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WEYL GROUP MULTIPLE DIRICHLET SERIES OF TYPE C

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We develop the theory of *Weyl group multiple Dirichlet series* for root systems of type C. For a root system of rank r and a positive integer n, these are Dirichlet series in r complex variables with analytic continuation and functional equations isomorphic to the associated Weyl group. They conjecturally arise as Whittaker coefficients of Eisenstein series on a metaplectic group with cover degree n. For type C and n odd, we construct an infinite family of Dirichlet series and prove they satisfy the above analytic properties in many cases. The coefficients are exponential sums built from Gelfand–Tsetlin bases of certain highest weight representations. Previous attempts to define such series by Brubaker, Bump, and Friedberg required n sufficiently large, so that coefficients were described by Weyl group orbits. We demonstrate that these two radically different descriptions match when both are defined. Moreover, for $n = 1$, we prove our series are Whittaker coefficients of Eisenstein series on $\mathrm{SO}(2r+1)$.

1. Introduction

Let Φ be a reduced root system of rank r. *Weyl group multiple Dirichlet series* (associated to Φ) are Dirichlet series in r complex variables which initially converge on a cone in $\mathbb{C}^r$, possess analytic continuation to a meromorphic function on the whole complex space, and satisfy functional equations whose action on $\mathbb{C}^r$ is isomorphic to the Weyl group of Φ.

For various choices of Φ and a positive integer n, infinite families of Weyl group multiple Dirichlet series defined over any number field F containing the $2n$-th roots of unity were introduced in [Chinta and Gunnells 2007; 2010; Brubaker et al. 2007; 2008]. The coefficients of these Dirichlet series are intimately related to the n-th power reciprocity law in F. It is further expected that these families are related to metaplectic Eisenstein series as follows. If one considers the split, semisimple, simply connected algebraic group G over F whose Langlands L-group

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has root system Φ, then it is conjectured that the families of multiple Dirichlet series associated to n and Φ (or the dual root system, depending on n) are precisely the Fourier–Whittaker coefficients of minimal parabolic Eisenstein series on the n-fold metaplectic cover of G. See Remark 3 for more details.

In light of this suggested relationship with Eisenstein series, one should be able to provide definitions of multiple Dirichlet series for any reduced root system Φ and any positive integer n having the desired analytic properties. However a satisfactory theory of the connections between various Dirichlet series and their relation to metaplectic Eisenstein series has only recently emerged for type A. This paper improves the current theory by developing some of the corresponding results for type C, suggesting that such representations of Eisenstein series should hold in great generality.

We begin by describing the basic shape of the Weyl group multiple Dirichlet series, which can be done uniformly for any reduced root system Φ of rank r. Given a number field F containing the $2n$-th roots of unity and a finite set of places S of F (chosen with certain restrictions described in Section 2.2), let $\mathbb{O}_S$ denote the ring of S-integers in F and $\mathbb{O}_S^\times$ the units in this ring. Then to any r-tuple of nonzero $\mathbb{O}_S$ integers $\boldsymbol{m} = (m_1, \ldots, m_r)$, we associate a *Weyl group multiple Dirichlet series* in r complex variables $\boldsymbol{s} = (s_1, \ldots, s_r)$ of the form

$$(1) \qquad Z_\Psi(s_1, \ldots, s_r; m_1, \ldots, m_r) = Z_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \sum_{\substack{\boldsymbol{c}=(c_1,\ldots,c_r) \\ \in(\mathbb{O}_S/\mathbb{O}_S^\times)^r}} \frac{H^{(n)}(\boldsymbol{c}; \boldsymbol{m})\Psi(\boldsymbol{c})}{|c_1|^{2s_1} \cdots |c_r|^{2s_r}},$$

where the coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ carry the main arithmetic content. The function $\Psi(\boldsymbol{c})$ guarantees the numerator of our series is well-defined up to $\mathbb{O}_S^\times$ units and is defined precisely in Section 2.3. Finally $|c_i| = |c_i|_S$ denotes the norm of the integer c_i as a product of local norms in $F_S = \prod_{v \in S} F_v$.

The coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ are not multiplicative, but nearly so and (as we will demonstrate in (17) and (19) of Section 2.4) can nevertheless be reconstructed from coefficients of the form

$$(2) \qquad H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) := H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}),$$

where p is a fixed prime in $\mathbb{O}_S$ and $k_i = \operatorname{ord}_p(c_i)$, $l_i = \operatorname{ord}_p(m_i)$.

There are two approaches to defining these prime-power contributions. Chinta and Gunnells [2007; 2010] use a remarkable action of the Weyl group to define the coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ as an average over elements of the Weyl group for any root system Φ and any integer $n \geq 1$, from which functional equations and analytic continuation of the series Z follow. By contrast, for Φ of type A and any $n \geq 1$, Brubaker, Bump, and Friedberg [2007] define the prime-power coefficients as a sum over basis vectors in a highest weight representation for $\mathrm{GL}(r+1, \mathbb{C})$ associated to the fixed r-tuple $\boldsymbol{l}$ in (2). They subsequently prove functional equations and analytic continuation for the multiple Dirichlet series via intricate combinatorial arguments in [Brubaker et al. 2009; 2011b]. It is therefore natural to ask whether a definition in the mold of [Brubaker et al. 2007] exists for the prime-power coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ for every root system Φ.

For Φ of type C, we present a positive answer to this question, in the form of the following conjecture and its subsequent proof in many special cases.

Conjecture. *For $\Phi = C_r$ for any r and for n odd, the Dirichlet series $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ described in (1), with coefficients of the form $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ as defined in Section 3, has the following properties*:

(I) *$Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ possesses analytic continuation to a meromorphic function on $\mathbb{C}^r$ and satisfies a group of functional equations isomorphic to $W(\mathrm{Sp}(2r))$, the Weyl group of* $\mathrm{Sp}(2r)$, *of the form (24), where the W action on $\mathbb{C}^r$ is as given in (21).*

(II) *$Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is the Whittaker coefficient of a minimal parabolic Eisenstein series on an n-fold metaplectic cover of* $\mathrm{SO}(2r+1, F_S)$.

Part (II) of this conjecture would imply part (I) according to the general Langlands–Selberg theory of Eisenstein series extended to metaplectic covers as in [Mœglin and Waldspurger 1995]. In practice, other methods to prove part (I) have resulted in sharp estimates for the scattering matrix involved in the functional equations that would be difficult to obtain from the general theory; see, for example, [Brubaker et al. 2006] .

In this paper, we make progress toward this general conjecture by proving the following two results, which will be restated more precisely in later sections once careful definitions have been given.

Theorem 1. *For n sufficiently large (as given in (41)), $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ matches the multiple Dirichlet series defined in [Brubaker et al. 2008] for the root system $\Phi = C_r$. Therefore, for such odd n, the multiple Dirichlet series possess the analytic properties cited in part (I) of the Conjecture.*

Theorem 2. *For $n = 1$, $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is a multiplicative function whose prime-power coefficients match those of the Casselman–Shalika formula for* $\mathrm{Sp}(2r)$. *Hence $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ agrees with the minimal parabolic (nonmetaplectic) Eisenstein series for* $\mathrm{SO}(2r+1, F_S)$. *Thus both parts of the Conjecture hold for $n = 1$.*

These theorems are symplectic analogs of those proven for type A in [Brubaker et al. 2007; 2008]. Theorem 2 is proved using a combinatorial identity from [Hamel and King 2002]. Theorem 1, our main result, also has a combinatorial proof using rather subtle connections between the Weyl group and Gelfand–Tsetlin patterns (henceforth GT-patterns) that parametrize basis vectors for highest weight representations of $\mathrm{Sp}(2r, \mathbb{C})$, the Langlands dual group of $\mathrm{SO}(2r+1)$.

Remark 3. The restriction that n must be odd is natural in light of earlier work by Savin [1988] showing that the structure of the Iwahori–Hecke algebra depends on the parity of the metaplectic cover and by Bump, Friedberg, and Ginzburg [2006] on conjectural dual groups for metaplectic covers. Indeed, though the construction of the Dirichlet series we propose in Section 3 makes sense for any n, attempts to prove functional equations for n even and $\boldsymbol{m}$ fixed using the techniques of [Beineke et al. 2010] suggest the coefficients have the wrong shape. In view of this evidence, we expect a similar combinatorial definition to hold for n even, but making use of the highest weight representation theory for $\mathrm{SO}(2r+1, \mathbb{C})$ (in contrast with the case n odd, and weights from $\mathrm{Sp}(2r, \mathbb{C})$ as in the Conjecture and the two subsequent theorems).

As noted above, the analog of the Conjecture is known for type A for any $n \geq 1$. Its proof, completed in [Brubaker et al. 2009; 2011b], makes critical use of the outer automorphism of the Dynkin diagram for type A. Thus mimicking the proof techniques to obtain results for type C is not possible. However, given any fixed $\boldsymbol{m}$ and n, one can verify the functional equations and meromorphic continuation with a finite amount of checking. See [Beineke et al. 2010] for the details of this argument in a small rank example.

The type A analog of part (II) of the Conjecture is proved in [Brubaker et al. 2011a] by computing the Fourier–Whittaker coefficients of Eisenstein series directly by inducing from successive maximal parabolics. The result is essentially a complicated recursion involving exponential sums and lower rank Eisenstein series. Then one checks the definition given in [Brubaker et al. 2007] satisfies the recursion. A similar approach should be possible in type C, and this will be the subject of future work. Such an approach depends critically on having a proposed solution to satisfy the recursion, so the methods of this paper are a necessary first step.

The precise definition of the prime-power coefficients (2) for type C is somewhat complicated, so we postpone it until Section 3. As alluded to earlier, coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ will be described in terms of basis vectors for highest weight representations of $\mathrm{Sp}(2r, \mathbb{C})$ with highest weight corresponding to $\boldsymbol{l}$. As noted in Remark 6, the definition produces Gauss sums which encode subtle information about Kashiwara raising/lowering operators in the crystal graph associated to the highest weight representation. As such, this paper offers the first evidence that mysterious connections between metaplectic Eisenstein series and crystal bases may hold in much greater generality, persisting beyond the type A theory in [Brubaker et al. 2007; 2011a; 2011b]. These connections may not be properly understood until a general solution to our problem for all root systems Φ is obtained.

Finally, the results of this paper give infinite classes of Dirichlet series with analytic continuation. One can then use standard Tauberian techniques to extract mean-value estimates for families of number-theoretic quantities appearing in the numerator of the series (or the numerator of polar residues of the series). For the n-cover of A_r, this method yielded the mean-value results of [Chinta 2005] for $r = 5, n = 2$ and [Brubaker and Bump 2006b] for $r = 3, n = 3$. It would be interesting to explore similar results in type C (remembering that our Conjecture may be verified for any given example with n, r, and $\boldsymbol{m}$ fixed with only a finite amount of checking, as sketched in [Beineke et al. 2010]).

Note. Since the initial submission of this paper, Chinta and Offen [2009] have given a proof in type A that the multiple Dirichlet series constructed by Chinta and Gunnells is in fact a metaplectic Whittaker coefficient. This argument has been extended in great generality by McNamara [2011]. Further, Ivanov [2010] has used the results of this paper to give an alternate definition of the prime-power coefficients (2) in terms of two-dimensional lattice models defined by Kuperberg [2002]. In the case $n = 1$, his methods give an alternate proof of Theorem 2. All of these results make a resolution of the Conjecture given above more desirable.

2. Definition of the multiple Dirichlet series

In this section, we present general notation for root systems and the corresponding Weyl group multiple Dirichlet series.

2.1. ***Root systems.*** Let Φ be a reduced root system contained in V, a real vector space of dimension r. The dual vector space $V^\vee$ contains a root system $\Phi^\vee$ in bijection with Φ, where the bijection switches long and short roots. Writing the dual pairing

$$V \times V^\vee \to \mathbb{R}, \quad (x, y) \mapsto B(x, y), \tag{3}$$

then $B(\alpha, \alpha^\vee) = 2$. Moreover, the simple reflection $\sigma_\alpha : V \to V$ corresponding to α is given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\alpha.$$

Note that σ_α preserves Φ. Similarly, define a dual reflection $\sigma_{\alpha^\vee} : V^\vee \to V^\vee$ by $\sigma_{\alpha^\vee}(x) = x - B(\alpha, x)\alpha^\vee$ with $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

For our purposes, without loss of generality, we may take Φ to be irreducible (that is, there do not exist orthogonal subspaces Φ_1, Φ_2 with $\Phi_1 \cup \Phi_2 = \Phi$). Then set $\langle \cdot , \cdot \rangle$ to be the Euclidean inner product on V and $\|\alpha\| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm, where we normalize so that $2\langle \alpha, \beta \rangle$ and $\|\alpha\|^2$ are integral for all $\alpha, \beta \in \Phi$.

With this notation,

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha \quad \text{for any } \alpha, \beta \in \Phi. \tag{4}$$

Partition Φ into positive roots Φ^+ and negative roots Φ^- and denote by $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ the subset of simple positive roots. Further, denote the fundamental dominant weights by ϵ_i for $i = 1, \ldots, r$ satisfying

$$\frac{2\langle \epsilon_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker delta. Any dominant weight λ is expressible in terms of the ϵ_i, and a distinguished role in the theory is played by the Weyl vector ρ, defined by

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha = \sum_{i=1}^{r} \epsilon_i. \tag{6}$$

2.2. ***Algebraic preliminaries.*** Keeping with the established foundations on Weyl group multiple Dirichlet series (see [Brubaker et al. 2006; 2008]), we define our Dirichlet series as indexed by integers rather than ideals. By using this approach, the coefficients of the Dirichlet series will closely resemble classical exponential sums, but some care needs to be taken to ensure the resulting series remains well-defined up to units.

Given a fixed positive odd integer n, let F be a number field containing the $2n$-th roots of unity, and let S be a finite set of places containing all ramified places over $\mathbb{Q}$, all archimedean places, and enough additional places so that the ring of S-integers $\mathbb{O}_S$ is a principal ideal domain. Recall that the $\mathbb{O}_S$ integers are defined as

$$\mathbb{O}_S = \{a \in F \mid a \in \mathbb{O}_v \ \forall v \notin S\}$$

and can be embedded diagonally in

$$F_S = \prod_{v\in S} F_v.$$

There exists a pairing

$$(\cdot\,,\cdot)_S : F_S^\times \times F_S^\times \to \mu_n \text{ defined by } (a, b)_S = \prod_{v\in S}(a, b)_v,$$

where the $(a, b)_v$ are local Hilbert symbols associated to n and v.

Further, to any $a \in \mathbb{O}_S$ and any ideal $\mathfrak{b} \subseteq \mathbb{O}_S$, we may associate the n-th power residue symbol $(\frac{a}{\mathfrak{b}})_n$ as follows. For prime ideals $\mathfrak{p}$, the expression $(\frac{a}{\mathfrak{p}})_n$ is the unique n-th root of unity satisfying the congruence

$$\left(\frac{a}{\mathfrak{p}}\right)_n \equiv a^{(N(\mathfrak{p})-1)/n} \ (\text{mod } \mathfrak{p}).$$

Extend the symbol to arbitrary ideals $\mathfrak{b}$ by multiplicativity, with the convention that the symbol is 0 whenever a and $\mathfrak{b}$ are not relatively prime. Since $\mathcal{O}_S$ is a principal ideal domain by assumption, we will write

$$\left(\frac{a}{b}\right)_n = \left(\frac{a}{\mathfrak{b}}\right)_n \qquad \text{for } \mathfrak{b} = b\mathcal{O}_S$$

and often drop the subscript n on the symbol when the power is understood from context.

Then if a, b are coprime integers in $\mathcal{O}_S$, we have the n-th power reciprocity law (see [Neukirch 1999, Theorem 6.8.3])

$$\left(\frac{a}{b}\right) = (b, a)_S \left(\frac{b}{a}\right), \tag{7}$$

which, in particular, implies that if $\epsilon \in \mathcal{O}_S^\times$ and $b \in \mathcal{O}_S$, then

$$\left(\frac{\epsilon}{b}\right) = (b, \epsilon)_S.$$

Finally, for a positive integer t and $a, c \in \mathcal{O}_S$ with $c \neq 0$, we define the Gauss sum $g_t(a, c)$ as follows. First, choose a nontrivial additive character ψ of F_S trivial on the $\mathcal{O}_S$ integers (see [Brubaker and Bump 2006a] for details). Then the n-th power Gauss sum is given by

$$g_t(a, c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)_n^t \psi\left(\frac{ad}{c}\right), \tag{8}$$

where we have suppressed the dependence on n in the notation on the left. The Gauss sum g_t is not multiplicative, but rather satisfies

$$g_t(a, cc') = \left(\frac{c}{c'}\right)_n^t \left(\frac{c'}{c}\right)_n^t g_t(a, c) g_t(a, c') \tag{9}$$

for any relatively prime pair $c, c' \in \mathcal{O}_S$.

2.3. ***Kubota's rank-1 Dirichlet series.*** Many of the definitions for Weyl group multiple Dirichlet series are natural extensions of those from the rank-1 case, so we begin with a brief description of these.

A subgroup $\Omega \subset F_S^\times$ is said to be *isotropic* if $(a, b)_S = 1$ for all $a, b \in \Omega$. In particular, $\Omega = \mathcal{O}_S (F_S^\times)^n$ is isotropic (where $(F_S^\times)^n$ denotes the n-th powers in $F_S^\times$). Let $\mathcal{M}_t(\Omega)$ be the space of functions $\Psi : F_S^\times \to \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon c) = (c, \epsilon)_S^{-t} \Psi(c) \quad \text{for any } \epsilon \in \Omega, c \in F_S^\times. \tag{10}$$

For $\Psi \in \mathcal{M}_t(\Omega)$, consider the generalization of Kubota's Dirichlet series:

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathcal{O}_s/\mathcal{O}_s^\times} \frac{g_t(a, c)\Psi(c)}{|c|^{2s}}. \tag{11}$$

Here $|c|$ is the order of $\mathcal{O}_S/c\mathcal{O}_S$, $g_t(a, c)$ is as in (8) and the term $g_t(a, c)\Psi(c)|c|^{-2s}$ is independent of the choice of representative c, modulo S-units. Standard estimates for Gauss sums show that the series is convergent if $\Re(s) > \frac{3}{4}$. Our functional equation computations will hinge on the functional equation for this Kubota Dirichlet series. Before stating this result, we require some additional notation. Let

$$\boldsymbol{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-2} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{12}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\boldsymbol{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s, \Psi, a) = \boldsymbol{G}_m(s)^{[F:\mathbb{Q}]/2} \zeta_F(2ms - m + 1)\mathcal{D}_t(s, \Psi, a), \tag{13}$$

where $m = n/\gcd(n, t)$, $\frac{1}{2}[F : \mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\text{fin}}$ let q_v denote the cardinality of the residue class field $\mathcal{O}_v/\mathcal{P}_v$, where $\mathcal{O}_v$ is the local ring in F_v and $\mathcal{P}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finite number of places in S_{fin}. If $\Psi \in \mathcal{M}_t(\Omega)$ and $\eta \in F_S^\times$, denote

$$\widetilde{\Psi}_\eta(c) = (\eta, c)_S\, \Psi(c^{-1}\eta^{-1}). \tag{14}$$

Then we have the next result, which follows from [Brubaker and Bump 2006a].

Theorem [Brubaker et al. 2008, Theorem 1]. *Let $\Psi \in \mathcal{M}_t(\Omega)$ and $a \in \mathcal{O}_S$. Let $m = n/\gcd(n, t)$. Then $\mathcal{D}_t^*(s, \Psi, a)$ has meromorphic continuation to all s, analytic except possibly at $s = 1/2 \pm 1/(2m)$, where it might have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ depending only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}_t^*(s, \Psi, a) = |a|^{1-2s} \sum_{\eta \in F_S^\times/(F_S^\times)^n} P_{a\eta}^t(s)\mathcal{D}_t^*(1 - s, \widetilde{\Psi}_\eta, a). \tag{15}$$

This result, based on ideas of Kubota [1969], relies on the theory of Eisenstein series. The case $t = 1$ is handled in [Brubaker and Bump 2006a]; the general case follows as discussed in the proof of [Brubaker et al. 2006, Proposition 5.2]. Notably, the factor $|a|^{1-2s}$ is independent of the value of t.

2.4. ***The form of higher rank multiple Dirichlet series.*** We now begin explicitly defining the multiple Dirichlet series, retaining our previous notation. By analogy with the rank-1 definition in (10), given an isotropic subgroup Ω, let $\mathcal{M}(\Omega^r)$ be the space of functions $\Psi : (F_S^\times)^r \to \mathbb{C}$ that satisfy the transformation property

$$\Psi(\boldsymbol{\epsilon c}) = \left(\prod_{i=1}^{r}(\epsilon_i, c_i)_S^{\|\alpha_i\|^2} \prod_{i<j}(\epsilon_i, c_j)_S^{2\langle\alpha_i,\alpha_j\rangle}\right)\Psi(\boldsymbol{c}) \tag{16}$$

for all $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in \Omega^r$ and all $\boldsymbol{c} = (c_1, \dots, c_r) \in (F_S^\times)^r$.

Recall from the introduction that, given a reduced root system Φ of fixed rank r, an integer $n \geq 1$, $\boldsymbol{m} \in \mathcal{O}_S^r$, and $\Psi \in \mathcal{M}(\Omega^r)$, we consider a function of r complex variables $\boldsymbol{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi(s_1, \dots, s_r; m_1, \dots, m_r) = Z_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \sum_{\substack{\boldsymbol{c}=(c_1,\dots,c_r)\\ \in(\mathcal{O}_S/\mathcal{O}_S^\times)^r}} \frac{H^{(n)}(\boldsymbol{c}; \boldsymbol{m})\Psi(\boldsymbol{c})}{|c_1|^{2s_1}\cdots|c_r|^{2s_r}}.$$

The function $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ carries the main arithmetic content. It is not defined as a multiplicative function, but rather a "twisted multiplicative" function. For us, this means that for S-integer vectors $\boldsymbol{c}, \boldsymbol{c}' \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r$ with $\gcd(c_1\cdots c_r, c_1'\cdots c_r') = 1$,

$$H^{(n)}(c_1c_1', \dots, c_rc_r'; \boldsymbol{m}) = \mu(\boldsymbol{c}, \boldsymbol{c}')H^{(n)}(\boldsymbol{c}; \boldsymbol{m})H^{(n)}(\boldsymbol{c}'; \boldsymbol{m}), \tag{17}$$

where $\mu(\boldsymbol{c}, \boldsymbol{c}')$ is an n-th root of unity depending on $\boldsymbol{c}, \boldsymbol{c}'$. It is given precisely by

$$\mu(\boldsymbol{c}, \boldsymbol{c}') = \prod_{i=1}^{r}\left(\frac{c_i}{c_i'}\right)_n^{\|\alpha_i\|^2}\left(\frac{c_i'}{c_i}\right)_n^{\|\alpha_i\|^2} \prod_{i<j}\left(\frac{c_i}{c_j'}\right)_n^{2\langle\alpha_i,\alpha_j\rangle}\left(\frac{c_i'}{c_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \tag{18}$$

where $(\frac{\cdot}{\cdot})_n$ is the n-th power residue symbol defined in Section 2.2. In the special case $\Phi = A_1$, the twisted multiplicativity in (17) and (18) agrees with the identity for Gauss sums in (9) in accordance with the numerator for the rank-1 case in (11).

Remark 4. We often think of twisted multiplicativity as the appropriate generalization of multiplicativity for the metaplectic group. In particular, for $n = 1$ we reduce to the usual multiplicativity on relatively prime coefficients. Moreover, many of the global properties of the Dirichlet series follow (upon careful analysis of the twisted multiplicativity and associated Hilbert symbols) from local properties, for example, functional equations as in [Brubaker et al. 2006; 2008]. For more on this perspective, see [Friedberg 2010].

The transformation property of functions in $\mathcal{M}(\Omega^r)$ in (16) is motivated by the identity

$$H^{(n)}(\boldsymbol{\epsilon c}; \boldsymbol{m})\Psi(\boldsymbol{\epsilon c}) = H^{(n)}(\boldsymbol{c}; \boldsymbol{m})\Psi(\boldsymbol{c}) \quad \text{for all } \boldsymbol{\epsilon} \in \mathcal{O}_S^r,\ \boldsymbol{c}, \boldsymbol{m} \in (F_S^\times)^r.$$

The proof can be verified using the n-th power reciprocity law from Section 2.2.

Now, given any $\boldsymbol{m}, \boldsymbol{m}', \boldsymbol{c} \in \mathcal{O}_S^r$ with $\gcd(m_1' \cdots m_r', c_1 \cdots c_r) = 1$, let

$$H^{(n)}(\boldsymbol{c}; m_1 m_1', \ldots, m_r m_r') = \prod_{i=1}^{r} \left(\frac{m_i'}{c_i}\right)_n^{-\|\alpha_i\|^2} H^{(n)}(\boldsymbol{c}; \boldsymbol{m}). \tag{19}$$

The definitions in (17) and (19) imply that it is enough to specify the coefficients $H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r})$ for any fixed prime p with $l_i = \mathrm{ord}_p(m_i)$ in order to completely determine $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ for any pair of S-integer vectors $\boldsymbol{m}$ and $\boldsymbol{c}$. These prime-power coefficients are described in terms of data from highest-weight representations associated to $(l_1, \ldots, l_r)$ and will be given precisely in Section 3.

2.5. *Weyl group actions.* In order to precisely state a functional equation for the Weyl group multiple Dirichlet series, we require an action of the Weyl group W of Φ on the complex parameters $(s_1, \ldots, s_r)$. This arises from the linear action of W, realized as the group generated by the simple reflections $\sigma_{\alpha^\vee}$, on $V^\vee$. From the perspective of Dirichlet series, it is more natural to consider this action shifted by $\rho^\vee$, half the sum of the positive coroots. Then each $w \in W$ induces a transformation $V_{\mathbb{C}}^\vee = V^\vee \otimes \mathbb{C} \to V_{\mathbb{C}}^\vee$ (still denoted by w) if we require that

$$B\big(w\alpha, w(\boldsymbol{s}) - \tfrac{1}{2}\rho^\vee\big) = B\big(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee\big).$$

We introduce coordinates on $V_{\mathbb{C}}^\vee$ using simple roots $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ as follows. Define an isomorphism $V_{\mathbb{C}}^\vee \to \mathbb{C}^r$ by

$$\boldsymbol{s} \mapsto (s_1, s_2, \ldots, s_r), \quad s_i = B(\alpha_i, \boldsymbol{s}). \tag{20}$$

This action allows us to identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$, and so the complex variables s_i that appear in the definition of the multiple Dirichlet series may be regarded as coordinates in either space. It is convenient to describe this action more explicitly in terms of the s_i, and it suffices to consider simple reflections which generate W. Using the action of the simple reflection σ_{α_i} on the root system Φ given in (4) in conjunction with (20) above gives:

Proposition 5. *The action of σ_{α_i} on $\boldsymbol{s} = (s_1, \ldots, s_r)$ defined implicitly in (20) is given by*

$$s_j \mapsto s_j - \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\big(s_i - \tfrac{1}{2}\big), \quad j = 1, \ldots, r. \tag{21}$$

In particular, $\sigma_{\alpha_i} : s_i \mapsto 1 - s_i$.

2.6. *Normalizing factors and functional equations.* The multiple Dirichlet series must also be normalized using Gamma and zeta factors in order to state precise functional equations. Let

$$n(\alpha) = \frac{n}{\gcd(n, \|\alpha\|^2)}, \quad \alpha \in \Phi^+.$$

For example, if $\Phi = C_r$ and we normalize short roots to have length 1, this implies that $n(\alpha) = n$ unless α is a long root and n is even (in which case $n(\alpha) = n/2$). By analogy with the zeta factor appearing in (13), for any $\alpha \in \Phi^+$, let

$$\zeta_\alpha(\boldsymbol{s}) = \zeta\left(1 + 2n(\alpha)B(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee)\right),$$

where ζ is the Dedekind zeta function attached to the number field F. Further, for $\boldsymbol{G}_n(s)$ as in (12), we may define

$$\boldsymbol{G}_\alpha(\boldsymbol{s}) = \boldsymbol{G}_{n(\alpha)}\left(\tfrac{1}{2} + B(\alpha, \boldsymbol{s} - \tfrac{1}{2}\rho^\vee)\right). \tag{22}$$

Then for any $\boldsymbol{m} \in \mathbb{O}_S^r$, the normalized multiple Dirichlet series is given by

$$Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m}) = \left(\prod_{\alpha \in \Phi^+} \boldsymbol{G}_\alpha(\boldsymbol{s})\zeta_\alpha(\boldsymbol{s})\right) Z_\Psi(\boldsymbol{s}, \boldsymbol{m}). \tag{23}$$

By considering the product over all positive roots, we guarantee that the other zeta and Gamma factors are permuted for each simple reflection $\sigma_i \in W$, and hence for all elements of the Weyl group.

Given any fixed n, $\boldsymbol{m}$ and root system Φ, we seek to define $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ (or equivalently, given twisted multiplicativity, to define H at prime-power coefficients) so that $Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m})$ satisfies functional equations of the form

$$Z^*_\Psi(\boldsymbol{s}; \boldsymbol{m}) = |m_i|^{1-2s_i} Z^*_{\sigma_i\Psi}(\sigma_i \boldsymbol{s}; \boldsymbol{m}) \tag{24}$$

for all simple reflections $\sigma_i \in W$. Here, $\sigma_i \boldsymbol{s}$ is as in (21) and the function $\sigma_i \Psi$, which essentially keeps track of the rather complicated scattering matrix in this functional equation, is defined as in [Brubaker et al. 2008, (37)]. As noted in [Brubaker et al. 2008, Section 7], given functional equations of this type, one can obtain analytic continuation to a meromorphic function of $\mathbb{C}^r$ with an explicit description of polar hyperplanes.

3. Definition of the prime-power coefficients

In this section, we give a precise definition of the coefficients $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ needed to complete the description of the multiple Dirichlet series for root systems of type C_r and n odd. All the previous definitions are stated in sufficient generality for application to multiple Dirichlet series for any reduced root system Φ and any positive integer n. Only the prime-power coefficients require specialization to our particular root system $\Phi = C_r$, though this remains somewhat complicated. We summarize the definition at the end of the section.

The vector $\boldsymbol{l} = (l_1, l_2, \ldots, l_r)$ appearing in $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ can be associated to a dominant weight for $\mathrm{Sp}_{2r}(\mathbb{C})$ of the form

$$\lambda = (l_1 + l_2 + \cdots + l_r, \ldots, l_1 + l_2, l_1). \tag{25}$$

The contributions to $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ will then be parametrized by basis vectors of the highest weight representation of highest weight $\lambda+\rho$, where ρ is the Weyl vector for C_r defined in (6), so that

$$\lambda+\rho=(l_1+l_2+\cdots+l_r+r,\ldots,l_1+l_2+2,l_1+1)=:(L_r,\ldots,L_1). \tag{26}$$

In [Brubaker et al. 2007], prime-power coefficients for multiple Dirichlet series of type A were attached to Gelfand–Tsetlin patterns, which parametrize highest weight vectors for $\mathrm{SL}_{r+1}(\mathbb{C})$ (see [Gelfand and Tsetlin 1950]). Here, we use an analogous basis for the symplectic group, according to branching rules given in [Zhelobenko 1962]. We will continue to refer to the objects comprising this basis as Gelfand–Tsetlin patterns, or GT-patterns.

More precisely, a GT-pattern P has the form

$$P=\begin{array}{ccccccccc} a_{0,1} & & a_{0,2} & & \cdots & & a_{0,r} & & \\ & b_{1,1} & & b_{1,2} & \cdots & b_{1,r-1} & & & b_{1,r} \\ & & a_{1,2} & & \cdots & & a_{1,r} & & \\ & & & \ddots & & \ddots & & & \vdots \\ & & & & & & a_{r-1,r} & & \\ & & & & & & & & b_{r,r} \end{array} \tag{27}$$

where the $a_{i,j}, b_{i,j}$ are nonnegative integers and the rows of the pattern interleave. That is, for all $a_{i,j}, b_{i,j}$ in the pattern P above,

$$\min(a_{i-1,j},a_{i,j})\geq b_{i,j}\geq\max(a_{i-1,j+1},a_{i,j+1}),$$
$$\min(b_{i+1,j-1},b_{i,j-1})\geq a_{i,j}\geq\max(b_{i+1,j},b_{i,j}).$$

The set of all patterns with top row $(a_{0,1},\ldots,a_{0,r})=(L_r,\ldots,L_1)$ form a basis for the highest weight representation with highest weight $\lambda+\rho$. Hence, we will consider GT-patterns with top row $(L_r,\ldots,L_1)$ as in (26), and refer to this set of patterns as $\mathrm{GT}(\lambda+\rho)$.

The contributions to each $H^{(n)}(p^{\boldsymbol{k}};\, p^{\boldsymbol{l}})$ with both $\boldsymbol{k}$ and $\boldsymbol{l}$ fixed come from a single weight space corresponding to $\boldsymbol{k}=(k_1,\ldots,k_r)$ in the highest weight representation $\lambda+\rho$ corresponding to $\boldsymbol{l}$. We first describe how to associate a weight vector to each GT-pattern. Let

$$s_a(i):=\sum_{m=i+1}^{r}a_{i,m}\quad\text{and}\quad s_b(i):=\sum_{m=i}^{r}b_{i,m} \tag{28}$$

be the row sums for the respective rows of a's and b's in P. (Here we understand that $s_a(r)=0$ corresponds to an empty sum.) Then define the weight vector $\mathrm{wt}(P)=(\mathrm{wt}_1(P),\ldots,\mathrm{wt}_r(P))$ by

$$\mathrm{wt}_i=\mathrm{wt}_i(P)=s_a(r-i)-2s_b(r+1-i)+s_a(r+1-i),\quad i=1,\ldots,r. \tag{29}$$

As the weights are generated in turn, we begin at the bottom of the pattern P and work our way up to the top. Our prime-power coefficients will then be supported at $(p^{k_1}, \dots, p^{k_r})$ with

$$k_1 = \frac{1}{2}\sum_{j=1}^{r} \mathrm{wt}_j + L_j, \quad k_i = \sum_{j=i}^{r} \left(\mathrm{wt}_j + L_j\right), \quad i = 2, \dots, r, \tag{30}$$

so that in particular, the k_i are nonnegative integers.

In terms of the GT-pattern P, the reader may check that

$$k(P) = (k_1(P), k_2(P), \dots, k_r(P)),$$

with

$$\begin{aligned} k_1(P) &= s_a(0) - \sum_{m=1}^{r} \left(s_b(m) - s_a(m)\right), \\ k_i(P) &= s_a(0) - 2\sum_{m=1}^{r+1-i} \left(s_b(m) - s_a(m)\right) - s_a(r+1-i) + \sum_{m=1}^{r+1-i} a_{0,m} \end{aligned} \tag{31}$$

for $1 < i \le r$.

Then we define

$$H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = H^{(n)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) = \sum_{\substack{P \in \mathrm{GT}(\lambda+\rho) \\ k(P) = (k_1, \dots, k_r)}} G(P), \tag{32}$$

where the sum is over all GT-patterns P with top row $(L_r, \dots, L_1)$ as in (26) satisfying the condition $\boldsymbol{k}(P) = (k_1, \dots, k_r)$ and $G(P)$ is a weighting function whose definition depends on the following elementary quantities. To each pattern P, define the corresponding data

$$v_{i,j} = \sum_{m=i}^{j} (a_{i-1,m} - b_{i,m}), \quad w_{i,j} = \sum_{m=j}^{r} (a_{i,m} - b_{i,m}), \quad u_{i,j} = v_{i,r} + w_{i,j}, \tag{33}$$

where we understand the entries $a_{i,j}$ or $b_{i,j}$ to be 0 if they do not appear in the pattern P.

Remark 6. The integers $u_{i,j}$ and $v_{i,j}$ have representation-theoretic meaning in terms of Kashiwara raising and lowering operators in the crystal graph associated to the highest weight representation of highest weight $\lambda + \rho$ for $U_q(\mathfrak{sp}(2r))$, the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sp}(2r)$. See [Littelmann 1998] for details, particularly Corollary 2 of Section 6. See also [Brubaker et al. 2011a; 2011b] for a more complete description in crystal language, focusing mainly on type A. We find this interpretation quite striking in light of the connection to Whittaker models on the metaplectic group. Ultimately, this can be seen as another instance of connections between quantum groups and principal series representations in the spirit of [Lusztig 2003]. This is not a perspective we emphasize here, but this line of inquiry is discussed further in [Beineke et al. 2010].

To each entry $b_{i,j}$ in P, associate

$$\gamma_b(i,j) = \begin{cases} g_{\delta_{jr}+1}(p^{v_{i,j}-1}, p^{v_{i,j}}) & \text{if } b_{i,j} = a_{i-1,j+1}, \\ \phi(p^{v_{i,j}}) & \text{if } a_{i-1,j} < b_{i,j} < a_{i-1,j+1}, n \mid v_{i,j}\cdot(\delta_{jr}+1), \\ 0 & \text{if } a_{i-1,j} < b_{i,j} < a_{i-1,j+1},\ n \nmid v_{i,j}\cdot(\delta_{jr}+1), \\ q^{v_{i,j}} & \text{if } b_{i,j} = a_{i-1,j}, \end{cases} \tag{34}$$

where $g_t(p^\alpha, p^\beta)$ is an n-th power Gauss sum as in (8), $\phi(p^a)$ is the Euler phi function for $\mathbb{O}_S/p^a\mathbb{O}_S$, $q = |\mathbb{O}_S/p\mathbb{O}_S|$, and δ_{jr} is the Kronecker delta function. These cases may be somewhat reduced, using elementary properties of Gauss sums, to

$$\gamma_b(i,j) = \begin{cases} q^{v_{i,j}} & \text{if } b_{i,j} = a_{i-1,j}, \\ g_{\delta_{jr}+1}(p^{v_{i,j}+b_{i,j}-a_{i-1,j+1}-1}, p^{v_{i,j}}) & \text{else.} \end{cases} \tag{35}$$

To each entry $a_{i,j}$ in P, with $i \geq 1$, we may associate

$$\gamma_a(i,j) = \begin{cases} g_1(p^{u_{i,j}-1}, p^{u_{i,j}}) & \text{if } a_{i,j} = b_{i,j-1}, \\ \phi(p^{u_{i,j}}) & \text{if } b_{i,j} < a_{i,j} < b_{i,j-1},\ n \mid u_{i,j}, \\ 0 & \text{if } b_{i,j} < a_{i,j} < b_{i,j-1},\ n \nmid u_{i,j}, \\ q^{u_{i,j}} & \text{if } a_{i,j} = b_{i,j}, \end{cases} \tag{36}$$

which can similarly be compacted to

$$\gamma_a(i,j) = \begin{cases} q^{u_{i,j}} & \text{if } a_{i,j} = b_{i,j}, \\ g_1(p^{u_{i,j}-a_{i,j}+b_{i,j-1}-1}, p^{u_{i,j}}) & \text{else.} \end{cases} \tag{37}$$

We introduce terminology to describe relationships between elements in a pattern P:

Definition 7. A GT-pattern P is *minimal at $b_{i,j}$* if $b_{i,j} = a_{i-1,j}$. It is *maximal at $b_{i,j}$* if $1 \leq j < r$ and $b_{i,j} = a_{i-1,j+1}$, or if $b_{i,r} = 0$. If none of these equalities holds, we say P is *generic at $b_{i,j}$*.

Likewise, P is *minimal at $a_{i,j}$* if $a_{i,j} = b_{i,j}$, and *maximal at $a_{i,j}$* if $a_{i,j} = b_{i,j-1}$. If neither equality holds, we say P is *generic at $a_{i,j}$*.

Definition 8. A GT-pattern P is *strict* if its entries are strictly decreasing across each horizontal row.

Define the coefficients

$$G(P) = \begin{cases} \prod_{1\le i\le j\le r} \gamma_a(i, j)\gamma_b(i, j) & \text{if } P \text{ is strict,} \\ 0 & \text{otherwise,} \end{cases} \tag{38}$$

where we again understand $\gamma_a(r, r)$ to be 1 since $a_{r,r}$ is not in the pattern P. Combining these definitions gives a definition of the prime-power coefficients in the series:

Definition 9 (summary of definitions for H). Given any prime p, define

$$H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \sum_{\substack{P\in\mathrm{GT}(\lambda+\rho) \\ k(P)=\boldsymbol{k}}} G(P), \tag{39}$$

where the sum is over all GT-patterns with top row corresponding to $\lambda + \rho$ and row sums fixed according to (31), and $G(P)$ is given as in (38) above with $\gamma_a(i, j)$ and $\gamma_b(i, j)$ of (37) and (35), respectively, defined in terms of $v_{i,j}$ and $u_{i,j}$ in (33).

In the right-hand side of (39), we have suppressed the dependence on n. This is appropriate since the expressions in (35) and (37) are given in terms of Gauss sums, which are defined uniformly for all n.

The coefficients $H^{(n)}(\boldsymbol{c}; \boldsymbol{m})$ appearing in (1) are now implicitly defined by (39) together with the twisted multiplicativity given in (17) and (19). The resulting multiple Dirichlet series $Z_\Psi(\boldsymbol{s}; \boldsymbol{m})$ is initially absolutely convergent for $\Re(s_i)$ sufficiently large. Indeed, if a pattern P has weight $\boldsymbol{k} = (k_1, \dots, k_r)$, then

$$|G(P)| < q^{k_1+\cdots+k_r},$$

and the number of patterns in a given weight space is bounded as a function of $\boldsymbol{m}$ corresponding to the highest weight vector.

4. Comparison in the stable case

We now compare our multiple Dirichlet series, having p-th-power coefficients as defined in (39), with the multiple Dirichlet series defined for arbitrary root systems Φ in [Brubaker et al. 2008], when n is sufficiently large. In this section, we determine the necessary lower bound on n explicitly, according to a *stability assumption* introduced in [Brubaker et al. 2006]. With this lower bound, we can then prove that for n odd, the two prescriptions agree.

Let $\boldsymbol{m} = (m_1, \dots, m_r)$ be a fixed r-tuple of nonzero $\mathbb{O}_S$ integers. To any fixed prime p in $\mathbb{O}_S$, set $l_i = \mathrm{ord}_p(m_i)$ for $i = 1, \dots, r$. Then define λ_p as in (25), so that in terms of the fundamental dominant weights ϵ_i, we have

$$\lambda_p = \sum_{i=1}^{r} l_i \epsilon_i.$$

Then we may define the function d_{λ_p} on the set of positive roots Φ^+ by

$$d_{\lambda_p}(\alpha) = \frac{2\langle \lambda_p + \rho, \alpha\rangle}{\langle \alpha, \alpha\rangle}. \tag{40}$$

For ease of computation in the results that follow, normalize the inner product $\langle\ ,\ \rangle$ so that $\|\alpha\|^2 = \langle \alpha, \alpha\rangle = 1$ if α is a short root, while $\|\alpha\|^2 = 2$ if α is a long root.

Stability Assumption. Let $\alpha = \sum_{i=1}^r t_i \alpha_i$ be the largest positive root in the partial ordering for Φ. Then for every prime p, we require that the positive integer n satisfies

$$n \geq \gcd(n, \|\alpha\|^2) \cdot d_{\lambda_p}(\alpha) = \gcd(n, \|\alpha\|^2) \cdot \sum_{i=1}^r t_i (l_i + 1). \tag{41}$$

When the Stability Assumption holds, we say we are "in the stable case." This is well-defined since $l_i = 0$ for all $i = 1, \ldots, r$ for all but finitely many primes p. For the remainder of this section, we work with a fixed prime p, and so write λ in place of λ_p when no confusion can arise.

For $\Phi = C_r$, let α_1 denote the long simple root, so the largest positive root is $\alpha_1 + \sum_{i=2}^r 2\alpha_i$. Moreover if n is odd, the condition (41) becomes

$$n \geq l_1 + 1 + \sum_{i=2}^r 2(l_i + 1). \tag{42}$$

For any $w \in W(\Phi)$, define the set $\Phi_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$. Following [Brubaker et al. 2006; 2008], the p-th-power coefficients of the multiple Dirichlet series in the stable case are given by

$$H_{\mathrm{st}}^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \ldots, p^{l_r}) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}), \tag{43}$$

where the dependence on n occurs only in the n-th-power residue symbol in the Gauss sums. In [Brubaker et al. 2008], it was established that the above definition of $H_{\mathrm{st}}^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ produces a Weyl group multiple Dirichlet series $Z^*(\boldsymbol{s}, \boldsymbol{m})$ with analytic continuation and functional equations (of the form in the Conjecture) provided the Stability Assumption on n holds. The proof works for any reduced root system Φ. In this section, we demonstrate that our definition $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ in terms of GT-patterns as in (39) matches that in (43) for n satisfying the (41) of the Stability Assumption.

Definition 10. If $P \in \mathrm{GT}(\lambda + \rho)$ is a GT-pattern and $G(P)$ is defined as in (38), then P is said to be *stable* if $G(P) \neq 0$ for some (odd) n satisfying (41) of the Stability Assumption.

As we will see in the following result, if P is stable for one such n, then $G(P)$ is nonzero for all n satisfying (41). These are the relevant patterns we must consider in establishing the equivalence of the two definitions $H_{\mathrm{st}}^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ and $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ in the stable case, and we begin by characterizing all such patterns.

Proposition 11. *A pattern $P \in \mathrm{GT}(\lambda + \rho)$ is stable if and only if, in each pair of rows in P with index i (that is, pattern entries $\{b_{i,j}, a_{i,j}\}_{j=i}^{r}$), the ordered set*

$$\{b_{i,i}, b_{i,i+1}, \ldots, b_{i,r}, a_{i,r}, a_{i,r-1}, \ldots, a_{i,i+1}\}$$

has an initial string in which all elements are minimal (as in Definition 7) *and all remaining elements are maximal.*

Proof. If any element $a_{i,j}$ or $b_{i,j}$ in the pattern P is neither maximal nor minimal, that is, is "generic" in the sense of Definition 7, then $\gamma_a(i, j)$ (or $\gamma_b(i, j)$, respectively) is nonzero if and only if $n \mid u_{i,j}$ according to (36) (or $n \mid v_{i,j}(\delta_{jr} + 1)$ according to (34), respectively). But one readily checks that n is precisely chosen in the Stability Assumption so that $n > \max_{i,j}\{u_{i,j}, (\delta_{jr}+1)v_{i,j}\}$ and hence neither divisibility condition can be satisfied. Therefore all entries of any stable P must be maximal or minimal. The additional necessary condition that P be strict (as in Definition 8) so that $G(P)$ is not always zero according to (38) guarantees that neighboring entries in the ordered set can never be of the form (maximal,minimal), which gives the result. □

The number of stable patterns P is thus $2^r r! = |W(C_r)|$, the order of the Weyl group of C_r.

4.1. ***Action of W on Euclidean space.*** In demonstrating the equality of the two prime-power descriptions, it was necessary to use an explicit coordinatization of the root system embedded in $\mathbb{R}^r$; it would be desirable to find a coordinate-free proof. Let $\boldsymbol{e}_i$ be the standard basis vector (1 in i-th component, 0 elsewhere) in $\mathbb{R}^r$. Choose the following coordinates for the simple roots of C_r:

$$\alpha_1 = 2\boldsymbol{e}_1, \quad \alpha_2 = \boldsymbol{e}_2 - \boldsymbol{e}_1, \quad \ldots, \quad \alpha_r = \boldsymbol{e}_r - \boldsymbol{e}_{r-1}. \tag{44}$$

Consider an element $w \in W(C_r)$, the Weyl group of C_r. As an action on $\mathbb{R}^r$, this group is generated by all permutations σ of the basis vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$ and all reflections $\boldsymbol{e}_i \mapsto -\boldsymbol{e}_i$ for $i = 1, \ldots, r$. Thus we may describe the action explicitly using $\varepsilon_w^{(i)} \in \{+1, -1\}$ for $i = 1, 2, \ldots, r$ so that

$$w(t_1, t_2, \ldots, t_r) = (\varepsilon_w^{(1)} t_{\sigma^{-1}(1)}, \varepsilon_w^{(2)} t_{\sigma^{-1}(2)}, \ldots, \varepsilon_w^{(r)} t_{\sigma^{-1}(r)}). \tag{45}$$

In the following proposition, we associate a unique Weyl group element w with each GT-pattern P that is stable. In this result, and in the remainder of this section, it will be convenient to refer to the rows of P beginning at the *bottom* rather than the top. We will therefore discuss rows a_{r-i}, for $1 \leq i \leq r$, for instance.

Proposition 12. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1}\cdots L_1$, hence with associated dominant weight vector $\lambda=\sum_{i=1}^r \ell_i\varepsilon_i$. Let nonnegative integers $k_1(P),\ldots,k_r(P)$ be defined as in (31), and let $k_{r+1}(P)=0$. Then there exists a unique element $w\in W(C_r)$ such that*

$$\lambda+\rho-w(\lambda+\rho)=(2k_1-k_2,k_2-k_3,\ldots,k_{r-1}-k_r,k_r)=\sum_{i=1}^r k_i\alpha_i. \tag{46}$$

In fact, for $i=2,\ldots,r$,

$$k_{i+1}-k_i+L_i=-\,\mathrm{wt}_i=\varepsilon_w^{(i)}L_{\sigma^{-1}(i)}, \tag{47}$$

where $L_{\sigma^{-1}(i)}$ is the unique element in row a_{r-i} that is not in row a_{r+1-i}, and the weight coordinate wt_i *is as in (29). Similarly,*

$$k_2-2k_1+L_1=-\,\mathrm{wt}_1=\varepsilon_w^{(1)}L_{\sigma^{-1}(1)}, \tag{48}$$

where $L_{\sigma^{-1}(1)}$ is the unique element in row a_{r-1} that is not in row a_r.

Proof. The definitions for ρ and λ give $\lambda+\rho=(L_1,\ldots,L_r)$ in Euclidean coordinates. Compute the coordinates of $(\lambda+\rho)-\sum_{i=1}^r k_i\alpha_i$ using (31) gives

$$L_1+k_2-2k_1=-\big(s_a(r-1)-2s_b(r)+s_a(r)\big)=-\,\mathrm{wt}_1 \tag{49}$$

and similarly, for $i=2,\ldots,r$,

$$L_i+k_{i+1}-k_i=-\big(s_a(r-i)-2s_b(r+1-i)+s_a(r+1-i)\big)=-\,\mathrm{wt}_i, \tag{50}$$

so that

$$\lambda+\rho-\sum_{i=1}^r k_i\alpha_i=-(\mathrm{wt}_1,\mathrm{wt}_2,\ldots,\mathrm{wt}_r). \tag{51}$$

Each pattern P has a unique weight vector. Since P is a stable pattern, it is easy to see that the i-th weight consists of the unique entry that is in row a_{r-i} but not in row a_{r+1-i}, with a negative sign if this entry is present in row b_{r+1-i}, or a positive sign if not. Thus the weight vector is simply a permutation of the entries in the top row, with a choice of sign in each entry. We may find a unique w (whose action is described above), for which

$$w(\lambda+\rho)=(\varepsilon_w^{(1)}L_{\sigma^{-1}(1)},\ldots,\varepsilon_w^{(r)}L_{\sigma^{-1}(r)})=-(\mathrm{wt}_1,\mathrm{wt}_2,\ldots,\mathrm{wt}_r). \tag{52}$$

Thus $L_{\sigma^{-1}(i)}$ is the unique element in row a_{r-i} that is not present in row a_{r+1-i}. $\square$

Corollary 13. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1}\cdots L_1$. For $1\le i\le r$, the set of elements in row a_{r-i} satisfies*

$$\{a_{r-i,r+1-i},a_{r-i,r+2-i},\ldots,a_{r-i,r}\}=\{L_{\sigma^{-1}(i)},L_{\sigma^{-1}(i-1)},\ldots,L_{\sigma^{-1}(1)}\}. \tag{53}$$

Proof. From Proposition 12, $L_{\sigma^{-1}(j)}$ is the unique element in row a_{r-j} that is not in row a_{r+1-j}. Working downwards, eliminate these elements for $j = i, i+1, \dots, r$, in order to reach row a_{r-j}. This leaves the remaining set. □

4.2. *Agreement of the multiple Dirichlet series.*

Theorem 1. *Let $\Phi = C_r$ and choose a positive integer n such that (41) of the Stability Assumption holds.*

(i) *Let P be a stable strict* GT*-pattern, and let $G(P)$ be the product of Gauss sums defined in (38) in Section 2. Let w be the Weyl group element associated to P as in Proposition 12. Then*

$$G(P) = \prod_{\alpha \in \Phi_w} g_{\|\alpha\|^2}(p^{d_\lambda(\alpha)-1}, p^{d_\lambda(\alpha)}),$$

matching the definition given in (43), with $d_\lambda(\alpha)$ as defined in (40).

(ii) $H_{\mathrm{st}}(c_1, \dots, c_r; m_1, \dots m_r) = H^{(n)}(c_1, \dots, c_r; m_1, \dots m_r)$.

That is, the Weyl group multiple Dirichlet series in the twisted stable case is identical to the series defined by the Gelfand–Tsetlin description for n sufficiently large.

Remark 14. The Conjecture presented in the introduction states that n should be odd. In fact, the proof of Theorem 1 works for any n satisfying the Stability Assumption, regardless of parity. However, we believe this is an artifact of the relative combinatorial simplicity of the "stable" coefficients. As noted in Remark 3, one expects a distinctly different combinatorial recipe than the one presented in this paper to hold uniformly for all even n.

Proof. It is clear that part (i) implies part (ii), since both coefficients are obtained from their prime-power parts by means of twisted multiplicativity.

In proving part (i), let P be the GT-pattern with top row $L_r\, L_{r-1} \cdots L_1$ associated to w by Proposition 12. Since P is stable, we have $u_{i,j} = 0$ if P is minimal at $a_{i,j}$, and $v_{i,j} = 0$ if P is minimal at $b_{i,j}$. Thus

$$G(P) = \prod_{a_{i,j} \text{ maximal}} g_1(p^{u_{i,j}-1}, p^{u_{i,j}}) \prod_{b_{i,j} \text{ maximal}} g_{\delta_{jr}+1}(p^{v_{i,j}-1}, p^{v_{i,j}}).$$

It suffices to show that the set of Gauss sum exponents $u_{i,j}$ and $v_{i,j}$ at maximal entries in P coincides with the set of $d_\lambda(\alpha)$ as α runs over Φ_w. (In fact, we show a slightly sharper statement, which matches Gauss sum exponents at maximal entries in pairs of rows of P with values of $d_\lambda(\alpha)$ as α runs over certain subsets of Φ_w.)

The number of maximal elements in a pair of rows b_{r+1-i} and a_{r+1-i} is described in the next result. First, we say that (i, j) is an *i-inversion* for w^{-1} if $j < i$ and $\sigma^{-1}(j) > \sigma^{-1}(i)$. The number of these pairs, as well as the number of those for which the inequality is preserved rather than inverted, will play an important role in counting Gauss sums. To this end, define the quantities

$$\begin{aligned} \mathrm{inv}_i(w^{-1}) &= \#\{(i,j) \mid \sigma^{-1}(j) > \sigma^{-1}(i) \text{ and } j < i\}, \\ \mathrm{pr}_i(w^{-1}) &= \#\{(i,j) \mid \sigma^{-1}(j) < \sigma^{-1}(i) \text{ and } j < i\}. \end{aligned} \tag{54}$$

Proposition 15. *Let P be a stable strict* GT*-pattern with top row $L_r\ L_{r-1} \cdots L_1$, and let $w \in W$ be the Weyl group element associated to P as in Proposition 12. Let* $\mathrm{inv}_i(w)$ *and* $\mathrm{pr}_i(w)$ *be as defined in (54), and let $m_i(P)$ denote the number of maximal entries in rows b_{r+1-i} and a_{r+1-i} together. Then,*

$$m_i(P) = \begin{cases} \mathrm{inv}_i(w^{-1}) & \text{if } \varepsilon_w^{(i)} = +1, \\ i + \mathrm{pr}_i(w^{-1}) & \text{if } \varepsilon_w^{(i)} = -1. \end{cases} \tag{55}$$

Proof. Recall from our means of associating w to P that $\varepsilon_w^{(i)}$ is opposite in sign from the i-th Gelfand–Tsetlin weight. Consider row b_{r+1-i} together with the rows immediately above and below:

$$\begin{array}{ccccccccc} a_{r-i,r+1-i} & & a_{r-1,r+2-i} & & \cdots & & \cdots & a_{r-i,r} & \\ & b_{r+1-i,r+1-i} & & \cdots & & \cdots & & & b_{r+1-i,r} \\ & & a_{r+1-i,r+2-i} & & \cdots & & \cdots & a_{r+1-i,r} & \end{array}$$

Suppose $\varepsilon_w^{(i)} = +1$, so $L_{\sigma^{-1}(i)}$ is missing from row a_{r+1-i} but present in row b_{r+1-i}. Then there are no maximal entries in row b_{r+1-i}, and m_i maximal entries in row a_{r+1-i}, so

$$b_{r+1-i,r+j-i} = a_{r-i,r+j-i} \quad \text{for } 1 \le j \le i, \tag{56}$$

$$a_{r+1-i,r+(j+1)-i} = \begin{cases} b_{r+1-i,r+j-i} & \text{for } 1 \le j \le m_i, \\ b_{r+1-i,r+(j+1)-i} & \text{for } m_i + 1 \le j \le i. \end{cases} \tag{57}$$

Moreover, the entry $L_{\sigma^{-1}(i)}$ in row b_{r+1-i} marks the switch from maximal to minimal as we move from left to right in row a_{r+1-i}. That is, all entries in row a_{r+1-i} to the left of $L_{\sigma^{-1}(i)}$ are maximal, while all those to the right are minimal. By Corollary 13, row a_{r+1-i} consists of the elements in the set $\{L_{\sigma^{-1}(j)} \mid j < i\}$. Since the rows of P are strictly decreasing, this means the maximal entries in row a_{r+1-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) > \sigma^{-1}(i)\}.$$

This set clearly has order $\mathrm{inv}_i(w^{-1})$.

Now suppose $\varepsilon_w^{(i)} = -1$, so that $L_{\sigma^{-1}(i)}$ is missing from both row a_{r+1-i} and row b_{r+1-i}. Then all entries in row a_{r+1-i} are maximal, and the last $m_i - i + 1$

entries in row b_{r+1-i} are maximal, so

$$a_{r+1-i,r+(j+1)-i} = b_{r+1-i,r+j-i} \quad \text{for } 1 \le j \le i-1, \tag{58}$$

$$b_{r+1-i,r+j-i} = \begin{cases} a_{r-i,r+j-i} & \text{for } 1 \le j \le 2i-1-m_i, \\ a_{r-i,r+(j+1)-i} & \text{for } 2i-m_i \le j \le i-1, \\ 0 & \text{for } j = i. \end{cases} \tag{59}$$

The entry $L_{\sigma^{-1}(i)}$ in row a_{r-i} marks the switch from minimal to maximal as we move to the right in row b_{r+1-i}. That is, all entries below and to the left of $L_{\sigma^{-1}(i)}$ are minimal, while those below and to the right are maximal. Since rows b_{r+1-i} and a_{r+1-i} are identical, the entries of row b_{r+1-i} are $\{L_{\sigma^{-1}(j)} \mid j < i\}$, by Corollary 13. Moreover, since rows are strictly decreasing, the maximal entries in row b_{r+1-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) < \sigma^{-1}(i)\} \cup \{0\}.$$

This set has order $\mathrm{pr}_i(w^{-1})+1$. Counting maximal entries in both rows, we obtain $m_i = (i-1) + \mathrm{pr}_i(w^{-1}) + 1 = i + \mathrm{pr}_i(w^{-1})$. □

Next, we establish a finer characterization of $\Phi_w = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}$. For $\Phi = C_r$, the roots in Φ^+ take different forms; the positive long roots are $2\boldsymbol{e}_\ell$ for $1 \le \ell \le r$, while the positive short roots are $\boldsymbol{e}_m \pm \boldsymbol{e}_\ell$ for $1 \le \ell < m \le r$. We will express Φ_w as a disjoint union of subsets indexed by $i \in \{1, 2, \ldots, r\}$. To this end, let i be fixed, and let j be any positive integer such that $j < i$. Consider positive roots of the following three types:

$$\begin{aligned} &\text{Type L:} && \alpha_{i,w} := 2\boldsymbol{e}_{\sigma^{-1}(i)}. \\ &\text{Type S}^+\text{:} && \alpha^+_{i,j,w} := \boldsymbol{e}_{\sigma^{-1}(j)} + \boldsymbol{e}_{\sigma^{-1}(i)}. \\ &\text{Type S}^-\text{:} && \alpha^-_{i,j,w} := \begin{cases} \boldsymbol{e}_{\sigma^{-1}(j)} - \boldsymbol{e}_{\sigma^{-1}(i)} & \text{if } \sigma^{-1}(j) > \sigma^{-1}(i), \\ \boldsymbol{e}_{\sigma^{-1}(i)} - \boldsymbol{e}_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i). \end{cases} \end{aligned}$$

Clearly we encounter each positive root exactly once as i and j vary as indicated. Let $\Phi^{(i)}_w \subseteq \Phi_w$ denote the set of all $\alpha_{i,w}$, $\alpha^+_{i,j,w}$, $\alpha^-_{i,j,w}$ belonging to Φ_w. The next lemma completely characterizes $\Phi^{(i)}_w$.

Lemma 16. *Let $i \in \{1, 2, \ldots, r\}$ be fixed, let j be any positive integer with $j < i$, and let $\Phi^{(i)}_w$ be as defined above.*

(1) $\alpha_{i,w} \in \Phi^{(i)}_w$ *if and only if* $\varepsilon^{(i)}_w = -1$.

(2) $\alpha^-_{i,j,w} \in \Phi^{(i)}_w$ *if and only if* $\sigma^{-1}(j) < \sigma^{-1}(i)$ *and* $\varepsilon^{(i)}_w = -1$, *or* $\sigma^{-1}(j) > \sigma^{-1}(i)$ *and* $\varepsilon^{(i)}_w = +1$.

(3) $\alpha^+_{i,j,w} \in \Phi^{(i)}_w$ *if and only if* $\varepsilon^{(i)}_w = -1$.

Consequently, $|\Phi^{(i)}_w| = m_i(P)$, *as defined in Proposition 15.*

Proof. As defined in (45), w acts on a basis vector $\boldsymbol{e}_\ell$ simply as $w(\boldsymbol{e}_\ell) = \varepsilon_w^{(\ell)} \boldsymbol{e}_{\sigma(\ell)}$, and this action extends linearly to each of the roots. Part (1) is immediate from the definition of Φ_w.

For part (2), if $\sigma^{-1}(j) < \sigma^{-1}(i)$ then

$$w(\alpha^-_{i,j,w}) = \varepsilon_w^{(i)} \boldsymbol{e}_i - \varepsilon_w^{(j)} \boldsymbol{e}_j.$$

If $\varepsilon_w^{(i)} = +1$, then since $j < i$, we have $w(\alpha^-_{i,j,w}) \in \Phi^+$ regardless of the value of $\varepsilon_w^{(j)}$. Thus $\alpha^-_{i,j,w} \notin \Phi_w^{(i)}$. Similarly, if $\varepsilon_w^{(i)} = -1$, then since $j < i$, we have $w(\alpha^-_{i,j,w}) \in \Phi^-$ regardless of the value of $\varepsilon_w^{(j)}$. Thus $\alpha^-_{i,j,w} \in \Phi_w^{(i)}$.

On the other hand, if $\sigma^{-1}(j) > \sigma^{-1}(i)$ then

$$w(\alpha^-_{i,j,w}) = \varepsilon_w^{(j)} \boldsymbol{e}_j - \varepsilon_w^{(i)} \boldsymbol{e}_i.$$

Considering the cases $\varepsilon_w^{(i)} = +1, -1$ in turn, we find that regardless of the value of $\varepsilon_w^{(j)}$, we have $w(\alpha^-_{i,j,w}) \in \Phi_w^{(i)}$ if and only if $\varepsilon_w^{(i)} = +1$.

For part (3), we have

$$w(\alpha^+_{i,j,w}) = \varepsilon_w^{(j)} \boldsymbol{e}_j + \varepsilon_w^{(i)} \boldsymbol{e}_i.$$

Using a similar argument, we see that independently of the value of $\varepsilon_w^{(j)}$, $w(\alpha^+_{i,j,w})$ is a negative root when $\varepsilon_w^{(i)}$ is negative, and a positive root otherwise.

Finally, we count elements in $\Phi_w^{(i)}$. If $\varepsilon_w^{(i)} = +1$, the conditions yield $\mathrm{inv}_i(w^{-1})$ elements of type S^-, and zero elements of types L and S^+. On the other hand, if $\varepsilon_w^{(i)} = -1$, there is one element of type L, $i-1$ elements of type S^+, and $\mathrm{pr}_i(w^{-1})$ elements of type S^-. In either case, $|\Phi_w^{(i)}| = m_i(P)$. □

For each of the roots in $\Phi_w^{(i)}$, we compute the corresponding d_λ (as defined in (40)) below.

Lemma 17. *With the notation as above, we have*

(1) $d_\lambda(\alpha_{i,w}) = L_{\sigma^{-1}(i)}$.

(2) $d_\lambda(\alpha^-_{i,j,w}) = \begin{cases} L_{\sigma^{-1}(j)} - L_{\sigma^{-1}(i)} & \text{if } \sigma^{-1}(j) > \sigma^{-1}(i), \\ L_{\sigma^{-1}(i)} - L_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) < \sigma^{-1}(i). \end{cases}$

(3) $d_\lambda(\alpha^+_{i,j,w}) = L_{\sigma^{-1}(j)} + L_{\sigma^{-1}(i)}$.

Proof. First, we compute $d_\lambda(\alpha_{i,w}) = d_\lambda(2\boldsymbol{e}_{\sigma^{-1}(i)})$. Using (44), we have

$$\alpha_{i,w} = \alpha_1 + \sum_{k=2}^{\sigma^{-1}(i)} 2\alpha_k, \tag{60}$$

where we regard the sum to be 0 if $\sigma^{-1}(i)=1$. Since $\langle\alpha_{i,w},\alpha_{i,w}\rangle=\langle\alpha_1,\alpha_1\rangle=2$ and $\langle\alpha_k,\alpha_k\rangle=1$ for $k=2,\dots,r$, we have

$$d_\lambda(\alpha_{i,w})=\frac{2\langle\lambda+\rho,\alpha_{i,w}\rangle}{\langle\alpha_{i,w},\alpha_{i,w}\rangle}=\sum_{m=1}^{r}(l_m+1)\sum_{k=1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}=L_{\sigma^{-1}(i)}. \tag{61}$$

Next, we compute $d_\lambda(\alpha^-_{i,j,w})=d_\lambda(\boldsymbol{e}_{\sigma^{-1}(i)}-\boldsymbol{e}_{\sigma^{-1}(j)})$ if $\sigma^{-1}(j)<\sigma^{-1}(i)$. (The computations if $\sigma^{-1}(j)>\sigma^{-1}(i)$ are analogous.) In this case, (44) gives

$$\alpha^-_{i,j,w}=\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\alpha_k, \tag{62}$$

where the sum is nonempty as $\sigma^{-1}(j)<\sigma^{-1}(i)$. Since $\langle\alpha^-_{i,j,w},\alpha^-_{i,j,w}\rangle=1$,

$$d_\lambda(\alpha^-_{i,j,w})=\sum_{m=1}^{r}(l_m+1)\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}=L_{\sigma^{-1}(i)}-L_{\sigma^{-1}(j)}. \tag{63}$$

Finally, we compute $d_\lambda(\alpha^+_{i,j,w})=d_\lambda(\boldsymbol{e}_{\sigma^{-1}(i)}+\boldsymbol{e}_{\sigma^{-1}(j)})$. Here, (44) gives

$$\alpha^+_{i,j,w}=\alpha_1+\sum_{k=2}^{\sigma^{-1}(j)}2\alpha_k+\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\alpha_k, \tag{64}$$

where the first sum is 0 if $\sigma^{-1}(j)=1$. Since $\langle\alpha^+_{i,j,w},\alpha^+_{i,j,w}\rangle=1$ as well, we have

$$\begin{aligned} d_\lambda(\alpha^+_{i,j,w})&=\sum_{m=1}^{r}(l_m+1)\left(\sum_{k=1}^{\sigma^{-1}(j)}\frac{4\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}+\sum_{k=\sigma^{-1}(j)+1}^{\sigma^{-1}(i)}\frac{2\langle\epsilon_m,\alpha_k\rangle}{\langle\alpha_k,\alpha_k\rangle}\right)\\ &=L_{\sigma^{-1}(i)}+L_{\sigma^{-1}(j)}, \end{aligned} \tag{65}$$

which completes the proof. □

Now let $D_i=\{d_\lambda(\alpha)\mid\alpha\in\Phi_w^{(i)}\}$. By Lemmas 16 and 17, we see that if $\epsilon_w^{(i)}=+1$, then

$$D_i=\{L_{\sigma^{-1}(j)}-L_{\sigma^{-1}(i)}\mid j<i\text{ and }\sigma^{-1}(j)>\sigma^{-1}(i)\}, \tag{66}$$

while if $\epsilon_w^{(i)}=-1$, then

$$\begin{aligned} D_i=\{L_{\sigma^{-1}(i)}\}\cup\{L_{\sigma^{-1}(j)}+L_{\sigma^{-1}(i)}\mid j<i\}\\ \cup\{L_{\sigma^{-1}(i)}-L_{\sigma^{-1}(j)}\mid j<i\text{ and }\sigma^{-1}(j)<\sigma^{-1}(i)\}. \end{aligned} \tag{67}$$

Now we examine the Gauss sums obtained from the GT-pattern P with top row $L_r\ L_{r-1}\cdots L_1$ associated to w. Suppose there are $m_i=m_i(P)$ maximal entries in rows b_{r+1-i} and a_{r+1-i} combined. First, suppose there are no maximal entries in row b_{r+1-i}. Then the first m_i entries in row a_{r+1-i} (reading from the left) are maximal. Since there are $i-1$ entries in row a_{r+1-i}, in this case we have $m_i < i$. We may apply (56) and (57) to compute the sums defining $u_{k,\ell}$ and $v_{k,\ell}$. These sums telescope, and we have

$$v_{r+1-i,r+j-i} = 0 \quad \text{for } 1 \le j \le i-1,$$

$$u_{r+1-i,r+(j+1)-i} = \begin{cases} 0 & \text{for } m_i + 1 \le j \le i, \\ a_{r-i,r+j-i} - b_{r+1-i,r+(m_i+1)-i} & \text{for } 1 \le j \le m_i. \end{cases}$$

By Proposition 12, $b_{r+1-i,r+(m_i+1)-i} = L_{\sigma^{-1}(i)}$, so to compute $u_{r+1-i,r+(j+1)-i}$ as j varies, we must determine the set of values for $a_{r-i,r+j-i}$ with $1 \le j \le m_i$. Recall that by Corollary 13, the entries in row a_{r-i} are given by

$$\{L_{\sigma^{-1}(j)} \mid 1 \le j \le i\}. \tag{68}$$

The rows are strictly decreasing, so the entries appearing left of $a_{r-1,r+(m_i+1)-i} = L_{\sigma^{-1}(i)}$ have an index greater than $\sigma^{-1}(i)$. That is,

$$\{a_{r-i,r+j-i} \mid 1 \le j \le m_i\} = \{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(j) > \sigma^{-1}(i)\}. \tag{69}$$

Thus the nonzero Gauss sum exponents for rows b_{r+1-i} and a_{r+1-i} are given by $u_{r+1-i,r+(j+1)-i} = L_{\sigma^{-1}(j)} - L_{\sigma^{-1}(i)}$ with $j < i$ and $\sigma^{-1}(j) > \sigma^{-1}(i)$. Finally, $\varepsilon_w^{(i)} = +1$, since there are no maximal entries in row b_{r+1-i} in this case. Thus our set of nonzero Gauss sum exponents matches the set D_i as given in (66).

Second, suppose there are maximal entries in row b_{r+1-i}. Consequently, all entries in row a_{r+1-i} are maximal, so there are $n_i := m_i - i + 1$ maximal entries in row b_{r+1-i}. We may apply (58) and (59) to compute the sums defining $u_{k,\ell}$ and $v_{k,\ell}$. These sums telescope, and we have

$$v_{r+1-i,r+j-i} = \begin{cases} 0 & \text{for } 1 \le j \le i - n_i, \\ a_{r-i,r+1-n_i} - a_{r-i,r+(j+1)-i} & \text{for } i+1-n_i \le j \le i-1, \\ a_{r-i,r+1-n_i} & \text{for } j = i, \end{cases}$$

$$u_{r+1-i,r+(j+1)-i} = a_{r-i,r+1-n_i} + a_{r+1-i,r+(j+1)-i} \quad \text{for } 1 \le j \le i-1.$$

By Proposition 12, $a_{r+1-i,r+1-n_i} = L_{\sigma^{-1}(i)}$, and thus $v_{r+1-i,r} = L_{\sigma^{-1}(i)}$. To compute the remaining exponents $v_{r+1-i,r+j-i}$ as j varies, we again appeal to (68). Since the rows are strictly decreasing, the entries appearing to the right of $L_{\sigma^{-1}(i)}$ in row a_{r-1} must have an index smaller than $\sigma^{-1}(i)$. That is,

$$\{a_{r-i,r+(j+1)-i} \mid i+1-n_i \le j \le i-1\} = \{L_{\sigma^{-1}(j)} \mid j < i \text{ and } \sigma^{-1}(i) > \sigma^{-1}(j)\}.$$

Thus $v_{r+1-i,r+j-i} = L_{\sigma^{-1}(i)} - L_{\sigma^{-1}(j)}$ with $i+1-n_i \le j < i$ and $\sigma^{-1}(i) > \sigma^{-1}(j)$.

To compute the exponents $u_{r+1-i,r+(j+1)-i}$, we note that by Corollary 13, the entries in row a_{r+1-i} are the $L_{\sigma^{-1}(j)}$ for which $1 \le j \le i-1$. Thus

$$u_{r+1-i,r+(j+1)-i} = L_{\sigma^{-1}(i)} + L_{\sigma^{-1}(j)}, \tag{70}$$

with $1 \le j \le i-1$. Finally, $\varepsilon_w^{(i)} = -1$, since there are maximal entries in row b_{r+1-i}. Combining the cases above, we match the set D_i given in (67).

This completes the proof of Theorem 1. □

5. Comparison with the Casselman–Shalika formula

The main focus of this section is the proof of Theorem 2, using a generating function identity given in [Hamel and King 2002]. This identity may be regarded as a deformation of the Weyl character formula for $\mathrm{Sp}(2r)$, though it is stated in the language of symplectic, shifted tableaux (whose definition we will soon recall) so we postpone the precise formulation. Recall that our multiple Dirichlet series take the form

$$Z_\Psi(\boldsymbol{s};\boldsymbol{m}) = \sum_{\boldsymbol{c}=(c_1,\dots,c_r)\in(\mathbb{O}_S/\mathbb{O}_S^\times)^r} \frac{H^{(n)}(\boldsymbol{c};\boldsymbol{m})\Psi(\boldsymbol{c})}{|c_1|^{2s_1}\cdots|c_r|^{2s_r}}.$$

In brief, we show that for $n=1$ our formulas for the prime-power supported contributions of $Z_\Psi(\boldsymbol{s},\boldsymbol{m})$ match one side of Hamel and King's identity, while the other side of the identity is given in terms of a character of a highest weight representation for $\mathrm{Sp}(2r)$. By combining the Casselman–Shalika formula with Hamel and King's result, we will establish Theorem 2.

5.1. ***Specialization of the multiple Dirichlet series for $n = 1$.*** Many aspects of the definition Z_Ψ are greatly simplified when $n=1$. First, we may take Ψ to be constant, since the Hilbert symbols appearing in the definition (16) are trivial for $n=1$. Moreover, the coefficients $H^{(n)}(\boldsymbol{c};\boldsymbol{m})$ for $n=1$ are perfectly multiplicative in both $\boldsymbol{c}$ and $\boldsymbol{m}$. That is, according to (18),

$$H^{(1)}(\boldsymbol{c}\cdot\boldsymbol{c}';\boldsymbol{m}) = H^{(1)}(\boldsymbol{c};\boldsymbol{m})H^{(1)}(\boldsymbol{c}';\boldsymbol{m}) \quad \text{when } \gcd(c_1\cdots c_r, c_1'\cdots c_r') = 1,$$

and according to (19),

$$H^{(1)}(\boldsymbol{c};\boldsymbol{m}\cdot\boldsymbol{m}') = H^{(1)}(\boldsymbol{c};\boldsymbol{m}) \quad \text{when } \gcd(m_1'\cdots m_r', c_1\cdots c_r) = 1.$$

Hence the global definition of $Z_\Psi(\boldsymbol{s};\boldsymbol{m})$ for fixed $\boldsymbol{m}$ is easily recovered from its prime-power supported contributions as follows:

$$Z_\Psi(\boldsymbol{s};\boldsymbol{m}) = \prod_{p\in\mathbb{O}_S}\Bigg(\sum_{\boldsymbol{k}=(k_1,\dots,k_r)} \frac{H^{(1)}(p^{\boldsymbol{k}};p^{\boldsymbol{l}})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}}\Bigg), \tag{71}$$

with $\boldsymbol{l} = (l_1, \dots, l_r)$ given by $\mathrm{ord}_p(m_i) = l_i$ for $i = 1, \dots, r$. The sum on the right-hand side runs over the finite number of vectors $\boldsymbol{k}$ for which $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ has nonzero support for fixed $\boldsymbol{l}$ according to (39).

We now simplify our formulas for $H^{(n)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ when $n = 1$. As before, set $q = |\mathbb{O}_S / p\mathbb{O}_S|$. With definitions as given in (34) and (36), let

$$\tilde{\gamma}_a(i, j) := q^{-u_{i,j}} \gamma_a(i, j) \quad \text{and} \quad \tilde{\gamma}_b(i, j) := q^{-v_{i,j}} \gamma_b(i, j).$$

Then by analogy with the definitions (38) and (39), define

$$\widetilde{G}(P) := \prod_{1 \le i \le j \le r} \tilde{\gamma}_a(i, j) \tilde{\gamma}_b(i, j),$$

$$\widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \widetilde{H}^{(1)}(p^{k_1}, \dots, p^{k_r}; p^{l_1}, \dots, p^{l_r}) := \sum_{k(P) = (k_1, \dots, k_r)} \widetilde{G}(P),$$

where again the sum is taken over GT-patterns P with fixed top row $(L_r, \dots, L_1)$ as in (26). By elementary properties of Gauss sums, when $n = 1$ we have, for a strict GT-pattern P,

$$\tilde{\gamma}_a(i, j) = \begin{cases} 1 & \text{if } P \text{ is minimal at } a_{i,j}, \\ 1 - 1/q & \text{if } P \text{ is generic at } a_{i,j}, \\ -1/q & \text{if } P \text{ is maximal at } a_{i,j}, \end{cases} \tag{72}$$

recalling the language of Definition 7 and similarly,

$$\tilde{\gamma}_b(i, j) = \begin{cases} 1 & \text{if } P \text{ is minimal at } b_{i,j}, \\ 1 - 1/q & \text{if } P \text{ is generic at } a_{i,j}, \\ -1/q & \text{if } P \text{ is maximal at } b_{i,j}. \end{cases} \tag{73}$$

When P is generic at $a_{i,j}$ (respectively $b_{i,j}$), the condition $n \mid u_{i,j}$ (respectively $n \mid v_{i,j}$) is trivially satisfied, since $n = 1$.

We claim that

$$H^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}}) = \widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})\, q^{k_1 + \cdots + k_r}. \tag{74}$$

This equality follows from the definitions of $H^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$ and $\widetilde{H}^{(1)}(p^{\boldsymbol{k}}; p^{\boldsymbol{l}})$, after matching powers of q on each side by applying the following combinatorial lemma.

Lemma 18. *For each* GT-*pattern* P,

$$\sum_{i=1}^{r} k_i(P) = \sum_{i=1}^{r} \left(\sum_{j=i}^{r} v_{i,j} + \sum_{j=i+1}^{r} u_{i,j} \right). \tag{75}$$

Proof. We proceed by expanding each side in terms of the entries $a_{i,j}$ and $b_{i,j}$ in the GT-pattern P, using the definitions above. Applying (31), we have

$$\sum_{i=1}^{r} k_i(P) = \left(r\, s_a(0) + \sum_{m=1}^{r-1} s_a(m) + \sum_{i=2}^{r} \sum_{m=1}^{r+1-i} (2\, s_a(m) + a_{0,m}) - \sum_{i=2}^{r} s_a(r+1-i) \right) - \left(\sum_{m=1}^{r} s_b(m) + \sum_{i=2}^{r} \sum_{m=i}^{r+1-i} 2\, s_b(m) \right).$$

Since $\sum_{i=2}^{r} s_a(r+1-i) = \sum_{m=1}^{r-1} s_a(m)$, the corresponding terms in the first parentheses cancel. After interchanging order of summation and evaluating sums over i, we obtain

$$\sum_{i=1}^{r} k_i(P) = r\, s_a(0) + \sum_{m=1}^{r} (r-m)\, a_{0,m} + \sum_{m=1}^{r-1} 2(r-m)\, s_a(m) - \sum_{m=1}^{r} (1+2(r-m))\, s_b(m).$$

Finally, applying (28) and combining the first two terms, we conclude that

$$\sum_{i=1}^{r} k_i(P) = \sum_{m=1}^{r} (2r-m)\, a_{0,m} + \sum_{m=1}^{r-1} \sum_{\ell=m+1}^{r} 2(r-m)\, a_{m,\ell} - \sum_{m=1}^{r} \sum_{\ell=m}^{r} (1+2(r-m))\, b_{m,\ell}. \tag{76}$$

On the other hand, from (33), after recombining terms we have

$$\sum_{i=1}^{r} \left(\sum_{j=i}^{r} v_{i,j} + \sum_{j=i+1}^{r} u_{i,j} \right) = -\sum_{i=1}^{r} \left(b_{i,i} + \sum_{j=i+1}^{r} \left(b_{i,j} + 2 \sum_{m=i}^{r} b_{i,m} \right) \right) + \sum_{i=1}^{r} \left(a_{i-1,i} + \sum_{j=i+1}^{r} \left(\sum_{m=i}^{j} 2\, a_{i-1,m} + \sum_{m=j+1}^{r} a_{i-1,m} + \sum_{m=j}^{r} a_{i,m} \right) \right).$$

After interchanging order of summation and evaluating sums on j, this equals

$$\sum_{i=1}^{r} \left((1+2(r-i))\, a_{i-1,i} + \sum_{m=i+1}^{r} (2r+1-(i+m))\, a_{i-1,m} + \sum_{m=i+1}^{r} (m-i)\, a_{i,m} \right) - \sum_{i=1}^{r} \sum_{m=i}^{r} (1+2(r-i))\, b_{i,m}.$$

The $i = 1$ terms from the first two summands in the big parentheses evaluate to $\sum_{m=1}^{r}(2r-m)\, a_{0,m}$, the first term in (76). After reindexing, the remaining terms in the parentheses give $\sum_{i=1}^{r-1}\sum_{m=i+1}^{r} 2(r-i)\, a_{i,m}$. Relabeling indices where needed gives the result. □

We now manipulate the prime-power supported contributions to the multiple Dirichlet series as in (71). Setting $y_i = |p|^{-2s_i}$ for $i = 1, \ldots, r$ and using (74) gives

$$\sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \frac{H^{(1)}(p^{k_1},\ldots,p^{k_r})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}} = \sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1},\ldots,p^{k_r})\,(qy_1)^{k_1}\cdots(qy_r)^{k_r}. \tag{77}$$

After making the change of variables

$$qy_1 \mapsto x_1^2, \quad qy_2 \mapsto x_1^{-1}x_2, \quad \ldots, \quad qy_r \mapsto x_{r-1}^{-1}x_r,$$

the right-hand side of (77) becomes

$$\sum_{(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1},\ldots,p^{k_r})\, x_1^{2k_1}(x_1^{-1}x_2)^{k_2}\cdots(x_{r-1}^{-1}x_r)^{k_r}.$$

By the relationship between the coordinates k_i and the weight coordinates wt_i given in (30), this is just

$$x_1^{L_1}\cdots x_r^{L_r}\sum_{(k_1,\ldots,k_r)} \widetilde{H}^{(1)}(p^{k_1},\ldots,p^{k_r})\, x_1^{\mathrm{wt}_1}\cdots x_r^{\mathrm{wt}_r},$$

where the L_i relate to l_i as in (26). Finally, letting

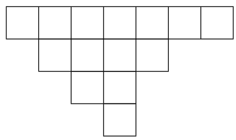
$$\mathrm{gen}(P) = \#\{\text{generic entries in } P\} \quad \text{and} \quad \max(P) = \#\{\text{maximal entries in } P\}$$

and using the simplifications for $n = 1$ in (72) and (73) for $\widetilde{H}^{(1)}$ in terms of $\widetilde{G}(P)$, then

$$\sum_{\boldsymbol{k}=(k_1,\ldots,k_r)} \frac{H^{(1)}(p^{k_1},\ldots,p^{k_r})}{|p|^{2k_1s_1}\cdots|p|^{2k_rs_r}} = x_1^{L_1}\cdots x_r^{L_r}\sum_{(k_1,\ldots,k_r)}\left(\frac{-1}{q}\right)^{\max(P)}\left(1-\frac{1}{q}\right)^{\mathrm{gen}(P)} x_1^{\mathrm{wt}_1}\cdots x_r^{\mathrm{wt}_r}, \tag{78}$$

with the x_i given in terms of $|p|^{-2s_i}$ by the composition of the above changes of variables. The right-hand side of (78) is now amenable to comparison with the identity of Hamel and King.

5.2. ***Symplectic shifted tableaux.*** In order to state the needed identity of Hamel and King, we introduce some additional terminology. To each strict GT-pattern P, we may associate an $\mathrm{Sp}(2r)$-*standard shifted tableau* of shape $\lambda+\rho$. Below, we follow [Hamel and King 2002], specializing Definition 2.5 to our circumstances. Consider the partition μ of $\lambda+\rho$, whose parts are given by $\mu_i = l_1 + \cdots l_i + r - i + 1$ for $i = 1, \dots, r$. (These are simply the entries in the top row of the pattern P in $\mathrm{GT}(\lambda+\rho)$.) Such a partition defines a *shifted Young diagram* constructed as follows: $|\mu|$ boxes are arranged in r rows of lengths $\mu_1, \mu_2, \dots, \mu_r$, and the rows are left-adjusted along a diagonal line. For instance, if $\mu = (7, 4, 2, 1)$, then our tableau has the following shape:



It remains to define how the tableau is to be filled. The alphabet will consist of the set $A = \{1, 2, \dots, r\} \cup \{\bar{1}, \bar{2}, \dots \bar{r}\}$, with ordering $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r$. We place an entry from A in each of the boxes of the tableau so that the entries are: (1) *weakly increasing* from left to right across each row and from top to bottom down each column, and (2) *strictly increasing* from top-left to bottom-right along each diagonal.

An explicit correspondence between $\mathrm{Sp}(2r)$-standard shifted tableaux and strict GT-patterns is given in [Hamel and King 2002, Definition 5.2]. Below we describe the prescription for determining S_P, the tableau corresponding to a given GT-pattern P, with notation as in (27).

(1) For $j = i, \dots, r$, the entries $a_{i-1,j}$ of P count, respectively, the number of boxes in the $(j-i+1)$–st row of S_P whose entries are less than or equal to the value $r-i+1$.

(2) For $j = i, \dots, r$, the entries $b_{i,j}$ of P count, respectively, the number of boxes in the $(j-i+1)$–st row of S_P whose entries are less than or equal to the value $\overline{r-i+1}$.

An example of this bijection is given in Figure 1.

We also associate the following statistics to any symplectic shifted tableau S:

(1) $\mathrm{wt}(S) = (\mathrm{wt}_1(S), \mathrm{wt}_2(S), \dots, \mathrm{wt}_r(S))$ for $\mathrm{wt}_i(S) = \#(i \text{ entries}) - \#(\bar{\imath} \text{ entries})$.

(2) $\mathrm{con}_k(S)$ is the number of connected components of the ribbon strip of S consisting of all the entries k.

(3) $\mathrm{row}_k(S)$ is the number of rows of S containing an entry k, and similarly $\mathrm{row}_{\bar{k}}(S)$ is the number of rows of S containing an entry $\bar{k}$.

$$
\begin{array}{cccccccccc}
9 & & 6 & & 5 & & 3 & & 2 & \\
 & 7 & & 6 & & 5 & & 3 & & 2 \\
 & & 7 & & 5 & & 4 & & 2 & \\
 & & & 5 & & 4 & & 3 & & 1 \\
 & & & & 5 & & 3 & & 1 & \\
 & & & & & 4 & & 2 & & 1 \\
 & & & & & & 4 & & 2 & \\
 & & & & & & & 3 & & 2 \\
 & & & & & & & & 3 & \\
 & & & & & & & & & 1
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
\bar{1} & 1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 \\
\hline
\multicolumn{1}{c|}{} & \bar{2} & \bar{2} & 3 & \bar{4} & 4 & \bar{5} & \multicolumn{2}{c}{} \\
\cline{2-7}
\multicolumn{2}{c|}{} & \bar{3} & \bar{4} & \bar{4} & 4 & \bar{5} & \multicolumn{2}{c}{} \\
\cline{3-7}
\multicolumn{3}{c|}{} & \bar{4} & 4 & \bar{5} & \multicolumn{3}{c}{} \\
\cline{4-6}
\multicolumn{4}{c|}{} & \bar{5} & \bar{5} & \multicolumn{3}{c}{} \\
\cline{5-6}
\end{array}
$$

Figure 1. The bijection between GT-patterns and symplectic shifted tableaux.

(4) $\mathrm{str}(S)$ is the total number of connected components of all ribbon strips of S.

(5) $\mathrm{bar}(S)$ is the total number of barred entries in S.

(6) $\mathrm{hgt}(S) = \sum_{k=1}^{r}(\mathrm{row}_k(S) - \mathrm{con}_k(S) - \mathrm{row}_{\bar{k}}(S))$.

It is easy to see that the weights associated with the tableaux S_P are identical to the previously defined weights associated with the pattern P.

Theorem [Hamel and King 2002, Theorem 1.2]. *Let λ be a partition into at most r parts, and let $\rho = (r, r-1, \ldots, 1)$. Then defining*

$$(79)\quad D_{\mathrm{Sp}(2r)}(\boldsymbol{x}; t) = \prod_{i=1}^{r} x_i^{r-i+1} \prod_{i=1}^{r}(1+tx_i^{-2}) \prod_{1\le i<j\le r}(1+tx_i^{-1}x_j)(1+tx_i^{-1}x_j^{-1}),$$

and letting $\mathrm{sp}_\lambda(\boldsymbol{x}) := \mathrm{sp}_\lambda(x_1, \ldots, x_r)$ be the character of the highest weight representation of $\mathrm{Sp}(2r)$ *with highest weight λ, we have*

$$(80)\qquad D_{\mathrm{Sp}(2r)}(t\boldsymbol{x}; t)\,\mathrm{sp}_\lambda(\boldsymbol{x}) = \sum_{S\in\mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))} t^{\mathrm{hgt}(S)+r(r+1)/2}(1+t)^{\mathrm{str}(S)-r}\boldsymbol{x}^{\mathrm{wt}(S)},$$

where $\mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))$ denotes the set of all $\mathrm{Sp}(2r)$-standard shifted tableaux of shape $\lambda+\rho$.

Remark 19. The identity appears in the theorem cited in the form

$$(81)\qquad D_{\mathrm{Sp}(2r)}(\boldsymbol{x}; t)\,\mathrm{sp}_\lambda(\boldsymbol{x}; t) = \sum_{S\in\mathcal{ST}^{\lambda+\rho}(\mathrm{Sp}(2r))} t^{\mathrm{hgt}(S)+2\,\mathrm{bar}(S)}(1+t)^{\mathrm{str}(S)-r}\boldsymbol{x}^{\mathrm{wt}(S)},$$

where $\mathrm{sp}_\lambda(\boldsymbol{x}; t)$ is a simple deformation of the usual symplectic character given in [Hamel and King 2002, (1.13)]. To relate (81) to (80), put $x_i \to tx_i$ for each $i = 1, \ldots, r$, which introduces a factor of $t^{\sum \mathrm{wt}_i(S)}$ on the right-hand side. From the definition of $\mathrm{wt}(S)$ and the correspondence with P, we see that

$$\sum_{i=1}^{r} \mathrm{wt}_i(S) = \frac{r(r+1)}{2} - 2\,\mathrm{bar}(S) + \sum_{i=1}^{r}(r-i+1)l_i. \tag{82}$$

Moreover, it is a simple exercise to show that

$$\mathrm{sp}_\lambda(t\boldsymbol{x}; t) = t^{\sum(r-i+1)l_i}\ \mathrm{sp}_\lambda(\boldsymbol{x}). \tag{83}$$

Applying the previous two identities to (81) gives (80).

We now show that the right-hand side of (80) may be expressed in terms of the right-hand side of (78), leading to an expression for the generating function for $H(p^{k_1}, \dots, p^{k_r})$ in terms of a symplectic character. The following lemma relates the exponents in this equation back to our GT-pattern P and the statistics of (78).

Lemma 20. *Let P be a strict* GT-*pattern of rank r and S_P its associated standard shifted tableau. Then we have the following relationships*:

(a) $\mathrm{gen}(P) = \mathrm{str}(S_P) - r$.

(b) $\max(P) = \mathrm{hgt}(S_P) + r(r+1)/2$.

This is stated without proof implicitly in [Hamel and King 2002, Corollary 5.3], using slightly different notation. The proof is elementary, but we include it in the next section for completeness. Assuming the lemma, letting $t = -1/q$ in (80), and using (78) with $|p| = q$, we see that

$$\begin{aligned}\sum_{(k_1,\dots,k_r)} & H(p^{k_1}, \dots, p^{k_r}) q^{-2k_1 s_1} \cdots q^{-2k_r s_r} \\ &= x_1^{L_1} \cdots x_r^{L_r}\ D_{\mathrm{Sp}(2r)}(-x_1/q, \dots, -x_r/q; -1/q)\ \mathrm{sp}_\lambda(x_1, \dots, x_r),\end{aligned} \tag{84}$$

with the identification

$$q^{1-2s_1} = x_1^2, \quad q^{1-2s_2} = x_1^{-1}x_2, \quad \dots, \quad q^{1-2s_r} = x_{r-1}^{-1}x_r. \tag{85}$$

One checks by induction on the rank r that, with x_i assigned as above,

$$x_1 x_2^2 \cdots x_r^r D_{\mathrm{Sp}(2r)}(-x_1/q, \dots, -x_r/q; -1/q) = \prod_{\alpha\in\Phi^+} (1 - q^{-(1+2B(\alpha, \boldsymbol{s}-(1/2)\rho^\vee))})$$

with $B(\alpha, \boldsymbol{s} - \frac{1}{2}\rho^\vee)$ as defined in (3). Moving this product to the left-hand side of (84), we can rewrite that equality as

$$\begin{aligned}\prod_{\alpha\in\Phi^+} (1 - q^{-(1+2B(\alpha, \boldsymbol{s}-(1/2)\rho^\vee))})^{-1} \sum_{(k_1,\dots,k_r)} & H(p^{k_1}, \dots, p^{k_r}) q^{-2k_1 s_1} \cdots q^{-2k_r s_r} \\ &= x_1^{L_1-1} \cdots x_r^{L_r-r}\ \mathrm{sp}_\lambda(x_1, \dots, x_r).\end{aligned} \tag{86}$$

The terms in the product are precisely the Euler factors for the normalizing zeta factors of $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ defined in (23) for the case $n=1$. Hence, the terms on the left-hand side of (86) constitute the complete set of terms in the multiple Dirichlet series $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ supported at monomials of the form $|p|^{-k_1 s_1-\cdots-k_r s_r}$.

Finally, we can restate and prove our second main result.

Theorem 2. *Let $\boldsymbol{m}=(m_1,\dots,m_r)\in\mathcal{O}_S$ with m_i nonzero for all i. For each prime $p\in\mathcal{O}_S$, let $\operatorname{ord}_p(m_i)=l_i$. Let $H^{(n)}(p^{k_1},\dots,p^{k_r};p^{l_1},\dots,p^{l_r})$ with $n=1$ be defined as in Section 5.1. Then the resulting multiple Dirichlet series $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ agrees with the $(m_1,\dots,m_r)$-th Fourier–Whittaker coefficient of a minimal parabolic Eisenstein series on $\mathrm{SO}_{2r+1}(F_S)$.*

Proof. In the case $n=1$, the multiple Dirichlet series $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ is Eulerian. Indeed, the power residue symbols used in the definition of twisted multiplicativity in (17) and (19) are all trivial. Hence it suffices to check that the Euler factors for Z^*_Ψ match those of the corresponding minimal parabolic Eisenstein series at each prime $p\in\mathcal{O}_S$.

The Euler factors for the minimal parabolic Eisenstein series can be computed using the Casselman–Shalika formula [1980, Theorem 5.4]. We briefly recall the form of this expression for a split, reductive group G over a local field F_v with usual Iwasawa decomposition $G=ANK=BK$. Let χ be an unramified character of the split maximal torus A and consider the induced representation $\operatorname{ind}_B^G(\chi)$. Given an unramified additive character ψ of the unipotent $N^-(F_v)$, opposite the unipotent N of B, there is an associated Whittaker functional

$$W_\psi(\phi)=\int_{N^-(F_v)}\phi(\bar{n})\psi(\bar{n})\,d\bar{n}, \tag{87}$$

where $\phi(ank):=\chi(a)\delta_B(a)^{1/2}$ is the normalized spherical vector with δ_B is the modular quasicharacter. The associated Whittaker function is given by setting $\mathcal{W}_\phi(g):=W(g\phi)$ and is determined by its value on $\pi^{-\lambda}$ for $\lambda\in X_*$, the coweight lattice and π a uniformizer for F_v. Then the Casselman–Shalika formula states that $\mathcal{W}_\phi(\pi^{-\lambda})=0$ unless λ is dominant, in which case

$$\delta_B(\pi^{-\lambda})^{1/2}\mathcal{W}_\phi(\pi^{-\lambda})=\Bigl(\prod_{\alpha\in\Phi^+}(1-q^{-1}\boldsymbol{t}^{-\alpha^\vee})\Bigr)\mathrm{ch}_\lambda(\boldsymbol{t}), \tag{88}$$

where ch_λ is the character of the irreducible representation of the Langlands dual group $G^\vee$ with highest weight λ and $\boldsymbol{t}$ denotes a diagonal representative of the semisimple conjugacy class in $G^\vee$ associated to $\operatorname{ind}_B^G(\chi)$ by Langlands via the Satake isomorphism (see [Borel 1979] for details). In the special case $G=\mathrm{SO}(2r+1)$, for relations with the above multiple Dirichlet series, we determine $\boldsymbol{t}=(x_1,\dots,x_r)$ according to (85) where $|\pi|_v^{-1}=q$. Since $G^\vee=\mathrm{Sp}(2r)$ in this case, the character $\mathrm{ch}_\lambda(\boldsymbol{t})$ in (88) is just $\mathrm{sp}_\lambda(x_1,\dots,x_r)$ as in the right-hand side of (86). Furthermore, the product over positive roots in (88) matches the Euler factors for the normalizing zeta factors of Z^*_Ψ appearing on the left-hand side of (86).

While the Casselman–Shalika formula is stated for principal series over a local field, because the global Whittaker coefficient is Eulerian, there is no obstacle to obtaining the analogous global result for F_S from the local result via passage to the adele group. Moreover, the minimal parabolic Eisenstein series Whittaker functional

$$\int_{N(\mathbb{A})/N(F)} E_\phi(ng)\psi_{\underline{m}}(n)\,dn = \int_{N(\mathbb{A})/N(F)} \sum_{\gamma\in B(F)\backslash G(F)} \phi(\gamma ng)\psi_{\underline{m}}(n)\,dn$$

can be shown to match the integral in (87) with $\psi = \psi_{\underline{m}}$ by the usual Bruhat decomposition for $G(F)$ and a standard unfolding argument.

Hence according to (86), the Euler factor for $Z^*_\Psi(\boldsymbol{s};\boldsymbol{m})$ matches that of the Fourier–Whittaker coefficient except possibly up to a monomial in the $|p|^{-2s_i}$ with $i = 1, \dots, r$. This disparity arises from the fact that the Whittaker functions in the Casselman–Shalika formula are normalized by the modular quasicharacter $\delta_B^{1/2}$, whereas our multiple Dirichlet series should correspond to unnormalized Whittaker coefficients in accordance with the functional equations σ_i as in (21) sending $s_i \mapsto 1 - s_i$. Hence, to check that the right-hand side of (86) exactly matches the unnormalized Whittaker coefficient of the Eisenstein series, it suffices to verify that

$$x_1^{L_1-1}\cdots x_r^{L_r-r}\,\mathrm{sp}_\lambda(x_1,\dots,x_r)$$

satisfies a local functional equation σ_j given in (21) as Dirichlet polynomials in $|p|^{-2s_i}$ for $i = 1, \dots, r$. $\square$

5.3. ***Proof of Lemma 20.*** For part (a) of the lemma, we induct on the rank. When $r = 2$, there are at most six connected components among all the ribbon strips of S_P, since 1 and $\bar{1}$ may only appear in the top row. Moreover, since P is strict there must be at least two connected components. Thus $0 \le \mathrm{str}(S_P) - 2 \le 4$. At each of the four entries in P below the top row, one shows that if the given entry is generic, it increases the count $\mathrm{str}(S_P)$ by 1.

Suppose that for a GT-pattern of rank $r-1$, each of the r^2 entries below the top row increases the count $\mathrm{str}(P)$ by 1. Then consider a GT-pattern P of rank r, and consider the collection of entries $a_{i,j}$, $b_{i,j}$ below the double line. These entries control the number of connected components consisting of copies of $\bar{1}, 1, \dots, \overline{r-1}$, and $r-1$ in P, in precisely the same way as the full collection of entries below the top row in a pattern of rank $r-1$. Thus inductively, for each generic entry $a_{i,j}$ with $2 \le i \le r-1$ and $3 \le j \le r$ or $b_{i,j}$ with $2 \le i, j \le r$, the count $\mathrm{str}(P)$ is increased by 1. Finally, for $i = 1$, one easily checks that the value of $\mathrm{str}(S_P)$ is increased by 1 for every generic $a_{1,j}$ or $b_{1,j}$.

For (b), we first establish the correct range for $\mathrm{hgt}(S_P)+r(r+1)/2$. For each k, it is clear that $0 \le \mathrm{row}_k(S_P)-\mathrm{con}_k(S_P) \le k-1$ and $0 \le \mathrm{row}_{\bar{k}}(S_P) \le k$. Combining these inequalities and summing over k, we have $0 \le \mathrm{hgt}(S_P)+r(r+1)/2 \le r^2$. We proceed by showing that each of the maximal entries increases the count $\mathrm{hgt}(S)$ by 1. The cases are as follows.

(1) If $a_{i,j}$ is maximal, then $a_{i,j} = b_{i,j-1}$, hence there are no $\overline{r+1-i}$ entries in row $j-i$ of the tableau. This decreases $\sum_{k=1}^{r} \mathrm{row}_{\bar{k}}(S_P)$ by 1, hence increasing $\mathrm{hgt}(S_P)$ by 1.

(2) If $b_{i,r}$ is maximal, then $b_{i,r} = 0$, which implies there are no $\overline{r+1-i}$ entries in row $r-i+1$. This similarly increases $\mathrm{hgt}(S_P)$ by 1.

(3) If $b_{i,j}$ is maximal with $1 \le j \le r-1$, then $b_{i,j} = a_{i-1,j+1}$. Since P is a strict pattern, it must follow that $b_{i,j} < a_{i-1,j}$ and $b_{i,j+1} < a_{i-1,j+1}$. By these strict inequalities, $r+1-i$ occurs in both row $j+1-i$ and row $j+2-i$. However, by the equality defining $b_{i,j}$ as maximal, the occurrences in each of these two rows form one connected component. (See, for instance, the $\bar{4}$ component in the example in Figure 1.) This decreases $\sum_{k=1}^{r} \mathrm{con}_k(S_P)$ by 1, hence increasing $\mathrm{hgt}(S_P)$ by 1. □

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