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EIGENVALUE ESTIMATES ON DOMAINS IN COMPLETE NONCOMPACT RIEMANNIAN MANIFOLDS

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In this paper, we obtain universal inequalities for eigenvalues of the Dirichlet eigenvalue problem of the Laplacian and the clamped plate problem on a bounded domain in an n -dimensional ($n \geq 3$) noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-K^2 \leq \text{Sec} \leq -k^2$, where $K \geq k \geq 0$ are constants. When M is $\mathbb{H}^n(-1)$ ($n \geq 3$), these inequalities become ones previously found by Cheng and Yang.

1. Introduction

Let M be an n -dimensional complete Riemannian manifold and $\Omega \subset M$ a bounded domain in M . The Dirichlet eigenvalue problem of the Laplacian is

$$(1-1) \quad \begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty,$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

A Dirichlet eigenvalue problem of the biharmonic operator or a clamped plate problem that describes the characteristic vibrations of a clamped plate is given by

$$(1-2) \quad \begin{cases} \Delta^2 u = \Gamma u & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator on M and ν denotes the outward normal derivative on $\partial\Omega$. We will denote eigenvalues and the corresponding real eigenfunctions by $\{\Gamma_i\}_{i=1}^\infty$ and $\{u_i\}_{i=1}^\infty$, respectively. The eigenvalues Γ_i satisfy

$$0 < \Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq \cdots \nearrow \infty.$$

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When M is a Euclidean space \mathbb{R}^n , these are estimates for the eigenvalues (1-1) that do not involve domain dependencies [Protter 1988]; see also [Ashbaugh 1999; 2002]. The main developments were obtained by Payne, Pólya, and Weinberger [Payne et al. 1956], Hile and Protter [1980], and Yang [1991]. More recently, for the Dirichlet eigenvalue problems of the Laplacian on a bounded domain in the n -dimensional unit sphere, complex projective space, and compact homogeneous Riemannian manifolds, Cheng and Yang [2005; 2006b; 2007] obtained the Yang-type inequalities for eigenvalues. For a bounded domain Ω in a complete Riemannian manifold M , the first author and Cheng [Chen and Cheng 2008] proved a Yang-type inequality by using the Nash embedding theorem (compare [El Soufi et al. 2009; Harrell 2007]).

By making use of estimates for eigenvalues of the eigenvalue problem of the Schrödinger like operator with a weight, Harrell and Michel [1994], Ashbaugh [2002], and Ashbaugh and Hermi [2007] have obtained several results. In fact, for $n = 2$, the Laplacian on $\mathbb{H}^2(-1)$ is like to the Laplacian on \mathbb{R}^2 with a weight. However, for $n > 2$, this property does not hold. Cheng and Yang [2009] found appropriate trial functions and obtained

$$(1-3) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \left(\lambda_i - \frac{(n-1)^2}{4} \right).$$

In this paper, we first treat the Dirichlet eigenvalue problem (1-1) of the Laplacian on a bounded domain of a complete noncompact Riemannian manifold M .

Theorem 1.1. *Assume that M^n ($n \geq 3$) is a noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-K^2 \leq \text{Sec} \leq -k^2$, where $K \geq k \geq 0$ are constants. For a bounded domain Ω in M , let λ_i be the i -th eigenvalue of the eigenvalue problem (1-1). Then we obtain*

$$(1-4) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (4\lambda_i - (n-1)^2 k^2 + 2(n-1)(K^2 - k^2)).$$

Remark. If $k = K = 1$, that is, M is a hyperbolic space $\mathbb{H}^n(-1)$, the eigenvalue inequality (1-4) agrees with (1-3) obtained by Cheng and Yang [2009].

The other purpose of this paper is to investigate estimates for eigenvalues of the clamped plate problem (1-2) on bounded domains Ω in a complete Riemannian manifold M^n .

For the universal inequalities for eigenvalues of the clamped plate problem in a bounded domain in \mathbb{R}^n , Payne et al. [1955; 1956] proved that

$$(1-5) \quad \Gamma_{k+1} - \Gamma_k \leq \frac{8(n+2)}{n^2 k} \sum_{i=1}^k \Gamma_i, \quad k = 1, 2, \dots$$

Hile and Yeh [1984] obtained

$$(1-6) \quad \sum_{i=1}^k \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1} - \Gamma_i} \geq \frac{n^2 k^{3/2}}{8(n+2)} \left(\sum_{i=1}^k \Gamma_i \right)^{-\frac{1}{2}}, \quad k = 1, 2, \dots$$

Hook [1990] and Chen and Qian [1990] independently proved

$$(1-7) \quad \frac{n^2 k^2}{8(n+2)} \leq \left(\sum_{i=1}^k \frac{\Gamma_i^{\frac{1}{2}}}{\Gamma_{k+1} - \Gamma_i} \right) \left(\sum_{i=1}^k \Gamma_i^{\frac{1}{2}} \right), \quad k = 1, 2, \dots$$

Cheng and Yang [2006a] gave an affirmative answer for a problem on universal inequalities for eigenvalues proposed by Ashbaugh [1999]: they proved that

$$(1-8) \quad \Gamma_{k+1} - \frac{1}{k} \sum_{i=1}^k \Gamma_i \leq \left(\frac{8(n+2)}{n^2} \right)^{\frac{1}{2}} \frac{1}{k} \left(\sum_{i=1}^k \Gamma_i (\Gamma_{k+1} - \Gamma_i) \right)^{\frac{1}{2}}, \quad k = 1, 2, \dots$$

For domains in a unit sphere, Wang and Xia [2007] gave a universal inequality for the clamped plate problem (1-2). They proved

$$(1-9) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{2n+4} \right) \left(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{4} \right).$$

For an n -dimensional complete manifold M , Cheng, Ichikawa, and Mametsuka [Cheng et al. 2010] obtained

$$(1-10) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq \frac{8(n+2)}{n^2} \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{2n+4} \sup_{\Omega} |H|^2 \right) \left(\Gamma_i^{\frac{1}{2}} + \frac{n^2}{4} \sup_{\Omega} |H|^2 \right).$$

For the real hyperbolic space $\mathbb{H}^n(-1)$, Cheng and Yang [2011] proved that

$$(1-11) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{4} \right) \left(\Gamma_i^{\frac{1}{2}} - \frac{(n-1)^2}{6} \right).$$

That paper motivated the present one, where we treat the clamped plate problem on a bounded domain of a noncompact simply connected complete Riemannian manifold M^n .

Theorem 1.2. *Assume that M^n ($n \geq 3$) is a noncompact simply connected complete Riemannian manifold with sectional curvature Sec satisfying $-K^2 \leq \text{Sec} \leq -k^2$, where $K \geq k \geq 0$ are constants. For a bounded domain Ω in M , let Γ_i be the i -th*

eigenvalue of the eigenvalue problem (1-2). Then we have

$$(1-12) \quad \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{4} ((n-1)k^2 - 2(K^2 - k^2)) \right) \\ \times \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6} ((n-1)k^2 - 2(K^2 - k^2)) \right).$$

Remark. If $k = K = 1$, that is, M^n is a hyperbolic space $\mathbb{H}^n(-1)$, then the eigenvalue inequality (1-12) agrees with (1-11) obtained by Cheng and Yang. Wang and Xia [2011] generalized (1-11) under the assumption that there exists some function whose norm of gradient is 1 and whose Laplacian is a constant.

From Theorem 1.2, we can immediately obtain the following.

Corollary 1.3. *Let Γ_i be the i -th eigenvalue of the eigenvalue problem (1-2). Then we have*

$$\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq 24 \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i - \frac{(n-1)^2}{16} ((n-1)k^2 - 2(K^2 - k^2))^2 \right).$$

2. Preliminaries

Let B and C be $(n-1) \times (n-1)$ real symmetrical matrixes. If all the eigenvalues of B are equal or greater than all the ones of C , then we write $B \succ C$.

Let (M, g) be an n -dimensional Riemannian manifold and D the Riemannian connection. The curvature tensor is a $(1,3)$ -tensor defined by

$$(2-1) \quad R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z$$

for all $X, Y, Z \in \Gamma(TM)$. Let $\gamma : [0, b) \rightarrow M$ be the minimal normal geodesic and $\{e_i(t)\}_{i=1}^n$ parallel orthonormal frame fields along $\gamma(t)$ such that $e_n(t) = \dot{\gamma}(t)$. Let

$$J_i(t) = \sum_{j=1}^{n-1} f_{ij}(t) e_j(t), \quad i = 1, \dots, n-1,$$

be the normal Jacobi fields along the geodesic $\gamma(t)$; that is

$$(2-2) \quad \ddot{f}_{ij} - f_{il} R_{nljt} = 0, \quad f_{ij}(0) = 0, \quad \dot{f}_{ij}(0) = \delta_{ij},$$

where

$$\dot{f}_{ij} = \frac{d}{dt} f_{ij}(t), \quad \ddot{f}_{ij} = \frac{d^2}{dt^2} f_{ij}(t), \quad R_{nljt} = g(R(e_n, e_l)e_n, e_j) = R_{nljn}.$$

Set

$$f(t) = (f_{ij}(t))_{(n-1) \times (n-1)}, \quad K(t) = (R_{nljn}(\gamma(t)))_{(n-1) \times (n-1)},$$

where $f_{ij}(t)$ is on column j and row i . Then (2-2) can be written as

$$(2-3) \quad \begin{cases} \ddot{f}(t) - f(t)K(t) = 0, & 0 < t < b, \\ f(0) = 0, \\ \dot{f}(0) = I_{n-1}, \end{cases}$$

where I_{n-1} is the $(n-1) \times (n-1)$ unit matrix.

Define the distance function $r(x) = \text{distance}(x, \gamma(0))$. Then

$$(2-4) \quad \text{Hess } r(\gamma(t)) = f(t)^{-1} \dot{f}(t), \quad \Delta r(\gamma(t)) = \text{tr}(f(t)^{-1} \dot{f}(t)).$$

Assume that Ω is a bounded domain in an n -dimensional noncompact simply connected complete Riemannian manifold (M, g) with section curvature Sec satisfying $-K^2 \leq \text{Sec} \leq -k^2$, where $0 \leq k \leq K$ are constants. For $p \notin \bar{\Omega}$ fixed, define the distance function $r(x) = \text{distance}(x, p)$. Then from the Hessian comparison theorem (cf. [Wu et al. 1989]), we have

$$(2-5) \quad K \frac{\cosh Kr}{\sinh Kr} I_{n-1} \succ \text{Hess } r \succ k \frac{\cosh kr}{\sinh kr} I_{n-1}.$$

From (2-4) and (2-5), we have

$$(2-6) \quad (n-1)k \frac{\cosh kr}{\sinh kr} \leq \Delta r \leq (n-1)K \frac{\cosh Kr}{\sinh Kr}.$$

Since $\partial_r \Delta r = -|\text{Hess } r|^2 - \text{Ric}(\partial_r, \partial_r)$ (cf. [Petersen 1998]), we have

$$(2-7) \quad -\partial_r \Delta r \leq (n-1)K^2 \frac{\cosh^2 Kr}{\sinh^2 Kr} - (n-1)k^2.$$

3. Proof of Theorem 1.1

Theorem 3.1 [Cheng and Yang 2006b]. *Let λ_i be the i -th eigenvalue of the above eigenvalue problem (1-1) and u_i the orthonormal eigenfunction corresponding to λ_i ; that is, u_i satisfies*

$$\begin{cases} u_i = -\lambda_i u_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} u_i u_j = \delta_{ij} & \text{for all } i, j = 1, 2, \dots \end{cases}$$

Then for any $f \in C^3(\Omega) \cap C^2(\partial\Omega)$, we have

$$(3-1) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} |\nabla f|^2 u_i^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \int_{\Omega} (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2.$$

Proof of Theorem 1.1. Taking $f = r$ in the formula (3-1), we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \int_{\Omega} |\nabla r|^2 u_i^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \int_{\Omega} (2\nabla r \cdot \nabla u_i + u_i \Delta r)^2.$$

Since $|\nabla r| = 1$, we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \int_{\Omega} (2\nabla r \cdot \nabla u_i + u_i \Delta r)^2.$$

From (2-6) and (2-7), we obtain

$$\begin{aligned} (3-2) \quad & \int_{\Omega} \left(2\nabla r \cdot \nabla u_i + u_i \Delta r \right)^2 \\ &= 4 \int_{\Omega} (\nabla r \cdot \nabla u_i)^2 + 4 \int_{\Omega} u_i \Delta r \nabla r \cdot \nabla u_i + \int_{\Omega} (u_i \Delta r)^2 \\ &\leq 4 \int_{\Omega} |\nabla u_i|^2 - \int_{\Omega} u_i^2 (\Delta r)^2 - 2 \int_{\Omega} u_i^2 \nabla r \cdot \nabla \Delta r \\ &= 4 \int_{\Omega} |\nabla u_i|^2 - \int_{\Omega} u_i^2 (\Delta r)^2 - 2 \int_{\Omega} u_i^2 \partial_r \Delta r \\ &= 4 \int_{\Omega} |\nabla u_i|^2 - \int_{\Omega} u_i^2 (\Delta r)^2 + 2 \int_{\Omega} u_i^2 (\text{Ric}(\partial_r, \partial_r) + |\text{Hess } r|^2) \\ &\leq 4\lambda_i - (n-1)^2 k^2 \int_{\Omega} u_i^2 \frac{\cosh^2 kr}{\sinh^2 kr} \\ &\quad - 2(n-1)k^2 + 2(n-1)K^2 \int_{\Omega} u_i^2 \frac{\cosh^2 Kr}{\sinh^2 Kr} \\ &= 4\lambda_i - (n-1)^2 k^2 + 2(n-1)(K^2 - k^2) \\ &\quad - (n-1)^2 \int_{\Omega} \frac{k^2}{\sinh^2 kr} u_i^2 + 2(n-1) \int_{\Omega} \frac{K^2}{\sinh^2 Kr} u_i^2. \end{aligned}$$

Since $K \geq k \geq 0$ and $r > 0$, we have

$$(3-3) \quad \frac{K}{\sinh Kr} \leq \frac{k}{\sinh kr}.$$

Since $n \geq 3$, we have

$$(3-4) \quad (n-1)^2 \frac{k^2}{\sinh^2 kr} - 2(n-1) \frac{K^2}{\sinh^2 Kr} \geq (n-1)(n-3) \frac{k^2}{\sinh^2 kr} \geq 0.$$

Finally, we have

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) (4\lambda_i - (n-1)^2 k^2 + 2(n-1)(K^2 - k^2)). \quad \square$$

4. Proof of Theorem 1.2

Let u_i be the i -th orthonormal eigenfunction corresponding to the eigenvalue Γ_i , $i = 1, \dots, k$; that is,

$$(4-1) \quad \begin{cases} \Delta^2 u_i = \Gamma_i u_i & \text{in } \Omega, \\ u_i = \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} u_i u_j = \delta_{ij} & \text{for any } i, j. \end{cases}$$

Defining the functions

$$\phi_i = r u_i - \sum_{j=1}^k a_{ij} u_j,$$

where

$$a_{ij} = \int_{\Omega} r u_i u_j,$$

we have

$$(4-2) \quad \phi_i|_{\partial\Omega} = \frac{\partial \phi_i}{\partial \nu}|_{\partial\Omega} = 0 \quad \text{and} \quad \int_{\Omega} \phi_i u_j = 0 \quad \text{for all } i, j = 1, \dots, k.$$

Therefore, we know that ϕ_i s are trial functions. From the Rayleigh–Ritz inequality [Chavel 1984], we have

$$(4-3) \quad \Gamma_{k+1} \leq \frac{1}{\|\phi_i\|^2} \int_{\Omega} (\Delta \phi_i)^2,$$

where

$$\|\phi_i\|^2 = \int_{\Omega} \phi_i^2.$$

From (4-1) and (4-2), we have

$$\begin{aligned} \Gamma_{k+1} \int_{\Omega} \phi_i^2 &\leq \int_{\Omega} (\Delta \phi_i)^2 = \int_{\Omega} \phi_i \Delta^2 \phi_i = \int_{\Omega} \phi_i \Delta^2 \left(r u_i - \sum_{j=1}^k a_{ij} u_j \right) \\ &= \int_{\Omega} \phi_i \Delta^2 (r u_i) = \int_{\Omega} \phi_i (\Delta^2 (r u_i) - \Gamma_i r u_i) + \Gamma_i \int_{\Omega} \phi_i^2, \end{aligned}$$

that is,

$$(\Gamma_{k+1} - \Gamma_i) \|\phi_i\|^2 \leq \int_{\Omega} \phi_i (\Delta^2 (r u_i) - \Gamma_i r u_i).$$

From the definition of ϕ_i and (4-2), we have

$$\begin{aligned}
(4-4) \quad & (\Gamma_{k+1} - \Gamma_i) \|\phi_i\|^2 \\
& \leq \int_{\Omega} (ru_i - \sum_{j=1}^k a_{ij} u_j) (\Delta^2(ru_i) - \Gamma_i ru_i) \\
& = \int_{\Omega} ru_i (\Delta^2(ru_i) - \Gamma_i ru_i) + \sum_{j=1}^k a_{ij}^2 (\Gamma_i - \Gamma_j) \\
& = \int_{\Omega} ru_i (\Delta(u_i \Delta r) + 2\Delta(\nabla r \cdot \nabla u_i) + 2\nabla r \cdot \nabla \Delta u_i + \Delta r \Delta u_i) \\
& \qquad \qquad \qquad + \sum_{j=1}^k a_{ij}^2 (\Gamma_i - \Gamma_j).
\end{aligned}$$

From (2-6), (2-7), and Stokes' theorem, by a direct calculation, we have

$$\begin{aligned}
(4-5) \quad & \int_{\Omega} ru_i (\Delta(u_i \Delta r) + 2\Delta(\nabla r \cdot \nabla u_i) + 2\nabla r \cdot \nabla \Delta u_i + \Delta r \Delta u_i) \\
& = \int_{\Omega} (\Delta(ru_i) (u_i \Delta r + 2\nabla r \cdot \nabla u_i) + u_i \nabla r^2 \cdot \nabla \Delta u_i + u_i r \Delta r \Delta u_i) \\
& = \int_{\Omega} ((u_i \Delta r + 2\nabla r \cdot \nabla u_i + r \Delta u_i) (u_i \Delta r + 2\nabla r \cdot \nabla u_i) \\
& \qquad \qquad \qquad + u_i \nabla r^2 \cdot \nabla \Delta u_i + u_i r \Delta r \Delta u_i) \\
& = \int_{\Omega} ((\Delta r)^2 u_i^2 + 2\nabla r \cdot \nabla u_i^2 \Delta r + 4(\nabla r \cdot \nabla u_i)^2 + 2ru_i \Delta r \Delta u_i + \nabla r^2 \cdot \nabla u_i \Delta u_i) \\
& \qquad \qquad \qquad + \int_{\Omega} u_i \nabla r^2 \cdot \nabla \Delta u_i \\
& = \int_{\Omega} ((\Delta r)^2 u_i^2 + 2\nabla r \cdot \nabla u_i^2 \Delta r + 4(\nabla r \cdot \nabla u_i)^2 + 2ru_i \Delta r \Delta u_i + \nabla r^2 \cdot \nabla (u_i \Delta u_i)) \\
& = \int_{\Omega} ((\Delta r)^2 u_i^2 + 2\nabla r \cdot \nabla u_i^2 \Delta r + 4(\nabla r \cdot \nabla u_i)^2 + (2r \Delta r - \Delta r^2) u_i \Delta u_i) \\
& = \int_{\Omega} (-(\Delta r)^2 u_i^2 - 2u_i^2 \nabla r \cdot \nabla \Delta r + 4(\nabla r \cdot \nabla u_i)^2 - 2u_i \Delta u_i) \\
& \leq \int_{\Omega} (4|\nabla u_i|^2 - 2u_i \Delta u_i) - \int_{\Omega} u_i^2 (2\nabla r \cdot \nabla \Delta r + (\Delta r)^2) \\
& \leq \int_{\Omega} u_i^2 \left(-(n-1)^2 k^2 \frac{\cosh^2 kr}{\sinh^2 kr} - 2(n-1)k^2 + 2(n-1)K^2 \frac{\cosh^2 Kr}{\sinh^2 Kr} \right) \\
& \qquad \qquad \qquad + \int_{\Omega} (4|\nabla u_i|^2 + 2u_i (-\Delta u_i))
\end{aligned}$$

Since $n \geq 3$, from (3-4), we have

$$\begin{aligned}
(4-6) \quad & \int_{\Omega} ru_i(\Delta(u_i \Delta r) + 2\Delta(\nabla r \cdot \nabla u_i) + 2\nabla r \cdot \nabla \Delta u_i + \Delta r \Delta u_i) \\
& \leq 6 \int_{\Omega} u_i(-\Delta u_i) - \int_{\Omega} u_i^2((n-1)^2 k^2 - 2(n-1)(K^2 - k^2)) \\
& \leq 6 \left(\int_{\Omega} (\Delta u_i)^2 \right)^{\frac{1}{2}} - (n-1)((n-1)k^2 - 2(K^2 - k^2)) \\
& = 6 \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6}((n-1)k^2 - 2(K^2 - k^2)) \right).
\end{aligned}$$

From (4-4) and (4-6), we deduce

$$\begin{aligned}
(4-7) \quad & (\Gamma_{k+1} - \Gamma_i) \|\phi_i\|^2 \\
& \leq 6 \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6}((n-1)k^2 - 2(K^2 - k^2)) \right) + \sum_{j=1}^k a_{ij}^2 (\Gamma_i - \Gamma_j).
\end{aligned}$$

Defining

$$b_{ij} = \int_{\Omega} (\nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r) u_j,$$

we have

$$b_{ij} = -b_{ji}.$$

From the definitions of b_{ij} and ϕ_i , we obtain

$$\begin{aligned}
(4-8) \quad & -2 \int_{\Omega} \phi_i (\nabla r \cdot \nabla u_i + \frac{1}{2} \Delta r u_i) \\
& = -2 \int_{\Omega} \left(ru_i - \sum_{j=1}^k a_{ij} u_j \right) (\nabla r \cdot \nabla u_i + \frac{1}{2} \Delta r u_i) \\
& = -2 \int_{\Omega} ru_i (\nabla r \cdot \nabla u_i + \frac{1}{2} \Delta r u_i) + 2 \sum_{j=1}^k a_{ij} b_{ij} \\
& = - \int_{\Omega} \left(\frac{1}{2} \nabla r^2 \cdot \nabla u_i^2 + r \Delta r u_i^2 \right) + 2 \sum_{j=1}^k a_{ij} b_{ij} \\
& = 1 + 2 \sum_{j=1}^k a_{ij} b_{ij}.
\end{aligned}$$

From (4-2), (4-8), and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
(4-9) \quad 1 + 2 \sum_{j=1}^k a_{ij} b_{ij} &= -2 \int_{\Omega} \phi_i (\nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r) \\
&= -2 \int_{\Omega} \phi_i \left(\nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r - \sum_{j=1}^k b_{ij} u_j \right) \\
&\leq \alpha_i \|\phi_i\|^2 + \frac{1}{\alpha_i} \left\| \nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r - \sum_{j=1}^k b_{ij} u_j \right\|^2 \\
&= \alpha_i \|\phi_i\|^2 + \frac{1}{\alpha_i} \left(\|\nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r\|^2 - \sum_{j=1}^k b_{ij}^2 \right),
\end{aligned}$$

where $\alpha_i > 0$ is a positive constant.

If $\Gamma_{k+1} - \Gamma_i > 0$, defining

$$\alpha_i = (\Gamma_{k+1} - \Gamma_i) \beta_i \quad \text{for } \beta_i > 0,$$

we infer that

$$\begin{aligned}
(4-10) \quad (\Gamma_{k+1} - \Gamma_i)^2 \left(1 + 2 \sum_{j=1}^k a_{ij} b_{ij} \right) \\
\leq (\Gamma_{k+1} - \Gamma_i)^3 \beta_i \|\phi_i\|^2 \\
+ m \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) \left(\|\nabla r \cdot \nabla u_i + \frac{1}{2} u_i \Delta r\|^2 - \sum_{j=1}^k b_{ij}^2 \right).
\end{aligned}$$

From (2-6), (2-7), and (3-4), we obtain

$$\begin{aligned}
(4-11) \quad \int_{\Omega} (2\nabla r \cdot \nabla u_i + u_i \Delta r)^2 \\
= 4 \int_{\Omega} (\nabla r \cdot \nabla u_i)^2 + 4 \int_{\Omega} u_i \Delta r \nabla r \cdot \nabla u_i + \int_{\Omega} (u_i \Delta r)^2 \\
\leq 4 \int_{\Omega} |\nabla u_i|^2 - \int_{\Omega} u_i^2 (\Delta r)^2 - 2 \int_{\Omega} u_i^2 \nabla r \cdot \nabla \Delta r \\
\leq 4\Gamma_i^{\frac{1}{2}} - (n-1)((n-1)k^2 - 2(K^2 - k^2)) \\
\leq 4\left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{4}((n-1)k^2 - 2(K^2 - k^2))\right).
\end{aligned}$$

Therefore, from (4-7), (4-9), (4-10), and (4-11), we obtain

$$\begin{aligned}
(4-12) \quad & (\Gamma_{k+1} - \Gamma_i)^2 \left(1 + 2 \sum_{j=1}^k a_{ij} b_{ij} \right) \\
& \leq (\Gamma_{k+1} - \Gamma_i)^2 \beta_i \left(6 \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6} ((n-1)k^2 - 2(K^2 - k^2)) \right) + \sum_{j=1}^k a_{ij}^2 (\Gamma_i - \Gamma_j) \right) \\
& \quad + \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{4} ((n-1)k^2 - 2(K^2 - k^2)) \right) \\
& \quad - \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) \sum_{j=1}^k b_{ij}^2.
\end{aligned}$$

From the antisymmetry of b_{ij} and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
2 \sum_{i,j=1}^k (\Gamma_{k+1} - \Gamma_i)^2 a_{ij} b_{ij} \\
- \sum_{i,j=1}^k (\Gamma_{k+1} - \Gamma_i) (\Gamma_i - \Gamma_j)^2 \beta_i a_{ij}^2 - \sum_{i,j=1}^k \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) b_{ij}^2 \leq 0.
\end{aligned}$$

From the above inequality and (4-12), we obtain

$$\begin{aligned}
(4-13) \quad & \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \\
& \leq 6 \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \beta_i \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6} ((n-1)k^2 - 2(K^2 - k^2)) \right) \\
& \quad + \sum_{i=1}^k \frac{1}{\beta_i} (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{4} ((n-1)k^2 - 2(K^2 - k^2)) \right) \\
& \quad + \sum_{i,j=1}^k (\Gamma_{k+1} - \Gamma_i) (\Gamma_{k+1} - \Gamma_j) (\Gamma_i - \Gamma_j) \beta_i a_{ij}^2.
\end{aligned}$$

From the variational principle, we can prove that

$$\Gamma_i \geq \lambda_i^2,$$

where λ_i denotes the i -th eigenvalue of the Dirichlet eigenvalue problem of the Laplacian on the same domain Ω . Since $4\lambda_1 \geq (n-1)^2 k^2 - 2(K^2 - k^2)$ from (3-2), setting

$$\beta_i = \beta \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6} ((n-1)k^2 - 2(K^2 - k^2)) \right)^{-1} \quad \text{for } \beta > 0$$

gives us

$$\begin{aligned}
& \sum_{i,j=1}^k (\Gamma_{k+1} - \Gamma_i)(\Gamma_{k+1} - \Gamma_j)(\Gamma_i - \Gamma_j)\beta_i a_{ij}^2 \\
&= \frac{1}{2} \sum_{i,j=1}^k (\Gamma_{k+1} - \Gamma_i)(\Gamma_{k+1} - \Gamma_j)(\Gamma_i - \Gamma_j)(\beta_i - \beta_j) a_{ij}^2 \\
&= -\frac{1}{2}\beta \sum_{i,j=1}^k \frac{(\Gamma_{k+1} - \Gamma_i)(\Gamma_{k+1} - \Gamma_j)(\Gamma_i - \Gamma_j)(\Gamma_i^{\frac{1}{2}} - \Gamma_j^{\frac{1}{2}})}{(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6}((n-1)k^2 - 2(K^2 - k^2)))} a_{ij}^2 \\
&\quad \times (\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6}((n-1)k^2 - 2(K^2 - k^2))) \\
&\leq 0.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 &\leq 6\beta \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \\
&+ \frac{1}{\beta} \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{4}((n-1)k^2 - 2(K^2 - k^2)) \right) \\
&\quad \times \left(\Gamma_i^{\frac{1}{2}} - \frac{n-1}{6}((n-1)k^2 - 2(K^2 - k^2)) \right).
\end{aligned}$$

Finally, taking $\beta = \frac{1}{12}$, we deduce (1-12). This completes the proof of Theorem 1.2.

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