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In an earlier article in coauthorship with G. P. Bessa, we obtained an estimate for the mean curvature of a cylindrically bounded proper submanifold in a product manifold where one factor is a Euclidean space. Here we extend this estimate to a general product ambient space endowed with a warped product structure.

Let (L^ℓ, g_L) and (P^n, g_P) be complete Riemannian manifolds of dimension ℓ and n, respectively, where L^ℓ is noncompact. Then let $N^{n+\ell} = L^\ell \times_\rho P^n$ be the product manifold $L^\ell \times P^n$ endowed with the warped product metric $ds^2 = dg_L + \rho^2 dg_P$ for some positive warping function $\rho \in C^\infty(L)$.

Let $B_P(r_0)$ denote the geodesic ball with radius r_0 centered at a reference point $o \in P^n$. Assume that the radial sectional curvatures in $B_P(r_0)$ along the geodesics issuing from o are bounded as $K_P^{\mathrm{rad}} \le b$ for some constant $b \in \mathbb{R}$, and that $0 < r_0 < \min\{\mathrm{inj}_P(o), \pi/2\sqrt{b}\}$, where $\mathrm{inj}_P(o)$ is the injectivity radius at o and $\pi/2\sqrt{b}$ is replaced by $+\infty$ if $b \le 0$. Then the mean curvature of the geodesic sphere $S_P(r_0) = \partial B_P(r_0)$ can be estimated from below by the mean curvature of a geodesic sphere of a space form of curvature b, that is,

$$C_b(t) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\,t) & \text{if } b > 0, \\ 1/t & \text{if } b = 0, \\ \sqrt{-b}\coth(\sqrt{-b}\,t) & \text{if } b < 0. \end{cases}$$

This is a direct consequence of the comparison theorems for the Riemannian distance, since the Hessian (respectively, Laplacian) of the distance function is nothing but the second fundamental form (respectively, mean curvature) of the geodesic spheres. A classical reference on this topic is [Greene and Wu 1979]. We also refer the reader to [Petersen 2006] or [Pigola et al. 2008] for a modern approach to the Hessian and Laplacian comparison theorems.

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By a *cylinder* in the warped space $N^{n+\ell}$, we mean a closed subset of the form

$$\mathscr{C}_{r_0} = \{(x, y) \in N^{n+\ell} : x \in L^\ell \text{ and } y \in B_P(r_0)\}.$$

Since the submanifolds $L^\ell \times \{p_0\} \subset N^{n+\ell}$ are totally geodesic, we have

$$|\rho H_{\mathscr{C}_{r_0}}| \geq \frac{n-1}{\ell+n-1} C_b(r_0),$$

where $H_{\mathscr{C}_{r_0}}$ is the mean curvature vector field of the hypersurface $L^\ell \times S_p(r_0)$.

The following theorem extends the result in [Alías et al. 2009], where the cylinders under consideration are contained in product spaces $\mathbb{R}^\ell \times P^n$. After the statement, we recall from [Alías et al. 2011] the concept of an Omori–Yau pair on a Riemannian manifold and discuss some implications of its existence.

Theorem 1. *Let $f\colon M^m \to L^\ell \times_\rho P^n$ be an isometric immersion, where L^ℓ carries an Omori–Yau pair for the Hessian and the functions ρ and $|\operatorname{grad} \log \rho|$ are bounded. If f is proper and $f(M) \subset \mathscr{C}_{r_0}$, then $\sup_M |H| = +\infty$ or*

$$\sup_M \rho|H| \geq \frac{m-\ell}{m} C_b(r_0), \tag{1}$$

where H is the mean curvature vector field of f.

In the proof, we see that the existence in L^ℓ of an Omori–Yau pair for the Hessian provides conditions, in a function-theoretic form, that guarantee the validity of the Omori–Yau maximum principle on M^m in terms of the corresponding property of L^ℓ and the geometry of the immersion.

Definition 2. The pair of functions (h, γ), for $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma\colon M \to \mathbb{R}_+$, is an *Omori–Yau pair for the Hessian* in M if

(a) $h(0) > 0$ and $h'(t) \geq 0$, for all $t \in \mathbb{R}_+$;

(b) $\limsup_{t\to+\infty} t h(\sqrt{t})/h(t) < +\infty$;

(c) $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}t}{\sqrt{h(t)}} = +\infty$;

(d) the function γ is proper;

(e) $|\operatorname{grad} \gamma| \leq c\sqrt{\gamma}$ for some $c > 0$ outside a compact subset of M; and

(f) $\operatorname{Hess} \gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

Similarly, the pair (h, γ) is an *Omori–Yau pair for the Laplacian* in M if it satisfies conditions (a)–(e) and

(f′) $\Delta\gamma \leq d\sqrt{\gamma h(\sqrt{\gamma})}$ for some $d > 0$ outside a compact subset of M.

We say that the *Omori–Yau maximum principle for the Hessian* holds for M if for any function $g \in C^\infty(M)$ bounded from above there exists a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in M such that

(a) $\lim_{k\to\infty} g(p_k) = \sup_M g$,

(b) $|\operatorname{grad} g(p_k)| \le 1/k$,

(c) $\operatorname{Hess} g(p_k)(X, X) \le (1/k) g_M(X, X)$ for all $X \in T_{p_k}M$.

Similarly, the *Omori–Yau maximum principle for the Laplacian* holds for M if these properties are satisfied with (c) replaced by

(c′) $\Delta g(p_k) \le 1/k$.

The following theorem of Pigola, Rigoli, and Setti gives sufficient conditions for an Omori–Yau maximum principle to hold for a Riemannian manifold.

Theorem 3 [Pigola et al. 2005]. *Assume that a Riemannian manifold M carries an Omori–Yau pair for the Hessian* (*resp. Laplacian*). *Then the Omori–Yau maximum principle for the Hessian* (*resp. Laplacian*) *holds in M.*

Example 4. Let M^m be a complete but noncompact Riemannian manifold, and write $r(y) = \operatorname{dist}_M(y, o)$ for some reference point $o \in M^m$. Assume that the radial sectional curvature of M^m satisfies $K^{\mathrm{rad}} \ge -h(r)$, where the smooth function h satisfies (a)–(c) in Definition 2 and is even at the origin, that is, $h^{(2k+1)}(0) = 0$ for $k \in \mathbb{N}$. Then, as shown in [Pigola et al. 2005], the functions (h, r^2) form an Omori–Yau pair for the Hessian. As for the function h, one can choose

$$h(t) = t^2 \prod_{j=1}^{N} (\log^{(j)}(t))^2, \quad t \gg 1,$$

where $\log^{(j)}$ stands for the j-th iterated logarithm.

To conclude this section, we observe that Theorem 1 is sharp. This is clear from (1) by taking as P^n a space-form and as M the hypersurface $L^\ell \times S_P(r_0)$ in $N^{n+\ell}$. In view of Example 4, it also follows that by taking $L^\ell = \mathbb{R}^\ell$ and constant ρ we recover the result in [Alías et al. 2009].

The proof

We first introduce some additional notations, and then recall a few basic facts on warped product manifolds.

Let $\langle\ ,\ \rangle$ denote the metrics in $N^{n+\ell}$, L^ℓ and M^m, while $(\ ,\ )$ stands for the metric in P^n. The corresponding norms are $|\ |$ and $\|\ \|$. In addition, let ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections in M^m and $N^{n+\ell}$, respectively, and ∇^L and ∇^P the ones in L^ℓ and P^n.

We always denote vector fields in TL by T, S and in TP by X, Y. Also, we identify vector fields in TL and TP with *basic* vector fields in TN by taking $T(x, y) = T(x)$ and $X(x, y) = X(y)$.

For the Lie-brackets of basic vector fields, we have that $[T, S] \in TL$ and $[X, Y] \in TP$ are basic and that $[X, T] = 0$. Then we have

$$\begin{aligned}\widetilde{\nabla}_S T &= \nabla^L_S T,\\ \widetilde{\nabla}_X T &= \widetilde{\nabla}_T X = T(\varrho)X,\\ \widetilde{\nabla}_X Y &= \nabla^P_X Y - \langle X, Y\rangle \operatorname{grad}^L \varrho,\end{aligned}$$

where the vector fields X, Y and T are basic and $\varrho = \log \rho$.

Our proof follows the main steps in [Alías et al. 2011], where the geometric situation considered differs from ours in that $f(M)$ there is contained in a *cylinder* of the form

$$\{(x, y) \in N^{n+\ell} : x \in B_L(r_0) \text{ and } y \in P^n\}.$$

In fact, a substantial part of the argument is to show that the Omori–Yau pair for the Hessian in L^ℓ induces an Omori–Yau pair for the Laplacian for a noncompact M^m when $|H|$ is bounded. Thus the Omori–Yau maximum principle for the Laplacian holds in M^m, and the proof follows from an application of the latter.

Suppose that M^m is noncompact, and let (h, Γ) be an Omori–Yau pair for the Hessian in L^ℓ. For $p \in M^m$, write $f(p) = (x(p), y(p))$. Set $\tilde{\Gamma}(x, y) = \Gamma(x)$ for $(x, y) \in N^{n+\ell}$ and

$$\gamma(p) = \tilde{\Gamma}(f(p)) = \Gamma(x(p)).$$

We show next that (h, γ) is an Omori–Yau pair for the Laplacian in M^m. First we argue that the function γ is proper. To see this, let $p_k \in M^m$ be a divergent sequence, that is, $p_k \to \infty$ in M^m as $k \to +\infty$. Thus, $f(p_k) \to \infty$ in $N^{n+\ell}$ because f is proper. Because $f(M)$ lies inside a cylinder, $x(p_k) \to \infty$ in L^ℓ. Hence, $\gamma(p_k) \to +\infty$ as $k \to +\infty$ because Γ is proper, and thus γ is proper.

It remains to verify conditions (e) and (f′) in Definition 2. We have from $\tilde{\Gamma}(x, y) = \Gamma(x)$ that

$$\langle \operatorname{grad}^N \tilde{\Gamma}(x, y), X\rangle = 0.$$

Thus

$$\operatorname{grad}^N \tilde{\Gamma}(x, y) = \operatorname{grad}^L \Gamma(x).$$

Since $\gamma = \tilde{\Gamma} \circ f$, we obtain

$$\operatorname{grad}^N \tilde{\Gamma}(f(p)) = \operatorname{grad}^M \gamma(p) + \operatorname{grad}^N \tilde{\Gamma}(f(p))^\perp, \tag{2}$$

where $(\,)^\perp$ denotes taking the normal component to f. Then

$$|\operatorname{grad}^M \gamma(p)| \le |\operatorname{grad}^N \tilde{\Gamma}(f(p))| = |\operatorname{grad}^L \Gamma(x(p))| \le c\sqrt{\Gamma(x(p))} = c\sqrt{\gamma(p)}$$

outside a compact subset of M^m, and thus (e) holds.

We have that

$$\widetilde{\nabla}_T \operatorname{grad}^N \tilde{\Gamma} = \nabla^L_T \operatorname{grad}^L \Gamma.$$

Hence Hess $\tilde{\Gamma}(T, S) =$ Hess $\Gamma(T, S)$ and Hess $\tilde{\Gamma}(T, X) = 0$. Also,

$$\widetilde{\nabla}_X \operatorname{grad}^N \tilde{\Gamma} = \widetilde{\nabla}_X \operatorname{grad}^L \Gamma = \operatorname{grad}^L \Gamma(\varrho) X.$$

Hence

$$\operatorname{Hess} \tilde{\Gamma}(X, Y) = \langle \operatorname{grad}^L \Gamma, \operatorname{grad}^L \varrho \rangle \langle X, Y \rangle.$$

For a unit vector $e \in T_p M$, set $e = e^L + e^P$, where $e^L \in T_{x(p)} L$ and $e^P \in T_{y(p)} P$. Then

$$\operatorname{Hess} \tilde{\Gamma}(f(p))(e, e) = \operatorname{Hess} \Gamma(x(p))(e^L, e^L) + \big\langle \operatorname{grad}^L \Gamma(x(p)), \operatorname{grad}^L \varrho(x(p)) \big\rangle |e^P|^2.$$

Also, an easy computation using (2) yields

$$\operatorname{Hess} \gamma(p)(e, e) = \operatorname{Hess} \tilde{\Gamma}(f(p))(e, e) + \big\langle \operatorname{grad}^L \Gamma(x(p)), \alpha(p)(e, e) \big\rangle,$$

where α denotes the second fundamental of f with values in the normal bundle. Thus,

$$\begin{aligned} \operatorname{Hess} \gamma(p)(e, e) = \operatorname{Hess} \Gamma(x(p))(e^L, e^L) &+ \big\langle \operatorname{grad}^L \Gamma(x(p)), \operatorname{grad}^L \varrho(x(p)) \big\rangle |e^P|^2 \\ &+ \big\langle \operatorname{grad}^L \Gamma(x(p)), \alpha(p)(e, e) \big\rangle. \end{aligned}$$

Since Hess $\Gamma \le d\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some positive constant d outside a compact subset of L^ℓ and the immersion is proper, we have

$$\operatorname{Hess} \Gamma(x(p))(e^L, e^L) \le d\sqrt{\gamma(p) h(\sqrt{\gamma(p)})} |e^L|^2 \le d\sqrt{\gamma(p) h(\sqrt{\gamma(p)})}$$

outside a compact subset of M^m. From $|\operatorname{grad}^L \Gamma| \le c\sqrt{\Gamma h(\sqrt{\Gamma})}$ for some c outside a compact subset of L^ℓ and $\sup_L |\operatorname{grad}^L \varrho| < +\infty$, we have

$$\big\langle \operatorname{grad}^L \Gamma(x(p)), \operatorname{grad}^L \varrho(x(p)) \big\rangle |e^P|^2 \le c' \sqrt{\gamma(p)}$$

for some positive constant c' outside a compact subset of M^m. Since γ is proper and h is unbounded, by (a) and (b) in Definition 2, we have

$$\sqrt{\gamma} \le \sqrt{\gamma h(\sqrt{\gamma}\,)}$$

outside a compact subset of M^m, because $\gamma \to +\infty$ as $p \to \infty$ and $\lim_{t \to +\infty} h(t) = +\infty$. Thus we obtain

$$\operatorname{Hess} \gamma(e, e) \le d_1 \sqrt{\gamma h(\sqrt{\gamma}\,)} + \big\langle \operatorname{grad}^L \Gamma(x), \alpha(e, e) \big\rangle \tag{3}$$

for some constant $d_1 > 0$, outside a compact subset of M^m.

On the other hand, we may assume that

$$|H| \le c\sqrt{h(\sqrt{\gamma}\,)} \tag{4}$$

for some constant $c > 0$, outside a compact subset of M^m. Otherwise, there exists a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that $p_k \to \infty$ as $k \to +\infty$ and

$$|H(p_k)| > k\sqrt{h(\sqrt{\gamma(p_k)})}.$$

With γ being proper and h unbounded from (a) and (b) in Definition 2, we conclude that $\sup_M |H| = +\infty$, in which case we are done with the proof of the theorem.

We obtain from (3) using (4) that $\Delta\gamma \leq c_1\sqrt{\gamma h(\sqrt{\gamma}\,)}$ for some constant $c_1 > 0$ outside a compact subset of M^m, and thus (f′) has been proved.

Consider the distance function $r(y) = \mathrm{dist}_P(y, o)$ in $B_P(r_0)$ and define $\tilde{r} \in C^\infty(N)$ by $\tilde{r}(x, y) = r(y)$. Then

$$\langle \mathrm{grad}^N \tilde{r}(x, y), T\rangle = 0.$$

Thus

$$\rho^2(x)\mathrm{grad}^N \tilde{r}(x, y) = \mathrm{grad}^P r(y).$$

We obtain that

$$\widetilde{\nabla}_T \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_T(\rho^{-2}\mathrm{grad}^P r) = -\rho^{-2}T(\varrho)\mathrm{grad}^P r.$$

Therefore

$$\mathrm{Hess}\ \tilde{r}(T, S) = 0$$

and

$$\mathrm{Hess}\ \tilde{r}(T, X) = -\rho^{-2}T(\varrho)\langle\mathrm{grad}^P r, X\rangle = -T(\varrho)(\mathrm{grad}^P r, X).$$

Also,

$$\widetilde{\nabla}_X \mathrm{grad}^N \tilde{r} = \widetilde{\nabla}_X(\rho^{-2}\mathrm{grad}^P r) = \rho^{-2}\big(\nabla^P_X \mathrm{grad}^P r - \langle X, \mathrm{grad}^P r\rangle \mathrm{grad}^L \varrho\big).$$

Hence

$$\mathrm{Hess}\ \tilde{r}(X, Y) = \rho^{-2}\langle\nabla^P_X \mathrm{grad}^P r, Y\rangle = (\nabla^P_X \mathrm{grad}^P r, Y) = \mathrm{Hess}\ r(X, Y).$$

For $e \in TM$, we have

$$\mathrm{Hess}\ \tilde{r}(e, e) = -2\langle\mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + \mathrm{Hess}\ r(e^P, e^P).$$

From the Hessian comparison theorem (see [Pigola et al. 2008, Chapter 2] for a modern approach) we obtain

$$\mathrm{Hess}\ r(e^P, e^P) \geq C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2).$$

Therefore,

$$\mathrm{Hess}\ \tilde{r}(e, e) \geq -2\langle\mathrm{grad}^L \varrho, e\rangle(\mathrm{grad}^P r, e^P) + C_b(r)(\|e^P\|^2 - (\mathrm{grad}^P r, e^P)^2). \tag{5}$$

We define $u \in C^\infty(M)$ by

$$u(p) = r(y(p)).$$

Thus, $u = \tilde{r} \circ f$ and

$$\operatorname{grad}^N \tilde{r}(f(p)) = \operatorname{grad}^M u(p) + \operatorname{grad}^N \tilde{r}(f(p))^{\perp}. \tag{6}$$

This gives

$$\operatorname{Hess} u(e_i, e_j) = \operatorname{Hess} \tilde{r}(e_i, e_j) + \langle \operatorname{grad}^N \tilde{r}, \alpha(e_i, e_j)\rangle,$$

where $e_1, \ldots, e_m$ is an orthonormal frame of TM. Thus

$$\Delta u = \sum_{j=1}^{m} \operatorname{Hess} \tilde{r}(e_j, e_j) + m\langle \operatorname{grad}^N \tilde{r}, H\rangle. \tag{7}$$

We have from $e_j = e_j^L + e_j^P$ that $1 = \langle e_j, e_j\rangle = \rho^2 \|e_j^P\|^2 + \sum_{k=1}^{\ell} \langle e_j, T_k\rangle^2$, where $T_1, \ldots, T_\ell$ is an orthonormal frame for TL. Hence

$$m = \rho^2 \sum_{j=1}^{m} \|e_j^P\|^2 + \sum_{k=1}^{\ell} |T_k^{\top}|^2,$$

where $T^{\top}$ is the tangent component of T. We obtain

$$\sum_{j=1}^{m} \|e_j^P\|^2 \geq (m-\ell)\rho^{-2}. \tag{8}$$

Since $(\operatorname{grad}^P r, e_j^P) = \langle \operatorname{grad}^N \tilde{r}, e_j^P\rangle = \langle \operatorname{grad}^N \tilde{r}, e_j\rangle = \langle \operatorname{grad}^M u, e_j\rangle$, we get from (5) that

$$\operatorname{Hess} \tilde{r}(e_j, e_j) \geq -2\langle \operatorname{grad}^L \varrho, e_j\rangle \langle \operatorname{grad}^M u, e_j\rangle + C_b(u)(\|e_j^P\|^2 - \langle \operatorname{grad}^M u, e_j\rangle^2).$$

Taking the trace and using (8) gives

$$\sum_{j=1}^{m} \operatorname{Hess} \tilde{r}(e_j, e_j) \geq -2\,\langle \operatorname{grad}^L \varrho, \operatorname{grad}^M u\rangle + C_b(u)\big((m-\ell)\rho^{-2} - |\operatorname{grad}^M u|^2\big).$$

Because $\langle \operatorname{grad}^N \tilde{r}, \operatorname{grad}^N \tilde{r}\rangle = \rho^2(\rho^{-2}\operatorname{grad}^P r, \rho^{-2}\operatorname{grad}^P r) = \rho^{-2}$, we have

$$\langle \operatorname{grad}^N \tilde{r}, H\rangle \geq -\rho^{-1}|H|.$$

Using (7), we conclude that

$$\Delta u \geq -2\langle \operatorname{grad}^L \varrho, \operatorname{grad}^M u\rangle + C_b(u)\left((m-\ell)\rho^{-2} - |\operatorname{grad}^M u|^2\right) - m\rho^{-1}|H|.$$

Thus

$$\rho|H| \geq \frac{m-\ell}{m} C_b(u) - \frac{\rho^2}{m}\big(\Delta u + 2|\operatorname{grad}^L \varrho||\operatorname{grad}^M u| + C_b(u)|\operatorname{grad}^M u|^2\big).$$

If M^m is compact, the proof follows easily by computing the inequality at a point of maximum of u. Thus, we may now assume that M^m is noncompact and that (4) holds.

Since $f(M) \subset \mathscr{C}_{r_0}$, we have $u^* = \sup_M u \le r_0 < +\infty$. By the Omori–Yau maximum principle, there is a sequence $\{p_k\}_{k\in\mathbb{N}}$ in M^m such that $u(p_k) > u^* - 1/k$, $|\mathrm{grad}^M u(p_k)| < 1/k$, and $\Delta u(p_k) < 1/k$. By assumption, we have $\sup_L \rho = K_1 < +\infty$ and $\sup_L |\mathrm{grad}^L \varrho| = K_2 < +\infty$. Hence

$$\sup_M \rho|H| \ge \rho(p_k)|H(p_k)| \ge \frac{m-\ell}{m} C_b(u(p_k)) - \frac{K_1^2}{m}\Big(\frac{1+2K_2}{k} + \frac{1}{k^2} C_b(u(p_k))\Big).$$

Letting $k \to +\infty$, we obtain

$$\sup_M \rho|H| \ge \frac{m-\ell}{m} C_b(u^*) \ge \frac{m-\ell}{m} C_b(r_0),$$

and this concludes the proof of the theorem.

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