuously as $\tilde{\phi} \in C_0(\mathbb{R}^n)$, and then define the harmonic functions

$$h_j(z) = \int_{\partial U_j} \tilde{\phi}(\zeta)\, d\omega_{U_j}(z, \zeta).$$

Because $\tilde{\phi}$ is continuous and $\omega_{U_j}(z, \cdot)$ converges weak* to $\omega_K(z, \cdot)$,

$$\lim_{j\to\infty} h_j(z) = \lim_{j\to\infty} \int_{\partial U_j} \tilde{\phi}(\zeta)\, d\omega_{U_j}(z, \zeta) = \int_{\partial K} \phi(\zeta)\, d\omega_K(z, \zeta) = \Phi(z).$$

Therefore, Φ is the pointwise limit of a sequence $\{h_j\}$ of functions harmonic in a neighborhood of K. Also, we can take the extension $\tilde{\phi}$ of ϕ in such a way that the sequence $\{h_j\}$ is uniformly bounded. It now easily follows that Φ is continuous on the interior of K. Indeed, consider a point z in the interior of K. Then there exists a ball B centered at z contained in the interior of K. The h_j are harmonic functions on B converging pointwise to Φ. Therefore, Φ is continuous on B by the Harnack principle, and so Φ is continuous on K. Thus we have a continuous function Φ with representation

$$\Phi(z) = \int_{\partial K} \phi(\zeta)\, d\omega_K(z, \zeta), \quad z \in K.$$

Since Φ is continuous on K by [Poletsky 1997], to check that $\Phi \in H(K)$, all that remains is to show that Φ is averaging with respect to Jensen measures, that is, the equivalence of the external definition of $H(K)$ and the definition by Jensen measures. So we need to see that $\Phi(z) = \mu_z(\Phi)$ for every $\mu_z \in \mathcal{J}_z(K)$ and for every $z \in K$. Because h_j is harmonic on U_j, we have $h_j(z) = \mu_z(h_j)$. However, by the Lebesgue dominated convergence theorem,

$$\mu_z(\Phi) = \lim_{j\to\infty} \mu_z(h_j) = \lim_{j\to\infty} h_j(z) = \Phi(z).$$

Thus $\Phi \in H(K)$.

For the converse, suppose $\partial_f K$ is not closed. Then there is a point $z_0 \in \partial K \setminus \partial_f K$. Since z_0 is not in $\partial_f K$, by Corollary 4.8, $\omega_K(z_0, \cdot)$ is not trivial. Therefore, we can find a set $E \subset \partial K$ such that $\omega_K(z_0, E) > 0$, with E in the complement of $B(z_0, r)$ for some $r > 0$. Consider a continuous function f on ∂K such that $f = 1$ on ∂K outside $B(z_0, r)$ and $f = 0$ on $B(z_0, r/2) \cap \partial K$. Then

$$\int_{\partial K} f(\zeta)\, d\omega_K(z_0, \zeta) > \omega_K(z_0, E), \quad z \in K.$$

However, $f(z_0) = 0$. Thus there can be no function in $H(K)$ that agrees with f on the boundary of K. □

Example 6.2. The following is a simple example of a compact set $K \subset \mathbb{R}^n$, $n \geq 3$, in which the fine boundary is not closed. The set K is obtained from the closed unit ball $\overline{B} \subset \mathbb{R}^n$ by deleting a sequence $\{B(z_n, r_n)\}_{n=1}^{\infty}$ of open balls whose centers and radii tend to zero. We take the centers to be $z_n = (2^{-n}, 0, \ldots, 0) \in \mathbb{R}^n$ and the radii $0 < r_n < 2^{-n-2}$. This is analogous to the "road runner" example of Gamelin [1969, Figure 2] and the Lebesgue spine [2001, p. 187].

By Theorem 6.1, one cannot expect a continuous solution for the Dirichlet problem on an arbitrary compact set, even with continuous boundary data. Therefore, at this point we consider the following broader class of solutions with weaker continuity requirement.

Definition 6.3. Let $fH^c(K)$ be the class of finely continuous functions on K that are finely harmonic on the fine interior of K and continuous and bounded on $\partial_f K$.

We saw in Definition 3.4 (via Jensen measures) that $H(K)$ consists of the functions in $C(K)$ satisfying the averaging property with respect to $\mathcal{J}(K)$, and by Definition 3.2 (interior) that it can also be seen as the $C(K)$ that are finely harmonic on the fine interior of K. Therefore, in the definition of $fH^c(K)$, we have maintained the finely harmonic requirement while requiring continuity only on the boundary $\partial_f K$ (to match the boundary data). In fact, Theorem 6.5 below shows that the functions in $fH^c(K)$ also satisfy the averaging property with respect to $\mathcal{J}(K)$.

Theorem 6.5 shows that the Dirichlet problem on compact sets $K \subset \mathbb{R}^n$ is solvable in the class of functions $fH^c(K)$ for boundary data that is continuous and bounded on $\partial_f K$. The functions that are continuous and bounded on $\partial_f K$ are denoted $C_b(\partial_f K)$. For this we need the following theorem.

Theorem 6.4 [Fuglede 1972, Theorem 11.9]. *The pointwise limit of a pointwise convergent sequence of finely harmonic functions u_m in U, a finely open subset of $\mathbb{R}^n$, is finely harmonic, provided that* $\sup_m |u_m|$ *is finely locally bounded in U.*

Theorem 6.5. *For every ϕ lying in $C_b(\partial_f K)$, that is, continuous and bounded on $\partial_f K$, there is a unique $h_\phi \in fH^c(K)$ equal to ϕ on $\partial_f K$. Moreover, h_ϕ satisfies the averaging property for $\mathcal{J}(K)$, and in particular,*

$$h_\phi(x) = \int_{\partial_f K} \phi(\zeta)\, d\omega_K(x, \zeta), \quad x \in K.$$

Proof. Let $\phi \in C_b(\partial_f K)$, and for $x \in \overline{\partial_f K}$ define

$$\tilde{\phi}(x) = \limsup_{y \to x,\, y \in \partial_f K} \phi(y).$$

Since ϕ is continuous on $\partial_f K$, if $x \in \partial_f K$, then $\tilde{\phi}(x) = \phi(x)$. Also, $\tilde{\phi}$ is upper semicontinuous, and thus we may find a decreasing sequence of functions $\{\phi_k\}$ that are continuous on $\overline{\partial_f K}$ and converge pointwise to $\tilde{\phi}$. Then we extend the ϕ_k to $C_0(\mathbb{R}^n)$ as $\hat{\phi}_k$. By taking $\tilde{\phi}_k = \min\{\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_k\}$, we can make the extensions be decreasing. Consider a decreasing sequence of regular domains U_j converging to K. Let $u_{j,k}$ be the solution of the Dirichlet problem on U_j for $\tilde{\phi}_k$. Since the measures $\omega_{U_j}(x, \cdot)$ weak* converge to $\omega_K(x, \cdot)$, we have that $\lim_j u_{j,k} = \int \tilde{\phi}_k \, d\omega_K := u_k$. Since the $\tilde{\phi}_k$ are decreasing, u_k must also be decreasing. Indeed, we let $h_\phi = \lim u_k$.

Take any $\mu \in \mathcal{J}(K)$. Then $\mu \in \mathcal{J}_{z_0}(U_j)$ for all j and some $z_0 \in K$. Since $u_{j,k}$ is harmonic, we have $\mu(u_{j,k}) = u_{j,k}(z_0)$. However, by the Lebesgue dominated convergence theorem, we have $\lim_j \mu(u_{j,k}) = \mu(u_k)$, and so $\mu(u_k) = u_k(z_0)$. Since the sequence $\{u_k\}$ is decreasing pointwise to h_ϕ, we have $\mu(h_\phi) = h_\phi(z_0)$, by the same theorem. Thus h_ϕ satisfies the averaging property on $\mathcal{J}(K)$. Since $\omega_K(z, \cdot)$ lies in $\mathcal{J}(K)$ for all $z \in K$, we see that

$$h_\phi(z) = \int_{\partial_f K} h_\phi(\zeta)\, d\omega_K(z, \zeta).$$

We now show that $h_\phi = \phi$ on $\partial_f K$. For any $x \in \mathcal{O}_k$, we know $\omega_K(x, \cdot) = \delta_x$, and

$$u_k(x) = \lim_{j \to \infty} u_{j,k}(x) = \int \tilde{\phi}_k(\zeta)\, d\omega_K(x, \zeta) = \tilde{\phi}_k(x).$$

Thus $u_k(x) = \tilde{\phi}_k(x)$ for all $x \in \partial_f K$, and so

$$h_\phi(x) = \lim_{k \to \infty} u_k(x) = \lim_{k \to \infty} \tilde{\phi}_k(x) = \phi(x)$$

for all $x \in \partial_f K$.

To see that h_ϕ is finely harmonic, we use Theorem 6.4. Observe that u_k is the pointwise limit of the harmonic (and therefore finely harmonic) functions $u_{j,k}$, and the solution h_ϕ is the pointwise limit of u_k. From the construction of these functions, it is clear that they are bounded. $\square$

Corollary 6.6. *The set $C_b(\partial_f K)$ is isometrically isomorphic to $fH^c(K)$.*

Proof. The previous theorem establishes the homomorphism taking $C_b(\partial_f K)$ to $fH^c(K)$. Observe that $h|_{\partial_f K} \in C_b(\partial_f K)$ for every $h \in fH^c(K)$. The uniqueness of the solution shows that $h|_{\partial_f K}$ extends as h. Furthermore, the isometry follows directly from the integral representation in the previous theorem. □

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EXTENSION OF AN ANALYTIC DISC AND DOMAINS IN $\mathbb{C}^2$ WITH NONCOMPACT AUTOMORPHISM GROUP

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Let Ω be a smoothly bounded domain in $\mathbb{C}^2$ such that the Bergman representative map near the boundary continues to be diffeomorphic up to the boundary. If such a domain admits a holomorphic automorphism group orbit accumulating at a boundary point of finite D'Angelo type $2m$, we show that the domain Ω is biholomorphic to the Thullen domain

$$\{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

This result refines the well-known theorem of E. Bedford and S. Pinchuk.

1. Introduction

Denote by $\mathrm{Aut}(\Omega)$ the set of biholomorphic self-maps of a domain (that is, an open connected set) Ω in the n-dimensional complex Euclidean space $\mathbb{C}^n$. By [Cartan 1932], $\mathrm{Aut}(\Omega)$ is a (real) Lie group with respect to the law of composition and the topology of uniform convergence on compact subsets. One of the traditional important questions is:

Which bounded domains admit a noncompact automorphism group?

There are several well-known results concerning this question; see, for example, [Wong 1977; Bedford and Pinchuk 1988; Kim 1992]. This paper also pertains to this line of research. Recall the following theorem:

Theorem 1.1 [Bedford and Pinchuk 1988]. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with a real analytic boundary. If Ω has a noncompact automorphism group, then Ω is biholomorphic to the Thullen domain*

$$E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}$$

for some positive integer m.

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The main thrust of this article is to try to *localize* this theorem. Theorem 1.1 and its generalizations and refinements (in [Bedford and Pinchuk 1991], for example) rely upon global assumptions (partly local but not local) that the boundary is globally real analytic (or, at least, of finite D'Angelo type). Such assumptions were needed in order use the orbit accumulation point not of the original noncompact automorphism orbit, but of a 1-parameter subgroup produced by the initial scaling method; the finite D'Angelo type of that orbit accumulation boundary point is that exponent $2m$ in Bedford and Pinchuk's theorem. Keeping this in mind, we state our main result here:

Theorem 1.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth* (C^∞) *boundary satisfying Condition BR* (see Definition 3.3). *Suppose there is a point $p_0 \in \partial\Omega$ of finite D'Angelo type $2m$, a point $q \in \Omega$, and a sequence $\{\varphi_j\} \subset \operatorname{Aut}(\Omega)$ such that*

$$\lim_{j\to\infty} \varphi_j(q) = p_0 \in \partial\Omega.$$

Then

$$\Omega \cong E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

The key step of the proof is showing the smooth extension of a certain holomorphic disc in the given domain. Since Fefferman's celebrated work [1974], analysis on the Bergman kernel function has been regarded as one of the most powerful tools in understanding the smooth extension of holomorphic mappings. In the equidimensional case, Bell and Ligocka [1980] introduced the so-called Conditions A and B on the Bergman kernel function, which guarantee the smooth extension of biholomorphic mappings. In contrast with the equidimensional case, Conditions A and B seem insufficient to prove the smooth extension of holomorphic discs in a bounded domain in $\mathbb{C}^2$. This is the reason why we define a new criterion for the smooth extension, which we call Condition BR.

According to [Ligocka 1980], Condition B holds if the Bergman representative maps, introduced by S. Bergman, form holomorphic coordinates near the boundary. Inspired by Ligocka's observation, we say that a domain with smooth boundary satisfies Condition BR if for every boundary point p, there is an interior point q at which the Bergman representative map gives rise to a smooth coordinate system in a relative open neighborhood of q that includes the boundary point p (see also Definition 3.3).

Outline of paper. In Section 2 we briefly explain Berteloot's argument on the Pinchuk scaling method without proof. The smooth extension of holomorphic disc under Condition BR is proved in Section 3 (see Proposition 3.7). The main theorem is proved in the last section.

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2. Berteloot's two-dimensional analysis on Pinchuk's scaling

Scaling. Let Ω be a domain in $\mathbb{C}^2$ and let p_0 belong to $\partial\Omega$. Assume that $\partial\Omega$ is of class C^∞, pseudoconvex and of finite type in a neighborhood of p_0. Let $2m$ be the type of $\partial\Omega$ at p_0 in the sense of [D'Angelo 1982]. We may assume that $p_0 = (0, 0)$ and that $\operatorname{Re}(\partial/\partial w)$ is the outward normal vector to $\partial\Omega$ at p_0.

Let $\{q_j\}$ be a sequence of points in Ω which converges to $(0, 0)$. For every j large enough, there exists a unique boundary point $p_j \in \partial\Omega$ which satisfies

$$q_j + (0, \varepsilon_j) = p_j, \quad \text{for some } \varepsilon_j > 0.$$

According to [Catlin 1989], if we let $2m$ be the D'Angelo type of $\partial\Omega$ at the origin, there exists a homogeneous subharmonic polynomial $H(z, \bar{z})$ of degree $2m$ with no harmonic terms such that, for a certain open neighborhood $\mathcal{U}$ of $(0, 0)$,

$$(z, w) \in \Omega \cap \mathcal{U} \iff \operatorname{Re} w + H(z, \bar{z}) + R(z, \operatorname{Im} w) < 0,$$

with $R(z, \operatorname{Im} w) := o(|z|^{2m} + \operatorname{Im} w)$.

Consider the sequence of maps $A_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$A_j(z, w) = (z - a_j, w - b_j + c_j(z - a_j)),$$

where $p_j = (a_j, b_j)$ and $c_j \in \mathbb{C}$ is chosen so that the complex tangent line of $\partial A_j(\Omega)$ at $(0, 0)$ is $\{(z, w) \in \mathbb{C}^2 : w = (0, 0)\}$. Then we have $A_j(p_j) = (0, 0)$, $A_j(q_j) = (0, -\varepsilon_j)$, and

$$(z, w) \in A_j(\Omega \cap \mathcal{U}) \iff \operatorname{Re} w + \sum_{k=2}^{2m} P_{k,j}(z, \bar{z}) + R_j(z, \bar{z}, \operatorname{Im} w) < 0,$$

where the $P_{k,j}(z, \bar{z})$ are homogeneous polynomials of degree k with no harmonic terms, and

$$R_j(z, \bar{z}, \operatorname{Im} w) = o(|z|^{2m+1} + |\operatorname{Im} w|), \qquad \lim_{j\to\infty} R_j(z, \operatorname{Im} w) = R(z, \operatorname{Im} w).$$

Since the set of polynomials of degree not exceeding k is a finite dimensional vector space, we simply give an inner product. Then choose $\delta_j > 0$ so that

$$\left\| \varepsilon_j^{-1} \sum_{k=2}^{2m} P_{k,j}(\delta_j z) \right\| = 1.$$

Since $\lim_{j\to\infty} P_{k,j} = 0$ for $k < 2m$ and $P_{2m,j}$ converges to some homogeneous subharmonic polynomial of degree $2m$ with no harmonic terms, it follows that $\delta_j^{2m} \le C\varepsilon_j$ for some constant C.

Then consider the dilation map $\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2$ defined by

$$\Lambda_j(z, w) = \left(\frac{z}{\delta_j}, \frac{w}{\varepsilon_j}\right).$$

Denote by $T_j : \Omega \cap \mathcal{U} \to \mathbb{C}^2$ the transformation defined by $T_j := \Lambda_j \circ A_j \circ \varphi_j$, for each j. This T_j is called the sequence of *scaling maps*. Note that

$$(z, w) \in T_j(\Omega \cap \mathcal{U}) \iff \operatorname{Re} w + \frac{1}{\varepsilon_j}\sum_{k=2}^{2m} P_{k,j}(\delta_j z, \delta_j \bar{z}) + \frac{1}{\varepsilon_j} R_j(\delta_j z, \delta_j \bar{z}, \varepsilon_j \operatorname{Im} w) < 0.$$

Note that the sequence of polynomials $\left\{\varepsilon_j^{-1} \sum_{k=2}^{2m} P_{k,j}(\delta_j z)\right\}$ is bounded in norm. Thus it contains a convergent subsequence, converging to some polynomial $H(z, \bar{z})$ of degree at most $2m$. Since the remainder term of the defining function tends to zero as $j \to \infty$, we see that the sequence of domains $T_j(\Omega \cap \mathcal{U})$ converges to a domain $M_H := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + H(z, \bar{z}) < 0\}$ with $\|H\| = 1$. According to [Berteloot 1994], the scaling sequence forms a normal family of holomorphic mappings and the polynomial $H(z, \bar{z})$ turns out to be a homogeneous polynomial. Moreover:

Theorem 2.1 [Berteloot 1994]. *Let Ω be a domain in $\mathbb{C}^2$, and let p_0 belong to $\partial\Omega$. Assume that there exists a sequence $\{\varphi_j\}$ in* $\operatorname{Aut}(\Omega)$ *and a point $q \in \Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p_0$. If $\partial\Omega$ is a pseudoconvex and finite D'Angelo type near p_0, then Ω is biholomorphically equivalent to the model domain*

$$M_H := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re} w + H(z, \bar{z}) < 0\},$$

where $H(z, \bar{z})$ is a homogeneous subharmonic polynomial that does not contain any harmonic terms.

From this point on, we denote the biholomorphism by $\Psi : \Omega \to M_H$. We have $\Psi(q) = (0, -1) \in \mathbb{C}^2$, since $T_j(q) = (0, -1)$ for every j.

Embedded totally geodesic disc. Consider the set $\{(0, w) \in \mathbb{C}^2\} \cap M_H \subset M_H$ which is just the left half plane in the complex plane $\{0\} \times \mathbb{C}$. Let $\widetilde{D}$ be the left half plane and D the open unit disc in $\mathbb{C}$. Hence we consider the biholomorphism $\mu : D \to \widetilde{D}$ defined by

$$\mu(\zeta) = \frac{\zeta + 1}{\zeta - 1},$$

and denote the injection map by $\iota : \widetilde{D} \hookrightarrow M_H$, that is, $\iota(\zeta) = (0, \zeta)$.

There are two families of automorphisms of M_H that preserve $\widetilde{D}$:

$$\begin{aligned}\tau_s &: (z, w) \mapsto (z, w + is) \qquad \text{with } s \in \mathbb{R},\\ \eta_t &: (z, w) \mapsto (t^{1/(2m)} z, tw) \quad \text{with } t > 0.\end{aligned}$$

Since D and Ω are bounded domains, they admit Bergman metrics. We denote them by β_D and β_Ω, respectively, and for unbounded domains M_H and $\widetilde{D}$, their Bergman metrics can be defined through pull-backs. Since the mappings Ψ and μ are biholomorphisms, we define the Bergman metric on M_H by $\beta_{M_H} := (\Psi^{-1})^* \beta_\Omega$ and the Bergman metric on $\widetilde{D}$ by $\beta_{\widetilde{D}} := (\mu^{-1})^* \beta_D$. We also have:

Proposition 2.2. *The inclusion ι is an isometric embedding up to a positive constant multiple, that is, $\iota^* \beta_{M_H} = \lambda \beta_{\widetilde{D}}$ for some constant $\lambda > 0$.*

Proof. Denote by $\Gamma_{\widetilde{D}}$ the set of automorphisms of M_H that preserve $\widetilde{D}$. Then by the observation above, the action $(\gamma, x) \mapsto \gamma(x) \colon \Gamma_{\widetilde{D}} \times \widetilde{D} \to \widetilde{D}$ is transitive. Furthermore, this action is isometric with respect to the restricted Bergman metric $\beta_{M_H}|_{\widetilde{D}}$, so $\beta_{M_H}|_{\widetilde{D}}$ has constant (negative) curvature. Also, $\beta_{\widetilde{D}}$ is a positive constant multiple of the Poincaré metric. Thus the assertion follows. $\square$

3. Extension of totally geodesic disc

In this section, we discuss the extension problem up to the boundary of the isometric embedding $g := \Psi^{-1} \circ \iota \circ \mu \colon D \to \Omega$ of the unit disc D into Ω.

This g is an injective proper holomorphic mapping. Since ι is an isometric embedding, g is also an isometric embedding (up to a constant multiple). Namely, $g^* \beta_\Omega = \lambda \cdot \beta_D$, for the same constant $\lambda > 0$ as above. Set $\widehat{D} := g(D)$, the image of D by g.

The Bergman representative map. For a bounded domain Ω in $\mathbb{C}^n$, let K_Ω denote the Bergman kernel function. Following S. Bergman's original exposition, we recite the definition of his "representative domain". Since this is actually a mapping, we call it the *Bergman representative map*. The definition we use in this article is as follows:

Definition 3.1. The *Bergman representative map* $b_{\Omega,p}$ is defined by

$$b_{\Omega,p}(z) = \big(b^1_{\Omega,p}(z), \ldots, b^n_{\Omega,p}(z)\big),$$

where

$$b^k_{\Omega,p}(z) = \frac{\partial}{\partial \overline{w}_k}\bigg|_{w=p} \log \frac{K_\Omega(z, w)}{K_\Omega(w, w)}.$$

This "mapping", if well-defined, maps Ω into $\mathbb{C}^n$. This map, for each $p \in \Omega$, is known to be a local biholomorphism of a neighborhood of p onto its image that is an open neighborhood of the origin in $\mathbb{C}^n$. In this regard, we shall frequently call this map the Bergman coordinate system throughout the rest of this article.

We should remark that our definition above is not the canonical Bergman representative "domain". However, the canonical Bergman representative "domain" differs from ours by a composition of an invertible complex-linear map.

The following proposition demonstrates the role of Bergman's representative map in our context.

Proposition 3.2. *Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively, and let $b_{\Omega_1,p}$ and $b_{\Omega_2,q}$ be the Bergman coordinate systems at p in Ω_1 and at q in Ω_2, respectively. If $f : \Omega_1 \to \Omega_2$ is a Bergman isometry (not necessarily onto) with $f(p) = q$, there exists a linear map $A : \mathbb{C}^n \to \mathbb{C}^m$ such that $b_{\Omega_1,p} = A \circ b_{\Omega_2,q} \circ f$.*

Proof. Let $(z_1, \dots, z_m)$ and $(w_1, \dots, w_m)$ represent the standard complex Euclidean coordinate expressions for points in $\mathbb{C}^m$ and

$$(Z_1, \dots, Z_n) \quad \text{and} \quad (W_1, \dots, W_n)$$

for points in $\mathbb{C}^n$. We write K_1 for the Bergman kernel K_{Ω_1} and K_2 for K_{Ω_2}.

Since $f^*\beta_{\Omega_2} = \beta_{\Omega_1}$,

$$\frac{\partial^2 \log K_1(z,z)}{\partial z_a \partial \bar{z}_b}\bigg|_{z=x} = \sum_{j,k=1}^{n} \left(\frac{\partial^2 \log K_2(Z,Z)}{\partial Z_j \partial \bar{Z}_k}\bigg|_{Z=f(x)} \right) \cdot \frac{\partial f_j}{\partial z_a}\bigg|_x \overline{\frac{\partial f_k}{\partial z_b}\bigg|_x}.$$

For each $x, y \in \Omega_1$, set $K(x,y) := K_2(f(x), f(y))$. Then,

$$\begin{aligned}\frac{\partial^2 \log K(z,z)}{\partial z_a \partial \bar{z}_b}\bigg|_{z=x} &= \frac{\partial^2 \log K_2(f(z), f(z))}{\partial z_a \partial \bar{z}_b}\bigg|_{z=x} \\ &= \sum_{j,k=1}^{n} \left(\frac{\partial^2 \log K_2(Z,Z)}{\partial Z_j \partial \bar{Z}_k}\bigg|_{Z=f(x)} \right) \cdot \frac{\partial f_j}{\partial z_a}\bigg|_x \overline{\frac{\partial f_k}{\partial z_b}\bigg|_x}.\end{aligned}$$

Hence, for each $a, b = 1, \dots, m$,

$$\frac{\partial^2}{\partial z_a \partial \bar{z}_b}\bigg|_{z=x} \{\log K_1(z,z) - \log K(z,z)\} = 0, \text{ for every } x \in \Omega_1,$$

or equivalently,

$$\log K_1(z,w) - \log K(z,w) = \varphi(z) + \overline{\varphi(w)}$$

for some holomorphic function $\varphi : \Omega_1 \to \mathbb{C}$.

Consequently we obtain

$$\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\left\{\log\frac{K_1(z,w)}{K_1(w,w)}-\log\frac{K(z,w)}{K(w,w)}\right\}$$
$$=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\big((\log K_1(z,w)-\log K(z,w))-(\log K_1(w,w)-\log K(w,w))\big)$$
$$=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\big(\varphi(z)+\overline{\varphi(w)}-(\varphi(w)+\overline{\varphi(w)})\big)=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}(\varphi(z)-\varphi(w))=0$$

for every $z,\, p\in\Omega_1$. In short,

$$\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_1(z,w)}{K_1(w,w)}=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K(z,w)}{K(w,w)}.$$

Altogether,

$$\begin{aligned}
b^a_{\Omega_1,p}(z)&=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_1(z,w)}{K_1(w,w)}=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K(z,w)}{K(w,w)}\\
&=\frac{\partial}{\partial \overline{w}_a}\bigg|_{w=p}\log\frac{K_2(f(z),f(w))}{K_2(f(w),f(w))}\\
&=\sum_{k=1}^{n}\left(\frac{\partial}{\partial \overline{W}_k}\bigg|_{W=f(p)}\log\frac{K_2(f(z),W)}{K_2(W,W)}\right)\cdot\overline{\frac{\partial f_k}{\partial z_a}\bigg|_p}\\
&=\sum_{k=1}^{n}\left(\overline{\frac{\partial f_k}{\partial z_a}\bigg|_p}\right)\cdot b^k_{\Omega_2,f(p)}(f(z)).
\end{aligned}$$

So it suffices to set

$$\overline{A}:=\begin{pmatrix}\frac{\partial f_1}{\partial z_1}(p) & \cdots & \frac{\partial f_n}{\partial z_1}(p)\\ \vdots & \ddots & \vdots\\ \frac{\partial f_1}{\partial z_m}(p) & \cdots & \frac{\partial f_n}{\partial z_m}(p)\end{pmatrix},$$

so that $b_{\Omega_1,p}=A\circ b_{\Omega_2,q}\circ f$. □

Now we present Condition BR precisely.

Definition 3.3. A domain $\Omega\Subset\mathbb{C}^n$ is said to satisfy Condition BR if, for any $q\in\partial\Omega$, there exists an open neighborhood $\mathcal{U}$ of q such that the Bergman representative map $b_{\Omega,p}$ centered at p is a C^∞- coordinate system on $\mathcal{U}\cap\overline{\Omega}$ for some $p\in\mathcal{U}\cap\Omega$.

Remark 3.4. Greene and Krantz [1982, Lemma 5.7] proved that every bounded domain with smooth strongly pseudoconvex boundary satisfies Condition BR by estimating the derivatives of the Bergman kernel function near the boundary. However, for a general bounded domain, it seems nontrivial to characterize Condition

BR in terms of other boundary (geometric) invariants. For instance, it is unknown whether bounded domains with real analytic boundary satisfy Condition BR.

Despite nontriviality for the characterization of the condition, the statement of the main theorem still makes sense, since the domain E_{2m} admits *global* Bergman representative coordinates: Let $\Omega^\alpha := \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2/\alpha} + |z_2|^2 < 1\}$ for a positive real number α. The explicit formula of the Bergman kernel of Ω^α is given by

$$K_{\Omega^\alpha}(z, w) = \frac{1}{\pi^2}(1 - z_2\overline{w}_2)^{\alpha-2}\frac{(\alpha+1)(1-z_2\overline{w}_2)^\alpha + (\alpha-1)z_1\overline{w}_1}{\big((1-z_2\overline{w}_2)^\alpha - z_1\overline{w}_1\big)^3}.$$

By a straightforward computation,

$$b_{\Omega^\alpha,0}(z) = \left(\frac{4\alpha+2}{\alpha+1}\,z_1,\, (\alpha+2)z_2\right),$$

and so

$$\det\left(\frac{\partial}{\partial z_j} b_{\Omega^\alpha,0}(z)\right) = \frac{2(2\alpha+1)(\alpha+2)}{\alpha+1} \neq 0.$$

In particular, the Bergman representative map of $E_{2m} = \Omega^{1/m}$ at the origin gives rise to a global coordinate system of the domain.

Remark 3.5. E. Ligocka [1980] showed that any bounded domain with smooth boundary satisfying Condition BR should satisfy Condition B, which says that Bell–Ligocka coordinates continue to be diffeomorphic up to the boundary. It may be reasonable to expect the converse to be true, but that has yet to be clarified as far as the author is aware.

We continue our proof of the extension of g to the boundary in the next section.

Proof of extension of g. Since $g^*\beta_\Omega = \lambda \cdot \beta_D$, Proposition 3.2 implies:

Corollary 3.6. *For $\zeta \in D$ and $g(\zeta) = \hat{\zeta} \in \Omega$,*

$$(\dagger)\qquad \lambda \cdot b_{D,\zeta}(z) = \overline{g_1'(\zeta)} \cdot b^1_{\Omega,\hat{\zeta}}(g(z)) + \overline{g_2'(\zeta)} \cdot b^2_{\Omega,\hat{\zeta}}(g(z)).$$

Now consider the reflection map $r : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $r(z, w) = (-z, w)$, which is an automorphism of M_H. The fixed point set in M_H of r is exactly equal to $\widetilde{D}$, that is, $\{p \in M_H : r(p) = p\} = \widetilde{D}$. If we set $\hat{r} := \Psi^{-1} \circ r \circ \Psi : \Omega \to \Omega$, then $\hat{r}$ is an automorphism of Ω and the fixed point set of $\hat{r}$ is equal to $\widehat{D}(= g(D))$. If we choose a particular point $\hat{\zeta} = g(\zeta)$, then $\hat{r}$ is a linear reflection with respect to the $b_{\Omega,\hat{\zeta}}$-coordinates. The definition of $\hat{r}$ implies that it has two eigenvalues, $+1$ and -1. Moreover, $\widehat{D}$ is a subset of a 1-dimensional linear subspace of $\mathbb{C}^2$. Thus there exists a linear isomorphism $L : \mathbb{C}^2 \to \mathbb{C}^2$ with the matrix representation $L = (L_{jk})_{j,k=1,2}$ such that

$$(\ddagger)\qquad L_{21} \cdot b^1_{\Omega,\hat{\zeta}}(g(z)) + L_{22} \cdot b^2_{\Omega,\hat{\zeta}}(g(z)) = 0.$$

Note that $\lambda \cdot b_{D,\zeta}(z)$ is never zero near the boundary of D. Thus $(\overline{g_1'(\zeta)}, \overline{g_2'(\zeta)})$ and (L_{21}, L_{22}) are linearly independent. Thus we may apply Cramer's rule to (†) and (‡) to deduce

$$b^1_{\Omega,\hat{\zeta}}(g(z)) = \frac{\lambda \cdot L_{22}}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}} b_{D,\zeta}(z),$$
$$b^2_{\Omega,\hat{\zeta}}(g(z)) = -\frac{\lambda \cdot L_{21}}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}} b_{D,\zeta}(z).$$

We may emphasize that $b_{\Omega,\hat{\zeta}}\big(g(z)\big)$ is equal to $b_{D,\zeta}(z)$ multiplied by the constant vector

$$\frac{\lambda}{L_{22} \cdot \overline{g_1'(\zeta)} - L_{21} \cdot \overline{g_2'(\zeta)}}(L_{22}, -L_{21})$$

in $\mathbb{C}^2$. This now yields what we wanted: the map $G := b_{\Omega,\hat{\zeta}} \circ g \circ b^{-1}_{D,\zeta}$ is linear and hence smooth everywhere. Consequently the map $g = b^{-1}_{\Omega,\hat{\zeta}} \circ G \circ b_{D,\zeta}$ extends smoothly up to the boundary of D. In summary, we have

Proposition 3.7. *Let D be a unit disc in $\mathbb{C}$, and define $g : D \to \Omega$ by*

$$g(\zeta) := \Psi^{-1}\Big(0, \frac{\zeta - 1}{\zeta + 1}\Big).$$

Then the map g can extends smoothly (C^∞) up to the boundary.

Remark 3.8. This proposition does not follow directly from the general extension theorems in several complex variables; notice that the dimensions of the domains involved are not equal. It may be worth noting the existence of an example by Globevnik and Stout [1986, Example III.5]. For the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$, there exists a proper holomorphic embedding $f : D \to \mathbb{B}^2$ such that the Hausdorff dimension of the boundary of $f(D)$ (precisely speaking the set of radial boundary limit values of f) is strictly larger than 1. In particular, f cannot even extend continuously to the boundary.

4. Application to the Bedford–Pinchuk theorem

We now present the proof of the main result of this article, restating it here for convenience:

Theorem 1.2. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^2$ with smooth (C^∞) boundary satisfying Condition BR* (see Definition 3.3). *Suppose there is a point $p_0 \in \partial\Omega$ of finite D'Angelo type $2m$, a point $q \in \Omega$, and a sequence $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ such that*

$$\lim_{j\to\infty} \varphi_j(q) = p_0 \in \partial\Omega.$$

Then

$$\Omega \cong E_{2m} := \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}.$$

Start with the biholomorphism $\Psi : \Omega \to M_H$ in Theorem 2.1, with $\Psi(q) = (0, -1)$, and recall the automorphisms τ_s and η_t of M_H defined as follows:

$$\tau_s(z, w) := (z, w + is) \quad \text{for } s \in \mathbb{R}$$

$$\eta_t(z, w) := (t^{1/2m} z, tw) \quad \text{for } t > 0.$$

Define the automorphism h_t of Ω by $h_t := \Psi^{-1} \circ \eta_t \circ \Psi$. Since η_t preserves $\widetilde{D}$, there exists $\ell_t \in \mathrm{Aut}(D)$ such that $g \circ \ell_t = h_t \circ g$. (Note here that every automorphism of the unit disc D extends holomorphically across the boundary of D.)

Lemma 4.2. *There exists a unique boundary point $\tilde{p}$ of Ω such that*

$$\lim_{t \to 0} h_t(q) = \tilde{p}.$$

Proof. Since $q = g(0)$, $h_t(q) = h_t(g(0))$. So

$$\lim_{t\to 0} h_t(q) = \lim_{t\to 0} h_t \circ g(0) = \lim_{t\to 0} g \circ \ell_t(0) = g\big(\lim_{t\to 0} \ell_t(0)\big) = g(1),$$

since $g : D \to \Omega$ extends to the boundary. Thus it suffices to let $g(1) = \tilde{p}$. Notice that $\tilde{p} \in \partial\Omega$ since g is proper. □

Note that h_t, for any $0 < t < 1$, fixes the boundary point $\tilde{p}$, and also that

$$h_t \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$$

due to Condition BR. Notice that $dh_t|_{\tilde{p}}$ has two eigenvalues, t and $t^{1/2m}$. Hence Lemma 4.2 implies that h_t is a contracting automorphism at $\tilde{p}$. At this step, note that whether $\partial\Omega$ is of D'Angelo finite type at $\tilde{p}$ is unclear. So we apply the following result:

Theorem 4.3 [Kim and Yoccoz 2011]. *Suppose that Ω is a bounded domain in $\mathbb{C}^n$ with a smooth boundary. If there exists $h \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$ that is contracting at a boundary point $\tilde{p}$, then $\partial\Omega$ at $\tilde{p}$ is of finite type in the sense of D'Angelo. Moreover, the boundary $\partial\Omega$ is defined by a weighted homogeneous polynomial determined completely by the resonance set of the contraction h.*

Therefore our $\tilde{p}$ is of finite type in the sense of D'Angelo and Ω is biholomorphic to the domain M_P defined by $M_P := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}\, w + P(z, \bar{z}) < 0\}$, where $P(z, \bar{z})$ is a weighted homogeneous polynomial. But since z is a single variable, our P is in fact homogeneous. According to Oeljeklaus [1993], deg $P = \deg H = 2m$. Therefore the domain Ω is biholomorphic to the domain which is defined by the homogeneous polynomial of degree $2m$.

It remains to show that the homogeneous polynomial P actually is equal to $|z|^{2m}$. For this purpose we shall follow the original method of Bedford and Pinchuk by constructing a parabolic automorphisms fixing $\tilde{p}$.

Define the automorphism k_s of Ω by $k_s := \Psi^{-1} \circ \tau_s \circ \Psi$. As before, there exists an automorphism m_s of D such that $g \circ m_s = k_s \circ g$.

Lemma 4.4. $\lim_{s\to\pm\infty} k_s(q)$ *is a single boundary point of* Ω. *Moreover, this limit point is the same as* $\tilde{p}$.

Proof. Since $q = g(0)$, $k_s(q) = k_s(g(0))$. So,

$$\lim_{s\to\pm\infty} k_s(q) = \lim_{s\to\pm\infty} k_s \circ g(0) = \lim_{s\to\pm\infty} g \circ m_s(0) = g\big(\lim_{s\to\pm\infty} m_s(0)\big) = g(1).$$

Hence the assertion follows. □

Notice again that $k_s \in \mathrm{Aut}(\Omega) \cap \mathrm{Diff}(\overline{\Omega})$ by Condition BR. Moreover, k_s preserves $\partial\Omega$ and fixes $\tilde{p}$. Hence Lemma 4.4 implies that k_s is parabolic with the limit point at $\tilde{p}$. Altogether, $\tilde{p}$ is the point fixed by the contraction h_t and the parabolic automorphisms k_s.

This allows to use the analysis of [Bedford and Pinchuk 1988] so that we may conclude that $H(z, \bar{z}) = c|z|^{2m}$. Therefore Ω is biholomorphic to the Thullen domain $E_{2m} := \{(z, w) \in \mathbb{C}^2 : \mathrm{Re}\, w + |z|^{2m} < 0\}$.

Remark 4.5. In Bedford and Pinchuk's result (Theorem 1.1), the exponent $2m$ for the Thullen domain in its conclusion is not clearly specified, since it comes from the type of the boundary point that arises as the limit point of the parabolic orbit produced in the proof. With the assumption of noncompactness of the automorphism group, Pinchuk's scaling produces a parabolic orbit. But the location of the limit point of this parabolic orbit is arbitrary. That is why the global finiteness of the D'Angelo type of the boundary (which follows in particular by the real analyticity) was assumed in the first place. In our case, on the other hand, we prove that the limit point of the parabolic orbit is also the limit point of a contraction — which follows by the extension theorem of the special totally geodesic disc (Proposition 3.7) — and hence the limit point has to be of D'Angelo finite type by the Kim–Yoccoz result (Theorem 4.3). Then we could further show, combining these results with that of a theorem of Oeljeklaus, that the exponent must actually be the D'Angelo type of the original boundary orbit accumulation point, as stated.

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REGULARITY OF THE FIRST EIGENVALUE OF THE p-LAPLACIAN AND YAMABE INVARIANT ALONG GEOMETRIC FLOWS

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We first prove that the first eigenvalue of the p-Laplace operator and the Yamabe invariant are both locally Lipschitz along geometric flows under weak assumptions without assumptions on curvature. Secondly, the Yamabe invariant is found to be directionally differentiable along geometric flows. As an application, an open question about the Yamabe metric and Einstein metric is partially answered.

1. Introduction

Motivated by the Hamilton's Ricci flow, the method of geometric flow has been widely used to deal with geometric and topological properties of manifolds. We often encounter the derivative of geometric quantities when applying the method of geometric flow. Cao [2007; 2008] and Li [2007] consider the monotonicity of the first eigenvalue of $-\Delta + cR$ ($c \geq \frac{1}{4}$) based on their derivatives along Ricci flow. Ling [2007] proved a comparison theorem for the eigenvalue of the Laplace operator based on its derivative along Ricci flow. Unfortunately, there are many geometric quantities about which we don't know whether they are differentiable along the flow. Chang and Lu [2007] derive a formula for the derivative of the Yamabe constant along Ricci flow under a crucial technical assumption. Recently Wu, Wang and Zheng [Wu et al. 2010] considered the first eigenvalue of the p-Laplace operator, whose differentiability along Ricci flow is unknown.

For the first eigenvalue of a linear operator, we may assume that there is a C^1-family of smooth eigenvalues and eigenfunctions along geometric flow by eigenvalue perturbation theory. We have no uniform method to deal with the smoothness of the first eigenvalue of a nonlinear operator — even the continuity is unknown.

As the first eigenvalue can be seen as a minimum of a functional, we consider the regularity of geometric quantities of this type along geometric flow. Inspired

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by the method used in [Hamilton 1986; Chow and Lu 2002] to prove the maximum principle for systems, we first study the relationship between the local Lipschitz property of continuous functions and their Dini derivatives.

Theorem 1.1. *Let $m(t)$ be a continuous function on an interval $\mathcal{I} \subset \mathbb{R}$. Suppose that for any $t \in \mathcal{I}$ there exists a C^1 function $M(t, s)$ of s defined on a neighborhood of t such that $M(t, t) = m(t)$ and $M(t, s) \geq m(s)$.*

(1) *If $(\partial M/\partial s)(t, t)$ is locally bounded, then $m(t)$ is locally Lipschitz.*

(2) *For any t in the interior of $\mathcal{I}$, if $m(t)$ is differentiable at t, then $m'(t) = (\partial M/\partial s)(t, t)$.*

Remark. By (2), if $m(t)$ is differentiable at an interior point t, then the derivative of $m(t)$ at this point is exactly $(\partial M/\partial s)(t, t)$, regardless of the choice of function $M(t, s)$.

Corollary 1.1.1. *In the same setting of Theorem 1.1, if $(\partial M/\partial s)(t, t)$ is locally bounded, then $m(t)$ is differentiable almost everywhere and $m'(t) = (\partial M/\partial s)(t, t)$ almost everywhere.*

Applying Theorem 1.1, we get the following results on the regularities of the first eigenvalues $\lambda_{1,p}$ of the p-Laplace operator and the Yamabe invariant along the general C^1 family of smooth geometric flows in this paper. We find that the first eigenvalue $\lambda_{1,p}$ of the p-Laplace operator is in general locally Lipschitz continuous. We also get local Lipschitz continuity of the Yamabe invariant and find its derivative with respect to t almost everywhere.

Theorem 1.2. *Let $g(x, t)$ be a C^1 family of smooth metrics on a n-dimensional compact Riemannian manifold M. Then the first eigenvalue $\lambda_{1,p}(g(t))$ of the p-Laplace operator is locally Lipschitz if $p \geq 2$ and M is closed or if $p > 1$ and M has nonempty boundary.*

Remark. In [Wu et al. 2010], a similar result on local Lipschitz continuity was obtained, but under some assumptions on curvature. Theorem 1.2 implies that local Lipschitz continuity should be available for more general smooth geometric flows without any curvature conditions.

Theorem 1.3. *Suppose M is an n-dimensional $(n \geq 3)$ closed connected Riemannian manifold, and $g(t)$, $t \in [0, T)$, is a C^1 family of smooth metrics on M. If $\bar{g}(t)$ is the Yamabe metric in the conformal class $[g(t)]$ for any $t \in [0, T)$, then the Yamabe invariant $\mathcal{Y}(g(t))$ is locally Lipschitz with respect to t, and*

$$\frac{d\mathcal{Y}(g(t))}{dt} \overset{\text{a.e.}}{=} -\int \frac{\bar{g}(t)}{g(t)} \left\langle \frac{\partial g}{\partial t}(t), \operatorname{Rc}^0(\bar{g}(t)) \right\rangle_{\bar{g}(t)} d\mu_{\bar{g}(t)} \operatorname{vol}(\bar{g}(t))^{-2/p}, \tag{1-1}$$

where a.e. stands for "almost everywhere", $p = \dfrac{2n}{n-2}$, and $\operatorname{Rc}^0(\bar{g}(t))$ is the trace-free part of Ricci curvature of $\bar{g}(t)$.

Let g_0 be a smooth metric on manifold M, $[g_0]$ be the conformal class of g_0, $\Lambda[g_0]$ be the collection of Yamabe metrics in $[g_0]$ and h be a smooth $(0,2)$-type symmetric tensor on M. Denote by $G_h(g_0, t)$ the collection of C^1 family of smooth metrics $g(t)$, $t \in [0, \varepsilon)$ with $g(0) = g_0$ and $(\partial g/\partial t)(0) = h$ for some $\varepsilon > 0$, we define $\Lambda_h[g_0]$ by

$$\Lambda_h[g_0] := \bigcup_{g(t)\in G_h(g_0,t)} \{\, \bar{g}_0 \in \Lambda[g_0] : \bar{g}_0 \text{ is an accumulation point of } \Lambda[g(t)] \text{ as } t \to 0 \,\},$$

where $\bar{g}_0$ generally exists by the compactness of $\Lambda[g_0]$ when $[g_0] \neq [g_{\mathrm{can}}]$, where g_{can} denotes the canonical metric on S^n [Anderson 2005]. (They prove that if $[g_i] \to [g_0] \neq [g_{\mathrm{can}}]$ smoothly, then every sequence of Yamabe metrics $(g^j)_i \in [g_i]$ has a subsequence converging smoothly to a Yamabe metric $[g^j] \in [g_0]$.) It is easy to see that if $\bar{g}_0 \in \Lambda_h[g_0]$ then $c\bar{g}_0 \in \Lambda_h[g_0]$ and $\bar{g}_0 \in \Lambda_{ch}[g_0]$ for any $c > 0$.

Recently, Brendle [2008] with Marques [2009] gave counterexamples to the compactness for a full set of solutions to the Yamabe equation if the dimension of the manifold greater than 24. Later, Khuri, Marques and Schoen [Khuri et al. 2009] proved compactness if the dimension equal or less than 24.

In addition to Theorem 1.3, we have the following derivative calculation at $t = 0$.

Theorem 1.4. *Let $g(t)$, $t \in [0, T)$, be a C^1 family of smooth metrics on a manifold M and g_{can} be the canonical metric on S^n. If $g(0) = g_0$ and $[g_0] \neq [g_{\mathrm{can}}]$, then*

$$\left.\frac{d\mathcal{Y}(g(t))}{dt}\right|_{t=0} = \min_{\tilde{g}_0\in\Lambda[g_0]} \left\{ -\int \frac{\tilde{g}_0}{g_0} \left\langle \frac{\partial g}{\partial t}(0), \mathrm{Rc}^0(\tilde{g}_0) \right\rangle_{\tilde{g}_0} d\mu_{\tilde{g}_0} \operatorname{vol}(\tilde{g}_0)^{-2/p} \right\} \tag{1-2}$$

$$= -\int \frac{\bar{g}_0}{g_0} \left\langle \frac{\partial g}{\partial t}(0), \mathrm{Rc}^0(\bar{g}_0) \right\rangle_{\bar{g}_0} d\mu_{\bar{g}_0} \operatorname{vol}(\bar{g}_0)^{-2/p},$$

where $p = 2n/(n-2)$, $\bar{g}_0 \in \Lambda_{(\partial g/\partial t)(0)}[g_0]$, $\mathrm{Rc}^0(\tilde{g}_0)$ is the trace-free part of Ricci curvature with respect to $\tilde{g}_0$ and $\operatorname{vol}(\tilde{g}_0)$ is the volume of M respect to $\tilde{g}_0$. In particular, $\mathcal{Y}$ is directionally differentiable at g_0.

Remark. This formula generalizes similar calculations in [Anderson 2005] where $\operatorname{tr}(\partial g/\partial t)(0) = 0$, $\operatorname{vol}(g(t)) = 1$, and g_0 has constant scalar curvature. Meanwhile, when $\bar{g}_0 \in \Lambda_{(\partial g/\partial t)(0)}[g_0]$, Equation (1-2) becomes more convenient to calculate, compared to the derivative calculation in [Anderson 2005] (in another form):

$$\min_{\tilde{g}_0} \left\{ -\int \frac{\tilde{g}_0}{g_0} \left\langle \mathrm{Rc}^0(\tilde{g}_0), \frac{\partial g}{\partial t}(0) \right\rangle_{\tilde{g}_0} d\mu_{\tilde{g}_0} \right\}, \tag{1-3}$$

where $\tilde{g}_0 \in \Lambda_1[g_0]$ is taken over all accumulation points of $\Lambda_1[g(t)]$ as $t \to 0$ for $\Lambda_1[g_0]$ the set of unit volume Yamabe metrics in $[g_0]$. The derivative is difficult to calculate using this formula, but by Theorem 1.4 we can calculate this derivative if we know a Yamabe metric $\bar{g}_0$ in $\Lambda_{(\partial g/\partial t)(0)}[g_0]$. Moreover, the set to minimize in (1-3) has only one element by the last equality in (1-2).

In addition to the local Lipschitz property of the Yamabe invariant, we have:

Corollary 1.4.1. *With the same assumptions as in Theorem 1.3, the Yamabe invariant $\mathcal{Y}(g(t))$ is directionally differentiable at all t where $[g(t)] \neq [g_{\mathrm{can}}]$.*

In particular, in formula (1-2), if $(\partial g/\partial t)(0) = -2\,\mathrm{Rc}(g_0)$ and g_0 is a Yamabe metric in $\Lambda_{-\mathrm{Rc}(g_0)}[g_0]$, then $g_0 \in \Lambda_{-2\,\mathrm{Rc}(g_0)}[g_0]$ and $R(g_0)$ is constant, hence

$$\left.\frac{d\mathcal{Y}(g(t))}{dt}\right|_{t=0} = \int \langle \mathrm{Rc}(g_0), \mathrm{Rc}^0(g_0)\rangle_{g_0}\, d\mu_{g_0}\, \mathrm{vol}(g_0)^{-2/p}$$
$$= \int \left|\mathrm{Rc}^0(g_0)\right|^2 d\mu_{g_0}\, \mathrm{vol}(g_0)^{-2/p} \geq 0.$$

Ricci flow evolves sphere to sphere, so we have the following conclusion along the Ricci flow.

Corollary 1.4.2. *Let M^n be a closed and connected manifold with $n \geq 3$ and $g(t)$, $t \in [0, T)$, be a solution of Ricci flow $\partial g/\partial t = -2\,\mathrm{Rc}$ on M with $g(0) = g_0$. If $g_0 \in \Lambda_{-\mathrm{Rc}}[g_0]$, then $d\mathcal{Y}(g(t))/dt|_{t=0} \geq 0$ and $d\mathcal{Y}(g(t))/dt|_{t=0} = 0$ if only if g_0 is a Einstein metric.*

Remark. There is a similar result in [Chang and Lu 2007] under the assumption that there exists a C^1 family of $\phi(t) > 0$ such that $\phi(t)^{4/(n-2)} g(t)$ is a Yamabe metric and $\phi(0)$ is constant. From the definition of $\Lambda_{-\mathrm{Rc}}[g_0]$ we can see that our assumption is weaker.

Let $\mathcal{C}$ denote the set of unit volume constant scalar curvature metrics on a connect closed manifold M; it is well-known (see [Besse 1987]) that generically $\mathcal{C}$ is an infinite-dimensional manifold. Let $s : \mathcal{C} \mapsto \mathbb{R}$ be the scalar curvature function. It has long been an open problem whether a Yamabe metric which is a local maximizer of s is necessarily an Einstein metric [Besse 1987]. Some progress on this question was made in [Bessieres et al. 2003] in dimension 3 and in [Anderson 2005] in any dimension. Let $\mathcal{M}$ be the collection of all smooth metrics on M and $\mathcal{Y} : \mathcal{M} \mapsto \mathbb{R}$ be the Yamabe invariant function. By the definition of the Yamabe invariant, $s(g) \geq \mathcal{Y}(g)$ for any $g \in \mathcal{C}$, hence if a Yamabe metric is a local maximizer of s, it must be a local maximizer of $\mathcal{Y}$. Now, we consider whether a Yamabe metric that is a local maximizer of the Yamabe invariant is necessarily an Einstein metric. Following from Corollary 1.4.2, the next result gives a partial answer.

Corollary 1.4.3. *Let M^n be a closed and connected manifold with $n \geq 3$ and suppose a Yamabe metric g is a local maximum of the Yamabe invariant functional $\mathcal{Y}(\,\cdot\,)$. If $g \in \Lambda_{-\mathrm{Rc}}[g]$, then g is Einstein.*

In Section 2, we give a basic introduction to Dini derivatives and the proof of Theorem 1.1. In Section 3, we prove the Lipschitz property of the first eigenvalue of the p-Laplace operator along geometric flows. In Section 4, we show that the Yamabe invariant is locally Lipschitz and directionally differentiable along geometric flows.

2. Dini derivatives and the proof of Theorem 1.1

In this section, we first recall the definitions of Dini derivatives and semicontinuity. Then we give some propositions about Dini derivatives. Lastly, we prove Theorem 1.1.

Hamilton [1986] studied properties of Lipschitz functions by means of their Dini derivatives, and from this derived the maximum principle for systems on closed manifolds. Chow [2002] proved similar results in weaker settings. Dini derivatives provide a powerful way to deal with nonregular functions.

These definitions of Dini derivatives and semicontinuity also appear in [Chow et al. 2008].

Definition 2.1 (Dini derivatives). Let $f(t)$ be a function on (a, b). The upper Dini derivative is the lim sup of forward difference quotients:

$$\frac{d^+f}{dt}(t) := \limsup_{h\to 0^+} \frac{f(t+h)-f(t)}{h}.$$

The lower Dini derivative is the lim inf of forward difference quotients:

$$\frac{d^-f}{dt}(t) := \liminf_{h\to 0^+} \frac{f(t+h)-f(t)}{h}.$$

The upper converse Dini derivative is the lim sup of backward difference quotients:

$$\frac{d_+f}{dt}(t) := \limsup_{h\to 0^+} \frac{f(t)-f(t-h)}{h}.$$

The lower converse Dini derivative is the lim inf of backward difference quotients:

$$\frac{d_-f}{dt}(t) := \liminf_{h\to 0^+} \frac{f(t)-f(t-h)}{h}.$$

If the function f is also defined at a, we can define its upper Dini derivative and lower Dini derivative at a; and if the function f is also defined at b, we can define its upper converse Dini derivative and lower converse Dini derivative at b.

Since we don't make any assumption on the function $f(t)$, it is possible that any one of the Dini derivatives of $f(t)$ above may take the value $+\infty$ or $-\infty$.

Definition 2.2 (semicontinuity). Let $f(t)$ be a function on an interval. We say f is right upper semicontinuous if $\limsup_{h\to 0^+} f(t+h) \le f(t)$; we say f is right lower semicontinuous if $\liminf_{h\to 0^+} f(t+h) \geq f(t)$; we say f is left upper semicontinuous if $\limsup_{h\to 0^+} f(t-h) \leq f(t)$; we say f is left lower semicontinuous if $\liminf_{h\to 0^+} f(t-h) \geq f(t)$.

Lemma 2.3. *If $f(t):(a,b)\to\mathbb{R}$ is left lower semicontinuous with $(d^+f/dt)(t)\leq 0$, then $f(t)$ is decreasing.*

Proof. Given $\varepsilon > 0$, define $f_\varepsilon(t) := f(t) - \varepsilon t$. We shall show that $f_\varepsilon(t) \leq f_\varepsilon(s)$ for any $a < s \leq t < b$. The lemma then follows from taking $\varepsilon \to 0$.

Since $(d^+f/dt)(s) \leq 0$, we have $(d^+f_\varepsilon/dt)(s) \leq -\varepsilon$, then there exists a number $\delta(s,\varepsilon) > 0$ such that $(f_\varepsilon(s+h) - f_\varepsilon(s))/h \leq -\varepsilon/2 < 0$ for all $h \in (0, \delta(s,\varepsilon))$, hence $f_\varepsilon(t) \leq f_\varepsilon(s)$ on $h \in [s, s+\delta(s,\varepsilon))$. Define $\tau(\varepsilon, s) \in [s,b]$ by

$$\tau := \sup\left\{\tau' \in [s,b] : f_\varepsilon(t) \leq f_\varepsilon(s) \text{ for all } t \in [s,\tau')\right\}.$$

then $\tau \geq s+\delta(s,\varepsilon) > s$. One can check that, in fact, $f_\varepsilon(t) \leq f_\varepsilon(s)$ for all $t \in [s,\tau)$. We now prove $\tau = b$ to complete the proof. If for some s and $\varepsilon > 0$, we have $\tau < b$, then there exists a sequence of times $\{\tau_i\} \nearrow \tau$, such that $f_\varepsilon(s) \geq f_\varepsilon(\tau_i - 1/2^i)$ when i is large enough. Hence

$$f_\varepsilon(s) \geq \liminf_{i\to\infty} f_\varepsilon(\tau_i - 1/2^i) \geq \liminf_{h\to 0^+} f_\varepsilon(\tau - h) \geq f_\varepsilon(\tau)$$

follows from the left lower semicontinuity of $f_\varepsilon(t)$. Applying the above procedure again by replacing s with τ gives $f_\varepsilon(t) \leq f_\varepsilon(\tau) \leq f_\varepsilon(s)$ when $t \in [\tau, \tau+\delta(\tau,\varepsilon))$, hence $f_\varepsilon(t) \leq f_\varepsilon(s)$ when $t \in [s, \tau+\delta(\tau,\varepsilon))$. This is a contradiction since the definition of τ implies $\delta(\tau,\varepsilon) \leq 0$. □

Note. A similar conclusion can be found in [Chow et al. 2008]. There, the domain of f is $[0,T)$, hence f must be both left lower semicontinuous and right upper semicontinuous. Here we choose the domain of f to be (a,b), so we can weaken the assumptions on f.

Proposition 2.4. (a) *If $f(t) : (a,b) \to \mathbb{R}$ is left lower semicontinuous, then $d^+f/dt \leq 0$ if and only if $f(t)$ is decreasing.*

(b) *If $f(t) : (a,b) \to \mathbb{R}$ is right upper semicontinuous, then $d_+f/dt \leq 0$ if and only if $f(t)$ is decreasing.*

(c) *If $f(t) : (a,b) \mapsto \mathbb{R}$ is left upper semicontinuous, then $d^-f/dt \geq 0$ if and only if $f(t)$ is increasing.*

(d) *If $f(t) : (a,b) \mapsto \mathbb{R}$ is right lower semicontinuous, then $d_-f/dt \geq 0$ if and only if $f(t)$ is increasing.*

Proof. (a) If $f(t)$ is decreasing then $d^+f/dt \leq 0$. The other direction follows from Lemma 2.3.

(b)–(d) follows from applying part (a) to the functions $-f(-t)$, $-f(t)$, and $f(-t)$, respectively. □

From this propositions, we see that a semicontinuous function is monotonic if certain types of its Dini derivatives have a definite sign. A further analysis shows that monotonicity can be a nice bridge between Dini derivatives of different type.

Claim 2.5. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $\mathring{\mathcal{I}}$ be its interior.*

(a) *If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right upper semicontinuous and left lower semicontinuous,*

$$\frac{d^+f}{dt} \le 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is decreasing on } \mathcal{I} \iff \frac{d_+f}{dt} \le 0 \text{ in } \mathring{\mathcal{I}}.$$

(b) *If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right lower semicontinuous and left upper semicontinuous,*

$$\frac{d^-f}{dt} \ge 0 \text{ in } \mathring{\mathcal{I}} \iff f(t) \text{ is increasing on } \mathcal{I} \iff \frac{d_-f}{dt} \ge 0 \text{ in } \mathring{\mathcal{I}}.$$

Proof. We prove the first part; the second is similar. If $f(t) : \mathcal{I} \mapsto \mathbb{R}$ is right upper semicontinuous and left lower semicontinuous, then $f(t)$ is decreasing on $\mathcal{I}$ if and only if $f(t)$ is decreasing on $\mathring{\mathcal{I}}$. The conclusion then follows from parts (a) and (b) of Proposition 2.4. □

Theorem 2.6. *If $f : (a, b) \mapsto \mathbb{R}$ is a continuous function with d^+f/dt or d_+f/dt locally bounded from above and d^-f/dt or d_-f/dt locally bounded from below, then f is locally Lipschitz.*

Proof. Given any $s \in (a, b)$, let $U(s)$ be a compact and connected neighborhood of s in (a, b). Then on $U(s)$, without loss of generality, we can assume $d^+f/dt \le A$ or $d_+f/dt \le A$ and $d^-f/dt \ge -A$ or $d_-f/dt \ge -A$, where $A > 0$ is a constant. Hence $d^+(f - At)/dt \le 0$ (or $d_+(f - At)/dt \le 0$) by parts (a) and (b) of Proposition 2.4, and $d_-(f + At)/dt \ge 0$ (or $d^-(f + At)/dt \ge 0$) by parts (c) and (d). Then $f - At$ is decreasing and $f + At$ is increasing on $U(s)$ by Claim 2.5. Thus $|f(t_2) - f(t_1)| \le A\,|t_2 - t_1|$ for any $t_1, t_2 \in U(s)$, so f is locally Lipschitz. □

Proof of Theorem 1.1. Since $M(t, t) = m(t)$ and $M(t, s) \ge m(s)$ in a neighborhood of t, we have

$$\tag{2-1} \begin{aligned} \frac{d^+m}{dt}(t) &= \limsup_{h\to 0^+} \frac{m(t+h)-m(t)}{h} \le \limsup_{h\to 0^+} \frac{M(t, t+h)-M(t, t)}{h} \\ &= \frac{\partial M}{\partial s}(t, t), \end{aligned}$$

$$\tag{2-2} \begin{aligned} \frac{d_-m}{dt}(t) &= \liminf_{h\to 0^+} \frac{m(t)-m(t-h)}{h} \ge \liminf_{h\to 0^+} \frac{M(t, t)-M(t, t-h)}{h} \\ &= \frac{\partial M}{\partial s}(t, t). \end{aligned}$$

Since $(\partial M/\partial s)(t,t)$ is locally bounded, $(d^+m/dt)(t)$ is bounded from above and $(d_-m/dt)(t)$ is bounded from below. Then by Theorem 2.6, the function $m(t)$ is locally Lipschitz in the interior of $\mathcal{I}$.

Let a be the left endpoint of $\mathcal{I}$, b be the right endpoint of $\mathcal{I}$. If $a \in \mathcal{I}$, let $c = \min\{a+1, (a+b)/2\}$. Since $(\partial M/\partial s)(t,t)$ is locally bounded on $\mathcal{I}$, we can assume that $|(\partial M/\partial s)(t,t)| \le A$ (A is a constant) on $[a,c]$. Then $d^+(m(t) - At)/dt \le 0$ on $[a,c)$ and $d_-(m(t) + At)/dt \ge 0$ on $(a,c]$ by (2-1) and (2-2). Hence by part (a) of Claim 2.5, the function $m(t) - At$, is decreasing on $[a,c]$, and by part (b), the function $m(t) + At$ is increasing on $[a,c]$. Then $|m(t_1) - m(t_2)| \le A\,|t_1 - t_2|$ for any $t_1, t_2 \in [a,c]$, so $m(t)$ is locally Lipschitz at $t = a$. Similarly, if $b \in \mathcal{I}$, then $m(t)$ is locally Lipschitz at $t = b$. In conclusion, $m(t)$ is locally Lipschitz on $\mathcal{I}$.

For any t in the interior of $\mathcal{I}$, if $m(t)$ is differentiable at this point, then by (2-1) we have $m'(t) = (d^+m/dt)(t) \le (\partial M/\partial s)(t,t)$, and by (2-2) we have $m'(t) = (d_-m/dt)(t) \ge (\partial M/\partial s)(t,t)$. Hence $m'(t) = (\partial M/\partial s)(t,t)$. □

3. First eigenvalue of the p-Laplacian

In this section we consider the local Lipschitz property of the p-Laplace operator along general geometric flows. Let (M, g) be a compact connected Riemannian manifold. Define

$$G(f,g) := \frac{\int_M |\nabla f|_g^p \, d\mu_g}{\int_M |f|^p \, d\mu_g},$$

where $\nabla f = df$ is a covariant vector. Recalling the definition of the first eigenvalue $\lambda_{1,p}(g)$ of the p-Laplace operator, it is known that if $\partial M \ne \varnothing$ then

$$\lambda_{1,p}(g) := \inf\bigl\{G(f,g) : f \in W_0^{1,p}(M),\ f \ne 0\bigr\}$$

and if M is closed then

$$\lambda_{1,p}(g) := \inf\bigl\{G(f,g) : f \in W^{1,p}(M),\ \int_M |f|^{p-2} f \, d\mu_g = 0,\ f \ne 0\bigr\}.$$

The minimum (a positive number) is achieved by a $C^{1,\alpha}$ $(0 < \alpha < 1)$ eigenfunction f (see [Serrin 1964; Tolksdorf 1984]). This eigenfunction f satisfies the Euler–Lagrange equation

$$\Delta_p f = -\lambda_{1,p}(g)|f|^{p-2} f,$$

where Δ_p $(p > 1)$ is the p-Laplace operator with respect to g given by

$$\Delta_p f = \operatorname{div}_g(|\nabla f|_g^{p-2} \nabla f).$$

The following theorem implies that $\lambda(g(t))$ is continuous with respect to t along general geometric flows.

Theorem 3.1 [Wu et al. 2010]. *If g_1 and g_2 are two metrics on M which satisfy $(1+\varepsilon)^{-1}g_1 \le g_2 \le (1+\varepsilon)g_1$, then for any $p > 1$, we have*

$$(1+\varepsilon)^{-(n+p/2)} \le \frac{\lambda_{1,p}(g_1)}{\lambda_{1,p}(g_2)} \le (1+\varepsilon)^{(n+p/2)}.$$

Let $f \in C^{1,\alpha}(M)$ be nonconstant and $g(x,t)$, $t \in [0,T)$, be a C^1 family of smooth metrics on M. Define a function of $c \in (-\infty,\infty)$ and $t \in [0,T)$:

$$P(c,t) := \int_M |f+c|^{p-2}\,(f+c)\,d\mu_{g(t)}, \quad p \ge 2.$$

The function $P(c,t)$ is C^1 with respect to c and t, since

$$\frac{\partial P}{\partial c} = (p-1)\int_M |f+c|^{p-2}\,d\mu_{g(t)} > 0.$$

Then by the implicit function theorem, given any c_0 and t_0 there exists a C^1 function $c(t)$ defined on a neighborhood of t_0 such that $P(c(t),t) = P(c_0,t_0)$.

In this and the next sections, if f is a real function on M, we simply write $\sup f$ instead of $\sup_{x\in M} f(x)$. Let $g(t)$ be a family of Riemannian metrics on manifold. If $\alpha(t)$ is a family of $(0,2)$-type tensors, we denote by $\operatorname{tr}\alpha(t) = g^{ij}(t)\alpha_{ij}(t)$ its trace with respect to $g(t)$ and by

$$|\alpha(t)|_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\alpha_{jl}(t)}$$

its norm with respect to $g(s)$; if $\beta(t)$ is also a family of $(0,2)$-type tensors, we denote by

$$\langle\alpha(t),\beta(t)\rangle_{g(s)} = \sqrt{g^{ij}(s)g^{kl}(s)\alpha_{ik}(t)\beta_{jl}(t)}$$

the inner product derived from the metric $g(s)$. Moreover, we use $|\alpha(t)|$ instead of $|\alpha(t)|_{g(t)}$ and $\langle\alpha(t),\beta(t)\rangle$ instead of $\langle\alpha(t),\beta(t)\rangle_{g(t)}$ for simplicity.

Proof of Theorem 1.2. For any t_0, let $f(t_0)$ be a minimizer of $G(\,\cdot\,,g(t_0))$. If M is closed and $p \ge 2$, then $f(t_0)$ is a nonconstant C^1 function on M with

$$\int_M |f(t_0)|^{p-2}\,f(t_0)\,d\mu_{g(t_0)} = 0.$$

Hence there is a continuous differentiable function $c(t_0,s)$ of s defined in a neighborhood of t_0 such that $c(t_0,t_0) = 0$ and

$$\int_M |f(t_0)+c(t_0,s)|^{p-2}\,(f(t_0)+c(t_0,s))\,d\mu_{g(s)} = 0.$$

Otherwise if $\partial M \ne \varnothing$ and $p > 1$, we can just take $c(t_0,s) \equiv 0$.

Let $N(t,s)=G(f(t)+c(t,s),g(s))$. Then $N(t,t)=\lambda_{1,p}(g(t))$ and $N(t,s)\geq\lambda_{1,p}(g(s))$. A simple calculation gives

$$
\begin{aligned}
&\frac{\partial N}{\partial s}(t,t)\\
&=\frac{\partial}{\partial s}\Big|_{s=t}\frac{\int_M|\nabla(f(t)+c(t,s))|^p_{g(s)}\,d\mu_{g(s)}}{\int_M|f(t)+c(t,s)|^p\,d\mu_{g(s)}}\\
&=\biggl(\int_M\Bigl(|\nabla f(t)|^p_{g(t)}\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}-\frac{p}{2}|\nabla f(t)|^{p-2}_{g(t)}\frac{\partial g}{\partial s}(t)(\nabla f(t),\nabla f(t))\Bigr)\,d\mu_{g(t)}\\
&\qquad-\lambda_{1,p}(g(t))\int_M\Bigl(|f(t)|^p\,\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}+p\,|f(t)|^{p-2}f(t)\frac{\partial c}{\partial s}(t,t)\Bigr)\,d\mu_{g(t)}\biggr)\\
&\qquad\qquad\times\Bigl(\int_M|f(t)|^p\,d\mu_{g(t)}\Bigr)^{-1}.
\end{aligned}
$$

To simplify this formula, we use that

$$\int_M p\,|f(t)|^{p-2}f(t)\frac{\partial c}{\partial s}(t,t)\,d\mu_{g(t)}=0.$$

When M is closed, this follows from $\int_M p\,|f(t)|^{p-2}f(t)\,d\mu_{g(t)}=0$, and when $\partial M\neq\varnothing$, from $c(t,s)\equiv 0$. Hence we get

$$
\begin{aligned}
\text{(3-1)}\quad\frac{\partial N}{\partial s}(t,t)=\biggl(\int_M\Bigl(&|\nabla f(t)|^p_{g(t)}\,\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}\\
&-\frac{p}{2}|\nabla f(t)|^{p-2}_{g(t)}\frac{\partial g}{\partial s}(t)(\nabla f(t),\nabla f(t))\Bigr)\,d\mu_{g(t)}\\
&-\lambda_{1,p}(g(t))\int_M|f(t)|^p\,\frac{\operatorname{tr}(\partial g/\partial s)(t)}{2}\,d\mu_{g(t)}\biggr)\\
&\times\Bigl(\int_M|f(t)|^p\,d\mu_{g(t)}\Bigr)^{-1}.
\end{aligned}
$$

Now apply the Cauchy–Schwarz formula

$$\Bigl|\frac{\partial g}{\partial s}(t)(\nabla f(t),\nabla f(t))\Bigr|\leq\Bigl|\frac{\partial g}{\partial s}(t)\Bigr|_{g(t)}|\nabla f(t)|^2$$

and the fact that

$$\Bigl|\operatorname{tr}\frac{\partial g}{\partial s}(t)\Bigr|=\Bigl|\Bigl\langle g(t),\frac{\partial g}{\partial s}(t)\Bigr\rangle\Bigr|\leq|g(t)|\,\Bigl|\frac{\partial g}{\partial s}(t)\Bigr|_{g(t)}=\sqrt{n}\,\Bigl|\frac{\partial g}{\partial s}(t)\Bigr|_{g(t)}$$

to obtain

$$\Bigl|\frac{\partial N}{\partial s}(t,t)\Bigr|\leq\Bigl(\sqrt{n}+\frac{p}{2}\Bigr)\lambda_{1,p}(g(t))\sup\Bigl|\frac{\partial g}{\partial s}(t)\Bigr|_{g(t)}.$$

By the compactness of M, $\lambda_{1,p}(g(t))$ and $\sup|(\partial g/\partial s)(t)|_{g(t)}$ are both continuous. Then $|(\partial N/\partial s)(t,t)|$ is locally bounded, so Theorem 1.1 implies Theorem 1.2. $\square$

4. The Yamabe invariant

In this section, we consider the local Lipschitz property of the Yamabe invariant along general geometric flows and use the constants

$$p=\frac{2n}{n-2},\quad a=\frac{4(n-1)}{n-2},\quad b=\frac{4}{n-2}.$$

With no specification, M is a n-dimensional ($n\geq 3$) connected closed smooth Riemannian manifold, g is a smooth metric on it. Denote by R its scalar curvature, by Rc its Ricci curvature, and by $\mathrm{Rc}^0=\mathrm{Rc}-\frac{1}{n}Rg$ its trace-free Ricci curvature. The conformal class $[g]$ of metric g is defined by

$$[g]:=\{\phi^b g:\phi\in C^\infty(M),\ \phi>0\},$$

and the homogeneous total scalar curvature $\mathbf{S}(g)$ is defined by

$$\mathbf{S}(g):=\int_M R\,d\mu_g\bigg/\int_M d\mu_g,$$

where $d\mu_g$ is the volume form with respect to metric g. Then the Yamabe invariant is defined by

$$\mathcal{Y}(g):=\inf_{\bar g\in[g]}\mathbf{S}(\bar g).\tag{4-1}$$

The minimizer metric is called a Yamabe metric. For the conformal transformation of the scalar curvature $R(g)$ and the trace-free Ricci curvature $\mathrm{Rc}^0(g)$, we have (see [Besse 1987])

$$\phi^{p-1}R(\phi^b g)=R(g)\phi-a\,\Delta\phi,\tag{4-2}$$

$$\mathrm{Rc}^0(\phi^2 g)=\mathrm{Rc}^0(g)+(n-2)\phi(\nabla\nabla\phi^{-1})^0,\tag{4-3}$$

where Δ is the Laplace–Beltrami operator with respect to the metric g, $\alpha^0=\alpha-\frac{1}{n}\operatorname{tr}(\alpha)\,\alpha$ is the trace-free part of (0, 2)-type tensor α. If we define

$$E(\phi,g):=\int(a\,|\nabla\phi|_g^2+R(g)\phi^2)\,d\mu_g,$$

$$Q(\phi,g):=\frac{E(\phi,g)}{\left(\int\phi^p\,d\mu_g\right)^{2/p}}=E(\phi,g)\,\|\phi\|_{p,g}^{-2},$$

where $\nabla\phi=d\phi$ is a covariant vector and $\|\phi\|_{p,g}=\left(\int\phi^p\,d\mu_g\right)^{1/p}$ is the L^p norm with respect to metric g. Then the Yamabe invariant $\mathcal{Y}(g)$ can also be defined by

$$\mathcal{Y}(g):=\inf\{Q(\phi,g):\phi\in C^\infty(M),\phi>0\}.\tag{4-4}$$

The existence of a minimizer u follows from the solution of the Yamabe problem (see [Lee and Parker 1987] for the history). Hence $u^b g$ is a Yamabe metric, moreover the minimizer u satisfies the Euler–Lagrange function

$$R(g)u - a\,\Delta u = \alpha\, u^{p-1}, \tag{4-5}$$

where

$$\alpha = E(u, g)\,\|u\|_{p,g}^{-p} = \mathcal{Y}(g)\,\|u\|_{p,g}^{2-p}\,.$$

Denote by g_{can} the canonical metric on S^n, and consider the set $\Lambda[g]$ of all smooth Yamabe metrics in a given conformal class $[g]$. By the solution to the Yamabe problem, the sets $\Lambda[g]$ as g varies are also compact in the following sense (see [Anderson 2005]): if $g_i \to g$ smoothly and $[g] \neq [g_{\mathrm{can}}]$, then any sequence of Yamabe metrics $\bar{g}_i \in \Lambda[g_i]$ has a subsequence converging smoothly to a Yamabe metric $\bar{g} \in \Lambda[g]$.

The Yamabe constant $\mathcal{Y}(g)$ is continuous with respect to g under the C^2-topology of the space of metrics on M (see [Besse 1987, Proposition 4.31]).

Proof of Theorem 1.3. Since each $\bar{g}(t) \in [g(t)]$ is a Yamabe metric, we can assume $\bar{g}(t) = \phi^b(t) g(t)$. Then $0 < \phi(t) \in C^1(M)$ and $\phi(t)$ minimizes $Q(\,\cdot\,, g(t))$. Defining $N(t,s) := Q(\phi(t), g(s))$, then $\mathcal{Y}(g(t)) = N(t,t)$ and $\mathcal{Y}(g(s)) \le N(t,s)$. We compute

$$\begin{aligned}
\frac{\partial N}{\partial s}(t,t)
&= \frac{\partial}{\partial s}\Big|_{s=t}\left(\frac{\int (a\,|\nabla\phi(t)|^2_{g(s)} + R(s)\phi(t)^2)\,d\mu_{g(s)}}{\left(\int \phi(t)^p\,d\mu_{g(s)}\right)^{2/p}}\right)\\
&= \int\left(-a\frac{\partial g}{\partial s}(t)(\nabla\phi(t), \nabla\phi(t)) + \frac{\partial R}{\partial s}(t)\phi(t)^2\right) d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-2/p}\\
&\quad + \int \frac{1}{2}\left(a\,|\nabla\phi(t)|^2_{g(s)} + R(t)\phi(t)^2\right)\operatorname{tr}\frac{\partial g}{\partial s}(t)d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-2/p}\\
&\qquad - \frac{1}{p}\mathcal{Y}(g(t))\int \phi(t)^p \operatorname{tr}\frac{\partial g}{\partial s}(t)d\mu_{g(t)}\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{-1},
\end{aligned} \tag{4-6}$$

so that

$$\begin{aligned}
\left|\frac{\partial N}{\partial s}(t,t)\right|
&\le \left(1 + \frac{\sqrt{n}}{2}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right| \cdot \frac{\int a\,|\nabla\phi(t)|^2_{g(t)}\,d\mu_{g(t)}}{\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{2/p}} + \frac{\sqrt{n}\,|\mathcal{Y}(g(t))|}{p}\sup\left|\frac{\partial g}{\partial s}(t)\right|\\
&\quad + \left(\sup\left|\frac{\partial R}{\partial s}(t)\right| + \frac{\sqrt{n}}{2}\sup|R(g(t))|\cdot\sup\left|\frac{\partial g}{\partial s}(t)\right|\right)\frac{\int \phi(t)^2\,d\mu_{g(t)}}{\left(\int \phi(t)^p\,d\mu_{g(t)}\right)^{2/p}}.
\end{aligned} \tag{4-7}$$

Next, we process the two integral terms in the above formula. Since $p > 2$, applying Hölder's inequality gives

$$\frac{\int \phi^2 \, d\mu_g}{\left(\int \phi^p \, d\mu_g\right)^{2/p}} \le \operatorname{vol}(g)^{1-2/p}. \tag{4-8}$$

By the definition of $Q(\phi, g)$, we have

$$\begin{aligned} \frac{\int a\,|\nabla\phi|_g^2 \, d\mu_g}{\left(\int \phi^p \, d\mu_g\right)^{2/p}} &= Q(\phi, g) - \frac{\int R\,\phi^2 \, d\mu_g}{\left(\int \phi^p \, d\mu_g\right)^{2/p}} \\ &\le Q(\phi, g) + \sup|R(g)| \,\frac{\int \phi^2 \, d\mu_g}{\left(\int \phi^p \, d\mu_g\right)^{2/p}} \\ &\le Q(\phi, g) + \sup|R(g)| \operatorname{vol}(g)^{1-2/p}. \end{aligned} \tag{4-9}$$

Substituting (4-8) and (4-9) into (4-7), we come to

$$\begin{aligned} \left|\frac{\partial N}{\partial s}(t,t)\right| \le \left(\left(1+\sqrt{n}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right| \cdot \sup|R(g(t))| + \sup\left|\frac{\partial R}{\partial s}(t)\right|\right) \operatorname{vol}(g(t))^{1-2/p} \\ + \left(1 + \frac{\sqrt{n}}{2} + \frac{\sqrt{n}}{p}\right)\sup\left|\frac{\partial g}{\partial s}(t)\right| \cdot |\mathcal{Y}(g(t))|. \end{aligned}$$

Since $\sup|\partial g/\partial s(t)|$, $\sup|\partial R/\partial s(t)|$, $\sup|R(g(t))|$, $\operatorname{vol}(g(t))^{1-2/p}$, and $|\mathcal{Y}(g(t))|$ are all continuous on the closed manifold M, we conclude that $(\partial N/\partial s)(t,t)$ is locally bounded, hence $\mathcal{Y}(g(t))$ is locally Lipschitz by Theorem 1.1.

Next, we simplify the formula (4-6). By (4-5) we have

$$R\phi(t) - a\,\Delta\phi(t) = \mathcal{Y}(g(t))\|\phi(t)\|_{p,g(t)}^{2-p}\phi(t)^{p-1}.$$

Multiplying both sides by $\phi(t)\operatorname{tr}(\partial g/\partial s)(t)$ and integrating by parts gives

$$\begin{aligned} \mathcal{Y}(g(t))\|\phi(t)\|_{p,g(t)}^{2-p} \int \phi(t)^p \operatorname{tr}\frac{\partial g}{\partial s}(t)\, d\mu_{g(t)} \\ = \int \left(R\phi(t)^2 + a\,|\nabla\phi(t)|^2 - \frac{a}{2}\,\Delta\left(\phi(t)^2\right)\right)\operatorname{tr}\frac{\partial g}{\partial s}(t)\, d\mu_{g(t)}. \end{aligned} \tag{4-10}$$

Substituting (4-10) into (4-6), we get

$$\begin{aligned} \frac{\partial N}{\partial s}(t,t) \\ = \Bigg(\int \frac{a}{2p}\Delta(\phi(t)^2)\operatorname{tr}\frac{\partial g}{\partial s}(t) + \phi(t)^2\left(\frac{\partial R}{\partial s}(t) + \frac{1}{n}R\operatorname{tr}\frac{\partial g}{\partial s}(t)\right) d\mu_{g(t)} \\ - \int a\left\langle\left(\frac{\partial g}{\partial s}(t)\right)^0, \nabla\phi(t)\otimes\nabla\phi(t)\right\rangle d\mu_{g(t)}\Bigg)\|\phi(t)\|_{p,g(t)}^{-2}. \end{aligned} \tag{4-11}$$

The evolution function of scalar curvature R is

$$\frac{\partial R}{\partial s} = \operatorname{div}\left(\operatorname{div}\frac{\partial g}{\partial s}\right) - \Delta \operatorname{tr}\frac{\partial g}{\partial s} - \left\langle \operatorname{Rc}, \frac{\partial g}{\partial s}\right\rangle.$$

Substituting this into (4-11) gives

$$\begin{aligned}
&\frac{\partial N}{\partial s}(t,t)\\
&\quad = \int \left\langle \frac{\partial g}{\partial s}(t), \big(\nabla\nabla\phi(t)^2 - \phi(t)^2 \operatorname{Rc} - a\nabla\phi(t)\otimes\nabla\phi(t)\big)^0 \right\rangle d\mu_{g(t)} \|\phi(t)\|^{-2}_{p,g(t)}\\
&\quad = -\int \left\langle \frac{\partial g}{\partial s}(t), \phi(t)^2\big(\operatorname{Rc} + (n-2)\phi(t)^{b/2}\nabla\nabla\phi(t)^{-b/2}\big)^0 \right\rangle d\mu_{g(t)} \|\phi(t)\|^{-2}_{p,g(t)}.
\end{aligned}$$

By the conformal transformation of trace-free Ricci curvature (4-3),

$$\frac{\partial N}{\partial s}(t,t) = -\int \phi^{-b}(t)\left\langle \frac{\partial g}{\partial s}(t), \operatorname{Rc}^0(\phi^b(t)g(t))\right\rangle d\mu_{\phi^b(t)g(t)} \operatorname{vol}(\phi^b(t)g(t))^{-2/p}.$$

Since $\phi^b(t) = \bar g(t)/g(t)$, we get

$$\frac{\partial N}{\partial s}(t,t) = -\int \frac{\bar g(t)}{g(t)}\left\langle \frac{\partial g}{\partial s}(t), \operatorname{Rc}^0(\bar g(t))\right\rangle_{\bar g(t)} d\mu_{\bar g(t)} \operatorname{vol}(\bar g(t))^{-2/p}. \tag{4-12}$$

Then the theorem follows from Corollary 1.1.1. □

Proof of Theorem 1.4. Let $\phi(t)$ be any minimizer of $Q(\,\cdot\,, g(t))$. Then

$$\tilde g(t) = \phi^b(t) g(t)$$

is the Yamabe metric in the conformal class $[g(t)]$. Define

$$\mathcal{N}(\tilde g(s), g(t)) := Q(\phi(s), g(t)).$$

Then

$$\mathcal{Y}(g(t)) = \mathcal{N}(\tilde g(t), g(t)), \quad \mathcal{Y}(g(t)) \le \mathcal{N}(\tilde g(s), g(t)).$$

Hence, when $t > 0$,

$$\begin{aligned}
\frac{\mathcal{N}(\tilde g(t), g(t)) - \mathcal{N}(\tilde g(t), g(0))}{t} &\le \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t}\\
&\le \frac{\mathcal{N}(\tilde g(0), g(t)) - \mathcal{N}(\tilde g(0), g(0))}{t}.
\end{aligned} \tag{4-13}$$

By (4-12) and the definitions of $\mathcal{N}$ and N, we get

$$\frac{\partial \mathcal{N}}{\partial t}(\tilde g(t), g(t)) = -\int \frac{\tilde g(t)}{g(t)}\left\langle \frac{\partial g}{\partial t}(t), \operatorname{Rc}^0(\tilde g(t))\right\rangle_{\tilde g(t)} d\mu_{\tilde g(t)} \operatorname{vol}(\tilde g(t))^{-2/p}. \tag{4-14}$$

It is easy to see that $\mathcal{N}(\tilde{g}(s), g(t))$ and $(\partial\mathcal{N}/\partial t)(\tilde{g}(s), g(t))$ as functionals of $\tilde{g}(s)$ and $g(t)$ are continuous with the C^2-topology on the space of metrics. Applying the mean value theorem to the variable t in the function $\mathcal{N}(\tilde{g}(s), g(t))$, there exists a number $0 < \beta(s,t) < t$ such that

$$\mathcal{N}(\tilde{g}(s), g(t)) - \mathcal{N}(\tilde{g}(s), g(0)) = t\,\frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(s), g(\beta(s,t))).$$

Substituting into (4-13), we come to

$$\frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t), g(\beta(t,t))) \le \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} \le \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(\beta(0,t))). \tag{4-15}$$

Letting $t \to 0$, then $\beta(0,t) \to 0$ and $\beta(t,t) \to 0$ follows from $0 < \beta(s,t) < t$. Hence

$$\limsup_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} \le \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(0), g(0)) \quad \text{for all } \tilde{g}(0) \in \Lambda[g(0)]. \tag{4-16}$$

Pick $t_i > 0$, $t_i \to 0$ such that

$$\liminf_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} = \lim_{i\to\infty} \frac{\mathcal{Y}(g(t_i)) - \mathcal{Y}(g(0))}{t_i}. \tag{4-17}$$

Using the compactness of $\Lambda[g_0]$, there exists a subsequence of t_i (denoted again by t_i for simplicity) and a Yamabe metric $\bar{g}_0 \in \Lambda[g_0]$ such that

$$\lim_{i\to\infty} \tilde{g}(t_i) = \bar{g}_0.$$

Then by the first inequality in (4-15),

$$\begin{aligned} \lim_{i\to\infty} \frac{\mathcal{Y}(g(t_i)) - \mathcal{Y}(g(0))}{t_i} &\ge \lim_{i\to\infty} \frac{\partial\mathcal{N}}{\partial t}(\tilde{g}(t_i), g(\beta(t_i, t_i))) \\ &= \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)). \end{aligned} \tag{4-18}$$

Hence by (4-16) and (4-17), $\mathcal{Y}(g(t))$ is differentiable at $t = 0$ and

$$\lim_{t\to 0} \frac{\mathcal{Y}(g(t)) - \mathcal{Y}(g(0))}{t} = \frac{\partial\mathcal{N}}{\partial t}(\bar{g}_0, g(0)). \tag{4-19}$$

This implies the first equality in (1-2) by (4-16) and (4-14). We now know that the t_i chosen after (4-17) can be any sequence of $t_i > 0, t_i \to 0$. Then the $\bar{g}_0$ in (4-19) can be any accumulation point of $\bar{g}(t)$ as $t \to 0$ in $\Lambda[g_0]$, hence any accumulation point of $\Lambda[g(t)]$ as $t \to 0$ in $\Lambda[g_0]$. The second equality in (1-2) follows from applying this to other metrics $g(t) \in G_{(\partial g/\partial t)(0)}(g_0, t)$. □

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