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INTERLACING LOG-CONCAVITY OF THE BOROS–MOLL POLYNOMIALS

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We say a sequence $\{P_m(x)\}_{m\geq 0}$ of polynomials of degree m with positive coefficients is interlacingly log-concave if the ratios of consecutive coefficients of $P_m(x)$ interlace the ratios of consecutive coefficients of $P_{m+1}(x)$ for any $m \geq 0$. Interlacing log-concavity of a sequence of polynomials is stronger than log-concavity of the polynomials themselves. We show that the Boros–Moll polynomials are interlacingly log-concave. Furthermore, we give a sufficient condition for interlacing log-concavity which implies that some classical combinatorial polynomials are interlacingly log-concave.

1. Introduction

Let $\{P_m(x)\}_{m\geq 0}$ be a sequence of polynomials, where

$$P_m(x) = \sum_{i=0}^{m} a_i(m)x^m$$

is a polynomial of degree m. Let

$$r_i(m) = \frac{a_i(m)}{a_{i+1}(m)}.$$

We say that the sequence of polynomials $\{P_m(x)\}_{m\geq 0}$ is *interlacingly log-concave* if the ratios $r_i(m)$ interlace the ratios $r_i(m+1)$, that is,

$$r_0(m+1) \leq r_0(m) \leq r_1(m+1) \leq r_1(m) \leq \cdots \leq r_{m-1}(m+1) \leq r_{m-1}(m) \leq r_m(m+1).$$

Recall that a sequence $\{a_i\}_{0\leq i\leq m}$ of positive numbers is said to be log-concave if

$$\frac{a_0}{a_1} \leq \frac{a_1}{a_2} \leq \cdots \leq \frac{a_{m-1}}{a_m}.$$

It is obvious that interlacing log-concavity implies log-concavity.

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The main objective of this paper is to prove the interlacing log-concavity of the Boros–Moll polynomials. For the background on these polynomials, see [Boros and Moll 1999a; 1999b; 1999c; 2001; 2004; Moll 2002; Amdeberhan and Moll 2009]. From now on, we use $P_m(x)$ to denote the Boros–Moll polynomial given by

$$P_m(x)=\sum_{j,k}\binom{2m+1}{2j}\binom{m-j}{k}\binom{2k+2j}{k+j}\frac{(x+1)^j(x-1)^k}{2^{3(k+j)}}. \tag{1}$$

Boros and Moll [1999b] derived the following formula for the coefficient $d_i(m)$ of x^i in $P_m(x)$:

$$d_i(m)=2^{-2m}\sum_{k=i}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{k}\binom{k}{i}. \tag{2}$$

In [Boros and Moll 1999c], they showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ is unimodal and that the maximum element appears in the middle. In other words,

$$d_0(m)<d_1(m)<\cdots<d_{[m/2]}(m)>d_{[m/2]-1}(m)>\cdots>d_m(m). \tag{3}$$

They also established the unimodality by a different approach [Boros and Moll 1999a]; see also [Alvarez et al. 2001].

Moll [2002] conjectured that the sequence $\{d_i(m)\}_{0\le i\le m}$ is log-concave. Kauers and Paule [2007] proved this conjecture based on recurrence relations found by using a computer algebra approach. Chen and Xia [2009] showed that the sequence $\{d_i(m)\}_{0\le i\le m}$ satisfies the ratio monotone property which implies log-concavity and the spiral property. A combinatorial proof of the log-concavity of $P_m(x)$ was found by Chen, Pang and Qu [$\geq$ 2011].

In addition to the Boros–Moll polynomials, we study polynomials whose coefficients satisfy triangular recurrence relations. It is easy to show that the binomial coefficients, the Narayana numbers and the Bessel numbers are interlacingly log-concave. We also give a sufficient condition for the interlacing log-concavity of a sequence of polynomials and prove that the rising factorials, the Bell polynomials and the Whitney polynomials are interlacingly log-concave.

2. The interlacing log-concavity of $d_i(m)$

In this section, we show that for $m\geq 2$, the Boros–Moll polynomials $P_m(x)$ are interlacingly log-concave.

Theorem 2.1. *For $m\geq 2$ and $0\leq i\leq m$, we have*

$$d_i(m)d_{i+1}(m+1)>d_{i+1}(m)d_i(m+1), \tag{4}$$

$$d_i(m)d_i(m+1)>d_{i-1}(m)d_{i+1}(m+1). \tag{5}$$

The proof relies on recurrence relations derived in [Kauers and Paule 2007]:

$$(6)\quad d_i(m+1) = \frac{m+i}{m+1}\, d_{i-1}(m) + \frac{(4m+2i+3)}{2(m+1)}\, d_i(m), \qquad 0 \le i \le m+1,$$

$$(7)\quad d_i(m+1) = \frac{(4m-2i+3)(m+i+1)}{2(m+1)(m+1-i)}\, d_i(m) - \frac{i(i+1)}{(m+1)(m+1-i)}\, d_{i+1}(m), \quad 0 \le i \le m,$$

$$(8)\quad d_i(m+2) = \frac{-4i^2+8m^2+24m+19}{2(m+2-i)(m+2)} d_i(m+1) - \frac{(m+i+1)(4m+3)(4m+5)}{4(m+2-i)(m+1)(m+2)} d_i(m), \quad 0 \le i \le m+1,$$

and for $0 \le i \le m+1$,

$$(9)\quad (m+2-i)(m+i-1)d_{i-2}(m) - (i-1)(2m+1)d_{i-1}(m) + i(i-1)d_i(m) = 0.$$

Moll [2007] independently derived the recurrence relations (6) and (9) from which the other two relations can be easily deduced.

To prove Theorem 2.1(4), we need the following lemma.

Lemma 2.2. *Assume that* $m \ge 2$. *For* $0 \le i \le m-2$, *we have*

$$(10)\qquad \frac{d_i(m)}{d_{i+1}(m)} < \frac{(4m+2i+3)d_{i+1}(m)}{(4m+2i+7)d_{i+2}(m)}.$$

Proof. We proceed by induction on m. When $m = 2$, it is easy to check that the result holds. Assume that the lemma is valid for n, namely,

$$(11)\qquad \frac{d_i(n)}{d_{i+1}(n)} < \frac{(4n+2i+3)d_{i+1}(n)}{(4n+2i+7)d_{i+2}(n)}, \quad 0 \le i \le n-2.$$

We aim to show that (10) holds for $n+1$, that is,

$$(12)\qquad \frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{(4n+2i+7)d_{i+1}(n+1)}{(4n+2i+11)d_{i+2}(n+1)}, \qquad 0 \le i \le n-1.$$

From the recurrence relation (6), we deduce that, for $0 \le i \le n-1$,

$$\begin{aligned}(2i+4n+7)&d_{i+1}^2(n+1) - (2i+4n+11)d_i(n+1)d_{i+2}(n+1)\\ &= (2i+4n+7)\left(\frac{i+n+1}{n+1}d_i(n) + \frac{2i+4n+5}{2(n+1)}d_{i+1}(n)\right)^2\\ &\quad - (2i+4n+11)\left(\frac{i+n+2}{n+1}d_{i+1}(n) + \frac{2i+4n+7}{2(n+1)}d_{i+2}(n)\right)\\ &\quad \times\left(\frac{n+i}{n+1}d_{i-1}(n) + \frac{2i+4n+3}{2(n+1)}d_i(n)\right)\end{aligned}$$

$$= \frac{A_1(n,i)+A_2(n,i)+A_3(n,i)}{4(n+1)^2},$$

where $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are given by

$$\begin{aligned} A_1(n,i) &= 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\ &\qquad -4(n+i)(2i+4n+11)(i+n+2)d_{i+1}(n)d_{i-1}(n), \\ A_2(n,i) &= (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\ &\qquad -(2i+4n+3)(2i+4n+11)(2i+4n+7)d_i(m)d_{i+2}(n), \\ A_3(n,i) &= (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8) \\ &\qquad \cdot d_{i+1}(n)d_i(n)-2(n+i)(2i+4n+11)(2i+4n+7)d_{i+2}(n)d_{i-1}(n). \end{aligned}$$

We will show that $A_1(n,i)$, $A_2(n,i)$ and $A_3(n,i)$ are all positive for $0 \le i \le n-2$. By the induction hypothesis (11), we find that for $0 \le i \le n-2$,

$$\begin{aligned} A_1(n,i) &> 4(2i+4n+7)(i+n+1)^2 d_i^2(n) \\ &\qquad -4(n+i)(2i+4n+11)(i+n+2)\frac{(4n+2i+1)}{(4n+2i+5)}\, d_i^2(n) \\ &= 4\,\frac{35+96n+72i+64ni+40n^2+28i^2}{2i+4n+5}\, d_i^2(n), \\ A_2(n,i) &> (2i+4n+7)(2i+4n+5)^2 d_{i+1}^2(n) \\ &\qquad -(2i+4n+3)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)}{(4n+2i+7)}\, d_{i+1}^2(n) \\ &= (40i+80n+76)d_{i+1}^2(n), \end{aligned}$$

which are both positive. Also by the induction hypothesis (11), we see that

$$d_i(n)d_{i+1}(n) > \frac{(2i+4n+5)(2i+4n+7)}{(2i+4n+3)(2i+4n+1)} d_{i-1}(n)d_{i+2}(n), \tag{13}$$

for $0 \le i \le n-2$. This implies that

$$\begin{aligned} &A_3(n,i) \\ &> (8i^3+40i^2+58i+32n^3+42n+80n^2+120ni+40i^2n+64n^2i+8)d_{i+1}(n)d_i(n) \\ &\qquad -2(n+i)(2i+4n+11)(2i+4n+7)\frac{(4n+2i+3)(4n+2i+1)}{(4n+2i+5)(4n+2i+7)}\, d_{i+1}(n)d_i(n) \\ &= 8\,\frac{5+22n+30i+44ni+24n^2+16i^2}{2i+4n+5}\, d_{i+1}(n)d_i(n), \end{aligned}$$

which is positive for $0 \le i \le n-2$. Hence the inequality (12) holds for $0 \le i \le n-2$. It remains to show that (12) is true for $i = n-1$, that is,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} < \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}. \tag{14}$$

From (2) it follows that

$$d_n(n+1) = 2^{-n-2}(2n+3)\binom{2n+2}{n+1}, \tag{15}$$

$$d_{n+1}(n+1) = \frac{1}{2^{n+1}}\binom{2n+2}{n+1}, \tag{16}$$

$$d_n(n+2) = \frac{(n+1)(4n^2+18n+21)}{2^{n+4}(2n+3)}\binom{2n+4}{n+2}. \tag{17}$$

Consequently,

$$\frac{d_{n-1}(n+1)}{d_n(n+1)} = \frac{n(4n^2+10n+7)}{2(2n+1)(2n+3)} < \frac{(2n+3)(6n+5)}{2(6n+9)} = \frac{(6n+5)d_n(n+1)}{(6n+9)d_{n+1}(n+1)}.$$

This completes the proof. □

We can now prove Theorem 2.1(4). In fact, we shall prove a stronger inequality.

Lemma 2.3. *Assume that $m \ge 2$. For $0 \le i \le m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} > \frac{(2i+4m+5)d_i(m+1)}{(2i+4m+3)d_{i+1}(m+1)}. \tag{18}$$

Proof. By Lemma 2.2, we have for $0 \le i \le m-1$,

$$d_i^2(m) > \frac{2i+4m+5}{2i+4m+1}\, d_{i-1}(m)d_{i+1}(m). \tag{19}$$

From (19) and the recurrence relation (6), for $0 \le i \le m-1$,

$$\begin{aligned}
&d_{i+1}(m+1)d_i(m) - \frac{2i+4m+5}{2i+4m+3}\, d_{i+1}(m)d_i(m+1)\\
&\quad= \frac{2i+4m+5}{2(m+1)}\, d_{i+1}(m)d_i(m) + \frac{i+m+1}{m+1}\, d_i(m)^2\\
&\qquad\qquad - \frac{2i+4m+5}{2i+4m+3}\left(\frac{2i+4m+3}{2(m+1)} d_i(m)d_{i+1}(m) + \frac{i+m}{m+1} d_{i-1}(m)d_{i+1}(m)\right)\\
&\quad= \frac{i+m+1}{m+1} d_i^2(m) - \frac{(4m+2i+5)(m+i)}{(4m+2i+3)(m+1)}\, d_{i-1}(m)d_{i+1}(m)\\
&\quad> \left(\frac{m+1+i}{m+1} - \frac{(4m+2i+1)(m+i)}{(4m+2i+3)(m+1)}\right) d_i^2(m)\\
&\quad= \frac{6m+4i+3}{(4m+2i+3)(m+1)}\, d_i^2(m) > 0,
\end{aligned}$$

which yields (18). □

We now turn to the proof of Theorem 2.1(5).

Lemma 2.4. *Assume that $m \geq 2$. For $0 \leq i \leq m-1$, we have*

$$\frac{d_i(m)}{d_{i+1}(m)} < \frac{d_{i+1}(m+1)}{d_{i+2}(m+1)}. \tag{20}$$

Proof. We proceed by induction on m. It is easy to check the lemma holds for $m = 2$. Assume that the lemma is true for $n \geq 2$, that is,

$$\frac{d_i(n)}{d_{i+1}(n)} < \frac{d_{i+1}(n+1)}{d_{i+2}(n+1)}, \quad 0 \leq i \leq n-1. \tag{21}$$

It will be shown that the theorem holds for $n+1$, that is,

$$\frac{d_i(n+1)}{d_{i+1}(n+1)} < \frac{d_{i+1}(n+2)}{d_{i+2}(n+2)}, \quad 0 \leq i \leq n. \tag{22}$$

Recall that the sequence $\{d_i(n+1)\}_{0 \leq i \leq n+1}$ is unimodal. Furthermore, from (3) or the ratio monotone property [Chen and Xia 2009], the maximum element appears in the middle, namely, $d_i(n+1) < d_{i+1}(n+1)$ when $0 \leq i \leq [(n+1)/2]-1$ and $d_i(n+1) > d_{i+1}(n+1)$ when $[(n+1)/2] \leq i \leq n$.

Showing (22) for $0 \leq i \leq n-1$ breaks into two cases.

The first case is $d_i(n+1) < d_{i+1}(n+1)$, namely, $0 \leq i \leq [(n+1)/2]-1$. From the recurrence relation (6), we find that for $0 \leq i \leq [(n+1)/2]-1$,

$$\begin{aligned}
&d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) \\
&= \frac{2i+4n+9}{2(n+2)}\, d_{i+1}^2(n+1) + \frac{i+n+2}{n+2} d_i(n+1)d_{i+1}(n+1) \\
&\qquad - \frac{2i+4n+11}{2(n+2)}\, d_i(n+1)d_{i+2}(n+1) - \frac{i+n+3}{n+2}\, d_i(n+1)d_{i+1}(n+1) \\
&= \frac{2i+4n+9}{2(n+2)}\, d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)}\, d_i(n+1)d_{i+2}(n+1) \\
&\qquad - \frac{1}{n+2}\, d_i(n+1)d_{i+1}(n+1) \\
&> \frac{2i+4n+7}{2(n+2)}\, d_{i+1}^2(n+1) - \frac{2i+4n+11}{2(n+2)}\, d_i(n+1)d_{i+2}(n+1),
\end{aligned}$$

which is positive by Lemma 2.2. It follows that for $0 \leq i \leq [(n+1)/2]-1$,

$$d_{i+1}(n+1)d_{i+1}(n+2) - d_{i+2}(n+2)d_i(n+1) > 0. \tag{23}$$

This completes the proof of the first case.

The second case is when $[(n+1)/2] \leq i \leq n-1$. From the recurrence relations (6) and (7), it follows that for $[(n+1)/2] \leq i \leq n-1$,

$$\begin{aligned}
&d_{i+1}(n+2)d_{i+1}(n+1)-d_{i+2}(n+2)d_i(n+1)\\
&\quad=\left(\frac{(4n-2i+5)(n+i+3)}{2(n+2)(n+1-i)}\,d_{i+1}(n+1)-\frac{(i+1)(i+2)}{(n+2)(n+1-i)}\,d_{i+2}(n+1)\right)\\
&\qquad\times\left(\frac{n+1+i}{n+1}d_i(n)+\frac{4n+2i+5}{2(n+1)}d_{i+1}(n)\right)\\
&\qquad\quad-\left(\frac{n+3+i}{n+2}\,d_{i+1}(n+1)+\frac{4n+2i+11}{2(n+2)}d_{i+2}(n+1)\right)\\
&\qquad\qquad\times\left(\frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)}\,d_i(n)-\frac{i(i+1)}{(n+1)(n+1-i)}\,d_{i+1}(n)\right)\\
&\quad=B_1(n,i)d_{i+1}(n+1)d_i(n)+B_2(n,i)d_{i+1}(n+1)d_{i+1}(n)\\
&\qquad\qquad+B_3(n,i)d_{i+2}(n+1)d_i(n)+B_4(n,i)d_{i+2}(n+1)d_{i+1}(n),
\end{aligned}$$

where $B_1(n,i)$, $B_2(n,i)$, $B_3(n,i)$ and $B_4(n,i)$ are given by

$$B_1(n,i)=\frac{(n+i+3)(n+1+i)}{(n+2)(n+1-i)(n+1)}, \tag{24}$$

$$B_2(n,i)=\frac{(n+i+3)(16n^2+40n+25+4i)}{4(n+2)(n+1-i)(n+1)}, \tag{25}$$

$$B_3(n,i)=-\frac{(n+1+i)(41+16n^2+56n-4i)}{4(n+2)(n+1-i)(n+1)}, \tag{26}$$

$$B_4(n,i)=-\frac{(i+1)(4n+5-i)}{(n+2)(n+1-i)(n+1)}. \tag{27}$$

Since $[(n+1)/2]\le i\le n-1$, it follows from (3) that $d_{i+1}(n+1)>d_{i+2}(n+1)$ and $d_i(n)>d_{i+1}(n)$. Thus we get

$$d_{i+1}(n+1)d_i(n)>d_{i+1}(n+1)d_{i+1}(n), \tag{28}$$

$$d_{i+1}(n+1)d_{i+1}(n)>d_{i+2}(n+1)d_{i+1}(n). \tag{29}$$

Observe that $B_1(n,i)$ and $B_2(n,i)$ are positive, and $B_3(n,i)$ and $B_4(n,i)$ are negative. By the induction hypothesis (21) and inequalities (28) and (29), we find that, for $[(n+1)/2]\le i\le n-1$,

$$\begin{aligned}
&d_{i+1}(n+2)d_{i+1}(n+1)-d_{i+2}(n+2)d_i(n+1)\\
&\qquad>\left(B_1(n,i)+B_2(n,i)+B_3(n,i)+B_4(n,i)\right)d_{i+1}(n+1)d_{i+1}(n)\\
&\qquad=\frac{24n+10n^2-8ni+8i^2+13}{2(n+2)(n+1-i)(n+1)}d_{i+1}(n+1)d_{i+1}(n)>0.
\end{aligned} \tag{30}$$

From the inequalities (23) and (30), it follows that (22) holds for $0\le i\le n-1$.

It is still necessary to show that (22) is true for $i = n$, that is,

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} < \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)}. \tag{31}$$

For the recurrence relation (9), setting $i = n+2$, we find that

$$\frac{d_n(n+1)}{d_{n+1}(n+1)} = \frac{2n+3}{2} < \frac{2n+5}{2} = \frac{d_{n+1}(n+2)}{d_{n+2}(n+2)},$$

as desired. Hence the proof is complete by induction. □

Lemmas 2.3 and 2.4 immediately imply the interlacing log-concavity of the Boros–Moll polynomials.

3. Polynomials with triangular relations on coefficients

Many combinatorial polynomials admit triangular relations on the coefficients. The log-concavity of polynomials of this kind has been extensively studied. We show that many classical polynomials of this kind are also interlacingly log-concave. For example, it is easy to check that the binomial coefficients, the Narayana numbers

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1},$$

and the Bessel numbers

$$B(n,k) = \frac{(2n-k-1)!}{2^k(n-k)!(k-1)!}$$

are interlacingly log-concave. Moreover, we give a criterion that applies to many combinatorial sequences such as the signless Stirling numbers of the first kind, the Stirling numbers of the second kind and the Whitney numbers.

Theorem 3.1. *Suppose that for any $n \geq 0$,*

$$G_n(x) = \sum_{k=0}^{n} T(n,k)x^k$$

is a polynomial of degree n which has only real zeros, and suppose that the coefficients $T(n,k)$ satisfy a recurrence relation of the form

$$T(n,k) = f(n,k)T(n-1,k) + g(n,k)T(n-1,k-1).$$

If

$$\frac{(n-k)k}{(n-k+1)(k+1)} f(n+1,k+1) \leq f(n+1,k) \leq f(n+1,k+1), \tag{32}$$

$$g(n+1,k+1) \leq g(n+1,k) \leq \frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1), \tag{33}$$

then the polynomials $G_n(x)$ are interlacingly log-concave.

Proof. Since the polynomial $G_n(x)$ has only real zeros, by Newton's inequality,

$$k(n-k)T(n,k)^2 \geq (k+1)(n-k+1)T(n,k-1)T(n,k+1).$$

Hence

$$\begin{aligned}
&T(n,k)T(n+1,k+1) - T(n+1,k)T(n,k+1)\\
&\quad = f(n+1,k+1)T(n,k)T(n,k+1) + g(n+1,k+1)T(n,k)^2\\
&\qquad\qquad - f(n+1,k)T(n,k)T(n,k+1) - g(n+1,k)T(n,k-1)T(n,k+1)\\
&\quad \geq (f(n+1,k+1) - f(n+1,k))\,T(n,k)T(n,k+1)\\
&\qquad + \left(\frac{(n-k+1)(k+1)}{(n-k)k} g(n+1,k+1) - g(n+1,k)\right) T(n,k-1)T(n,k+1),
\end{aligned}$$

which is positive by (32) and (33). It follows that

$$\frac{T(n,k)}{T(n,k+1)} \geq \frac{T(n+1,k)}{T(n+1,k+1)}. \tag{34}$$

On the other hand, we have

$$\begin{aligned}
&T(n,k+1)T(n+1,k+1) - T(n,k)T(n+1,k+2)\\
&\quad = f(n+1,k+1)T(n,k+1)^2 + g(n+1,k+1)T(n,k)T(n,k+1)\\
&\qquad\qquad - f(n+1,k+2)T(n,k)T(n,k+2) - g(n+1,k+2)T(n,k+1)T(n,k)\\
&\quad \geq \left(f(n+1,k+1) - \frac{(n-k-1)(k+1)}{(n-k)(k+2)} f(n+1,k+2)\right) T(n,k+1)^2\\
&\qquad\qquad + (g(n+1,k+1) - g(n+1,k+2))T(n,k+1)T(n,k).
\end{aligned}$$

It follows from (32) that

$$\frac{T(n,k)}{T(n,k+1)} \leq \frac{T(n+1,k+1)}{T(n+1,k+2)}. \tag{35}$$

This completes the proof. □

Employing Theorem 3.1, we can show that many combinatorial polynomials which have only real zeros are interlacingly log-concave, for example,

(1) the polynomials

$$x(x+1)(x+2)\cdots(x+n-1),$$

whose coefficients are the signless Stirling numbers of the first kind, which satisfy the recurrence relation

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1);$$

(2) the Bell polynomials whose coefficients are the Stirling numbers of the second kind $S(n, k)$, which satisfy the recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k);$$

(3) the Whitney polynomials

$$W_n(x) = \sum_{k=0}^{n} W_m(n, k)x^k,$$

which have only real zeros; see [Benoumhani 1997; 1999]. The coefficients $W_m(n, k)$ satisfy the recurrence relation

$$W_m(n, k) = (1+mk)W_m(n-1, k) + W_m(n-1, k-1).$$

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