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**A FORMULA EQUATING OPEN AND CLOSED  
GROMOV-WITTEN INVARIANTS AND ITS APPLICATIONS  
TO MIRROR SYMMETRY**

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# A FORMULA EQUATING OPEN AND CLOSED GROMOV–WITTEN INVARIANTS AND ITS APPLICATIONS TO MIRROR SYMMETRY

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We prove that open Gromov–Witten invariants for semi-Fano toric manifolds of the form  $X = \mathbb{P}(K_Y \oplus \mathbb{O}_Y)$ , where  $Y$  is a toric Fano manifold, are equal to certain 1-pointed closed Gromov–Witten invariants of  $X$ . As applications, we compute the mirror superpotentials for these manifolds. In particular, this gives a simple proof for the formula of the mirror superpotential for the Hirzebruch surface  $F_2$ .

## 1. Introduction

Let  $X$  be a compact complex  $n$ -dimensional toric manifold equipped with a toric Kähler structure  $\omega$ . Let  $L$  be a Lagrangian torus fiber of the moment map associated to  $(X, \omega)$ . Fukaya, Oh, Ohta and Ono define open Gromov–Witten invariants for  $(X, L)$  as follows (see [Fukaya et al. 2010a]). Let  $\beta \in \pi_2(X, L)$  be a relative homotopy class with Maslov index  $\mu(\beta) = 2$ . Let  $\mathcal{M}_1(L, \beta)$  be the moduli space of holomorphic disks in  $X$  with boundaries lying in  $L$  and with one boundary marked point representing the class  $\beta$ . A compactification of  $\mathcal{M}_1(L, \beta)$  is given by the moduli space  $\overline{\mathcal{M}}_1(L, \beta)$  of stable maps from genus 0 bordered Riemann surfaces  $(\Sigma, \partial\Sigma)$  to  $(X, L)$  with one boundary marked point representing the class  $\beta$ . As shown in their monumental work [Fukaya et al. 2009] by the same authors,  $\overline{\mathcal{M}}_1(L, \beta)$  is a Kuranishi space with real virtual dimension  $n$ . By Corollary 11.5 in [Fukaya et al. 2010a], there exists a virtual fundamental  $n$ -cycle  $[\overline{\mathcal{M}}_1(L, \beta)]^{\text{vir}}$ . The pushforward of this cycle by the evaluation map  $\text{ev} : \overline{\mathcal{M}}_1(L, \beta) \rightarrow L$  at the boundary marked point then gives

$$c_\beta = \text{ev}_*([\overline{\mathcal{M}}_1(L, \beta)]^{\text{vir}}) \in H_n(L, \mathbb{Q}) \cong \mathbb{Q}.$$

By Lemma 11.7 in the same reference, the homology class  $c_\beta$  is independent of the perturbation data (transversal multisections) used to define  $[\overline{\mathcal{M}}_1(L, \beta)]^{\text{vir}}$ . Hence,  $c_\beta$  is an open Gromov–Witten invariant for  $(X, L)$ .

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Let  $Y$  be an  $(n-1)$ -dimensional toric Fano manifold. Consider the  $\mathbb{P}^1$ -bundle  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$  over  $Y$ , where  $K_Y$  denotes the anticanonical bundle of  $Y$ . Then  $X$  is an  $n$ -dimensional toric manifold which is semi-Fano, i.e., the anticanonical bundle  $K_X^{-1}$  is nef. Let  $h \in H_2(X, \mathbb{Z})$  be the fiber class. Let  $\alpha \in H_2(X, \mathbb{Z})$  be an effective class with  $c_1(\alpha) = c_1(X) \cdot \alpha = 0$ . Consider the moduli space  $\overline{\mathcal{M}}_{0,1}(X, h+\alpha)$  of genus-0 stable maps to  $X$  with one marked point representing the class  $h+\alpha$ .<sup>1</sup> By [Fukaya and Ono 1999],  $\overline{\mathcal{M}}_{0,1}(X, h+\alpha)$  is a Kuranishi space with complex virtual dimension  $n$ . The pushforward of the virtual fundamental cycle  $[\overline{\mathcal{M}}_{0,1}(X, h+\alpha)]^{\text{vir}}$  by the evaluation map  $\text{ev} : \overline{\mathcal{M}}_{0,1}(X, h+\alpha) \rightarrow X$  gives a 1-pointed closed Gromov–Witten invariant of  $X$ :

$$\text{GW}_{0,1}^{X,h+\alpha}([\text{pt}]) = \text{ev}_*([\overline{\mathcal{M}}_{0,1}(X, h+\alpha)]^{\text{vir}}) \in H_{2n}(X, \mathbb{Q}) \cong \mathbb{Q},$$

where  $[\text{pt}]$  denotes the Poincaré dual of a point in  $X$ .

Now let  $\iota_0 : Y \hookrightarrow X$  be the closed embedding of  $Y$  as the zero section of  $K_Y$ . The image is a toric prime divisor  $D_0 = \iota_0(Y) \subset X$ . As above, we equip  $X$  with a toric Kähler structure  $\omega$  and fix a Lagrangian torus fiber  $L$  in  $X$ . Corresponding to  $D_0$  is a relative homotopy class  $\beta_0 \in \pi_2(X, L)$ . More precisely,  $\beta_0 \in \pi_2(X, L)$  is the class such that  $D_i \cdot \beta_0 = \delta_{i0}$  for any toric prime divisor  $D_i$  in  $X$ . The main result of this paper is the following formula.

**Theorem 1.1.** *For the  $\mathbb{P}^1$ -bundle  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$  over a toric Fano manifold  $Y$  and for any effective class  $\alpha \in H_2(X, \mathbb{Z})$  with  $c_1(\alpha) = 0$ , we have*

$$c_{\beta_0+\alpha} = \text{GW}_{0,1}^{X,h+\alpha}([\text{pt}]).$$

Note that  $\beta_0 + \alpha \in \pi_2(X, L)$  is a Maslov index two class since  $c_1(\alpha) = 0$ . We will prove this formula in Section 4 by comparing the Kuranishi structures of  $\overline{\mathcal{M}}_1(L, \beta_0 + \alpha)$  and  $\overline{\mathcal{M}}_{0,1}(X, h + \alpha)$ .

We can apply this formula to study mirror symmetry. Recall that the mirror of a compact toric  $n$ -fold  $X$  is given by a Landau–Ginzburg model  $(X^\vee, W)$  consisting of a bounded domain  $X^\vee \subset (\mathbb{C}^*)^n$  and a holomorphic function  $W : X^\vee \rightarrow \mathbb{C}$  called the mirror superpotential. In [Fukaya et al. 2010a] (see also [Cho and Oh 2006; Auroux 2007; 2009; Chan and Leung 2010a; 2010b]), it is shown that the mirror superpotential can be expressed as a power series whose coefficients are the open Gromov–Witten invariants defined above. However, when  $X$  is non-Fano, these invariants are in general very hard to compute. The only known examples are the mirror superpotentials for the Hirzebruch surfaces  $F_2$  and  $F_3$ , first computed in [Auroux 2009] using degeneration methods and wall-crossing formulas. Fukaya, Oh, Ohta and Ono gave a different proof for the  $F_2$  case in [Fukaya et al. 2010b].

<sup>1</sup>In  $\overline{\mathcal{M}}_{0,1}$ , subscript 0 and 1 denote the genus and number of marked points respectively.

As an immediate application of our formula, we can express the mirror superpotential of  $X = \mathbb{P}(K_Y \oplus \mathbb{O}_Y)$  in terms of 1-point closed Gromov–Witten invariants (see [Theorem 5.2](#)). In particular, since  $F_2 = \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1})$  and its Gromov–Witten invariants are easy to compute as it is symplectomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , this gives a very simple proof of the formula for the mirror superpotential of  $F_2$ . See the example in [Section 5](#). Our formula has since then been applied to study mirror symmetry for various classes of toric manifolds. See [[Lau et al. 2011](#); [2010](#); [Chan et al. 2010](#); [Chan and Lau 2010](#)] more details.

The rest of this paper is organized as follows. In [Section 2](#), we briefly review Kuranishi spaces and recall the results that we need in this paper. In [Section 3](#), we establish several preliminary results concerning the toric manifolds  $X = \mathbb{P}(K_Y \oplus \mathbb{O}_Y)$ . In [Section 4](#) we prove our formula by a direct comparison of Kuranishi structures. In [Section 5](#), we discuss applications of our formula to mirror symmetry.

## 2. Kuranishi structures

In this section, we briefly review the theory of Kuranishi spaces and recall some of their properties for later use. We follow Appendix A1 in the book [[Fukaya et al. 2009](#)]. See also Section 3 in [[Fukaya and Ono 1999](#)].

Let  $\mathcal{M}$  be a compact metrizable space.

**Definition 2.1** [[Fukaya et al. 2009](#), Definitions A1.1, A1.3, A1.5]. A Kuranishi structure on  $\mathcal{M}$  of (real) virtual dimension  $d$  consists of the following data:

- (1) For each point  $\sigma \in \mathcal{M}$ ,
  - (1.1) A smooth manifold  $V_\sigma$  (with boundary or corners) and a finite group  $\Gamma_\sigma$  acting smoothly and effectively on  $V_\sigma$ .
  - (1.2) A real vector space  $E_\sigma$  on which  $\Gamma_\sigma$  has a linear representation and such that  $\dim V_\sigma - \dim E_\sigma = d$ .
  - (1.3) A  $\Gamma_\sigma$ -equivariant smooth map  $s_\sigma : V_\sigma \rightarrow E_\sigma$ .
  - (1.4) A homeomorphism  $\psi_\sigma$  from  $s_\sigma^{-1}(0)/\Gamma_\sigma$  onto a neighborhood of  $\sigma$  in  $\mathcal{M}$ .
- (2) For each  $\sigma \in \mathcal{M}$  and for each  $\tau \in \text{Im } \psi_\sigma$ ,
  - (2.1) A  $\Gamma_\tau$ -invariant open subset  $V_{\sigma\tau} \subset V_\tau$  containing  $\psi_\tau^{-1}(\tau)$ .<sup>2</sup>
  - (2.2) A homomorphism  $h_{\sigma\tau} : \Gamma_\tau \rightarrow \Gamma_\sigma$ .
  - (2.3) An  $h_{\sigma\tau}$ -equivariant embedding  $\varphi_{\sigma\tau} : V_{\sigma\tau} \rightarrow V_\sigma$  and an injective  $h_{\sigma\tau}$ -equivariant bundle map  $\hat{\varphi}_{\sigma\tau} : E_\tau \times V_{\sigma\tau} \rightarrow E_\sigma \times V_\sigma$  covering  $\varphi_{\sigma\tau}$ .

Moreover, these data should satisfy the following conditions:

- (i)  $\hat{\varphi}_{\sigma\tau} \circ s_\tau = s_\sigma \circ \varphi_{\sigma\tau}$ .<sup>3</sup>

<sup>2</sup>Here and in C2 below, we regard  $\psi_\tau$  as a map from  $s_\tau^{-1}(0)$  to  $\mathcal{M}$  by composing with the quotient map  $V_\tau \rightarrow V_\tau/\Gamma_\tau$ .

<sup>3</sup>Here and after, we also regard  $s_\sigma$  as a section  $s_\sigma : V_\sigma \rightarrow E_\sigma \times V_\sigma$ .

- (ii)  $\psi_\tau = \psi_\sigma \circ \varphi_{\sigma\tau}$ .
- (iii) If  $\xi \in \psi_\tau(s_\tau^{-1}(0)) \cap V_{\sigma\tau} / \Gamma_\tau$  then, in a sufficiently small neighborhood of  $\xi$ ,

$$\varphi_{\sigma\tau} \circ \varphi_{\tau\xi} = \varphi_{\sigma\xi} \quad \text{and} \quad \hat{\varphi}_{\sigma\tau} \circ \hat{\varphi}_{\tau\xi} = \hat{\varphi}_{\sigma\xi}.$$

The spaces  $E_\sigma$  are called obstruction spaces (or obstruction bundles), the maps  $\{s_\sigma : V_\sigma \rightarrow E_\sigma\}$  are called Kuranishi maps, and  $(V_\sigma, E_\sigma, \Gamma_\sigma, s_\sigma, \psi_\sigma)$  is called a Kuranishi neighborhood of  $\sigma \in \mathcal{M}$ .

To define virtual fundamental chains, we need Kuranishi spaces with extra structures.

**Definition 2.2** [Fukaya et al. 2009, Definitions A1.14, A1.17]. A Kuranishi space is said to have a tangent bundle if the differential of  $s_\sigma$  in the direction of the normal bundle induces a bundle isomorphism

$$(2-1) \quad ds_\sigma : \frac{\varphi_{\sigma\tau}^* T V_\sigma}{T V_{\sigma\tau}} \cong \frac{E_\sigma \times V_{\sigma\tau}}{\hat{\varphi}_{\sigma\tau}(E_\tau \times V_{\sigma\tau})}$$

as  $\Gamma_\tau$ -equivariant bundles on  $V_{\sigma\tau}$ .

For a Kuranishi space with tangent bundle, an orientation consists of trivializations of  $\bigwedge^{\text{top}} E_\sigma^* \otimes \bigwedge^{\text{top}} T V_\sigma$  compatible with the isomorphisms (2-1).

We will not give the precise definition of multisections here. See Definitions A1.19, A1.21 in [Fukaya et al. 2009] for details. Roughly speaking, a multisection  $\mathfrak{s}$  is a system of multivalued perturbations  $\{s'_\sigma : V_\sigma \rightarrow E_\sigma\}$  of the Kuranishi maps  $\{s_\sigma : V_\sigma \rightarrow E_\sigma\}$  satisfying certain compatibility conditions. For a Kuranishi space  $\mathcal{M}$  with tangent bundle, there exist (a family of) multisections  $\mathfrak{s}$  which are transversal to 0 (Theorem A1.23 in [Fukaya et al. 2009]). Furthermore, suppose that  $\mathcal{M}$  is oriented. Let  $\text{ev} : \mathcal{M} \rightarrow Y$  be a strongly smooth map to a smooth manifold  $Y$ , i.e., a family of  $\Gamma_\sigma$ -invariant smooth maps  $\{\text{ev}_\sigma : V_\sigma \rightarrow Y\}$  such that  $\text{ev}_\sigma \circ \varphi_{\sigma\tau} = \text{ev}_\tau$  on  $V_{\sigma\tau}$ . Then, using these transversal multisections, one can define the virtual fundamental chain  $\text{ev}_*([\mathcal{M}]^{\text{vir}})$  as a  $\mathbb{Q}$ -singular chain in  $Y$  (Definition A1.28 in [Fukaya et al. 2009]).

We will also need the notion of fiber products of Kuranishi spaces. See Appendix A1.2 in [Fukaya et al. 2009] for more details. As before, let  $\text{ev} : \mathcal{M} \rightarrow Y$  be a strongly smooth map from a Kuranishi space  $\mathcal{M}$  to a smooth manifold  $Y$ . Suppose that  $\text{ev}$  is weakly submersive, i.e., each  $\text{ev}_\sigma : V_\sigma \rightarrow Y$  is a submersion. Let  $W$  be another manifold and  $g : W \rightarrow Y$  be a smooth map. Consider the fiber product

$$\mathcal{X} = \mathcal{M} \times_Y W = \{(\sigma, p) \in \mathcal{M} \times W : \text{ev}(\sigma) = g(p)\}.$$

**Definition 2.3** [Fukaya et al. 2009, Definition A1.37]. Let  $(\sigma, p) \in \mathcal{X}$  and let  $(V_\sigma, E_\sigma, \Gamma_\sigma, s_\sigma, \psi_\sigma)$  be a Kuranishi neighborhood of  $\sigma \in \mathcal{M}$ . We set

$$V_{(\sigma,p)} = \{(\tau, q) \in V_\sigma \times W : \text{ev}_\sigma(\tau) = g(q)\}.$$

Then  $V_{(\sigma,p)}$  is a smooth manifold since  $\text{ev}_\sigma$  is a submersion. We also set  $E_{(\sigma,p)} = E_\sigma$ ,  $\Gamma_{(\sigma,p)} = \Gamma_\sigma$  and define  $s_{(\sigma,p)}$ ,  $\psi_{(\sigma,p)}$  in the obvious way. This defines a Kuranishi neighborhood of  $(\sigma, p) \in \mathcal{X}$ , and they glue together to give a Kuranishi structure on  $\mathcal{X}$ .

**Lemma 2.4** [Fukaya et al. 2009, Lemma A1.39]. *If the Kuranishi space  $\mathcal{M}$  has a tangent bundle, so does the Kuranishi structure on  $\mathcal{X}$ . If the Kuranishi structure on  $\mathcal{M}$  and the manifolds  $Y, W$  are all oriented, so is the Kuranishi structure on  $\mathcal{X}$ .*

Let  $\hat{\text{ev}} : \mathcal{X} \rightarrow W$  be the projection map. We remark that this is a strongly smooth map. The following lemma is crucial to the proof of our main result.

**Lemma 2.5** [Fukaya et al. 2009, Lemma A1.43]. *Suppose that  $Y$  and  $W$  are oriented and compact without boundary, and  $\partial\mathcal{M} = \emptyset$ . Then we have*

$$\text{PD}(\hat{\text{ev}}_*([\mathcal{X}]^{\text{vir}})) = g^*(\text{PD}(\text{ev}_*([\mathcal{M}]^{\text{vir}}))),$$

where PD denotes Poincaré dual.

### 3. A class of semi-Fano toric manifolds

Let  $Y$  be an  $(n - 1)$ -dimensional toric Fano manifold. Denote by  $K_Y$  its canonical line bundle. Consider the  $\mathbb{P}^1$ -bundle  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$  over  $Y$ . In this section, we shall establish some elementary properties of the toric manifold  $X$  which will be of use later.

Let  $e_1, \dots, e_n$  be the standard basis of a rank  $n$  lattice  $N \cong \mathbb{Z}^n$ , and let

$$N' = \left\{ v = \sum_{j=1}^n v^j e_j \in N \mid v^n = 0 \right\} \cong \mathbb{Z}^{n-1}.$$

Let  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$  and  $N'_{\mathbb{R}} = N' \otimes_{\mathbb{Z}} \mathbb{R}$ . Without loss of generality, we can choose the primitive generators of the 1-dimensional cones of the fan  $\Delta$  in  $N_{\mathbb{R}}$  defining  $X$  to be

$$v_0 = e_n, \quad v_1 = w_1 + e_n, \quad \dots, \quad v_m = w_m + e_n, \quad v_{m+1} = -e_n,$$

where  $w_1, \dots, w_m \in N'$  are the primitive generators of the 1-dimensional cones of a fan  $\Delta'$  in  $N'_{\mathbb{R}}$  defining  $Y$ .

**Lemma 3.1.** *Let  $h \in H_2(X, \mathbb{Z})$  be the fiber class of the  $\mathbb{P}^1$ -bundle  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$ . Let  $\iota_0 : Y \hookrightarrow X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$  be the closed embedding which maps  $Y$  to the zero section of  $K_Y$ .  $\iota_0$  induces an embedding  $\iota_{0*} : H_2(Y, \mathbb{Z}) \hookrightarrow H_2(X, \mathbb{Z})$ . Then we have*

$$H_2^{\text{eff}}(X, \mathbb{Z}) \cong \mathbb{Z}_{\geq 0} h \oplus \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z}).$$

Here, superscript “eff” refers to effective classes. Moreover, we have  $c_1(h) = 2$  and  $c_1(\alpha) = 0$  for any  $\alpha \in \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z})$ . In particular,  $X$  is semi-Fano, i.e., the anticanonical bundle  $K_X^{-1}$  is nef.

*Proof.* Recall that a subset  $\mathcal{P} = \{v_{i_1}, \dots, v_{i_p}\} \subset \{v_0, \dots, v_{m+1}\}$  is called a *primitive collection* if for each  $1 \leq k \leq p$ , the elements of  $\mathcal{P} \setminus \{x_{i_k}\}$  generate a  $(p - 1)$ -dimensional cone in  $\Delta$  but  $\mathcal{P}$  itself does not generate a cone in  $\Delta$ ; see [Batyrev 1991]. The *focus*  $\sigma(\mathcal{P})$  of  $\mathcal{P}$  is the cone in  $\Delta$  of the smallest dimension which contains  $v_{i_1} + \dots + v_{i_p}$ . Let  $v_{j_1}, \dots, v_{j_q}$  be the generators of  $\sigma(\mathcal{P})$ . Then there exists positive integers  $n_1, \dots, n_q$  such that

$$v_{i_1} + \dots + v_{i_p} = n_1 v_{j_1} + \dots + n_q v_{j_q}.$$

This is known as a *primitive relation*. Recall that the homology group  $H_2(X, \mathbb{Z})$  is given by the kernel of the surjective map  $\mathbb{Z}^d \rightarrow N$ ,  $E_i \mapsto v_i$ , where  $\{E_1, \dots, E_d\}$  is the standard basis of  $\mathbb{Z}^d$ . Also, the effective cone  $H_2^{\text{eff}}(X, \mathbb{Z})$  is generated by primitive relations.

In our case,  $\mathcal{P}_0 := \{v_0, v_{m+1}\}$  is obviously a primitive collection for  $\Delta$ . The primitive relation  $v_0 + v_{m+1} = 0$  corresponds to the fiber class  $h$  of the  $\mathbb{P}^1$ -bundle  $X \rightarrow Y$ . It is obvious that we have  $c_1(h) = c_1(X) \cdot h = 2$ .

By Proposition 4.1 in [Batyrev 1991], we have  $\mathcal{P} \cap \mathcal{P}_0 = \emptyset$  for any other primitive collection  $\mathcal{P} \neq \mathcal{P}_0$ . Suppose that  $\mathcal{P} \neq \mathcal{P}_0$  is a primitive collection consisting of the elements  $v_{i_1} = w_{i_1} + e_n, \dots, v_{i_p} = w_{i_p} + e_n$ , where  $1 \leq i_1 < \dots < i_p \leq m$ . Then  $\mathcal{P}' := \{w_{i_1}, \dots, w_{i_p}\}$  is obviously a primitive collection for the fan  $\Delta'$  defining  $Y$ . Now, let  $w_{j_1}, \dots, w_{j_q}$  be the generators of the focus  $\sigma(\mathcal{P}')$  of  $\mathcal{P}'$ . The primitive relation for  $\Delta'$  is given by

$$(3-1) \quad w_{i_1} + \dots + w_{i_p} = n_1 w_{j_1} + \dots + n_q w_{j_q},$$

for some  $n_1, \dots, n_q \in \mathbb{Z}_{>0}$ . Let  $\gamma \in H_2^{\text{eff}}(Y, \mathbb{Z})$  be the corresponding effective class. Since  $Y$  is Fano, we have  $p - n_1 - \dots - n_q = c_1(Y) \cdot \gamma > 0$ . In terms of the  $v_i$ 's, (3-1) becomes a primitive relation

$$v_{i_1} + \dots + v_{i_p} = n_1 v_{j_1} + \dots + n_q v_{j_q} + (p - n_1 - \dots - n_q) v_0$$

for  $\Delta$ . This corresponds to the class  $\alpha := \iota_{0*}(\gamma) \in H_2^{\text{eff}}(X, \mathbb{Z})$ , whose Chern number is given by  $c_1(\alpha) = p - n_1 - \dots - n_q - (p - n_1 - \dots - n_q) = 0$ . □

As usual, denote by  $D_0, D_1, \dots, D_m, D_{m+1}$  the toric prime divisors corresponding to the primitive generators  $v_0, v_1, \dots, v_m, v_{m+1}$  respectively. Note that  $D_0 = \iota_0(Y)$ .

**Lemma 3.2.** *Let  $\varphi : \mathbb{P}^1 \rightarrow X$  be a nonconstant holomorphic map from  $\mathbb{P}^1$  to  $X$ .*

- (1) *Suppose that  $[\varphi(\mathbb{P}^1)] = h + \alpha \in H_2(X, \mathbb{Z})$  for some  $\alpha \in \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z})$ . Then  $\varphi(\mathbb{P}^1)$  is contained in one of the toric prime divisors  $D_0, D_1, \dots, D_m$ .*
- (2) *Suppose that  $[\varphi(\mathbb{P}^1)] = \alpha \in \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z})$ . Then  $\varphi(\mathbb{P}^1)$  is contained in the toric prime divisor  $D_0$ .*

*Proof.* Suppose that  $\varphi : \mathbb{P}^1 \rightarrow X$  is a nonconstant holomorphic map with class  $h + \alpha$  for some  $\alpha \in \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z})$ . From the proof of the above lemma, we know that the class  $h + \alpha$  corresponds to the primitive relation

$$\left(1 - \sum_{i=1}^m a_i\right) v_0 + \sum_{i=1}^m a_i v_i + v_{m+1} = 0.$$

Moreover, we have  $\sum_{i=1}^m a_i \geq 1$ , and if  $\sum_{i=1}^m a_i = 1$ , then there exists  $1 \leq i \leq m$  such that  $a_i < 0$ . Hence there exists  $0 \leq i \leq m$  such that  $D_i \cdot \varphi(\mathbb{P}^1) = D_i \cdot (h + \alpha) < 0$ . This implies that  $\varphi(\mathbb{P}^1)$  is contained in  $D_i$ . This proves (1). (2) can be proved in the same way. □

### 4. Proof of Theorem 1.1

We equip  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$  with a toric Kähler structure  $\omega$ . Let  $L \subset X$  be a Lagrangian torus fiber of the associated moment map. For  $i = 0, 1, \dots, m, m + 1$ , let  $\beta_i \in \pi_2(X, L)$  be the relative homotopy class such that  $D_j \cdot \beta_i = \delta_{ij}$ . Then each  $\beta_i$  is a Maslov index two class with  $\partial\beta_i = v_i$ , where  $\partial : \pi_2(X, L) \rightarrow \pi_1(L)$  is the boundary map, and  $\pi_2(X, L)$  is generated by  $\beta_0, \beta_1, \dots, \beta_m, \beta_{m+1}$ . Moreover, each  $\beta_i$  is represented by a family of holomorphic disks  $\varphi_i : (D^2, \partial D^2) \rightarrow (X, L)$ . Here,  $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$  is the unit disk.

Fix a nonzero effective class  $\alpha \in H_2^{\text{eff}}(X, \mathbb{Z})$  with  $c_1(\alpha) = 0$ . Let  $\overline{\mathcal{M}}_1(L, \beta_0 + \alpha)$  be the moduli space of stable maps from genus 0 bordered Riemann surfaces to  $(X, L)$  with one boundary marked point representing the class  $\beta_0 + \alpha$ . To simplify notation, we denote  $\overline{\mathcal{M}}_1(L, \beta_0 + \alpha)$  by  $\mathcal{M}^L$ . Similarly, we denote by  $\mathcal{M}^X$  the moduli space  $\overline{\mathcal{M}}_{0,1}(X, h + \alpha)$  of genus 0 stable maps to  $X$  with one marked point representing the class  $h + \alpha$ . We have evaluation maps<sup>4</sup>

$$\text{ev} : \mathcal{M}^L \rightarrow L \quad \text{and} \quad \text{ev} : \mathcal{M}^X \rightarrow X.$$

By [Fukaya et al. 2009], both  $\mathcal{M}^L$  and  $\mathcal{M}^X$  are oriented Kuranishi spaces with tangent bundles, and the evaluation maps are both strongly smooth and weakly submersive. The real virtual dimensions of  $\mathcal{M}^L$  and  $\mathcal{M}^X$  are  $n$  and  $2n$  respectively. Moreover, since  $\mu(\beta_0 + \alpha) = 2$ , we have  $\partial\mathcal{M}^L = \emptyset$  by Corollary 11.5 in [Fukaya et al. 2010a]. It is also well-known that  $\mathcal{M}^X$  has no boundary. Hence, they define virtual fundamental cycles

$$\text{ev}_*([\mathcal{M}^L]^{\text{vir}}) \in H_n(L, \mathbb{Q}) \quad \text{and} \quad \text{ev}_*([\mathcal{M}^X]^{\text{vir}}) \in H_{2n}(X, \mathbb{Q}).$$

Fix a point  $p \in L \subset X$ . Let  $\iota : \{p\} \hookrightarrow L$  (resp.  $\iota : \{p\} \hookrightarrow X$ ) be the inclusion of the point  $p$ . We can then apply the construction in Definition 2.3 and Lemma 2.4

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<sup>4</sup>By a slight abuse of notation, we use  $\text{ev}$  to denote both evaluation maps. It should clear from the context which one we are referring to.



to give oriented Kuranishi structures with tangent bundles on the spaces:

$$\mathcal{M}_p^L := \mathcal{M}^L \times_L \{p\}, \quad \mathcal{M}_p^X := \mathcal{M}^X \times_X \{p\}.$$

Both have real virtual dimension 0. Let  $\hat{e}v : \mathcal{M}_p^L \rightarrow \{p\}$ ,  $\hat{e}v : \mathcal{M}_p^X \rightarrow \{p\}$  be the induced (constant) maps. Then we have virtual fundamental cycles

$$\hat{e}v_*([\mathcal{M}_p^L]^{\text{vir}}), \quad \hat{e}v_*([\mathcal{M}_p^X]^{\text{vir}}) \in H_0(\{p\}, \mathbb{Q}) \cong \mathbb{Q}.$$

Now [Lemma 2.5](#) says that

**Proposition 4.1.** *We have*

$$\begin{aligned} \text{PD}(\hat{e}v_*([\mathcal{M}_p^L]^{\text{vir}})) &= \iota^* \text{PD}(\text{ev}_*([\mathcal{M}^L]^{\text{vir}})), \\ \text{PD}(\hat{e}v_*([\mathcal{M}_p^X]^{\text{vir}})) &= \iota^* \text{PD}(\text{ev}_*([\mathcal{M}^X]^{\text{vir}})) \end{aligned}$$

in  $H^0(\{p\}, \mathbb{Q}) \cong \mathbb{Q}$ .

Therefore, to prove [Theorem 1.1](#), it suffices to show that  $\mathcal{M}_p^L$  and  $\mathcal{M}_p^X$  have the same Kuranishi structures.

To do this, we first show that  $\mathcal{M}_p^L$  can naturally be identified with  $\mathcal{M}_p^X$  as a set. Let us recall the following results proved in [\[Cho and Oh 2006\]](#), which holds for general toric manifolds.

**Theorem 4.2** ([\[Cho and Oh 2006, Theorem 5.2\]](#); see also [\[Fukaya et al. 2010a, Theorem 11.1\]](#)). *Let  $(X, \omega)$  be a toric Kähler manifold and  $L$  be a Lagrangian torus fiber of its moment map. Let  $D_1, \dots, D_d$  be all the toric prime divisors in  $X$  and  $\beta_1, \dots, \beta_d \in \pi_2(X, L)$  be the relative homotopy classes such that  $D_j \cdot \beta_i = \delta_{ij}$ .*

- (1) *If  $\varphi : (D^2, \partial D^2) \rightarrow (X, L)$  is a holomorphic map from a disk representing a Maslov index two class  $\beta \in \pi_2(X, L)$ , then  $\beta = \beta_i$  for some  $i \in \{1, \dots, d\}$ .*
- (2) *For  $i = 1, \dots, d$ , let  $\bar{\mathcal{M}}_1(L, \beta_i)$  be the moduli space of stable maps from genus 0 bordered Riemann surfaces to  $(X, L)$  with one boundary marked point representing the class  $\beta_i$ . Then the evaluation map  $\text{ev} : \bar{\mathcal{M}}_1(L, \beta_i) \rightarrow L$  is an orientation preserving diffeomorphism. In particular, for any  $p \in L$  and any  $i \in \{1, \dots, d\}$ , there is a unique (up to automorphisms of the domain) genus 0 bordered stable map whose boundary passes through  $p$  and whose domain is a disk which represents the class  $\beta_i$ .*

Now, let  $\sigma^L = ((\Sigma^L, z), \varphi)$  represent a point in  $\mathcal{M}_p^L$ . This consists of a genus 0 bordered Riemann surface  $\Sigma^L$  with a boundary marked point  $z \in \partial \Sigma^L$  and a stable map  $\varphi : (\Sigma^L, \partial \Sigma^L) \rightarrow (X, L)$  such that  $\varphi(z) = p$ .

**Proposition 4.3.**  *$\Sigma^L$  can be decomposed as  $\Sigma^L = \Sigma_0^L \cup \Sigma_1$ , where  $\Sigma_0^L = D^2$  is a disk and  $\Sigma_1$  is a genus 0 nodal curve, such that the restrictions  $\varphi_0 := \varphi|_{\Sigma_0^L}$  and  $\varphi_1 := \varphi|_{\Sigma_1}$  represent the classes  $\beta_0$  and  $\alpha$  respectively.*

*Proof.* The Maslov index of  $\beta_0 + \alpha$  is  $\mu(\beta_0 + \alpha) = 2$  since  $c_1(\alpha) = 0$ . By [Theorem 4.2](#) (1), there does not exist any nonconstant holomorphic map from a disk to  $(X, L)$  with class  $\beta_0 + \alpha$ , so  $\Sigma^L$  must be singular. Decompose  $\Sigma^L$  into irreducible components. Let  $\varphi_j : (D^2, \partial D^2) \rightarrow (X, L)$  and  $\varphi_k : \mathbb{P}^1 \rightarrow X$  be the restriction of  $\varphi$  to the disk and sphere components respectively. Then  $\beta_0 + \alpha = \sum_j [\varphi_j] + \sum_k [\varphi_k]$ . Notice that, by the proof of [Lemma 3.1](#), any  $\alpha \in H_2(X, \mathbb{Z})$  with  $c_1(\alpha) = 0$  cannot be expressed as a  $\mathbb{Z}$ -linear combination of  $\beta_i$ 's with positive coefficients. Hence, there must be only one disk component in  $\Sigma$ . Therefore, we can decompose  $\Sigma$  into  $\Sigma_0^L \cup \Sigma_1$ , where  $\Sigma_0^L = D^2$  is a disk and  $\Sigma_1$  is a genus 0 nodal curve (i.e., a tree of  $\mathbb{P}^1$ 's). Now, the restriction  $\varphi_0 := \varphi|_{\Sigma_0^L}$  is a nonconstant holomorphic map from  $(D^2, \partial D^2)$  to  $(X, L)$ . By [Theorem 4.2](#) (1) again, the class of  $\varphi_0$  must be  $\beta_0$ . Hence  $\varphi_1 := \varphi|_{\Sigma_1}$  represents  $\alpha$ .  $\square$

**Proposition 4.4.** *There exists a unique holomorphic map*

$$\varphi_{m+1} : (D^2, \partial D^2) \rightarrow (X, L)$$

*representing the class  $\beta_{m+1}$  such that its boundary  $\partial\varphi_{m+1} := \varphi_{m+1}|_{\partial D^2}$  is exactly given by  $\partial\varphi_0 := \varphi_0|_{\partial D^2}$  with the opposite orientation, where  $\varphi_0$  is the map obtained in [Proposition 4.3](#)*

*Proof.* Let  $\varphi_{m+1} : (D^2, \partial D^2) \rightarrow (X, L)$  be a holomorphic map representing the class  $\beta_{m+1}$  such that  $p \in \varphi(\partial D^2)$ . By [Theorem 4.2](#) (2), there exists one and only one such map up to automorphisms of  $D^2$ . Consider the moduli space  $\overline{\mathcal{M}}_{0,1}(X, h)$  of genus 0 stable maps to  $X$  with one marked point which represent the fiber class  $h$ . Since  $X \rightarrow Y$  is a  $\mathbb{P}^1$ -bundle, the evaluation map  $\text{ev} : \overline{\mathcal{M}}_{0,1}(X, h) \rightarrow X$  is an isomorphism. Hence, there exists a unique (up to automorphisms of the domain) holomorphic map  $\phi : \mathbb{P}^1 \rightarrow X$  representing the class  $h$  which passes through  $p \in L \subset X$ . The image of this map is the fiber  $C_p \cong \mathbb{P}^1$  of  $X \rightarrow Y$  which contains  $p$ . Now, the intersection  $C_p \cap L \cong S^1$  splits the fiber  $C_p$  into two disks. This gives two holomorphic maps  $\varphi'_0 : (D^2, \partial D^2) \rightarrow (X, L)$  and  $\varphi'_{m+1} : (D^2, \partial D^2) \rightarrow (X, L)$  with classes  $\beta_0$  and  $\beta_{m+1}$  respectively. By [Theorem 4.2](#) (2), they must be the same as  $\varphi_0, \varphi_{m+1}$  up to automorphisms of  $D^2$ . Hence, by composing  $\varphi_{m+1}$  with an automorphism of  $D^2$ , which is uniquely determined by  $\varphi_0$ , we get the desired unique holomorphic map representing the class  $\beta_{m+1}$ .  $\square$

By [Proposition 4.4](#), we can glue the maps  $\varphi : (\Sigma^L, \partial\Sigma^L) \rightarrow (X, L)$  and  $\varphi_{m+1} : (D^2, \partial D^2) \rightarrow (X, L)$  together to give a holomorphic map  $\varphi' : \Sigma \rightarrow X$  which represents the class  $\beta_0 + \beta_{m+1} + \alpha = h + \alpha$ , where  $\Sigma$  is the union of  $\Sigma^L$  and  $D^2$  with their boundaries identified in the obvious way. It is easy to see that this map is stable. Hence,  $\sigma^X := ((\Sigma, z), \varphi')$  represents a point in  $\mathcal{M}^X = \overline{\mathcal{M}}_{0,1}(X, h + \alpha)$  and we have  $\text{ev}(\sigma) = p$ . This defines a map

$$j : \mathcal{M}_p^L \rightarrow \mathcal{M}_p^X, \quad [\sigma^L] \mapsto [\sigma^X].$$

This is well-defined: Any automorphism of  $\sigma^L = ((\Sigma^L, z), \varphi)$  acts trivially on the component  $\Sigma_0^L$  because  $\varphi$  is nonconstant on this component. So any representative of  $[\sigma^L]$  is mapped to the same isomorphism class in  $\mathcal{M}_p^X$ . We need to show that  $j$  is bijective.

Let  $\sigma^X = ((\Sigma, z), \varphi)$  be representing a point in  $\mathcal{M}_p^X$ . This consists of a genus 0 nodal curve  $\Sigma$  with a marked point  $z \in \Sigma$  and a stable map  $\varphi : \Sigma \rightarrow X$  representing the class  $h + \alpha$  such that  $\varphi(z) = p$ . The following is an analog of Proposition 4.3.

**Proposition 4.5.**  *$\Sigma$  can be decomposed as  $\Sigma = \Sigma_0 \cup \Sigma_1$ , where  $\Sigma_0 \cong \mathbb{P}^1$  is irreducible, such that the restrictions  $\varphi_0 := \varphi|_{\Sigma_0}$  and  $\varphi_1 := \varphi|_{\Sigma_1}$  represent the classes  $h$  and  $\alpha$  respectively.*

*Proof.* By Lemma 3.2 (1), there does not exist any nonconstant holomorphic map from  $\mathbb{P}^1$  to  $X$  representing the class  $h + \alpha$  whose image is not contained entirely in the toric divisors. Hence,  $\Sigma$  must be singular. Decompose  $\Sigma$  into components  $\Sigma = \bigcup_a \Sigma_a$ , where each  $\Sigma_a \cong \mathbb{P}^1$  is irreducible. Then we have

$$\sum_a [\varphi(\Sigma_a)] = h + \alpha.$$

Since  $h$  is primitive, there exists  $a_0$  such that  $\varphi(\Sigma_{a_0}) = h + \alpha'$  and  $\sum_{a \neq a_0} [\varphi(\Sigma_a)] = \alpha''$  for some  $\alpha', \alpha'' \in \iota_{0*} H_2^{\text{eff}}(Y, \mathbb{Z}) \subset H_2^{\text{eff}}(X, \mathbb{Z})$  with  $\alpha = \alpha' + \alpha''$ . By Lemma 3.1, we have  $c_1(\alpha') = c_1(\alpha'') = 0$ . Then, by Lemma 3.2 (2), the images of  $\bigcup_{a \neq a_0} \Sigma_a$  is contained entirely in the zero section  $D_0$ . So the image of  $\Sigma_{a_0}$  must be intersecting with  $L$  at  $p$ . Applying Lemma 3.2 (1) again, we see that  $\alpha'$  must be zero. The result follows. □

Note that  $\varphi_0$  is a nonconstant holomorphic map from  $\mathbb{P}^1$  to  $X$  whose image contains  $p$ . Arguing as in the proof of Proposition 4.4, we see that the image of  $\varphi_0$  is the fiber  $C_p$  of the  $\mathbb{P}^1$ -bundle  $X \rightarrow Y$  which contains  $p$ , and  $\varphi_0(\mathbb{P}^1) \cap L = S^1$ . We can then split  $\Sigma_0 \cong \mathbb{P}^1$  into two disks  $\Sigma_0 = \Sigma'_0 \cup \Sigma''_0 \cong D^2 \cup D^2$ , and split  $\varphi_0$  into two holomorphic maps  $\varphi'_0 : (\Sigma'_0, \partial \Sigma'_0) \rightarrow (X, L)$  and  $\varphi'_{m+1} : (\Sigma''_0, \partial \Sigma''_0) \rightarrow (X, L)$  which represent the classes  $\beta_0$  and  $\beta_{m+1}$  respectively. Now, let  $\Sigma^L := \Sigma'_0 \cup \Sigma_1$  and  $\varphi' := \varphi|_{\Sigma^L}$ . Then  $\varphi' : (\Sigma^L, \partial \Sigma^L) \rightarrow (X, L)$  is a genus 0 bordered stable map such that  $\varphi(\partial \Sigma^L)$  contains  $p$ , and  $\sigma^L := ((\Sigma^L, z), \varphi')$  represents a point in  $\mathcal{M}_p^L$ . By our constructions,  $j([\sigma^L]) = [\sigma^X]$ . This defines a map  $j^{-1} : \mathcal{M}_p^X \rightarrow \mathcal{M}_p^L$ . Again, since any automorphism of  $\sigma^X = ((\Sigma, z), \varphi)$  acts trivially on the component  $\Sigma_0$ , the map  $j^{-1}$  is well-defined. It is obvious that this is the inverse map of  $j$ . Hence,  $j$  is a bijective map.

**Proposition 4.6.** *Under the bijective map  $j : \mathcal{M}_p^L \rightarrow \mathcal{M}_p^X$ , the Kuranishi structures on  $\mathcal{M}_p^L$  and  $\mathcal{M}_p^X$  can be naturally identified.*

*Proof.* We shall first briefly recall the constructions of Kuranishi neighborhoods from [Fukaya and Ono 1999] and [Fukaya et al. 2009].

We begin with  $\mathcal{M}_p^L$ . Let  $\sigma^L = ((\Sigma^L, z), \varphi)$  be representing a point in  $\mathcal{M}_p^L$ . By Proposition 4.3, we can decompose  $\Sigma^L$  into irreducible components  $\Sigma^L = \Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_k$ , where  $\Sigma_0 = D^2$  is a disk and  $\Sigma_1, \dots, \Sigma_k$  are copies of  $\mathbb{P}^1$ , such that the restrictions of  $\varphi$  to  $\Sigma_0$  and  $\bigcup_{a=1}^k \Sigma_a$  represent the classes  $\beta_0$  and  $\alpha$  respectively.

For each  $a = 0, 1, \dots, k$ , let  $W^{1,p}(\Sigma_a; \varphi^*(TX); L)$  be the space of sections  $v$  of  $\varphi^*(TX)$  of  $W^{1,p}$  class such that the restriction of  $v$  to  $\partial \Sigma_a$  lies in  $\varphi^*(TL)$ , and  $W^{0,p}(\Sigma_a; \varphi^*(TX) \otimes \wedge^{0,1})$  be the space of sections of  $\varphi^*(TX) \otimes \wedge^{0,1}$  of  $W^{0,p}$  class. Note that  $L$  does not play a role in the definition of  $W^{1,p}(\Sigma_a; \varphi^*(TX); L)$  for  $a = 1, \dots, k$ . Then, let  $W^{1,p}(\Sigma^L; \varphi^*(TX); L)$  be the subspace of

$$\bigoplus_{a=0}^k W^{1,p}(\Sigma_a; \varphi^*(TX); L)$$

consisting of elements

$$\{u = (u_a)\} \in \bigoplus_{a=0}^k W^{1,p}(\Sigma_a; \varphi^*(TX); L)$$

such that for any singular point  $w \in \Sigma^L$  which is the intersection of two irreducible components  $\Sigma_a$  and  $\Sigma_b$ , we have  $u_a(w) = u_b(w)$ . Also let

$$W^{0,p}(\Sigma^L; \varphi^*(TX) \otimes \wedge^{0,1}) = \bigoplus_{a=0}^k W^{0,p}(\Sigma_a; \varphi^*(TX) \otimes \wedge^{0,1}).$$

Consider the linearization of the Cauchy–Riemann operator  $\bar{\partial}$ :

$$D_\varphi \bar{\partial} : W^{1,p}(\Sigma^L; \varphi^*(TX); L) \rightarrow W^{0,p}(\Sigma^L; \varphi^*(TX) \otimes \wedge^{0,1}).$$

This is a Fredholm operator by ellipticity.

To construct the obstruction space, choose open subsets  $W_a$  of  $\Sigma_a$  whose closure is disjoint from the boundary of each of  $\Sigma_a$  and from the singular and marked points. Then, for each  $a = 0, 1, \dots, k$ , by the unique continuation theorem, we can choose a finite dimensional subset  $E_a$  of  $C_0^\infty(W_a; \varphi^*(TX))$  such that

$$\text{Im } D_\varphi \bar{\partial} + \bigoplus_{a=0}^k E_a = W^{0,p}(\Sigma^L; \varphi^*(TX) \otimes \wedge^{0,1}).$$

We also choose  $\bigoplus_{a=0}^k E_a$  to be invariant under the group  $\Gamma_{\sigma^L}$  of automorphisms of  $\sigma^L$ . We set  $E_{\sigma^L} = \bigoplus_{a=0}^k E_a$ .

Let

$$\Pi : W^{0,p}(\Sigma^L; \varphi^*(TX) \otimes \wedge^{0,1}) \rightarrow W^{0,p}(\Sigma^L; \varphi^*(TX) \otimes \wedge^{0,1})/E_{\sigma^L}$$

be the projection map. Let  $V_{\text{map}, \sigma^L}$  be the kernel of the operator  $\Pi \circ D_\varphi \bar{\partial}$ . Now, consider the automorphism group  $\text{Aut}(\Sigma^L, z)$  of the marked bordered Riemann surface

$(\Sigma^L, z)$ . The group  $\text{Aut}(\Sigma^L, z)$  may not be finite since some components may be unstable. However, we can naturally embed the Lie algebra  $\text{Lie}(\text{Aut}(\Sigma^L, z))$  into  $V_{\text{map}, \sigma^L}$ . Take its  $L^2$  orthogonal complement (with respect to a certain metric). Then let  $V'_{\text{map}, \sigma^L}$  be a small neighborhood of the zero of it.

On the other hand, let  $V_{\text{deform}, \sigma^L}$  be a small neighborhood of the origin in the space of first order deformations of the stable components of  $(\Sigma^L, z)$ . Also let  $V_{\text{resolve}, \sigma^L}$  be a small neighborhood of the origin in the space  $\bigoplus_w (T_w \Sigma_a \otimes T_w \Sigma_b)$ , where the sum is over singular points  $w \in \Sigma^L \setminus \Sigma_0$  and  $\Sigma_a, \Sigma_b$  are the two components such that  $\Sigma_a \cap \Sigma_b = \{w\}$ . There is a family of marked semistable bordered Riemann surfaces  $\{(\Sigma_\zeta^L, z) : \zeta \in V_{\text{deform}, \sigma^L} \times V_{\text{resolve}, \sigma^L}\}$  over the product  $V_{\text{deform}, \sigma^L} \times V_{\text{resolve}, \sigma^L}$ . We remark that, since we do not deform the singular point in  $\Sigma_0$ , each  $\Sigma_\zeta^L$  is singular and can be decomposed as  $\Sigma_\zeta^L = \Sigma_0 \cup \Sigma'_\zeta$ .

Let  $V'_{\sigma^L} = V'_{\text{map}, \sigma^L} \times V_{\text{deform}, \sigma^L} \times V_{\text{resolve}, \sigma^L}$ . By the proof of Proposition 12.23 in [Fukaya and Ono 1999], there exist a  $\Gamma_{\sigma^L}$ -equivariant smooth map

$$s_{\sigma^L} : V'_{\sigma^L} \rightarrow E_{\sigma^L}$$

and a family of smooth maps

$$\varphi_{u, \zeta} : (\Sigma_\zeta^L, \partial \Sigma_\zeta^L) \rightarrow (X, L)$$

for  $(u, \zeta) \in V'_{\sigma^L}$  such that  $\bar{\partial} \varphi_{u, \zeta} = s_{\sigma^L}(u, \zeta)$ . Now we set  $V_{\sigma^L} = \{(u, \zeta) \in V'_{\sigma^L} : \varphi_{u, \zeta}(z) = p\}$ . By abuse of notation, denote the restriction of  $s_{\sigma^L}$  to  $V_{\sigma^L}$  also by  $s_{\sigma^L}$ . Then by [Fukaya and Ono 1999], there is a map  $\psi_{\sigma^L}$  mapping  $s_{\sigma^L}^{-1}(0)/\Gamma_{\sigma^L}$  onto a neighborhood of  $[\sigma^L]$  in  $\mathcal{M}_p^L$ . This finishes the review of the construction of a Kuranishi neighborhood  $(V_{\sigma^L}, E_{\sigma^L}, \Gamma_{\sigma^L}, s_{\sigma^L}, \psi_{\sigma^L})$  of  $[\sigma^L] \in \mathcal{M}_p^L$ .

For a point in  $\mathcal{M}_p^X$  represented by  $\sigma^X = ((\Sigma, z), \varphi)$ , using Proposition 4.5, we decompose  $\Sigma$  into irreducible components  $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \dots \cup \Sigma_k$ , where  $\Sigma_0, \Sigma_1, \dots, \Sigma_k$  are all copies of  $\mathbb{P}^1$ , such that the restrictions of  $\varphi$  to  $\Sigma_0$  and  $\bigcup_{a=1}^k \Sigma_a$  represent the classes  $h$  and  $\alpha$  respectively. The construction of a Kuranishi neighborhood  $(V_{\sigma^X}, E_{\sigma^X}, \Gamma_{\sigma^X}, s_{\sigma^X}, \psi_{\sigma^X})$  of  $[\sigma^X] \in \mathcal{M}_p^X$  is more or less the same as above, except that  $W^{1,p}(\Sigma_0; \varphi^*(TX); L)$  is replaced by the space  $W^{1,p}(\Sigma_0; \varphi^*(TX))$  of sections  $v$  of  $\varphi^*(TX)$  of class  $W^{1,p}$ .

We can now go back to the proof of the proposition.

Let  $[\sigma^L] \in \mathcal{M}_p^L, [\sigma^X] \in \mathcal{M}_p^X$  be such that  $j([\sigma^L]) = [\sigma^X]$ . First of all, it is obvious that the automorphism groups  $\Gamma_{\sigma^L}$  and  $\Gamma_{\sigma^X}$  are the same. Next, since the moduli space of maps from  $(D^2, \partial D^2)$  to  $(X, L)$  with class  $\beta_0$  is unobstructed, we can choose  $E_0 = 0$  for the obstruction space  $E_{\sigma^L}$ . Similarly, since the moduli space of maps from  $\mathbb{P}^1$  to  $X$  with class  $h$  is unobstructed, we can also choose  $E_0 = 0$  for the obstruction space  $E_{\sigma^X}$ . Hence, the obstruction spaces  $E_{\sigma^L}$  and  $E_{\sigma^X}$  are both of the form  $0 \oplus E_1 \oplus \dots \oplus E_k$  and can be identified naturally.

We can identify  $V_{\text{deform}, \sigma^L}$  with  $V_{\text{deform}, \sigma^X}$  since the component  $\Sigma_0$  in  $\Sigma^L$  has no nontrivial deformations and the component  $\Sigma_0$  in  $\Sigma$  is unstable. It is also clear that we can identify  $V_{\text{resolve}, \sigma^L}$  with  $V_{\text{resolve}, \sigma^X}$ . Now, let  $(u = (u_0, u_1, \dots, u_k), \zeta) \in V_{\sigma^L}$ . Because  $E_0 = 0$ , we have  $D_\varphi \bar{\partial} u_0 = 0$ . From the construction of the family of smooth maps  $\varphi_{u, \zeta} : (\Sigma_\zeta^L, \partial \Sigma_\zeta^L) \rightarrow (X, L)$ , it follows that the restriction of  $\varphi_{u, \zeta}$  to the component  $\Sigma_0$  is a holomorphic map with class  $\beta_0$ . We also have  $\varphi_{u, \zeta}(z) = p$ . But there is a unique (up to automorphisms of the domain) holomorphic map from  $(D^2, \partial D^2)$  to  $(X, L)$  with class  $\beta_0$  whose boundary passes through  $p$ , which is given by  $\varphi|_{\Sigma_0}$ . So we must have  $u_0 = 0$ . By a similar argument, all  $(u, \zeta) \in V_{\sigma^X}$  also have  $u_0 = 0$ . Therefore, we can naturally identify  $V_{\sigma^L}$  and  $V_{\sigma^X}$ .

Finally, we can identify the families of maps  $\{\varphi_{u, \zeta} : (\Sigma_\zeta^L, \partial \Sigma_\zeta^L) \rightarrow (X, L) : (u, \zeta) \in V_{\sigma^L}\}$  with  $\{\varphi_{u, \zeta} : \Sigma_\zeta \rightarrow X : (u, \zeta) \in V_{\sigma^X}\}$  by the gluing construction that we used in the definition of the map  $j$ . Hence, the maps  $s_{\sigma^L}$  and  $\psi_{\sigma^L}$  can also be naturally identified with the maps  $s_{\sigma^X}$  and  $\psi_{\sigma^X}$  respectively.

This completes the proof of the proposition.  $\square$

[Theorem 1.1](#) now follows from [Propositions 4.1](#) and [4.6](#).

## 5. Applications to mirror symmetry

In this section, we apply [Theorem 1.1](#) to study mirror symmetry for the toric manifolds  $X = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$ . We shall first briefly review the constructions of the mirrors for toric manifolds, following [[Cho and Oh 2006](#); [Auroux 2007](#); [2009](#); [Fukaya et al. 2010a](#); [2011](#); [Chan and Leung 2010a](#); [2010b](#)].

As usual,  $N \cong \mathbb{Z}^n$  is a rank  $n$  lattice,  $M = \text{Hom}(N, \mathbb{Z})$  is the dual lattice and  $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$  is the dual pairing. Also let  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ ,  $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ , and denote by  $T_N$  and  $T_M$  the real tori  $N_{\mathbb{R}}/N$  and  $M_{\mathbb{R}}/M$  respectively.

Let  $X = X_\Delta$  be an  $n$ -dimensional smooth projective toric variety defined by a fan  $\Delta$  in  $N_{\mathbb{R}}$ . Let  $v_1, \dots, v_d$  be the primitive generators of the 1-dimensional cones in  $\Delta$ . We equip  $X$  with a toric Kähler structure  $\omega$ . Let  $P$  be the corresponding moment polytope and  $\mu : X \rightarrow P$  be the moment map.  $P$  is defined by a set of inequalities

$$P = \{x \in M_{\mathbb{R}} \mid \langle x, v_i \rangle \geq \lambda_i \text{ for } i = 1, \dots, d\},$$

for some  $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ . For  $i = 1, \dots, d$ , we let  $l_i : M_{\mathbb{R}} \rightarrow \mathbb{R}$  be the affine linear function defined by  $l_i(x) = \langle x, v_i \rangle - \lambda_i$ .

We are interested in the mirror symmetry for the Kähler manifold  $X$ , equipped with the toric Kähler structure  $\omega$  and the nowhere zero meromorphic  $n$ -form  $\Omega = d \log w_1 \wedge \dots \wedge d \log w_n$ , where  $w_1, \dots, w_n$  are the standard complex coordinates on the open dense orbit  $U = N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n \subset X$ . From the point of view of [[Auroux 2007](#)], we are looking at the mirror symmetry for  $X$  relative to the toric divisor  $D_\infty = \bigcup_{i=1}^d D_i = X \setminus U$ . As before,  $D_i$  is the toric prime divisor in  $X$

corresponding to  $v_i$ . The mirror geometry is given by a Landau–Ginzburg model  $(X^\vee, W)$  consisting of a bounded domain  $X^\vee \subset (\mathbb{C}^*)^n$  and a holomorphic function  $W : X^\vee \rightarrow \mathbb{C}$  called the mirror superpotential.

As discussed in [Auroux 2007; Chan and Leung 2010a; 2010b], the mirror manifold  $X^\vee$  can be obtained by dualizing Lagrangian torus fibrations (so-called T-duality) as follows. Restricting the moment map  $\mu : X \rightarrow P$  to the open dense orbit  $U \subset X$  gives a torus bundle  $\mu : U \rightarrow \text{Int}(P)$ , where  $\text{Int}(P)$  denotes the interior of the polytope  $P$ . In fact this bundle is trivial, so we have  $U = \text{Int}(P) \times \sqrt{-1} T_N$ . The mirror manifold  $X^\vee$  is given by the total space of the dual torus bundle, i.e.,

$$X^\vee = \text{Int}(P) \times \sqrt{-1} T_N^\vee = \text{Int}(P) \times \sqrt{-1} T_M.$$

$X^\vee$  comes with a natural Kähler structure. In particular, as a complex manifold,  $X^\vee$  is biholomorphic to a bounded domain in  $(\mathbb{C}^*)^n = M_{\mathbb{R}} \times \sqrt{-1} T_M$ . If  $y = (y_1, \dots, y_n) \in (\mathbb{R}/2\pi\mathbb{Z})^n$  are the fiber coordinates on  $T_M$  and the complex coordinates on  $(\mathbb{C}^*)^n$  are given by  $z_j = \exp(-x_j - \sqrt{-1} y_j)$ ,  $j = 1, \dots, n$ , where  $x = (x_1, \dots, x_n) \in \text{Int}(P)$ , then  $X^\vee \subset (\mathbb{C}^*)^n$  can be written as

$$X^\vee = \{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n : |e^{\lambda_i} z_i| < 1, i = 1, \dots, d\}.$$

Geometrically,  $X^\vee$  should be viewed as the moduli space of pairs  $(L, \nabla)$  consisting of a (special) Lagrangian torus fiber of the moment map  $\mu : X \rightarrow P$  together with a flat  $U(1)$ -connection  $\nabla$  on the trivial line bundle  $\underline{\mathbb{C}}$  over  $L$ . More precisely, to a point

$$z = (z_1 = \exp(x_1 + \sqrt{-1} y_1), \dots, z_n = \exp(x_n + \sqrt{-1} y_n)) \in X^\vee,$$

we associate the flat  $U(1)$ -connection  $\nabla_y = d + (\sqrt{-1}/2) \sum_{j=1}^n y_j du_j$  on the trivial line bundle  $\underline{\mathbb{C}}$  over the Lagrangian torus  $L_x = \mu^{-1}(x) \cong T_N$ , where  $u = (u_1, \dots, u_n) \in (\mathbb{R}/2\pi\mathbb{Z})^n$  are the fiber coordinates on  $T_N$ . This picture is motivated by the SYZ conjecture for mirror Calabi–Yau manifolds proposed by Strominger, Yau and Zaslow in [Strominger et al. 1996].

On the other hand, it turns out that the mirror superpotential  $W : X^\vee \rightarrow \mathbb{C}$  acts as the mirror of the obstruction  $m_0$  to the Floer homology of Lagrangian torus fibers in  $X$ .<sup>5</sup> As shown in [Fukaya et al. 2009],  $m_0$  comes from the virtual counting of Maslov index two holomorphic disks in  $X$  with boundary in the Lagrangian torus fibers  $L$ . This leads to the following expression for  $W$ : For  $\beta \in \pi_2(X, L)$ , we define a holomorphic function  $Z_\beta$  on  $X^\vee$  by

$$Z_\beta(L, \nabla) = \exp\left(-\frac{1}{2\pi} \int_\beta \omega\right) \text{hol}_\nabla(\partial\beta).$$

---

<sup>5</sup>Fukaya, Oh, Ohta and Ono call  $W$  the potential function and they define it over the Novikov ring  $\Lambda_0$  instead of  $\mathbb{C}$ ; see [Fukaya et al. 2009; 2010a; 2011].

Then the mirror superpotential  $W : X^\vee \rightarrow \mathbb{C}$  is given by the following holomorphic function

$$(5-1) \quad W(L, \nabla) = \sum_{\beta \in \pi_2(X, L), \mu(\beta)=2} c_\beta Z_\beta(L, \nabla),$$

assuming that the sum converges. See [Cho and Oh 2006; Auroux 2007; 2009; Fukaya et al. 2010a; 2011] for more details.

For  $i = 1, \dots, d$ , let  $\beta_i \in \pi_2(X, L)$  be the relative homotopy class such that  $D_j \cdot \beta_i = \delta_{ij}$ . Then, by the symplectic area formula of Cho and Oh (Theorem 8.1 in [Cho and Oh 2006]), we have

$$\int_{\beta_i} \omega = 2\pi l_i(x) = 2\pi(\langle x, v_i \rangle - \lambda_i),$$

where  $x \in \text{Int}(P)$  is the image of  $L$  under the moment map (i.e.,  $L = \mu^{-1}(x)$ ). Hence, for the basic classes  $\beta_i$ , the function  $Z_{\beta_i}$  is given in local coordinates by

$$Z_{\beta_i}(L_x, \nabla_y) = \exp(-l_i(x)) \exp(-\sqrt{-1}\langle y, v_i \rangle) = e^{\lambda_i} z^{v_i},$$

where  $z^v$  denotes the monomial  $z_1^{v_1} \dots z_n^{v_n}$ .

By Theorem 4.2, we have  $c_{\beta_i} = 1$  for  $i = 1, \dots, d$ . In particular, when  $X$  is Fano (i.e., the anticanonical bundle  $K_X^{-1}$  is ample),  $\beta_1, \dots, \beta_d \in \pi_2(X, L)$  are the only Maslov index two classes. Hence, the mirror superpotential is given explicitly by

$$W = Z_{\beta_1} + \dots + Z_{\beta_d} = e^{\lambda_1} z^{v_1} + \dots + e^{\lambda_d} z^{v_d}.$$

However, in the non-Fano cases, the invariants  $c_\beta$  and hence  $W$  are in general very hard to compute. The only non-Fano examples whose mirror superpotentials are explicitly computed are the Hirzebruch surfaces  $F_2$  and  $F_3$ , first computed in [Auroux 2009]. Later, Fukaya, Oh, Ohta and Ono gave a different proof for the  $F_2$  case in [Fukaya et al. 2010b].

Let's go back to our toric manifolds  $X = \mathbb{P}(K_Y \oplus \mathbb{C}_Y)$ . We want to compute their mirror superpotentials using Theorem 1.1.

**Lemma 5.1.** *If  $\beta \in \pi_2(X, L)$  is a Maslov index two class with  $c_\beta \neq 0$ , then  $\beta$  must either be one of  $\beta_1, \dots, \beta_m, \beta_{m+1}$  or of the form  $\beta_0 + \alpha$  for some effective class  $\alpha \in H_2(X, \mathbb{Z})$  with  $c_1(\alpha) = 0$ .*

*Proof.* First of all, since  $X$  is semi-Fano,  $c_1(\alpha) \geq 0$  for any effective class  $\alpha \in H_2(X, \mathbb{Z})$ . Hence, if  $\beta \in \pi_2(X, L)$  is a Maslov index two class, then it must be of the form  $\beta_i + \alpha$  for some  $i = 0, 1, \dots, m, m + 1$  and some effective class  $\alpha \in H_2(X, \mathbb{Z})$  with  $c_1(\alpha) = 0$ . Let  $\varphi : (\Sigma^L, \partial \Sigma^L) \rightarrow (X, L)$  be a stable map from a genus 0 bordered Riemann surface  $(\Sigma^L, \partial \Sigma^L)$  to  $(X, L)$  representing the class  $\beta_i + \alpha$ . Suppose that  $\alpha \neq 0$ . Then, by the proof of Proposition 4.3, we can decompose



$\Sigma^L$  into  $\Sigma_0^L \cup \Sigma_1$ , where  $\Sigma_0^L = D^2$  is a disk and  $\Sigma_1$  is a genus 0 nodal curve, such that the restrictions  $\varphi_0 := \varphi|_{\Sigma_0^L}$  and  $\varphi_1 := \varphi|_{\Sigma_1}$  represent the classes  $\beta_i$  and  $\alpha$  respectively. However, by Lemma 3.2 (2), the image of  $\varphi_1$  must be contained entirely in the toric prime divisor  $D_0$ . Since  $\varphi(\Sigma_0^L) \cdot D_0 = \delta_{0i}$  and the domain of  $\varphi$  is connected, we must have  $i = 0$ . Hence  $c_{\beta_i + \alpha} = 0$  unless  $i = 0$  or  $\alpha = 0$ .  $\square$

**Theorem 5.2.** For the  $\mathbb{P}^1$ -bundle  $X = \mathbb{P}(K_Y \oplus \mathbb{O}_Y)$  over a toric Fano manifold  $Y$ , the mirror superpotential  $W : X^\vee \rightarrow \mathbb{C}$  is given by

$$W = CZ_{\beta_0} + Z_{\beta_1} + \cdots + Z_{\beta_m} + Z_{\beta_{m+1}},$$

where

$$C = 1 + \sum_{\substack{\alpha \in H_2^{\text{eff}}(X, \mathbb{Z}), \\ \alpha \neq 0, c_1(\alpha) = 0}} \text{GW}_{0,1}^{X, h+\alpha}([\text{pt}]) q^\alpha,$$

and  $q^\alpha = \exp(-\frac{1}{2\pi} \int_\alpha \omega)$ .

*Proof.* This is a consequence of formula (5-1), Lemma 5.1 and Theorem 1.1.  $\square$

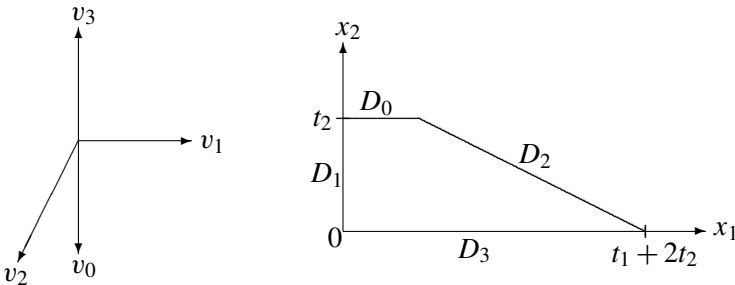
**Example: The Hirzebruch surface  $F_2$ .** Consider  $X = F_2 = \mathbb{P}(K_{\mathbb{P}^1} \oplus \mathbb{O}_{\mathbb{P}^1})$ . We choose the primitive generators of the 1-dimensional cones in the fan  $\Delta$  defining  $F_2$  to be<sup>6</sup>

$$v_0 = (0, -1), \quad v_1 = (1, 0), \quad v_2 = (-1, -2), \quad v_3 = (0, 1)$$

in  $N = \mathbb{Z}^2$ . We equip  $F_2$  with a toric Kähler structure so that moment polytope  $P$  is given by

$$P = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_2 \leq t_2, x_1 + 2x_2 \leq t_1 + 2t_2\},$$

where  $t_1, t_2 > 0$ . Here is a depiction of the fan  $\Delta$  defining  $F_2$  (left) and its moment polytope  $P$  (right).



<sup>6</sup>This choice of generators is different from the one in Section 3. This does not alter any of our results. We make this choice just to make our notation consistent with that of [Auroux 2009; Fukaya et al. 2010b].

The effective cone  $H_2^{\text{eff}}(F_2, \mathbb{Z})$  is generated by two primitive relations

$$v_0 + v_3 = 0 \quad \text{and} \quad v_1 + v_2 - 2v_0 = 0.$$

Let  $h := (1, 0, 0, 1)$ ,  $\alpha := (-2, 1, 1, 0) \in H_2^{\text{eff}}(F_2, \mathbb{Z})$  be the corresponding homology classes, which represent the fiber and the base of  $F_2$  respectively. Then

$$t_1 = \int_{\alpha} \omega_X \quad \text{and} \quad t_2 = \int_h \omega_X.$$

Let  $q_i = \exp(-t_i)$  for  $i = 1, 2$ . We also have  $c_1(h) = 2$  and  $c_1(\alpha) = 0$ .

Now, the mirror manifold  $X^\vee$  is a bounded domain in  $(\mathbb{C}^*)^2$ . By [Theorem 5.2](#), the mirror superpotential  $W : X^\vee \rightarrow \mathbb{C}$  is given by

$$W = CZ_{\beta_0} + Z_{\beta_1} + Z_{\beta_2} + Z_{\beta_3} = C \frac{q_2}{z_2} + z_1 + \frac{q_1 q_2^2}{z_1 z_2^2} + z_2,$$

where

$$C = \sum_{k=0}^{\infty} \text{GW}_{0,1}^{F_2, h+k\alpha}(\text{PD}[\text{pt}]) q_1^k,$$

and  $z_1, z_2$  are the standard coordinates on  $(\mathbb{C}^*)^2$ .  $F_2$  is symplectomorphic to  $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$  with induced isomorphism on degree-2 homology given by

$$H_2(F_2, \mathbb{Z}) \rightarrow H_2(F_0, \mathbb{Z}) : \quad \alpha \mapsto l_1 - l_2 \quad \text{and} \quad h \mapsto l_2,$$

where  $l_1, l_2 \in H_2(F_0, \mathbb{Z})$  are the line classes in the two  $\mathbb{P}^1$  factors. Since Gromov–Witten invariants are symplectic invariants, the Gromov–Witten invariants of  $F_2$  are all equal to those of  $F_0$ . So we have

$$\text{GW}_{0,1}^{F_2, h+k\alpha}(\text{PD}[\text{pt}]) = \text{GW}_{0,1}^{F_0, kl_1 + (1-k)l_2}(\text{PD}[\text{pt}]) = \begin{cases} 1, & \text{if } k = 0 \text{ or } k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence,  $c_{\beta_0+k\alpha} = 0$  for  $k \geq 2$  and  $c_{\beta_0+\alpha} = c_{\beta_0} = 1$ . We conclude that  $C = 1 + q_1$  and the mirror superpotential is given by

$$W = z_1 + z_2 + \frac{q_1 q_2^2}{z_1 z_2^2} + \frac{q_2 + q_1 q_2}{z_2}.$$

This agrees with the formula in [Proposition 3.1](#) in [\[Auroux 2007\]](#). □

The formula in [Theorem 1.1](#) has been applied to investigate mirror symmetry for various classes of toric manifolds. In [\[Lau et al. 2011\]](#), the formula was generalized and used to compute open Gromov–Witten invariants for toric Calabi–Yau 3-folds. In [\[Chan et al. 2010\]](#) and [\[Lau et al. 2010\]](#), the formula and its generalization in [\[Lau et al. 2011\]](#) were used to obtain an enumerative meaning for the (inverse) mirror maps for toric Calabi–Yau 2- and 3-folds. In particular, this explains why we always get integral coefficients for the Taylor expansions of these mirror maps.

In [Chan and Lau 2010], the formula was used to compute mirror superpotentials for all semi-Fano toric surfaces.

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