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## THE PRINCIPLE OF STATIONARY PHASE FOR THE FOURIER TRANSFORM OF *D*-MODULES


JIANGXUE FANG



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# THE PRINCIPLE OF STATIONARY PHASE FOR THE FOURIER TRANSFORM OF $D$-MODULES

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**We show that the formal germ at the infinity of the Fourier transform of a holonomic $D$-module depends only on the formal germ of the $D$-module at its singular points and at the infinity.**

## 1. Introduction

The stationary phase approximation is a basic principle of asymptotic analysis, exemplified by the oscillatory integral

$$I(t') = \int g(t)e^{it'f(t)}dt.$$

If the derivative of $f(t)$ does not vanish at any point in $\mathrm{Supp}(f)$, then $I(t')$ is rapidly decreasing at $\infty$. If $f(t)$ has only finitely many critical points in $\mathrm{Supp}(f)$, the major contribution to the value of the integral $I(t')$ for large $t'$ comes from neighborhoods of those critical points. More generally, consider the integral

$$I(t') = \int_{a(t')}^{b(t')} g(t,t')e^{if(t,t')}dt,$$

where all the functions are real-valued. Under certain conditions, for $t' \to \infty$,

$$I(t') = \sum_{f_t(t,t')=0} \left( g(t,t')\sqrt{\frac{2\pi}{|f_{tt}(t,t')|}}e^{if(t,t')+\frac{i\pi}{4}\mathrm{sgn} f_{tt}(t,t')} + o\left(\frac{g(t,t')}{\sqrt{|f_{tt}(t,t')|}}\right)\right).$$

The classical principle of stationary phase outlined above relates to the real Fourier transform. To study Deligne's $\ell$-adic Fourier transform, Gérard Laumon [1987] introduced a corresponding principle of stationary phase and the local $\ell$-adic Fourier transform. (See [Katz 1988] for a good exposition.) We are interested in the $D$-module case.

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We fix a field $k$ of characteristic 0 and use the following notations:

(1) Let $p_1,\ p_2$ be the projections $\operatorname{Spec} k[t,t'] = \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$, and let $\bar{p}_1,\ \bar{p}_2$ be the projections $\mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^1_k$. Let $\alpha : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$ and $\mu : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ be the inclusions.

(2) For any $x \in k$, let $t_x = t - x$ and $t'_x = t' - x$. Let $t_\infty = 1/t = z$, $t'_\infty = 1/t' = z'$ and $\eta' = \operatorname{Spec} k(t')$. For any $x \in k \cup \{\infty\}$, let $\eta_x = \operatorname{Spec} k((t_x))$, $\eta'_x = \operatorname{Spec} k((t'_x))$.

(3) For any $x,\ y \in k \cup \{\infty\}$, let $k((t_x, t'_y))$ be the field of the formal Laurent series $\sum_{i,j \gg -\infty} a_{ij} t_x^i t_y'^j$, $a_{ij} \in k$. For any $k((t_x))$-vector space $M$, let
$$M((t'_y)) = M \otimes_{k((t_x))} k((t_x, t'_y)).$$

(4) Denote by $\mathcal{L}$ the rank-one connection $(\mathbb{O}_{\mathbb{A}^1_k}, d + dt)$ on $\mathbb{A}^1_k$. Then $\mathcal{L}$ corresponds to the $D$-module $\mathbb{O}_{\mathbb{A}^1_k} \cdot e^t$ on $\mathbb{A}^1_k$. So $\mathcal{L}$ is a substitute of $e^{it}$ in classical Fourier analysis. Let $X$ be a scheme. Any section $f \in \mathbb{O}(X)$ defines a morphism $\phi : X \to \mathbb{A}^1_k$ and let $\mathcal{L}_f = \phi^* \mathcal{L}$.

Let $\mathcal{M}$ be a vector bundle with a connection $\nabla$ on a nonempty open subscheme $U$ of $\mathbb{A}^1_k$ and let $i : U \hookrightarrow \mathbb{A}^1$ and $j : U \to \mathbb{P}^1_k$ be the inclusions. The connection $\nabla$ on $\mathcal{M}$ can be extended to a connection $i_*\nabla$ on $i_*\mathcal{M}$ and a connection $j_*\nabla$ on $j_*\mathcal{M}$. The global (geometric) Fourier transform of the $D$-module $i_*\mathcal{M}$ on $\mathbb{A}^1_k$ is defined to be
$$\mathcal{F}(i_*\mathcal{M}) = p_{2+}(p_1^* i_*\mathcal{M} \otimes_{\mathbb{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}} \mathcal{L}_{tt'})[1],$$
where $\otimes$ and $p_{2+}$ are derived functors of $D$-modules. This definition is analogous to
$$\widehat{f}(t') = \int f(t) e^{itt'} dt.$$
More precisely, we have
$$\begin{aligned}
\mathcal{F}(i_*\mathcal{M}) &\cong R^1 p_{2*}\big(p_1^* i_*\mathcal{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathbb{O}_{\mathbb{A}^1_k}} i_*\mathcal{M})\big) \\
&\cong \alpha^*\alpha_* R^1 p_{2*}\big(p_1^* i_*\mathcal{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathbb{O}_{\mathbb{A}^1_k}} i_*\mathcal{M})\big) \\
&\cong \alpha^* R^1 \bar{p}_{2*}\mu_*\big(p_1^* i_*\mathcal{M} \xrightarrow{p_1^* i_*\nabla + t'dt} p_1^*(\Omega^1_{\mathbb{A}^1_k} \otimes_{\mathbb{O}_{\mathbb{A}^1_k}} i_*\mathcal{M})\big) \\
&\cong \alpha^* R^1 \bar{p}_{2*}\big(\bar{p}_1^* j_*\mathcal{M} \otimes \mu_*\mathbb{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k} \otimes j_*\mathcal{M}) \otimes \mu_*\mathbb{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}\big).
\end{aligned}$$
Consider the complex
$$(*) \qquad \big(\bar{p}_1^* j_*\mathcal{M} \otimes \mu_*\mathbb{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k} \otimes j_*\mathcal{M}) \otimes \mu_*\mathbb{O}_{\mathbb{A}^1_k \times \mathbb{A}^1_k}\big).$$
We have
$$\mathcal{F}(i_*\mathcal{M})|_{\eta_{\infty'}} = R^1 \bar{p}_{2*}(*)|_{\eta_{\infty'}}.$$

To study $\mathcal{F}(i_*\mathcal{M})|_{\eta_{\infty'}}$, one needs to study $R^1\bar{p}_{2*}(*)|_{\mathrm{Spf}\,k[[z']]}$. The complex $(*)$ involves quasicoherent sheaves that may not be coherent sheaves. To study the localization of $(*)$ on $\mathrm{Spf}\,k[[z']]$, we need to transform them into coherent sheaves. For this reason, Bloch and Esnault [2004] rewrote $(*)$ in terms of the cohomology of a complex of coherent modules. They found a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of the connection $j_*\mathcal{M}$ such that $(\bar{p}_1^* j_*\nabla + t'dt)(\bar{p}_1^*\mathcal{V}) \subset \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k}(T)\otimes\mathcal{W})$ and the inclusion of complexes

$$\bigl(\bar{p}_1^*\mathcal{V} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*(\Omega^1_{\mathbb{P}^1_k}(T)\otimes\mathcal{W})\bigr) \subset (*)$$

is a quasi-isomorphism. Here $T=\mathbb{P}^1_k - U$. However, for any good lattice pair $\mathcal{V}$, $\mathcal{W}$ of the connection $j_*\mathcal{M}$, the conditions above do not hold, because the differential form $t'dt$ is singular on $\mathbb{P}^1_k\times\{\infty\}\cup\{\infty\}\times\mathbb{P}^1_k$. We only have

$$(\bar{p}_1^* j_*\nabla + t'dt)(\bar{p}_1^*\mathcal{V}) \subset \bar{p}_1^*\bigl(\Omega^1_{\mathbb{P}^1_k}(T)\otimes(\mathcal{W}+\mathcal{V}(\{\infty\}))\bigr)(\mathbb{P}^1_k\times\{\infty\})$$

and a subcomplex

$$\bigl(\bar{p}_1^*\mathcal{V} \xrightarrow{\bar{p}_1^* j_*\nabla + t'dt} \bar{p}_1^*\bigl(\Omega^1_{\mathbb{P}^1_k}(T)\otimes(\mathcal{W}+\mathcal{V}(\{\infty\}))\bigr)(\mathbb{P}^1_k\times\{\infty\})\bigr) \tag{1-1}$$

of $(*)$. This inclusion of complexes (1-1) $\subset (*)$ is still not a quasi-isomorphism. Using Deligne's construction of good lattice pairs, we find a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of $j_*\mathcal{M}$ in Lemma 2.3 such that $(1\text{-}1)|_{\mathbb{P}^1_k\otimes_k k(t')} \subset (*)|_{\mathbb{P}^1_k\otimes_k k(t')}$ is a quasi-isomorphism. From this, we get the following stationary phase formula.

**Theorem 1.1.** *Let $\mathcal{M}$ be a vector bundle with a connection $\nabla$ on a nonempty open subscheme $U$ of $\mathbb{A}^1_k$, and let $i : U \hookrightarrow \mathbb{A}^1$ be the inclusion. Suppose all points in $\mathbb{A}^1_k - U$ are $k$-rational. Then the natural map*

$$\begin{aligned}\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} \to \bigoplus_{x\in\mathbb{A}^1_k-U} &\operatorname{coker}\bigl((\mathcal{M}|_{\eta_x})((z')) \xrightarrow{z'\partial_{t_x}+1} (\mathcal{M}|_{\eta_x})((z'))\bigr)\\ &\oplus \operatorname{coker}\bigl((\mathcal{M}|_{\eta_\infty})((z')) \xrightarrow{z'\partial_z-\frac{1}{z^2}} (\mathcal{M}|_{\eta_\infty})((z'))\bigr)\end{aligned} \tag{1-2}$$

*is an isomorphism of formal connections on $k((z'))$.*

The direct summands on the right side of (1-2) induce the definition of local Fourier transforms for formal connections.

The paper is organized as follows. In Section 2, we discuss the good lattice pairs of connections on a smooth curve. Passing to the stalks, we discuss the good lattice pairs of connections on a discrete valuation field. In Section 3, we prove the stationary phase formula using proper base change theorem between formal schemes.

## 2. Good lattice pairs

Let $X$ be a smooth algebraic curve over $k$ and $j : X \hookrightarrow \bar{X}$ the smooth compactification. Let $\mathcal{F}$ be a vector bundle on $X$ with a connection $\nabla$. Set $\Sigma = \bar{X} - X$. A *pair of good lattices* $\mathcal{V}, \mathcal{W}$ *of* $j_*\mathcal{F}$ is a pair of vector bundles on $\bar{X}$ which extends $\mathcal{F}$ and satisfies the following conditions:

(1) $\mathcal{V} \subset \mathcal{W} \subset j_*\mathcal{F}$.

(2) $\nabla(\mathcal{V}) \subset \Omega^1_{\bar{X}}(\Sigma) \otimes \mathcal{W}$.

(3) For any effective divisor $D$ supported on $\Sigma$, the inclusion of complexes

$$\big(\mathcal{V} \xrightarrow{\nabla} \Omega^1_{\bar{X}}(\Sigma) \otimes \mathcal{W}\big) \to \big(\mathcal{V}(D) \xrightarrow{\nabla} \Omega^1_{\bar{X}}(\Sigma) \otimes \mathcal{W}(D)\big)$$

is a quasi-isomorphism. Taking the direct limit with respect to $D$, we get a quasi-isomorphism:

$$\big(\mathcal{V} \xrightarrow{\nabla} \Omega^1_{\bar{X}}(\Sigma) \otimes \mathcal{W})\big) \to \big(j_*\mathcal{F} \xrightarrow{\nabla} \Omega^1_{\bar{X}} \otimes j_*\mathcal{F}\big).$$

The existence of good lattice pairs can be passed to the stalks. So we only need to consider the local case: good lattice pairs of connections on a discrete valuation field.

Let $K$ be a discrete valuation field with the valuation $v$. Let $A$ be the valuation ring, $t$ a uniformizer, and $\partial$ a continuous derivation on $K$ such that $\partial(t) = 1$ and $\partial(A) \subseteq A$.

**Definition 2.1.** A *connection* on $K$ (of rank $k$, where $k$ is finite) is a $k$-dimensional vector space $M$ over $K$ with an additive map $\partial : M \to M$ satisfying $\partial(fm) = f\partial(m) + \partial(f)m$ for any $f \in K$ and $m \in M$.

Let $r$ be the rank of the connection $M$. Set $\tau = t\partial$. There exists a cyclic element $v \in M$, in the sense that the elements $\tau^i v$, for $0 \le i \le r-1$, form a basis of $M$ over $K$. We have

$$\tau^r v = \sum_{0 \le i \le r-1} a_i \tau^i v$$

for some $a_i \in K$. The *Newton polygon* $N(M)$ of $M$ is the convex hull of

$$\{(u, v) \mid u \le i,\, v \ge v(a_i)\}$$

in the plane $\mathbb{R}^2$. The *slopes* of $M$ are the slopes of nonvertical edges of $N(M)$, and we eliminate the slope 0 if the horizontal edge is situated in $u \le 0$. The slopes are independent of the choice of the cyclic elements. The sum of all the slopes of $M$ is called the *irregularity* of $M$, and is denoted by $i(M)$. Then

$$i(M) = \max_{0 \le i \le r} (0, -v(a_i)).$$

A *lattice* of $M$ is a finitely generated $A$-submodule $V$ of $M$ that spans $M$. For any artinian $A$-module $V$, the length of $V$ is denoted by $\ell(V)$.

**Definition 2.2.** A pair of lattices $V$, $W$ of $(M, \partial)$ is called *good* if the following conditions are satisfied

(1) $V \subset W \subset M$.

(2) $\partial V \subset (1/t)W$.

(3) For any $i \in \mathbb{N}$, the natural inclusion of complexes

$$\left(V \xrightarrow{\partial} \frac{1}{t}W\right) \to \left(\frac{1}{t^i}V \xrightarrow{\partial} \frac{1}{t^{i+1}}W\right)$$

is a quasi-isomorphism.

Note that if $V$, $W$ is a good lattice pair, so is $(1/t^i)W$, $(1/t^i)W$ for any $i \in \mathbb{N}$.

Condition (3) above is equivalent to the following:

(3′) For any $i \in \mathbb{N}$, the map

$$\frac{1}{t^i}V \Big/ \frac{1}{t^{i-1}}V \xrightarrow{\mathrm{gr}_i \partial} \frac{1}{t^{i+1}}W \Big/ \frac{1}{t^i}W$$

induced by $\partial$ is an isomorphism.

One can show that $i(M) = \ell(W/V)$.

**Lemma 2.3.** *Let $k \hookrightarrow k'$ be an extension of fields of characteristic 0. Let $\partial_t$ be the natural derivation on $k(t)$ and on $k'(t)$. The variable $t$ defines a discrete valuation $v$ on $k(t)$ and $k'(t)$. Let $A$ and $A'$ be their discrete valuation rings, respectively. Suppose $c$ is an element in $k'$ which is not algebraic over $k$. Let $M$ be a connection on $k(t)$, and let $M_c$ be the connection on $k'(t)$ whose underlying space is the $k'(t)$-vector space $M \otimes_{k(t)} k'(t)$, and with the operation $\partial_t$ defined by*

$$\partial_t(m \otimes f) = \partial_t(m) \otimes f + m \otimes \partial_t(f) - m \otimes \frac{c}{t^2}$$

*for any $m \in M$ and any $f \in k'(t)$. Then there exists a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of $M$, such that $\mathcal{V} \otimes_A A'$, $(\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A'$ is also a good lattice pair of the connection $M_c$ on $k'(t)$.*

*Proof.* Set $r = \mathrm{rk} M$. Choose a cyclic element $v$ of $M$. Let $\varepsilon$ be the basis $\{\tau^i v \mid 0 \le i \le r-1\}$ of $M$ over $k(t')$. We have $\tau^r v = \sum_{0 \le i < r} a_i \tau^i v$ for some $a_i \in K$. The irregularity $i(M)$ of $M$ is $\max_{0 \le i < r}(0, -v(a_i))$. Consider the Newton polygon of the differential operator $\tau^r - \sum_{0 \le i \le r-1} a_i \tau^i$. Let $j$ be the integer such that $(j, v(a_j))$ is a vertex of this Newton polygon, and such that the slopes of this Newton polygon on the right side (respectively left side) of $(j, v(a_j))$ is $> 1$ (respectively $\leq 1$). Set $a_r = 1$. Then we have

$$v(a_{j+i}) - v(a_j) > i \text{ for any } 1 \leq i \leq r - j,$$
$$v(a_{j-i}) - v(a_j) \geq -i \text{ for any } 0 \leq i \leq j.$$

Then

$$v(a_j) - j = \min_{0\leq i\leq r}(v(a_i) - i). \tag{2-1}$$

The matrix of the differential operator $\tau$ with respect to the basis $\varepsilon$ is

$$\Gamma = \begin{pmatrix} 0 & & & a_0 \\ 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_{r-1} \end{pmatrix}.$$

The characteristic polynomial of $\Gamma$ is $\lambda^r - \sum_{0\leq i\leq r-1} a_i\lambda^i$. Let

$$\Lambda = \operatorname{diag}\{1, \dots, 1, t, \dots, t^{r-j+i(M)+v(a_j)}\},$$

and let $e = \varepsilon\Lambda = \{e_i \mid 0 \leq i < r\}$. Set $l = j - v(a_j) - i(M) \geq 0$. Then the matrix of the differential operator $\tau$ with respect to the basis $e$ is

$$\Gamma' = \begin{pmatrix} 0 & & & & & & t^{r-l}a_0 \\ 1 & & & & & & t^{r-l}a_1 \\ & \ddots & & & & & \vdots \\ & & 1 & & & & t^{r-l}a_{l-1} \\ & & & \frac{1}{t} & & & t^{r-l-1}a_l \\ & & & & \ddots & & \vdots \\ & & & & & \frac{1}{t} & a_{r-1} \end{pmatrix} + \operatorname{diag}\{0, \cdots, 0, 1, \cdots, r-l\}.$$

Let $P(\lambda) = \lambda^r - \sum_{0\leq i\leq r-1} a'_i\lambda^i$ be the characteristic polynomial of $\Gamma'$. Since

$$\Gamma' = \Lambda^{-1}\Gamma\Lambda + \operatorname{diag}\{0, \dots, 0, 1, \dots, r-l\},$$

we have

$$a'_i - a_i \in \sum_{i<j<r} \mathbb{Z}a_j + \mathbb{Z} \subset K.$$

So

$$\max\{0, -v(a'_i) \mid 0 \leq i < r\} = \max\{0, -v(a_i) \mid 0 \leq i < r\} = i(M).$$

Write $P(\lambda) = t^{-i(M)} \sum_i b_i\lambda^i$, $b_i \in K$. Then $b_i \in A$ and $v(b_i) = 0$ for at least one $i$. The residue polynomial $\sum_i \bar{b}_i\lambda^i$ of $\sum_i b_i\lambda^i$ is nonzero. For almost all $n \in \mathbb{Z}$,

$\sum_i \bar{b}_i(-n)^i \neq 0$. In this case, we have

$$-v(\det(n+\Gamma')) = -v((-1)^r P(-n)) = -v\Big(t^{-i(M)}\Big(\sum_i b_i(-n)^i\Big)\Big) = i(M).$$

Then, for almost all $n \in \mathbb{Z}$,

$$i(M) = -v(\det(n+\Gamma')). \tag{2-2}$$

Let $V$ be the lattice of $M$ generated by $e$. Define

$$[(n+\Gamma')V : V] = \ell((n+\Gamma')V + V/V) - \ell((n+\Gamma')V + V/(n+\Gamma')V). \tag{2-3}$$

By [Deligne 1970, p. 48, Proposition 2], we have

$$[(n+\Gamma')V : V] = -v(\det(\mathrm{n}+\Gamma')). \tag{2-4}$$

Let $W$ be the lattice of $M$ generated by

$$e_0, \dots, e_{l-1}, \frac{1}{t}e_l, \dots, \frac{1}{t}e_{r-1}.$$

Then $\ell(W/V) = r-l$. Since $((n+\Gamma')V+V)/W$ is an artinian $A$-module generated by the single element

$$x = \sum_{0\le i\le l-1} a_i t^{r-l} e_i + \sum_{l\le i\le r-1} a_i t^{r-1-i} e_i = \sum_{0\le i\le l-1} a_i t^{r-l} e_i + \sum_{l\le i\le r-1} a_i t^{r-i}\frac{1}{t} e_i.$$

For any $i$, we have $i(M) \geq -v(a_i)$ and $v(a_j) - j \leq v(a_i) - i$. Then

$$v(t^{i(M)+l-r} a_i t^{r-l}) \geq 0 \quad \text{and} \quad v(t^{i(M)+l-r} a_i t^{r-i}) \geq v(t^{i(M)+l-r} a_j t^{r-j}) = 0.$$

Then the annihilator of $x$ in $((n+\Gamma')V+V)/W$ is $t^{i(M)+l-r}$. So

$$\ell((n+\Gamma')V + V/W) = i(M) + l - r.$$

Then

$$\ell((n+\Gamma')V + V/V) = \ell(W/V) + \ell((n+\Gamma')V + V/W) = i(M). \tag{2-5}$$

Comparing this equality with (2-2), (2-3), and (2-4), we get

$$\ell((n+\Gamma')V + V/(n+\Gamma')V) = 0$$

for almost $n \in \mathbb{Z}$, that is, $(n+\Gamma')V \supset V$ for almost all $n \in \mathbb{Z}$.

The $A$-module

$$(n+\Gamma')V + \frac{1}{t}V \Big/ \frac{1}{t}V$$

is artinian and is generated by one element $x$ whose annihilator is

$$t^{i(M)+l-r} = t^{j-v(a_j)-r}.$$

Then

$$\ell\Big((n+\Gamma')V+\frac{1}{t}V\Big/V\Big)=\ell\Big((n+\Gamma')V+\frac{1}{t}V\Big/\frac{1}{t}V\Big)+\ell\frac{1}{t}V\Big/V \tag{2-6}$$
$$=j-v(a_j)=\sum_{\lambda:\text{slope of }M}\max(\lambda,1).$$

The matrix of the differential operator $\tau$ with respect to the basis $\varepsilon$ of $M_c$ is $\Gamma-c/t$. The characteristic polynomial $P'(\lambda)$ of $\Gamma-c/t$ is

$$P'(\lambda)=\Big(\lambda+\frac{c}{t}\Big)^r-\sum_{0\le i<r}a_i\Big(\lambda+\frac{c}{t}\Big)^i.$$

Write $P'(\lambda)=\lambda^r+\sum_{0\le i<r}b_i\lambda^i$ for some $b_i\in k'(t)$. Then

$$b_0=\Big(\frac{c}{t}\Big)^r-\sum_{0\le i<r}a_i\Big(\frac{c}{t}\Big)^i=\frac{a_j}{t^j}\bigg(\frac{1}{a_jt^{r-j}}c^r-\sum_{0\le i<r}\frac{a_i}{a_jt^{i-j}}c^i\bigg).$$

By (2-1), we have

$$\frac{1}{a_jt^{r-j}}c^r-\sum_{0\le i<r}\frac{a_i}{a_jt^{i-j}}c^i\in A[c],$$

and its residue in $k'$ is a nonzero polynomial over $k$ of $c$. Since $c$ is not algebraic over $k$, this residue is nonzero. Then we have

$$v(b_0)=v\Big(\frac{a_j}{t^j}\Big)=v(a_j)-j.$$

Also by (2-1), we have $v(b_i)\ge v(b_0)$ for any $0\le i<r$. So

$$\max\nolimits_{0\le i<r}(0,-v(b_i))=j-v(a_j)=i(M_c).$$

The matrix of the differential operator $\tau$ with respect to the basis $e$ of $M_c$ is $\Gamma''=\Gamma'-c/t$. Write the characteristic polynomial of $\Gamma''$ as $\lambda^r+\sum_{0\le i<r}b_i'\lambda^i$ for some $b_i'\in k'(t)$. By a similar proof as above, we have

$$\max\nolimits_{0\le i<r}(0,-v(b_i'))=\max\nolimits_{0\le i<r}(0,-v(b_i))=i(M_c).$$

For almost $n\in\mathbb{Z}$, we have

$$-v(\det(n+\Gamma''))=i(M_c).$$

Let $V'=V\otimes_A A'$. We have $(n+\Gamma'')V'+V'\subseteq\dfrac{1}{t}V'+\Gamma'V'$; therefore So

$$\ell((n+\Gamma'')V'+V'/V')\le\ell\Big(\frac{1}{t}V'+\Gamma'V'\Big/V'\Big). \tag{2-7}$$

Since $A\to A'$ is flat and $k\otimes_A A'=k'$, for any artinian $A$-module $M$, one can prove $\ell(M)=\ell(M\otimes_A A')$. Since $(1/t)V+\Gamma'V/V$ is an artinian $A$-module, by (2-6), we have

$$\ell\Big(\frac{1}{t}V' + \Gamma' V' \Big/ V'\Big) = \ell\Big(\frac{1}{t}V + \Gamma' V \Big/ V\Big) = j - v(a_j). \tag{2-8}$$

By (2-4), we have, for almost $n \in \mathbb{Z}$,

$$\begin{aligned}\ell((n+\Gamma'')V' + V'/V') &\geq \ell((n+\Gamma'')V' + V'/V') - \ell((n+\Gamma'')V' + V'/(n+\Gamma'')V') \\ &= -v(\det(n+\Gamma'')) = j - v(a_j).\end{aligned}$$

Comparing this inequality with (2-7) and (2-8), we have for almost $n \in \mathbb{Z}$,

$$\ell((n+\Gamma'')V' + V'/V') = j - v(a_j);$$

$$\ell((n+\Gamma'')V' + V'/(n+\Gamma'')V') = 0;$$

$$(n+\Gamma'')V' + V' = \frac{1}{t}V' + \Gamma' V' = \Big(\frac{1}{t}V + \Gamma' V\Big) \otimes_A A'. \tag{2-9}$$

So for almost $n \in \mathbb{Z}$, $(n+\Gamma'')V' \supseteq V'$. Let $e' = (1/t^N)e$. The matrix of $\tau$ with respect to the basis $e'$ of $M$ (respectively $M'_c$) is $\Gamma_1 := \Gamma' - N$ (respectively $\Gamma_2 := \Gamma'' - N$). Let $\mathcal{V} = (1/t^N)V$ and let $\mathcal{V}' = (1/t^N)V'$. Choose $N$ large enough so that for any $n \leq 0$, we have

$$(n+\Gamma_1)\mathcal{V} \supset \mathcal{V} \quad \text{and} \quad (n+\Gamma_2)\mathcal{V}' \supseteq \mathcal{V}'.$$

Let $\mathcal{W} = \Gamma_1\mathcal{V}$. By (2-9), we have $\Gamma_2\mathcal{V}' = (\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A'$. Let's prove $\mathcal{V}$, $\mathcal{W}$ is a good lattice of $M$ now. We only need to verify condition (3′) for any $i \in \mathbb{N}$. Conjugating by $1/t^i$, the $A$-linear map

$$\mathrm{gr}_i\tau : \frac{1}{t^i}\mathcal{V} \Big/ \frac{1}{t^{i-1}}\mathcal{V} \to \frac{1}{t^i}\mathcal{W} \Big/ \frac{1}{t^{i-1}}\mathcal{W}$$

can be identified with the $A$-linear map

$$\mathrm{gr}_0\tau - i = \Gamma_1 - i : \mathcal{V}/t\mathcal{V} \to \mathcal{W}/t\mathcal{W}.$$

Since $(\Gamma_1 - i)\mathcal{V} \supset \mathcal{V}$, we have

$$(\Gamma_1 - i)\mathcal{V} = (\Gamma_1 - i)\mathcal{V} + \mathcal{V} \supset \Gamma_1\mathcal{V} = \mathcal{W}.$$

So $\Gamma_1 - i : \mathcal{V}/t\mathcal{V} \to \mathcal{W}/t\mathcal{W}$ is surjective. But the domain and the range of $\mathrm{gr}_i\tau$ are artinian $A$-modules of the same length $r$, so $\mathrm{gr}_0\tau - i$ is an isomorphism and so is $\mathrm{gr}_i\tau$. This proves $\mathcal{V}$, $\mathcal{W}$ is a good lattice pair of $M$. Repeating the proof, we conclude that $\mathcal{V} \otimes_A A'$, $(\mathcal{W} + (1/t)\mathcal{V}) \otimes_A A'$ is a good lattice pair of $M_c$. □

**Remark 2.4.** Lemma 2.3 is the main technical lemma for the proof of the stationary phase principle in the next section. Lemma 2.3 also allows us to choose a good lattice pair $\mathcal{V}$, $\mathcal{W}$ of $M$ such that

$$\dim_k\Big(\mathcal{W} + \frac{1}{t}\mathcal{V} \Big/ \mathcal{V}\Big) = \sum_{\lambda:\text{slope of } M} \max(\lambda, 1). \tag{2-10}$$

Formula (2-10) is easily seen to give a new proof of the following result:

**Lemma 2.5** [Bloch and Esnault 2004, Lemma 3.3]. *Let $M$ be a connection on $K$. The slopes of $M$ are all $\leq 1$ (respectively $\geq 1$) if and only if there exists a good lattice pair $\mathcal{V}$, $\mathcal{W}$ such that $\mathcal{W} \subseteq (1/t)\mathcal{V}$ (respectively $\mathcal{W} \supseteq (1/t)\mathcal{V}$).*

(Note that the original proof by Bloch and Esnault needs the assumption that $K$ is complete.)

## 3. Stationary phase principle

Let $K = k(t')$. For any scheme $X$ over $k$ and any $\mathcal{O}_X$-modules $\mathcal{F}$, let $X_K = X \otimes_k K$ and $\mathcal{F}_K = \mathcal{F}|_{X_K}$. For any $k$-morphism $f : X \to Y$, let $f_K : X_K \to Y_K$ be the base change of $f$.

We keep the notation used in Section 1. In this section we prove Theorem 1.1.

For any $x \in T_K = T$, $(\mathcal{V}_K)_x$, $(\mathcal{W}_K)_x$ is a good lattice pair of the connection $(j_{K*}\mathcal{M}_K)_x$ on $K(t_x)$. Since $t'$ is not algebraic over $k$, by Lemma 2.3, we may assume that

$$\mathcal{V}_\infty \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty}, \ \Big(\mathcal{W}_\infty + \frac{1}{z}\mathcal{V}_\infty\Big) \otimes_{\mathcal{O}_{\mathbb{P}^1_k,\infty}} \mathcal{O}_{\mathbb{P}^1_K,\infty}$$

is a good lattice pair of the connection

$$\partial_z - \frac{t'}{z^2} : (j_{K*}\mathcal{M}_K)_\infty \to (j_{K*}\mathcal{M}_K)_\infty.$$

**Lemma 3.1.** *The inclusion of complexes* (1-1) $\subset$ $(*)$ *induces a quasi-isomorphism*

$$(1\text{-}1)|_{\mathbb{P}^1_K} \simeq (*)|_{\mathbb{P}^1_K}.$$

*Proof.* We have

$$(1\text{-}1)|_{\mathbb{P}^1_K} = \big(\mathcal{V}_K \xrightarrow{j_{K*}\nabla_K + t'dt} \Omega^1_{\mathbb{P}^1_K}(T_K) \otimes (\mathcal{W}_K + \mathcal{V}_K(\{\infty\}))\big),$$

$$(*)|_{\mathbb{P}^1_K} = \big(j_{K*}\mathcal{M}_K \xrightarrow{j_{K*}\nabla_K + t'dt} \Omega^1_{\mathbb{P}^1_K} \otimes j_{K*}\mathcal{M}_K\big).$$

First we have $(1\text{-}1)|_{U_K} = (*)|_{U_K}$. For any $x \in S_K$, let's prove $(1\text{-}1)|_{\mathbb{P}^1_K} \subset (*)|_{\mathbb{P}^1_K}$ induces a quasi-isomorphism on the stalks at $x$. It suffices to show that

$$\left(\frac{1}{t_x^i}(\mathcal{V}_K)_x \Big/ \frac{1}{t_x^{i-1}}(\mathcal{V}_K)_x\right) \xrightarrow{\mathrm{gr}_i(\partial_{t_x} + t')} \left(\frac{1}{t_x^{i+1}}(\mathcal{W}_K)_x \Big/ \frac{1}{t_x^{i}}(\mathcal{W}_K)_x\right)$$

is an isomorphism for any $i \geq 1$. As $(\mathcal{V}_K)_x \subset (\mathcal{W}_K)_x$, the map $\mathrm{gr}_i(\partial_{t_x} + t')$ is equal to $\mathrm{gr}_i(\partial_{t_x})$, which is an isomorphism by the definition of good lattices. The inclusion

$$\big((1\text{-}1)|_{\mathbb{P}^1_K}\big)_\infty \to \big((*)|_{\mathbb{P}^1_K}\big)_\infty$$

can be written as

$$\Big(\mathcal{V}_\infty \otimes_{\mathbb{O}_{\mathbb{P}^1_k,\infty}} \mathbb{O}_{\mathbb{P}^1_K,\infty} \xrightarrow{\partial_z - \frac{t'}{z^2}} \frac{1}{z}\Big(\mathcal{W}_\infty + \frac{1}{z}\mathcal{V}_\infty\Big) \otimes_{\mathbb{O}_{\mathbb{P}^1_k,\infty}} \mathbb{O}_{\mathbb{P}^1_K,\infty}\Big) \subset \Big((j_{K*}\mathcal{M}_K)_\infty \xrightarrow{\partial_z - \frac{t'}{z^2}} (j_{K*}\mathcal{M}_K)_\infty\Big).$$

It is a quasi-isomorphism by the assumption on $\mathcal{V}_\infty$ and $\mathcal{W}_\infty$. □

**Lemma 3.2.** $$R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'} \cong R^1\bar{p}_{2*}(*)|_{\eta'}.$$

*Proof.* Consider the Cartesian diagram

(3-1)
$$\begin{array}{ccc} \mathbb{P}^1_K & \longrightarrow & \eta' = \operatorname{Spec} K \\ \downarrow & & \downarrow \\ \mathbb{P}^1_k \times \mathbb{P}^1_k & \xrightarrow{\bar{p}_2} & \mathbb{P}^1_k. \end{array}$$

By Lemma 3.1, we have

$$R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'} \cong H^1(\mathbb{P}^1_K,\ (1\text{-}1)|_{\mathbb{P}^1_K}) \cong H^1(\mathbb{P}^1_K,\ (*)|_{\mathbb{P}^1_K}) \cong R^1\bar{p}_{2*}(*)|_{\eta'}.$$ □

**Corollary 3.3.** $$\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} = R^1\bar{p}_{2*}(1\text{-}1)|_{\eta'_\infty}.$$

Denote by $\mathbb{P}^1_k[[z']]$ the formal completion of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ along its closed subscheme $\mathbb{P}^1_k \times \{\infty\}$. For any coherent sheaf $\mathcal{K}$ on $\mathbb{P}^1_k$, let $\mathcal{K}[[z']] = \mathcal{K}|_{\mathbb{P}^1_k[[z']]}$.

**Lemma 3.4** [Bloch and Esnault 2004, Corollary 2.2].

$$R^1\bar{p}_{2*}(1\text{-}1) \otimes_{\mathbb{O}_{\mathbb{P}^1_k}} k[[z']] \cong H^1\Big(\mathbb{P}^1[[z']], \mathcal{V}[[z']] \xrightarrow{z'\nabla + dt} \Omega^1_{\mathbb{P}^1}(T) \otimes \mathcal{W}[[z']]\Big).$$

**Lemma 3.5** [Bloch and Esnault 2004, Lemma 2.4 and Corollary 2.5]. *Let $\mathcal{H}$ be the complex*

$$\mathcal{V}[[z']] \xrightarrow{z'\nabla + dt} \big(\Omega^1_{\mathbb{P}^1_k}(T) \otimes (\mathcal{W} + \mathcal{V}(\{\infty\}))\big)[[z']].$$

*Then $\mathcal{H}^0$ equals* (0) *and $\mathcal{H}^1$ is supported on $T \subset \mathbb{P}^1_k = \mathbb{P}^1_k[[z']]$. For any $x \in T$, let $\widehat{\mathcal{V}_x} = \mathcal{V}_x \otimes_{\mathbb{O}_{\mathbb{P}^1_{k,x}}} k[[t_x]]$ and $\widehat{\mathcal{W}_x} = \mathcal{W}_x \otimes_{\mathbb{O}_{\mathbb{P}^1_{k,x}}} k[[t_x]]$. We have*

$$\mathcal{H}^1_x = \operatorname{coker}\Big(\widehat{\mathcal{V}_x}[[z']] \xrightarrow{z'\nabla + dt} \Omega^1_{\mathbb{P}^1_k}(T) \otimes (\widehat{\mathcal{W}_x} + \widehat{\mathcal{V}(\{\infty\})_x})[[z']]\Big).$$ □

**Corollary 3.6.**

$$H^1(\mathbb{P}^1_k[[z']], \mathcal{H}) = \bigoplus_{x \in S} \operatorname{coker}\Big(\widehat{\mathcal{V}_x}[[z']] \xrightarrow{z'\partial_{t_x} + 1} \frac{1}{t_x}\widehat{\mathcal{W}_x}[[z']]\Big) \oplus \operatorname{coker}\Big(\widehat{\mathcal{V}_\infty}[[z']] \xrightarrow{z'\partial_z - \frac{1}{z^2}} \frac{1}{z}\Big(\widehat{\mathcal{W}_\infty} + \frac{1}{z}\widehat{\mathcal{V}_\infty}\Big)[[z']]\Big).$$

Combining Corollary 3.3, Lemma 3.4 and Corollary 3.6, we have

$$\begin{aligned}\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} &= R^1\bar{p}_{2*}(1\text{-}1)\otimes_{\mathbb{O}_{\mathbb{P}^1_k}} k[[z']]\otimes_{k[[z']]} k((z'))\\ &= \bigoplus_{x\in S}\operatorname{coker}\Big(\widehat{\mathcal{V}_x}((z'))\xrightarrow{z'\partial_{t_x}+1}\frac{1}{t_x}\widehat{\mathcal{W}_x}((z'))\Big)\\ &\qquad\oplus\operatorname{coker}\Big(\widehat{\mathcal{V}_\infty lpz'})\xrightarrow{z'\partial_z-\frac{1}{z^2}}\frac{1}{z}(\widehat{\mathcal{W}_\infty}+\frac{1}{z}\widehat{\mathcal{V}_\infty})((z'))\Big).\end{aligned}$$

The left side of this equality is independent of the choice $\mathcal{V}$ and $\mathcal{W}$. For any $i\in\mathbb{N}$, $\mathcal{V}(iT)$ and $\mathcal{W}(iT)$ still satisfy the condition of Lemma 3.1. Then the above equality holds if we replace $\mathcal{V}$ and $\mathcal{W}$ by $\mathcal{V}(iT)$ and $\mathcal{W}(iT)$, respectively. Taking the direct limit on $i$, we have

$$\begin{aligned}\mathcal{F}(i_*\mathcal{M})|_{\eta'_\infty} &= \bigoplus_{x\in S}\operatorname{coker}\big((\mathcal{M}|_{\eta_x})((z'))\xrightarrow{z'\partial_{t_x}+1}(\mathcal{M}|_{\eta_x})((z'))\big)\\ &\qquad\oplus\operatorname{coker}\Big((\mathcal{M}|_{\eta_\infty})((z'))\xrightarrow{z'\partial_z-\frac{1}{z^2}}(\mathcal{M}|_{\eta_\infty})((z'))\Big).\end{aligned}$$

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JIANGXUE FANG
DEPARTMENT OF MATHEMATICS
CAPITAL NORMAL UNIVERSITY
BEIJING 100148
CHINA
fangjiangx@yahoo.com.cn

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