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SIMPLE CLOSED CURVES, WORD LENGTH, AND NILPOTENT QUOTIENTS OF FREE GROUPS

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We consider the fundamental group π of a surface of finite type equipped with the infinite generating set consisting of all simple closed curves. We show that every nilpotent quotient of π has finite diameter with respect to the word metric given by this set. This is in contrast with a result of Danny Calegari that shows that π has infinite diameter with respect to this set. We also give a general criterion for a finitely generated group equipped with a generating set to have this property.

1. Introduction

A surface of finite type is a surface whose fundamental group is finitely generated. Given such a surface, there is no canonical choice of generating set. If one wishes to define a suitably canonical generating set of a geometric nature, then it becomes necessary to consider infinite generating sets. One such set is the set of all elements whose conjugacy class can be represented by a simple closed curve. These are in some sense the simplest elements of the fundamental group, and are thus a natural choice for a generating set.

Benson Farb posed the question whether the fundamental group, endowed with the word metric given by this set, has finite diameter. This question was answered negatively by Danny Calegari [2008]. In this paper, our goal is to investigate the same question for some quotients of the fundamental group. In contrast with Calegari's result, we find the following.

Theorem 1.1. *Let Σ be a surface of finite type, $\pi = \pi_1(\Sigma)$, and let $\mathcal{S} \subset \pi$ be any generating set containing at least one element in each conjugacy class that is represented by a nonseparating simple closed curve. Let $\rho : \pi \to N$ be a homomorphism into any nilpotent group. Then $\rho(\pi)$ has finite diameter in the word metric with respect to the set $\rho(\mathcal{S})$.*

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In surfaces of genus greater than 1, π has many nilpotent quotients of every degree of nilpotency. Furthermore, it is residually nilpotent; that is, for every $x \in \pi$, there is some nilpotent quotient $q : \pi \to N$ such that $q(x) \neq 1$.

We say that a group G is *nilpotent-bounded with respect to the set S* if any nilpotent quotient of G has finite diameter with respect to the word metric given by the image of S. As part of the proof, we prove the following more general result.

Theorem 1.2. *Let G be a finitely generated group, and let $S \subset G$ be a generating set such that $G/[G, G]$ has finite diameter with respect to the word metric given by S. Then G is nilpotent-bounded with respect to S.*

2. Nilpotent groups and lower central series

Given a group Γ, we define a decreasing sequence of subgroups of Γ called the *lower central series of* Γ by the following rule:

$$\Gamma_0 = \Gamma, \quad \Gamma_{n+1} = [\Gamma, \Gamma_n].$$

A group is nilpotent if $\Gamma_n = \langle 1 \rangle$ for some n. A group is called n-step nilpotent if $\Gamma_n = 1$ and $\Gamma_{n-1} \neq 1$. For every n, the group $L_n := \Gamma/\Gamma_n$ is a nilpotent group. These groups have the property that any nilpotent quotient of G factors through one of the projections $\Gamma \to L_n$.

Put $A_n := \Gamma_{n-1}/\Gamma_n$. It is a standard fact that $A_n < Z(L_n)$, the center of L_n. Also, if Γ is finitely generated, then A_n is also finitely generated. Given a generating set S of Γ, the group A_n is generated by the images of elements of the form $[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n \in S$ and $[a_1, \ldots, a_n]$ denotes a generalized commutator, that is,

$$[a_1, \ldots, a_n] = [\ldots [a_1, a_2], a_3], \ldots, a_n].$$

In the course of the proof, we require the following technical lemma about generalized commutators in nilpotent groups.

Lemma 2.1. *Let Γ be any group, let $n, k \in \mathbb{N}$, and let $a_1, \ldots, a_n \in \Gamma$. Then*

$$[a_1, \ldots, a_n]^k \equiv_{n+1} ([a_1^k, \ldots, a_n]),$$

where $\equiv_i$ is understood as having equal images in L_i.

Proof. First, recall that $A_n < Z(L_{n+1})$. Let $x \in \Gamma_{n-1}$ and $y \in \Gamma$. Note that $[x, y] \in \Gamma_n$. Thus we have that

$$[x^k, y] \equiv_{n+1} x^k y x^{-k} y^{-1} \equiv_{n+1} x^k y [x, y]^k y^{-1} x^{-k} \equiv_{n+1} [x, y]^k.$$

The last equality stems from the fact that $[x, y]^k$ is central in L_{n+1}, and thus is invariant under conjugation. This proves the claim for the case $n = 1$. We now proceed by induction.

By the case $n = 1$, we have that:

$$[a_1, \dots, a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}], a_n]^k \equiv_{n+1} [[a_1, \dots, a_{n-1}]^k, a_n].$$

By induction, we can write:

$$[a_1, \dots, a_{n-1}]^k \equiv_{n+1} [[a_1, \dots, a_{n-2}]^k, a_{n-1}]\gamma_n,$$

where $\gamma_n \in \Gamma_n$. Since the image of Γ_n is central in L_{n+1}, we have that

$$[[a_1, \dots, a_{n-1}]^k \gamma_n^{-1}, a_n] \equiv_{n+1} [a_1, \dots, a_{n-1}]^k, a_n].$$

Proceeding similarly, we get the claim of the lemma. □

3. Proof of the main theorems

Lemma 3.1. *Let $n \in \mathbb{N}$ and let $e_1, \dots, e_{2n}$ be the standard basis for $\mathbb{Z}^{2n}$. Then the set $\mathcal{S} = \mathrm{Sp}_{2n}(\mathbb{Z}) \cdot e_1$ generates $\mathbb{Z}^{2n}$ with finite diameter.*

Proof. We prove this fact first for $n = 1$. In this case, $\mathrm{Sp}_{2n}(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$. Given a vector $v = \binom{a}{b} \in \mathbb{Z}^2$ such that $\gcd(a, b) = 1$, there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$. In this case,

$$A = \begin{pmatrix} a & -y \\ b & x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and $A \cdot e_1 = v$, and thus $v \in \mathcal{S}$. For a general vector $v = \binom{a}{b}$, notice that

$$v = \binom{a-1}{1} + \binom{1}{b-1}$$

and that $\gcd(1, a-1) = \gcd(1, b-1) = 1$, and thus $v \in \mathcal{S} + \mathcal{S}$.

Now consider the case $n > 1$. In this case, we have that $D < \mathrm{Sp}_{2n}(\mathbb{Z})$, where $D \cong \prod_{i=1}^{n} \mathrm{SL}_2(\mathbb{Z})$ is the group of matrices containing n copies of $\mathrm{SL}_2(\mathbb{Z})$ along the diagonal and zeroes in all other entries. Also, $\hat{e} = e_1 + e_3 + \cdots + e_{2n-1}$ is in $\mathcal{S}$. Given $\binom{a_i}{b_i}_{i=1}^{n} \in \mathbb{Z}^{2n}$, by the case $n = 1$ there are $2n$ matrices $A_1, \dots, A_n, B_1, \dots B_n \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$A_i \cdot e_1 = \binom{a_i - 1}{1}, \quad B_i \cdot e_1 = \binom{1}{b_i - 1}.$$

Let $A = \mathrm{diag}(A_1, \dots, A_n)$ and $B = \mathrm{diag}(B_1, \dots, B_n)$. Then

$$v = A \cdot \hat{e} + B \cdot \hat{e}.$$

Thus $\mathbb{Z}^{2n}$ is generated by $\mathcal{S}$ with finite diameter. □

Lemma 3.2. *Let Γ be a finitely generated group, and let $n \in \mathbb{N}$. Suppose that $\mathcal{S} \subset \Gamma$ generates Γ and generates L_n with finite diameter. Then $\mathcal{S}$ generates L_{n+1} with finite diameter.*

Proof. By assumption, there exists an N_0 such that for any $w \in \Gamma$, there exist $s_1, \dots s_m \in \mathcal{S}$ (with $m < N_0$) such that

$$(s_1 \dots s_m)^{-1} w \in \Gamma_n.$$

Thus, it is enough to show that the image of $\mathcal{S}$ in L_{n+1} generates A_n with finite diameter. The group A_n is a finitely generated abelian group that is generated by elements of the form $[s_1, \dots, s_n]$, where $s_1, \dots s_n, \in \mathcal{S}$. Choose such a generating set: $\gamma_1, \dots, \gamma_p$. Consider $\gamma_1 = [s_1, \dots, s_n]$. Given any $k \in \mathbb{N}$, by Lemma 2.1, we have that $\gamma_1^k \equiv_{n+1} [s_1^k, \dots, s_n]$. Further, there exist elements $\sigma_1, \dots, \sigma_m \in \mathcal{S}$ with $m < N_0$ and an element $\gamma \in \Gamma_n$ such that

$$s_1^k = \sigma_1 \cdots \sigma_m \gamma.$$

The elements $\sigma_1, \dots, \sigma_m, \gamma$ depend on γ_1 and k, but their number does not. Thus

$$\gamma_1^k \equiv_{n+1} [\sigma_1 \cdots \sigma_m \gamma, \dots, s_n] \equiv_{n+1} [\sigma_1 \cdots \sigma_m, \dots, s_n],$$

where the last equality stems from the centrality of Γ_n. The last expression is a word in the elements of $\mathcal{S}$, whose length is bounded from above by a number that does not depend on k. This is true not just for γ_1, but for $\gamma_2, \dots, \gamma_p$. Since the group A_n is abelian, and every element in it can be written as a product of powers of $\gamma_1, \dots, \gamma_p$, we get that A_n is generated by $\mathcal{S}$ with finite diameter, as required. □

Proof of Theorem 1.2. It is a direct consequence of Lemma 3.2 and induction. □

Proof of Theorem 1.1. Let $H = H_1(S, \mathbb{Z})$. There exists a simple closed curve in π that is mapped to e_1 under this mapping. The mapping class group acts on H, and it is well-known that this action induces a surjective homomorphism onto $\mathrm{Sp}_{2g}(\mathbb{Z})$ [Farb and Margalit 2012, Proposition 8.4]. Furthermore, the nonseparating simple closed curves form a single mapping class group orbit. Thus, by Lemma 3.1 and Theorem 1.2, π is nilpotent-bounded with respect to $\mathcal{S}$. □

4. Finding smaller generating sets

Using Theorem 1.2, it is possible to find smaller generating sets for which π is nilpotent-bounded. We give one such set here, but it is relatively simple to find many of them. In order to do so, we need a simple corollary.

Corollary 4.1. *Let G be a finitely generated group. Let $H = H_1(G, \mathbb{Z}) \cong G/[G, G]$. Suppose that $H \cong H_1 \oplus \cdots \oplus H_k$, and that for each $i = 1, \dots, k$ we are given a set $S_i \subset \Sigma$ whose projection to H is contained in H_i and generates H_i with finite diameter. Then G is nilpotent-bounded with respect to $S_1 \cup \cdots \cup S_k$.*

Proof of Corollary 4.1. This is a direct result of Theorem 1.2 and the fact that any element of $x \in H$ can be written as $x = h_1 + \cdots + h_k$ with $h_i \in H_i$. □

An example of an application of Corollary 4.1 is the following. Let Σ be an orientable surface of genus $g > 1$. It is common to choose a generating set for $\pi = \pi_1(\Sigma)$ of the form $S' = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, where all of the above are represented by simple closed curves, the geometric intersection number of α_i and β_i is one, and they can be realized disjointly from all the other curves. Let $\Gamma_i = \langle \alpha_i, \beta_i \rangle$. The group Γ_i is the fundamental group of an embedded torus with one boundary component. Let $H = H_1(\Sigma)$, and let H_i be the projection to H of Γ_i. Then $H = H_1 \oplus \cdots \oplus H_g$. Thus, if we let $\mathcal{S}$ be any set containing at least one representative in each conjugacy class of a simple closed curve that lies in one of the g tori described above, then π is nilpotent-bounded with respect to $\mathcal{S}$.

5. Further questions

The contrast between the result in this paper and Calegari's result that π has infinite diameter with respect to $\mathcal{S}$ gives rise to several questions.

Question 1. Recall that $L_n = \pi/\pi_n$. By Theorem 1.1, L_n has finite diameter with respect to $\mathcal{S}$. Call this diameter d_n. The sequence $\{d_n\}_{n=1}^{\infty}$ is nondecreasing. Is this sequence bounded? If so, by what value? If not, what is its asymptotic growth rate?

If the sequence $\{d_n\}_{n=1}^{\infty}$ were indeed unbounded, that would imply that π has infinite diameter with respect to $\mathcal{S}$. However, the converse implication is not necessarily true. One way to see this is to consider the following example: Suppose that π is a free group. Choose a free generating set for π, and let $|\,.\,|$ be the word metric given by this set. The set $\bigcup_{i=1}^{\infty} L_i$ is countable. Choose an enumeration of all of its elements: $\{\ell_i\}_{i=1}^{\infty}$. Each of the ℓ_i's is a coset of an infinite subgroup of π. For each i, choose an element $l_i \in \ell_i$ such that $|l_{i+1}| > 2^{|l_i|}$. Let $\mathcal{L} = \{l_i\}_{i=1}^{\infty}$. The group π is nilpotent-bounded with respect to the set $\mathcal{L}$. Indeed, by construction, $\mathcal{L}$ surjects onto every nilpotent quotient, and thus generates each nilpotent quotient with diameter 1. However, by using the triangle inequality for $|\,.\,|$, it is simple to see that $\mathcal{L}$ cannot generate π with finite diameter.

Question 2. The lower central series is but one of the important series of nested subgroups of π. Another such series is the derived series, whose elements are quotients of surjections onto solvable groups. This sequence is defined by

$$\Gamma^{(0)} = \Gamma, \qquad \Gamma^{(n+1)} = [\Gamma^{(n)}, \Gamma^{(n)}].$$

Is the conclusion of Theorem 1.1 true if we replace the word nilpotent with the word solvable?

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