Pacific Journal of Mathematics

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Volume 255 No. 1 January 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 255, No. 1, 2012

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From the Riemannian geometric point of view, one of the most fundamental problems in the study of Lagrangian submanifolds is the classification of Lagrangian submanifolds with parallel second fundamental form. In 1980's, H. Naitoh completely classified the Lagrangian submanifolds with parallel second fundamental form and without Euclidean factor in complex projective space, by using the theory of Lie groups and symmetric spaces. He showed that such a submanifold is always locally symmetric and is one of the symmetric spaces: $\mathrm{SO}(k+1)/\mathrm{SO}(k)$ $(k \geq 2)$, $\mathrm{SU}(k)/\mathrm{SO}(k)$ $(k \geq 3)$, $\mathrm{SU}(k)$ $(k \geq 3)$, $\mathrm{SU}(2k)/\mathrm{Sp}(k)$ $(k \geq 3)$, $\mathrm{E}_6/\mathrm{F}_4$.

In this paper, we completely classify the Lagrangian submanifolds in complex projective space with parallel second fundamental form by an elementary geometrical method. We prove that such a Lagrangian submanifold is either totally geodesic, or the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, or the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form, or one of the standard symmetric spaces: $\mathrm{SU}(k)/\mathrm{SO}(k)$, $\mathrm{SU}(k)$, $\mathrm{SU}(2k)/\mathrm{Sp}(k)$ $(k \geq 3)$, $\mathrm{E}_6/\mathrm{F}_4$.

As the arguments are of a local nature, at the same time, due to the correspondence between C-parallel Lagrangian submanifolds in Sasakian space forms and parallel Lagrangian submanifolds in complex space forms, we can also give a complete classification of all C-parallel submanifolds of S^{2n+1} equipped with its standard Sasakian structure.

1. Introduction

One of the first studies of Lagrangian submanifolds of complex space forms was done by Chen and Ogiue [1974]. Since then such submanifolds have been studied

This research was supported by Tsinghua University–Katholieke Universiteit Leuven Bilateral Scientific Cooperation Fund. Li was supported by NSFC grant number 10971110. Wang was supported by NSFC grant number 11171175 and the Fundamental Research Funds for the Central Universities. Wang is the corresponding author.

MSC2010: primary 53B25; secondary 53C42.

Keywords: Lagrangian submanifolds, complex space forms, complex projective space, parallel second fundamental form.

by many authors and a lot of progress has been made in order to understand them properly. Notwithstanding, several open problems remain.

One of the first questions asked and solved by Naitoh in a series of papers [1980; 1981a; 1981b; 1982; 1983a] was the classification of the parallel Lagrangian submanifolds of the complex projective space. The classification relies heavily on the study of symmetric spaces (and Lie groups), and whereas in the irreducible case the classification is clear, little information is given on how to construct all reducible examples. In this paper, we use the techniques developed in [Hu et al. 2009; 2011] in order to obtain a complete and explicit classification of the Lagrangian submanifolds in complex projective space with parallel second fundamental form by an elementary geometric method. The advantage of this approach is that it also allows the study of related problems in this area, such as:

(i) Which are the biharmonic parallel submanifolds of the complex projective space?

(ii) Which are the second order parallel submanifolds (in the sense of Lumiste [2009])?

(iii) Which are the semiparallel submanifolds?

The main result we show is the following:

Classification theorem. *Let M be a Lagrangian submanifold in $\mathbb{CP}^n(4)$ with constant holomorphic sectional curvature* 4. *Suppose that M has parallel second fundamental form*, *then either M is totally geodesic*, *or*

(i) *M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form*, *or*

(ii) *M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form*, *or*

(iii) $n=\frac{1}{2}k(k+1)-1$, $k\geq 3$, *and M is congruent with* $\mathrm{SU}(k)/\mathrm{SO}(k)$, *or*

(iv) $n=k^2-1$, $k\geq 3$, *and M is congruent with* $\mathrm{SU}(k)$, *or*

(v) $n=2k^2-k-1$, $k\geq 3$, *and M is congruent with* $\mathrm{SU}(2k)/\mathrm{Sp}(k)$, *or*

(vi) $n=26$ *and M is congruent with* $\mathrm{E}_6/\mathrm{F}_4$.

The Calabi product is a standard technique [Bolton et al. 2009; Castro et al. 2006; Hu et al. 2008; Li and Wang 2011; Rodriguez Montealegre and Vrancken 2009]. It allows one to construct with one (or two) Lagrangian immersions a new Lagrangian immersion. It is recalled in detail in Section 4 of the paper.

The paper is organized as follows. In Section 2, we recall the basic formulas for Lagrangian submanifolds of complex space forms. In Section 3, we give a construction of an appropriate basis and hence decompose the tangent space into 3 orthogonal distributions $\mathcal{D}_1$, which is 1-dimensional, $\mathcal{D}_2$ and $\mathcal{D}_3$. According to the dimension of $\mathfrak{D}_2$, we have n cases $\{\mathfrak{C}_m\}_{1\le m\le n}$ to consider, where $m = \dim \mathfrak{D}_2 + 1$. We show that the case $\{\mathfrak{C}_n\}$ does not occur. In order to get the components of the second fundamental form, we define a bilinear map L from $\mathfrak{D}_2 \times \mathfrak{D}_2$ to $\mathfrak{D}_3$ and give some properties of L. In Section 4, we introduce for any unit vector $v \in \mathfrak{D}_2$ a linear map $P_v : \mathfrak{D}_2 \to \mathfrak{D}_2$ and study its properties. We use the previous results to obtain a direct sum decomposition for $\mathfrak{D}_2$. We prove that there exists an integer k_0 and unit vectors $v_1, \dots, v_{k_0} \in \mathfrak{D}_2$ such that

$$\mathfrak{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \cdots \oplus \{v_{k_0}\} \oplus V_{v_{k_0}}(0),$$

where $V_{v_j}(0)$ is the eigenspace of P_{v_j} with eigenvalue 0. We remark that we always mean an orthogonal sum of vector spaces when we speak of a direct sum. We also find that $\dim V_{v_1}(0) = \cdots = \dim V_{v_{k_0}}(0)$ and the dimension which we denote by $\mathfrak{p}$ can only be equal to 0, 1, 3 or 7 when $k_0 \ge 2$. Note that up to this point all results remain valid assuming only that M is semiparallel. We also recall some characterizations of the Calabi product Lagrangian immersions in $\mathbb{CP}^n(4)$, whose application gives that M is the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form for case $\{\mathfrak{C}_1\}$. In Section 5, we discuss case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 = 1$. In Sections 6–9, we consider each of the four cases: case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 \ge 2$ and $\mathfrak{p} = 0, 1, 3, 7$ separately and in each case we obtain a complete classification of the Lagrangian submanifolds in $\mathbb{CP}^n(4)$ with parallel second fundamental form. In Section 10, we complete the proof of the Classification theorem.

2. Preliminaries

In this section, M will always denote an n-dimensional Lagrangian submanifold of $\bar{M}^n(4\varepsilon)$, an n-dimensional complex space form with constant holomorphic sectional curvature 4ε. We denote the Levi-Civita connections on M, $\bar{M}^n(4\varepsilon)$ and the normal bundle by ∇, D and $\nabla^\perp_X$ respectively. The formulas of Gauss and Weingarten are given by (see [Chen 1973; 1997a; 1997b; Castro et al. 2006])

$$D_X Y = \nabla_X Y + h(X, Y) \quad \text{and} \quad D_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where X and Y are tangent vector fields and ξ is a normal vector field on M.

As M is Lagrangian, we have (see [Chen 2001; 2005; Li and Vrancken 2005])

$$\nabla^\perp_X JY = J\nabla_X Y \quad \text{and} \quad A_{JX} Y = -Jh(X, Y) = A_{JY} X, \tag{2-1}$$

where h and A denote respectively the second fundamental form and the shape operator.

We denote the curvature tensors of ∇ and $\nabla^\perp_X$ by R and $R^\perp$, respectively. The first and second covariant derivatives of h are defined by

$$\begin{aligned}(\nabla h)(X,Y,Z) &= \nabla^\perp_X h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y),\\ (\nabla^2 h)(X,Y,Z,W) &= \nabla^\perp_X((\nabla h)(Y,Z,W)) - (\nabla h)(\nabla_X Y, Z, W)\\ &\quad - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W),\end{aligned}$$

where X, Y, Z and W are tangent vector fields.

The equations of Gauss, Codazzi and Ricci for a Lagrangian submanifold of $\bar{M}^n(4\varepsilon)$ are given by (see [Chen and Ogiue 1974; Chen 1997a; 1997b; 2001])

$$\begin{aligned}R(X,Y)Z &= \varepsilon(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + [A_{JX}, A_{JY}]Z,\\ (\nabla h)(X,Y,Z) &= (\nabla h)(Y,X,Z),\\ R^\perp(X,Y)JZ &= \varepsilon(\langle Y,Z\rangle JX - \langle X,Z\rangle JY) + J[A_{JX}, A_{JY}]Z,\end{aligned} \tag{2-2}$$

where X, Y and Z are tangent vector fields. Note that for a Lagrangian submanifold the equations of Gauss and Ricci are mutually equivalent.

We have the following Ricci identity (see [Montiel and Urbano 1988]):

$$\begin{aligned}(\nabla^2 h)(X,Y,Z,W) &= (\nabla^2 h)(Y,X,Z,W)\\ &\quad + JR(X,Y)A_{JZ}W - h(R(X,Y)Z,W) - h(R(X,Y)W,Z),\end{aligned} \tag{2-3}$$

where X, Y, Z and W are tangent vector fields.

The Lagrangian condition implies that

$$\begin{aligned}\langle R^\perp(X,Y)JZ, JW\rangle &= \langle R(X,Y)Z, W\rangle,\\ \langle h(X,Y), JZ\rangle &= \langle h(X,Z), JY\rangle,\end{aligned}$$

for tangent vector fields X, Y, Z and W.

From now on, we will also assume that M has parallel fundamental form, that is, in each point p of M, $\nabla h = 0$.

Note that the vanishing of ∇h together with the Ricci identity (2-3) imply that

$$\begin{aligned}&(R(X,Y)h)(Z,W)\\ &\quad = R^\perp(X,Y)\,h(Z,W) - h(R(X,Y)Z,W) - h(Z,R(X,Y)W) \equiv 0,\end{aligned} \tag{2-4}$$

for tangent vector fields X, Y, Z and W. Lagrangian submanifolds satisfying the above property are called semiparallel. Using this property, following an idea first introduced by Ejiri [1981], and since then widely applied and very useful for solving various problems in submanifold theory, a special orthonormal basis can be constructed.

3. The construction of an appropriate orthonormal basis

In this section, we will always assume that M is a Lagrangian submanifold of $\bar{M}^n(4\varepsilon)$ with semiparallel second fundamental form, where $\bar{M}^n(4\varepsilon)$ is an n-dimensional complex space form with constant holomorphic sectional curvature 4ε.

Throughout this section, we fix $p \in M$ and let $UM_p = \{u \in T_pM \mid \|u\| = 1\}$. Note that totally geodesic submanifolds in symmetric spaces have been classified completely by Chen and Nagano [1977; 1978], we will assume that p is a non-totally geodesic point and we define $f(u) = \langle h(u,u), Ju\rangle$ for $u \in UM_p$ and take e_1 as a vector in which f attains its maximum. The following lemma can be found in [Li and Vrancken 2005], [Li and Wang 2009] and [Montiel and Urbano 1988].

Lemma 3.1. *There exists an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM satisfying*:

(i) $h(e_1, e_i) = \lambda_i J e_i$ *for* $i = 1, \dots, n$, *where* λ_1 *is the maximum of* f.

(ii) $\lambda_i \le \frac{1}{2}\lambda_1$ *for* $i = 2, \dots, n$, *and if* $\lambda_j = \frac{1}{2}\lambda_1$ *for some* j, *then* $f(e_j) = 0$.

Furthermore, by taking $X = Z = W = e_1$, $Y = e_j$ for $j \ge 2$ in (2-4), by Lemma 3.1.(i) there exists a unique m with $1 \le m \le n$ such that

$$\lambda_2 = \lambda_3 = \cdots = \lambda_m = \tfrac{1}{2}\lambda_1 \quad \text{and} \quad \lambda_{m+1} = \cdots = \lambda_n = \mu, \tag{3-1}$$

where

$$\mu := \frac{\lambda_1 - \sqrt{\lambda_1^2 + 4\varepsilon}}{2}.$$

We define $\mathfrak{D}_2 := \operatorname{span}\{e_2, \dots, e_m\}$ and $\mathfrak{D}_3 := \operatorname{span}\{e_{m+1}, \dots, e_n\}$.

Lemma 3.2. *The tangent space T_pM can be decomposed as a direct sum of* 3 *orthogonal vector spaces, that is, $T_pM = \mathfrak{D}_1 \oplus \mathfrak{D}_2 \oplus \mathfrak{D}_3$, where*

(i) $\mathfrak{D}_1$ *is a* 1*-dimensional vector space spanned by the unit tangent vector* e_1,

(ii) $h(e_1, v) = \frac{1}{2}\lambda_1 v$, *for any* $v \in \mathfrak{D}_2$,

(iii) $h(e_1, w) = \mu w$, *for any* $w \in \mathfrak{D}_3$,

(iv) $h(v_1, v_2) - \frac{1}{2}\lambda_1\langle v_1, v_2\rangle J e_1 \in J\mathfrak{D}_3$, *for any* $v_1, v_2 \in \mathfrak{D}_2$.

We have n cases $\{\mathfrak{C}_m\}_{1 \le m \le n}$ as follows:

Case $\mathfrak{C}_1$: $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \mu$.

Case $\mathfrak{C}_n$: $\lambda_2 = \lambda_3 = \cdots = \lambda_n = \frac{1}{2}\lambda_1$.

Case $\mathfrak{C}_m$: $\lambda_2 = \cdots = \lambda_m = \frac{1}{2}\lambda_1$ and $\lambda_{m+1} = \cdots = \lambda_n = \mu$ for $2 \le m \le n-1$.

Our aim in the next sections is to describe explicitly the second fundamental form h when restricted to vectors belonging to $\mathfrak{D}_2$. In view of Lemma 3.2 this is trivial in case that $m = 1$ or $m = n$. We first state:

Theorem 3.3. *Case $\{\mathfrak{C}_n\}$ does not occur.*

Proof. Suppose that it did. We use (2-4), and we choose $X = e_1$, $Y = v$, $Z = v$ and $W = v$, with v a unit vector belonging to $\mathfrak{D}_2$. Taking also into account, from the previous lemma, that

$$h(e_1, e_1) = \lambda_1 J e_1, \quad h(e_1, v) = \tfrac{1}{2}\lambda_1 J v \quad \text{and} \quad h(v, v) = \tfrac{1}{2}\lambda_1 J e_1,$$

we find that $\lambda_1 = 0$. This is a contradiction. □

By applying Theorem 4.12 (see also [Li and Wang 2011, Theorem 1.6]), we conclude that M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form for case $\{\mathfrak{C}_1\}$. We will now restrict ourselves in the remainder of this section, as well as in the next sections, to the case $\{\mathfrak{C}_m\}$ when $1 < m < n$. Surprisingly enough it is the form of the second fundamental form restricted to $\mathfrak{D}_2$ which will play a crucial role and in some sense completely describe the immersion. For convenience we write

$$\eta = \tfrac{1}{2}\sqrt{\lambda_1^2 + 4\varepsilon}$$

and without loss of generality we may assume that $\eta \neq 0$.

By Lemma 3.2 we can introduce a bilinear map $L : \mathfrak{D}_2 \times \mathfrak{D}_2 \to \mathfrak{D}_3$ by

$$L(v_1, v_2) := -J\big(h(v_1, v_2) - \tfrac{1}{2}\lambda_1 \langle v_1, v_2\rangle J e_1\big), \quad v_1, v_2 \in \mathfrak{D}_2. \tag{3-2}$$

We will now distinguish vectors belonging to the different vector spaces and so we use the notations $v, v_j \in \mathfrak{D}_2$, $w, w_r \in \mathfrak{D}_3$.

Lemma 3.4. *We have $\langle h(\mathfrak{D}_3, \mathfrak{D}_3), J\mathfrak{D}_2\rangle = 0$. The tensor L is an isotropic tensor in the sense of O'Neill [1965], that is,*

$$\langle L(v, v), L(v, v)\rangle = \tfrac{1}{2}\lambda_1 \eta \|v\|^2, \quad v \in \mathfrak{D}_2. \tag{3-3}$$

Linearizing this expression, it follows for arbitrary vectors $v_1, v_2, v_3, v_4 \in \mathfrak{D}_2$ that

$$\begin{aligned}\langle L(v_1, v_2), L(v_3, v_4)\rangle + \langle L(v_1, v_3), L(v_2, v_4)\rangle + \langle L(v_1, v_4), L(v_2, v_3)\rangle \\ = \tfrac{1}{2}\lambda_1 \eta\big(\langle v_1, v_2\rangle\langle v_3, v_4\rangle + \langle v_1, v_3\rangle\langle v_2, v_4\rangle + \langle v_1, v_4\rangle\langle v_2, v_3\rangle\big).\end{aligned} \tag{3-4}$$

Proof. By taking $Z = W = e_1$ in (2-4) we immediately obtain that for arbitrary vectors x and y, $R(x, y)e_1$ is an eigenvector of A_{Je_1} with eigenvalue $\frac{1}{2}\lambda_1$. So $R(x, y)e_1 \in \mathfrak{D}_2$. Moreover taking $v \in \mathfrak{D}_2$ and $w \in \mathfrak{D}_3$, by the Gauss equation (2-2) we have

$$R(v, w)e_1 = (\mu - \tfrac{1}{2}\lambda_1) A_{Jv} w = -\eta A_{Jv} w,$$

so we have

$$A_{Jv} w \in \mathfrak{D}_2, \quad \text{for all} \quad v \in \mathfrak{D}_2,\ w \in \mathfrak{D}_3, \tag{3-5}$$

which gives the first claim of the lemma.

In order to prove the second claim, we use again (2-4), and we choose $X = e_1$, $Y = v_1$, $Z = v_2$ and $W = v_3$, all belonging to $\mathfrak{D}_2$. By using (2-2) and the definition of L, it follows immediately that

$$h(v_1, L(v_2, v_3)) + h(v_2, L(v_1, v_3)) + h(v_3, L(v_1, v_2)) = \tfrac{1}{2}\lambda_1\eta(\langle v_2, v_3\rangle Jv_1 + \langle v_1, v_3\rangle Jv_2 + \langle v_1, v_2\rangle Jv_3). \tag{3-6}$$

Taking the inner product with v_4 and using the complete symmetry of the cubic form completes the proof. □

We now decompose $\mathfrak{D}_3$ as a direct sum of two orthogonal vector spaces. We define $\mathfrak{D}_{31}$ to be the vector space $\mathrm{vect}\{L(\mathfrak{D}_2, \mathfrak{D}_2)\}$ generated by vectors $L(X, Y)$ where $X, Y \in \mathfrak{D}_2$, and $\mathfrak{D}_{32}$ as its orthogonal complement in $\mathfrak{D}_3$. Then by taking $X = e_1$, $Y = v_1$, $Z = v_2$ and $W = w$ in (2-4) where $v_1, v_2 \in \mathfrak{D}_2$ and $w \in \mathfrak{D}_{32}$ and using the fact that $h(v_2, w) = 0$ we get:

Lemma 3.5. *Let $v_1, v_2 \in \mathfrak{D}_2$ and $w \in \mathfrak{D}_{32}$. Then*

$$h(L(v_1, v_2), w) = \mu\eta\langle v_1, v_2\rangle Jw. \tag{3-7}$$

Similarly, we also have:

Lemma 3.6. *Let $v_1, v_2, v_3, v_4 \in \mathfrak{D}_2$ and let $\{u_1, \ldots, u_{m-1}\}$ be an orthonormal basis of $\mathfrak{D}_2$, then we have*

$$\begin{aligned} h(L(v_1, v_2), L(v_3, v_4)) &= \mu\langle L(v_1, v_2), L(v_3, v_4)\rangle Je_1 + \mu\eta\langle v_1, v_2\rangle JL(v_3, v_4) \\ &+ \sum_{i=1}^{m-1}\langle L(v_1, u_i), L(v_3, v_4)\rangle JL(v_2, u_i) + \sum_{i=1}^{m-1}\langle L(v_2, u_i), L(v_3, v_4)\rangle JL(v_1, u_i). \end{aligned} \tag{3-8}$$

Proof. By (2-2), we have for $v, \tilde{v} \in \mathfrak{D}_2$ that

$$R(e_1, v)\tilde{v} = (\varepsilon + \tfrac{1}{4}\lambda_1^2)\,\langle v, \tilde{v}\rangle e_1 - \eta L(v, \tilde{v}) = \eta^2\langle v, \tilde{v}\rangle e_1 - \eta L(v, \tilde{v}). \tag{3-9}$$

Similarly, we have for $v \in \mathfrak{D}_2$ and $w \in \mathfrak{D}_3$ that $R(e_1, v)w = \eta A_{Jv}w$.

As M is semiparallel, we have from (2-4) that

$$R^{\perp}(e_1, v_1)h(v_2, L(v_3, v_4)) = h(R(e_1, v_1)v_2, L(v_3, v_4)) + h(v_2, R(e_1, v_1)L(v_3, v_4)). \tag{3-10}$$

We now compute each of the terms in the above equation separately. Since, by

Lemma 3.4, $h(v_j, L(v_k, v_l)) \in J\mathfrak{D}_2$, we can write

$$h(v_2, L(v_3, v_4)) = \sum_{i=1}^{m-1} \langle h(v_2, L(v_3, v_4)), Ju_i\rangle Ju_i = \sum_{i=1}^{m-1} \langle L(v_2, u_i), L(v_3, v_4)\rangle Ju_i.$$

Therefore, we get

$$\begin{aligned} R^\perp(e_1, v_1)\, h(v_2, L(v_3, v_4)) &= \sum_{i=1}^{m-1} \langle L(v_2, u_i), L(v_3, v_4)\rangle R^\perp(e_1, v_1) Ju_i \\ &= \eta^2 \langle L(v_1, v_2), L(v_3, v_4)\rangle Je_1 - \eta \sum_{i=1}^{m-1} \langle L(v_2, u_i), L(v_3, v_4)\rangle JL(v_1, u_i). \end{aligned}$$

Next, as $L(v_3, v_4) \in \mathfrak{D}_3$, we have

$$R(e_1, v_1)L(v_3, v_4) = \eta A_{Jv_1} L(v_3, v_4) = \eta \sum_{i=1}^{m-1} \langle L(v_1, u_i), L(v_3, v_4)\rangle u_i.$$

Hence

$$\begin{aligned} h(v_2, R(e_1, v_1)L(v_3, v_4)) &= \eta \sum_{i=1}^{m-1} \langle L(v_1, u_i), L(v_3, v_4)\rangle h(v_2, u_i) \\ &= \frac{\lambda_1}{2}\eta \langle L(v_1, v_2), L(v_3, v_4)\rangle Je_1 + \eta \sum_{i=1}^{m-1} \langle L(v_1, u_i), L(v_3, v_4)\rangle JL(v_2, u_i). \end{aligned}$$

Finally the last term of (3-10) can be computed as follows:

$$h(R(e_1, v_1)v_2, L(v_3, v_4)) = \eta^2 \mu \langle v_1, v_2\rangle JL(v_3, v_4) - \eta h(L(v_1, v_2), L(v_3, v_4)).$$

Combining all three terms now completes the proof of the lemma. $\square$

We note that Equation (3-8) has very important consequences which will be used in sequel sections. For example:

Lemma 3.7. *Assume that $m \geq 3$ and let $\{u_1, \dots, u_{m-1}\}$ be an orthonormal basis of $\mathfrak{D}_2$, then for $p \neq j$, we have*

$$\begin{aligned} 0 &= \Big(\eta(\eta + \tfrac{1}{2}\lambda_1) - 4\langle L(u_j, u_p), L(u_j, u_p)\rangle\Big) L(u_p, u_j) \\ &\quad + \sum_{i \neq p} \Big(\langle L(u_p, u_i), L(u_j, u_j)\rangle - 2\langle L(u_j, u_i), L(u_p, u_j)\rangle\Big) L(u_j, u_i). \end{aligned} \tag{3-11}$$

In particular, if $L(u_1, u_2) \neq 0$ and $L(u_1, u_i)$ is orthogonal to $L(u_1, u_2)$ for all $i \neq 2$, then

$$\langle L(u_1, u_2), L(u_1, u_2)\rangle = \tfrac{1}{4}\eta(\eta + \tfrac{1}{2}\lambda_1) =: \tau. \tag{3-12}$$

Proof. We use (3-8). Interchanging the couples of indices $\{1, 2\}$ and $\{3, 4\}$ we find the following condition:

$$
\begin{aligned}
(3\text{-}13)\quad 0 = {} & \eta\mu\big(\langle v_1, v_2\rangle L(v_3, v_4) - \langle v_3, v_4\rangle L(v_1, v_2)\big) \\
& + \sum_{i=1}^{m-1} \langle L(v_1, u_i), L(v_3, v_4)\rangle L(v_2, u_i) + \sum_{i=1}^{m-1} \langle L(v_2, u_i), L(v_3, v_4)\rangle L(v_1, u_i) \\
& - \sum_{i=1}^{m-1} \langle L(v_3, u_i), L(v_1, v_2)\rangle L(v_4, u_i) - \sum_{i=1}^{m-1} \langle L(v_4, u_i), L(v_1, v_2)\rangle L(v_3, u_i).
\end{aligned}
$$

If we take $v_2 = v_3 = v_4 = u_j$ and $v_1 = u_p$ with j and p different, then by using also the isotropy condition, (3-13) reduces to

$$
\begin{aligned}
0 = {} & \big(\eta(\eta + \tfrac{1}{2}\lambda_1) - 4\langle L(u_j, u_p), L(u_j, u_p)\rangle\big) L(u_p, u_j) \\
& + \sum_{i \neq p} \big(\langle L(u_p, u_i), L(u_j, u_j)\rangle - 2\langle L(u_j, u_i), L(u_p, u_j)\rangle\big) L(u_j, u_i).
\end{aligned}
$$

Finally (3-12) follows by taking $j = 1$ and $p = 2$ in the (3-11), and by using Lemma 3.4. □

4. A map $P_v : \mathfrak{D}_2 \to \mathfrak{D}_2$ for unit vector $v \in \mathfrak{D}_2$ and a decomposition of $\mathfrak{D}_2$

We now define for any given unit vector $v \in \mathfrak{D}_2$ a linear map $P_v : \mathfrak{D}_2 \to \mathfrak{D}_2$ by

$$
(4\text{-}1)\qquad P_v \tilde{v} = A_{Jv} L(v, \tilde{v}) \quad \text{for } \tilde{v} \in \mathfrak{D}_2.
$$

It is easily seen that P_v is well defined and a symmetric linear operator satisfying

$$
(4\text{-}2)\qquad \langle P_v \tilde{v}, v^*\rangle = \langle A_{Jv} L(v, \tilde{v}), v^*\rangle = \langle L(v, \tilde{v}), L(v, v^*)\rangle = \langle P_v v^*, \tilde{v}\rangle
$$

for all $\tilde{v}, v^* \in \mathfrak{D}_2$. Moreover, we have:

Lemma 4.1. *For all unit $v \in \mathfrak{D}_2$, the operator $P_v : \mathfrak{D}_2 \to \mathfrak{D}_2$ has $\sigma = \frac{1}{2}\lambda_1\eta$ as an eigenvalue with eigenvector v. In the orthogonal complement of $\{v\}$ the operator has two eigenvalues, namely τ and 0, where τ is defined in (3-12).*

Proof. According to (3-2) and (3-3), we have

$$
\langle v, P_v v\rangle = \langle L(v, v), L(v, v)\rangle = \tfrac{1}{2}\lambda_1\eta,
$$

and if $v^* \perp v$, then

$$
\langle v^*, P_v v\rangle = \langle L(v, v^*), L(v, v)\rangle = 0.
$$

This implies that $P_v v = \frac{1}{2}\lambda_1\eta\, v$.

Next, we take an orthonormal basis $\{u_i\}_{i=1}^{m-1}$ of $\mathfrak{D}_2$ consisting of eigenvectors of P_v such that $P_v u_i = \sigma_i u_i$ for $1 \leq i \leq m-1$, with $u_1 = v$ and $\sigma_1 = \frac{1}{2}\lambda_1\eta$. We take the inner product of formula (3-11) for $j=1$ and any $p \geq 2$ with $L(v, u_p)$. We have

$$\langle L(u_1, u_p), L(u_1, u_p)\rangle \big(\tau - \langle L(u_1, u_p), L(u_1, u_p)\rangle\big) = 0. \tag{4-3}$$

Here we have used that

$$\langle L(u_1, u_p), L(u_1, u_i)\rangle = \langle u_p, P_{u_1} u_i\rangle = \langle u_p, \sigma_i u_i\rangle = 0 \quad \text{for all } i \neq p.$$

By (4-3), we get either

$$\sigma_p = \langle L(v, u_p), L(v, u_p)\rangle = 0 \quad \text{or} \quad \sigma_p = \langle L(v, u_p), L(v, u_p)\rangle = \tau. \qquad \square$$

In the following we denote by $V_v(0)$ and $V_v(\tau)$ the eigenspaces of P_v (in the orthogonal complement of $\{v\}$) with respect to the eigenvalues 0 and τ, respectively. Note that in exceptional cases it can happen that $\tau = \sigma$.

Lemma 4.2. *Let $u, v \in \mathcal{D}_2$ be two unit orthogonal vectors. The following statements are equivalent*:

(i) $u \in V_v(0)$.

(ii) $L(u, v) = 0$.

(iii) $L(u, u) = L(v, v)$.

(iv) $v \in V_u(0)$.

Moreover any of the previous statements implies that

(v) $P_u = P_v$ *on* $\{u, v\}^\perp$.

Proof. As $\langle v_1, P_v v_2\rangle = \langle L(v, v_1), L(v, v_2)\rangle$, the equivalence of (i), (ii) and (iv) follows immediately. As u and v are orthogonal, the isotropy condition implies that

$$\langle L(u, u), L(v, v)\rangle + 2\langle L(u, v), L(u, v)\rangle = \tfrac{1}{2}\lambda_1 \eta.$$

Because $\langle L(u, u), L(u, u)\rangle = \langle L(v, v), L(v, v)\rangle = \frac{1}{2}\lambda_1\eta$, the equivalence of (ii) and (iii) now follows from the Cauchy–Schwarz inequality.

Now in order to prove (v), we may assume that (i), (ii), (iii) and (iv) are satisfied. As the space spanned by $\{u, v\}$ is invariant by P_u and P_v, also its orthogonal complement is invariant. By taking v_1, v_2 in this orthogonal complement and using the isotropy condition, we find

$$\begin{aligned}\langle v_1, P_v v_2\rangle &= \langle L(v, v_1), L(v, v_2)\rangle \\ &= -\tfrac{1}{2}\langle L(v, v), L(v_1, v_2)\rangle + \tfrac{1}{4}\lambda_1\eta\langle v_1, v_2\rangle \\ &= -\tfrac{1}{2}\langle L(u, u), L(v_1, v_2)\rangle + \tfrac{1}{4}\lambda_1\eta\langle v_1, v_2\rangle \\ &= \langle L(u, v_1), L(u, v_2)\rangle = \langle v_1, P_u v_2\rangle. \qquad \square\end{aligned}$$

Lemma 4.3. *Let $v, \tilde{v} \in \mathfrak{D}_2$ be two unit orthogonal vectors. Then the equality*

$$\langle L(v, \tilde{v}), L(v, \tilde{v})\rangle = \tau$$

holds if and only if $\tilde{v} \in V_v(\tau)$.

Moreover, if we assume $u \in V_v(0)$ and the equality holds, then $u \in V_{\tilde{v}}(\tau)$.

Proof. If $\tilde{v} \in V_v(\tau)$, then $\langle L(v, \tilde{v}), L(v, \tilde{v})\rangle = \langle \tilde{v}, P_v\tilde{v}\rangle = \tau$.

Conversely, if $\langle L(v, \tilde{v}), L(v, \tilde{v})\rangle = \tau$, we can write

$$\tilde{v} = \cos\theta v_0 + \sin\theta v_1, \quad |v_0| = |v_1| = 1,$$

where $v_0 \in V_v(0)$ and $v_1 \in V_v(\tau)$. Then we get

$$\tau = \langle L(v, \tilde{v}), L(v, \tilde{v})\rangle = \langle P_v\tilde{v}, \tilde{v}\rangle = \cos^2\theta\tau,$$

which means that $\sin\theta = 0$ and $\tilde{v} = \cos\theta v_1 \in V_v(\tau)$.

Now assume the equality holds. If $u \in V_v(0)$, then as $v \in V_v(\sigma)$ and $\tilde{v} \in V_v(\tau)$, we see that $u, v, \tilde{v}$ are orthonormal vectors. Therefore $P_u\tilde{v} = P_v\tilde{v} = \tau\tilde{v}$ by Lemma 4.2, which means that $\tilde{v} \in V_u(\tau)$. Applying the first part of the lemma now shows that we have $u \in V_{\tilde{v}}(\tau)$. □

Lemma 4.4. *Let $v_1, v_2, v_3 \in \mathfrak{D}_2$ be orthonormal vectors satisfying $v_1, v_2 \in V_{v_3}(\tau)$. Then for any vector $v \in \mathfrak{D}_2$, we have $\langle L(v_1, v_2), L(v_3, v)\rangle = 0$.*

Proof. Using the linearity of the assertion, we may assume that v is an eigenvector of P_{v_3}. By Lemma 4.2 we only have to consider the case $v \notin V_{v_3}(0)$.

We choose an orthonormal basis $\{u_i\}_{i=1}^{m-1}$ of $\mathfrak{D}_2$ consisting of eigenvectors of P_{v_3} such that $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$. We now use (3-13) for $p = 1$, $j = 2$, $k = l = 3$ to obtain

$$\begin{aligned}(4\text{-}4)\quad 0 = -\mu\eta L(v_1, v_2) &+ \sum_{i=1}^{m-1}\langle L(v_1, u_i), L(v_3, v_3)\rangle L(v_2, u_i)\\ &+\sum_{i=1}^{m-1}\langle L(v_2, u_i), L(v_3, v_3)\rangle L(v_1, u_i) - 2\sum_{i=1}^{m-1}\langle L(v_3, u_i), L(v_1, v_2)\rangle L(v_3, u_i).\end{aligned}$$

Remark that if $i = 3$ and $k = 1, 2$, it follows that$\langle L(v_k, u_i), L(v_3, v_3)\rangle = 0$, and if $k = 1, 2$ and $i \neq k, 3$, we have that $\langle L(v_k, u_i), L(v_3, v_3)\rangle = -2\langle v_k, P_{v_3}u_i\rangle = 0$. Using this, together with (3-4) and the assumption we see that (4-4) reduces to

$$(4\text{-}5)\qquad \sum_{i=1}^{m-1}\langle L(v_3, u_i), L(v_1, v_2)\rangle L(v_3, u_i) = 0.$$

Note that we have

$$\langle L(v_3, u_p), L(v_3, u_q)\rangle = \langle u_p, P_{v_3}u_q\rangle = 0 \quad \text{if } p \neq q.$$

Thus (4-5) implies that $\langle L(v_1, v_2), L(v_3, u_i)\rangle = 0$ for all u_i, which immediately implies that for any vector $v \in \mathcal{D}_2$, we have $\langle L(v_1, v_2), L(v_3, v)\rangle = 0$. □

Using the above lemmas, we can introduce a direct sum decomposition for $\mathcal{D}_2$, which turns out crucial for our purpose.

Pick any unit vector $v_1 \in \mathcal{D}_2$ and recall that $\tau = \frac{1}{4}\eta(\eta + \frac{1}{2}\lambda_1)$, then by Lemma 4.1, we have a direct sum decomposition for $\mathcal{D}_2$

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus V_{v_1}(\tau),$$

where here and later on, we denote also by $\{\cdot\}$ the vector space spanned by its elements. If $V_{v_1}(\tau) \neq \{0\}$, we take an arbitrary unit vector $v_2 \in V_{v_1}(\tau)$. Then by Lemma 4.3 we have:

$$v_1 \in V_{v_2}(\tau), \quad V_{v_1}(0) \subset V_{v_2}(\tau) \quad \text{and} \quad V_{v_2}(0) \subset V_{v_1}(\tau).$$

From this we deduce that

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \{v_2\} \oplus V_{v_2}(0) \oplus \big(V_{v_1}(\tau) \cap V_{v_2}(\tau)\big).$$

If $V_{v_1}(\tau) \cap V_{v_2}(\tau) \neq \{0\}$, we further pick a unit vector $v_3 \in V_{v_1}(\tau) \cap V_{v_2}(\tau)$. Then

$$\mathcal{D}_2 = \{v_3\} \oplus V_{v_3}(0) \oplus V_{v_3}(\tau),$$

and by Lemma 4.3 we have

$$v_1, v_2 \in V_{v_3}(\tau) \quad \text{and} \quad V_{v_1}(0), V_{v_2}(0) \subset V_{v_3}(\tau).$$

It follows that

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \{v_2\} \oplus V_{v_2}(0) \oplus \{v_3\} \oplus V_{v_3}(0) \\ \oplus \big(V_{v_1}(\tau) \cap V_{v_2}(\tau) \cap V_{v_3}(\tau)\big).$$

Considering that $\dim \mathcal{D}_2 = m - 1$ is finite, we easily obtain by induction:

Proposition 4.5. *There exists an integer k_0 and unit vectors $v_1, \dots, v_{k_0} \in \mathcal{D}_2$ so*

$$\mathcal{D}_2 = \{v_1\} \oplus V_{v_1}(0) \oplus \cdots \oplus \{v_{k_0}\} \oplus V_{v_{k_0}}(0). \tag{4-6}$$

We denote $\{v_k\} \oplus V_{v_k}(0)$ by V_k. In what follows, we will now study the decomposition (4-6) in more detail.

Lemma 4.6. (i) *For any unit vector $u_1 \in \{v_1\} \oplus V_{v_1}(0)$, we have*

$$\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0).$$

(ii) *For any unit vectors $u_1, \tilde{u}_1 \in \{v_1\} \oplus V_{v_1}(0)$ and $u_1 \perp \tilde{u}_1$, we have $L(u_1, \tilde{u}_1) = 0$.*

Proof. (i) We first assume that u_1 is orthogonal to v_1. As then $u_1 \in V_{v_1}(0)$, we have that $L(u_1, v_1) = 0$ and $v_1 \in V_{u_1}(0)$. Also on $\{u_1, v_1\}^\perp$ we have that $P_{u_1} = P_{v_1}$, which implies that the orthogonal complement of u_1 in $V_{v_1}(0)$ coincides with the orthogonal complement of v_1 in $V_{u_1}(0)$. This completes the proof in this case.

Now we consider the general case. If $\dim(V_{v_1}(0)) = 0$, there is nothing to prove. If $\dim(V_{v_1}(0)) \geq 2$, we can take a vector $\tilde{u}$ in that space which is orthogonal to both u_1 and v_1. Applying twice the previous result then completes the proof. If $\dim(V_{v_1}(0)) = 1$, there exists $v_0 \in V_{v_1}(0)$ such that $V_{v_1}(0) = \{v_0\}$. Denote $u_1 = \cos\theta v_1 + \sin\theta v_0$. By Lemma 4.2, we see that

$$L(\cos\theta v_1 + \sin\theta v_0, -\sin\theta v_1 + \cos\theta v_0) = 0$$

and hence $-\sin\theta v_1 + \cos\theta v_0 \in V_{u_1}(0)$. Therefore $\{v_1\} \oplus V_{v_1}(0) \subset \{u_1\} \oplus V_{u_1}(0)$. If we do not have the equality, we can find a vector in the second space which is orthogonal to both v_1 and u_1. As $\{v_1\} \oplus V_{v_1}(0) = \{x\} \oplus V_x(0) = \{u_1\} \oplus V_{u_1}(0)$, we get a contradiction.

In order to prove (ii), we have by (i) that

$$\{v_1\} \oplus V_{v_1}(0) = \{u_1\} \oplus V_{u_1}(0).$$

As u_1 and $\tilde{u}_1$ are orthogonal this implies that $\tilde{u}_1 \in V_{u_1}(0)$. Consequently we see that $L(u_1, \tilde{u}_1) = 0$. □

Lemma 4.7. *In the decomposition (4-6), if we pick a unit vector $u_2 \in V_{v_2}(0)$, then there exists a unique vector $u_1 \in v_1 \oplus V_{v_1}(0)$ such that $L(v_1, u_2) = L(v_2, u_1)$. Moreover u_1 is a unit vector belonging to $V_{v_1}(0)$ and $L(v_1, v_2) = -L(u_2, u_1)$.*

Proof. Let $\{u_1^l, \dots, u_{p_l}^l\}$ be an orthonormal basis of $V_{v_l}(0)$, $1 \leq l \leq k_0$, such that $u_1^2 = u_2$. Then

$$\bigl\{v_1, \dots, v_{k_0}, u_1^1, \dots, u_{p_1}^1, \dots, u_1^{k_0}, \dots, u_{p_{k_0}}^{k_0}\bigr\} =: \{\tilde{u}_i\}_{1 \leq i \leq m-1}$$

forms an orthonormal basis of $\mathfrak{D}_2$. Now we use (3-8) with the vectors v_2, u_2, v_1, u_1. As by Lemma 4.2 $L(v_2, u_2) = 0$, and by our decomposition $v_1 \in V_{v_2}(\tau)$, we obtain

$$\begin{aligned}
0 &= h(L(v_2, u_2), L(v_1, v_2)) \\
&= \mu\langle L(u_2, v_2), L(v_1, v_2)\rangle Je_1 + \sum_{i=1}^{m-1}\langle L(v_2, \tilde{u}_i), L(v_1, v_2)\rangle JL(u_2, \tilde{u}_i) \\
&\qquad\qquad\qquad\qquad + \sum_{i=1}^{m-1}\langle L(u_2, \tilde{u}_i), L(v_1, v_2)\rangle JL(v_2, \tilde{u}_i) \\
&= \sum_{i=1}^{m-1}\langle P_{v_2}v_1, \tilde{u}_i\rangle JL(u_2, \tilde{u}_i) + \sum_{i=1}^{m-1}\langle L(u_2, \tilde{u}_i), L(v_1, v_2)\rangle JL(v_2, \tilde{u}_i) \\
&= \tau JL(u_2, v_1) + \sum_{i=1}^{m-1}\langle L(u_2, \tilde{u}_i), L(v_1, v_2)\rangle JL(v_2, \tilde{u}_i).
\end{aligned}$$

From this we see that we can put

$$u_1 = -\frac{1}{\tau}\sum_{i=1}^{m-1}\langle L(u_2, \tilde{u}_i), L(v_1, v_2)\rangle \tilde{u}_i. \tag{4-7}$$

Noting that $u_2 \in V_{v_1}(\tau)$, and applying Lemma 4.4 and Lemma 4.6, we see that the above sum is nonzero only if $\tilde{u}_i = u_2$ and $\tilde{u}_i = v_1$ or if $\tilde{u}_i \in V_{v_1}(0)$.

If $\tilde{u}_i = u_2$, using Lemma 4.2, we get that

$$\langle L(u_2, u_2), L(v_1, v_2)\rangle = \langle L(v_2, v_2), L(v_1, v_2)\rangle = 0,$$

whereas if $\tilde{u}_i = v_1$, we have that

$$\langle L(u_2, v_1), L(v_1, v_2)\rangle = \langle u_2, P_{v_1} v_2\rangle = \tau\langle u_2, v_2\rangle = 0.$$

Consequently $u_1 \in V_{v_1}(0)$.

In order to prove the uniqueness in $v_1 \oplus V_{v_1}(0)$, suppose that $\tilde{u}_1 \in v_1 \oplus V_{v_1}(0)$ such that $L(v_1, u_2) = L(v_2, \tilde{u}_1)$. Then we have $L(v_2, u_1 - \tilde{u}_1) = 0$, hence by Lemma 4.2 we have $u_1 - \tilde{u}_1 \in V_{v_2}(0)$. But we also have $u_1 - \tilde{u}_1 \in v_1 \oplus V_{v_1}(0) \subset V_{v_2}(\tau)$, so we must have $u_1 - \tilde{u}_1 = 0$.

To show that vector $u_1 \in V_{v_1}(0)$ satisfying $L(v_1, u_2) = L(v_2, u_1)$ must be of unit length, we note that as $u_2 \in V_{v_2}(0) \subset V_{v_1}(\tau)$ and $u_1 \in V_{v_1}(0) \subset V_{v_2}(\tau)$, then

$$\langle L(v_1, u_2), L(v_1, u_2)\rangle = \tau \quad \text{and} \quad \langle L(v_2, u_1), L(v_2, u_1)\rangle = \|u_1\|^2\tau.$$

Hence $\|u_1\|^2 = 1$ and u_1 is a unit vector.

In order to prove $L(v_1, u_2) = L(v_2, u_1)$ and $L(v_1, v_2) = -L(u_2, u_1)$ are equivalent, we use the isotropic condition (3-4) and the Cauchy–Schwarz inequality.

Suppose now that $L(v_1, u_2) = L(v_2, u_1)$. We have $v_1,\ u_1 \in V_{v_2}(\tau) = V_{u_2}(\tau)$ by Lemma 4.6, so we get $\langle L(v_1, v_2), L(v_1, v_2)\rangle = \tau$, $\langle L(u_1, u_2), L(u_1, u_2)\rangle = \tau$. As the isotropy condition gives

$$\langle L(v_1, v_2), -L(u_1, u_2)\rangle = \langle L(v_1, u_2), L(v_2, u_1)\rangle = \langle L(v_2, u_1), L(v_2, u_1)\rangle = \tau,$$

then by using the Cauchy–Schwarz inequality we get $L(v_1, v_2) = -L(u_2, u_1)$. The other hand side can be proved in a similar way. $\square$

Lemma 4.8. *In the decomposition (4-6), we write* $V_l = \{v_l\} \oplus V_{v_l}(0)$, $1 \le l \le k_0$.

(1) *For any unit vector* $a \in V_j$,

$$h(L(a, a), L(a, a)) = \tfrac{1}{2}\lambda_1\mu\eta J e_1 + \eta(\mu + \lambda_1) J L(a, a). \tag{4-8}$$

(2) *For any unit vectors* $a \in V_j,\ b \in V_l,\ j \neq l$,

$$h(L(a,a), L(a,b)) = \tfrac{1}{2}\eta(\mu+\lambda_1)JL(a,b), \tag{4-9}$$

$$h(L(a,a), L(b,b)) = \tfrac{1}{2}\eta\mu^2 Je_1 + \eta\mu J(L(a,a)+L(b,b)), \tag{4-10}$$

$$h(L(a,b), L(a,b)) = \mu\tau Je_1 + \tau J(L(a,a)+L(b,b)). \tag{4-11}$$

(3) *For unit vectors* $a \in V_j,\ b, b' \in V_l,\ c \in V_q, d \in V_s$ *and* $j,\ l,\ q, s$ *being distinct*, b *and* b' *being orthogonal*,

$$h(L(a,b), L(a,c)) = \tau JL(b,c), \tag{4-12}$$

$$h(L(a,a), L(b,c)) = \eta\mu JL(b,c), \tag{4-13}$$

$$h(L(a,b), L(a,b')) = 0, \tag{4-14}$$

$$h(L(a,b), L(c,d)) = 0. \tag{4-15}$$

(4) *For orthogonal unit vectors* $a_1,\ a_2 \in V_j$ *and unit vectors* $b \in V_l,\ c \in V_q$ *with* $j,\ l,\ q$ *being distinct*, *we have*

$$h(L(a_1,b), L(a_2,c)) = \tau JL(b,c'), \tag{4-16}$$

where $c' \in V_q$ *is the unique unit vector satisfying* $L(a_2,c) = L(a_1,c')$.

Proof. We take an orthonormal basis of $\mathfrak{D}_2$ in such a way so that it consists of all the orthonormal basis of $V_j,\ 1 \le j \le k_0$. Then the conclusions are direct consequences of Lemma 3.6. For example, to prove (4-12) we combine Lemma 3.6 with the fact $\langle L(a,b), L(a,c)\rangle = \langle b, P_a c\rangle = \tau\langle b,c\rangle = 0$ and the isotropic properties of L. From (4-12) and Lemmas 4.6 and 4.7 we get (4-16). □

Proposition 4.9. *In the decomposition* (4-6), *if* $k_0 = 1$, *then* $\dim(\operatorname{Im} L) = 1$. *If* $k_0 \ge 2$, *then* $\dim V_{v_1}(0) = \cdots = \dim V_{v_{k_0}}(0)$. *We denote the dimension by* $\mathfrak{p}$, *then* $\dim \mathfrak{D}_2 = m-1 = k_0(\mathfrak{p}+1)$. *Moreover*, $\mathfrak{p}$ *can only be equal to* 0, 1, 3 *or* 7.

Proof. When $k_0 = 1$, from Lemma 4.2 and Lemma 4.6 we get that $L(v_1, v_1)$ is a basis of $\operatorname{Im} L$, hence $\dim(\operatorname{Im} L) = 1$. As a direct consequence of Lemma 4.7, for any $j \neq l$, we can define a one to one linear map from $V_{v_j}(0)$ to $V_{v_l}(0)$, which preserves the length of vectors. Hence $V_{v_j}(0)$ and $V_{v_l}(0)$ are isomorphic and have the same dimension which we denote by $\mathfrak{p}$. To make the following discussion meaningful, we now assume $\mathfrak{p} \ge 1$.

Set $V_l = \{v_l\} \oplus V_{v_l}(0),\ 1 \le l \le k_0$. Let $\{v_l, u_1^l, \ldots, u_{\mathfrak{p}}^l\}$ be an orthonormal basis of V_l. For each $j = 1, \ldots, \mathfrak{p}$, Lemmas 4.6 and 4.7 show that we can define a linear map $\mathfrak{T}_j : V_1 \to V_1$ such that the image $\mathfrak{T}_j(v)$ of any unit vector $v \in V_1$ satisfies

$$L(v, u_j^2) = L(v_2, \mathfrak{T}_j(v)). \tag{4-17}$$

The linear map $\mathfrak{T}_j : V_1 \to V_1$ has these fundamental properties:

(P1) $\langle \mathfrak{T}_j(v), \mathfrak{T}_j(v)\rangle = \langle v, v\rangle$, that is, $\mathfrak{T}_j$ preserves the length of vectors.

(P2) For all $v \in V_1$, we have $\mathfrak{T}_j(v) \perp v$.

(P3) $\mathfrak{T}_j^2 = -\,\mathrm{id}$.

(P4) For all $j \neq l$ and $v \in V_1$, we have $\langle \mathfrak{T}_j(v), \mathfrak{T}_l(v)\rangle = 0$.

Since (P1) and (P2) can be easily seen from Lemma 4.7 and the definition of $\mathfrak{T}_j$, we need only to verify explicitly (P3) and (P4).

For any unit vector $v \in V_1$, we have

$$L(v_2, \mathfrak{T}_j^2(v)) = L(\mathfrak{T}_j(v), u_j^2). \tag{4-18}$$

Using the fact $\{\mathfrak{T}_j(v)\} \oplus V_{\mathfrak{T}_j(v)}(0) = V_1$ and $u_j^2 \in V_{v_2}(0) \subset V_{\mathfrak{T}_j(v)}(\tau)$, we have

$$\begin{aligned}\langle L(\mathfrak{T}_j(v), u_j^2), L(\mathfrak{T}_j(v), u_j^2)\rangle &= \langle L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\rangle \\ &= \langle L(v, v_2), L(v, v_2)\rangle = \tau.\end{aligned}$$

Since $v, \mathfrak{T}_j(v), v_2, u_j^2$ are orthonormal vectors, by (3-4), (4-17) and $L(v_2, u_j^2) = 0$ we see that

$$\begin{aligned}0 &= \langle L(v, v_2), L(\mathfrak{T}_j(v), u_j^2)\rangle + \langle L(v, \mathfrak{T}_j(v)), L(v_2, u_j^2)\rangle + \langle L(v, u_j^2), L(v_2, \mathfrak{T}_j(v))\rangle \\ &= \langle L(v, v_2), L(\mathfrak{T}_j(v), u_j^2)\rangle + \langle L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\rangle.\end{aligned}$$

Applying (4-12) and the Cauchy–Schwarz inequality we deduce

$$L(\mathfrak{T}_j(v), u_j^2) = -L(v, v_2). \tag{4-19}$$

Combining (4-18) and (4-19), we get $L(\mathfrak{T}_j^2(v) + v, v_2) = 0$, which implies that $\mathfrak{T}_j^2(v) + v \in V_{v_2}(0)$. As $\mathfrak{T}_j^2(v) + v \in V_1 \subset V_{v_2}(\tau)$, it follows that $\mathfrak{T}_j^2(v) = -v$ for a unit vector v and then by linearity for all $v \in V_1$, as claimed by (P3).

To verify (P4), we note that, if $j \neq l$ and $\mathfrak{T}_j(v), \mathfrak{T}_l(v) \in V_v(0)$, then by definition

$$L(v_2, \mathfrak{T}_j(v)) = L(v, u_j^2) \perp L(v, u_l^2) = L(v_2, \mathfrak{T}_l(v)).$$

If we assume $\mathfrak{T}_l(v) = a\mathfrak{T}_j(v) + x$, where $x \perp \mathfrak{T}_j(v)$ and $x \in V_v(0)$, then

$$\begin{aligned}0 &= \langle L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_l(v))\rangle \\ &= \langle L(v_2, \mathfrak{T}_j(v)), aL(v_2, \mathfrak{T}_j(v)) + L(v_2, x)\rangle \\ &= a\langle L(v_2, \mathfrak{T}_j(v)), L(v_2, \mathfrak{T}_j(v))\rangle = a\tau.\end{aligned}$$

Thus $a = 0$ and therefore $\mathfrak{T}_j(v) \perp \mathfrak{T}_l(v)$, as claimed.

We now look at the unit hypersphere $S^{\mathfrak{p}}(1) \subset V_1$, properties (P1)–(P4) above show that at $v \in S^{\mathfrak{p}}(1)$ one has

$$T_v S^{\mathfrak{p}}(1) = \big\{\mathfrak{T}_1(v), \dots, \mathfrak{T}_{\mathfrak{p}}(v)\big\}.$$

Hence, by the properties (P1)–(P4), $S^{\mathfrak{p}}(1)$ is parallelizable. Then, according to Bott and Milnor [1958] and Kervaire [1958], the dimension $\mathfrak{p}$ can only be equal to $1, 3$ or 7. □

From now on we will restrict ourselves to the complex projective case, that is, we will assume that $\epsilon = 1$. From Proposition 4.9 we see that, in order to complete the proof of the Classification theorem, it is sufficient to deal with case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with either $k_0 = 1$ or $k_0 \ge 2$ and $\mathfrak{p} = 0,\ 1,\ 3,\ 7$. In most cases the classification will reduce to a Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, or a Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form. These are respectively constructed in the following way, see [Bolton et al. 2009; Castro et al. 2006; Hu et al. 2008; Li and Wang 2011; Rodriguez Montealegre and Vrancken 2009].

Definition 4.10 [Bolton et al. 2009]. Let $\psi_i : (M_i, g_i) \to \mathbb{CP}^{n_i}(4)$, $i = 1, 2$, be two Lagrangian immersions and let $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2) : I \to \mathbb{S}^3(1) \subset \mathbb{C}^2$ be a Legendre curve. Then

$$\psi = \Pi(\tilde{\gamma}_1\tilde{\psi}_1; \tilde{\gamma}_2\tilde{\psi}_2) : I \times M_1 \times M_2 \to \mathbb{CP}^n(4)$$

is a Lagrangian immersion, where $n = n_1 + n_2 + 1$, $\tilde{\psi}_i : M_i \to \mathbb{S}^{2n_i+1}(1)$ are horizontal lifts of ψ_i, $i = 1, 2$, respectively and Π is the Hopf fibration. We call ψ a *warped product* Lagrangian immersion of ψ_1 and ψ_2. When n_1 (or n_2) is zero, we call ψ a *warped product* Lagrangian immersion of ψ_2 (or ψ_1) and a point.

Definition 4.11 [Li and Wang 2011]. In Definition 4.10, when

$$\tilde{\gamma}(t) = \big(r_1 e^{i\frac{r_2}{r_1}at}, r_2 e^{-i\frac{r_1}{r_2}at}\big),$$

where r_1, r_2, and a are positive constants with $r_1^2 + r_2^2 = 1$, we call ψ a *Calabi product* Lagrangian immersion of ψ_1 and ψ_2. When n_1 (or n_2) is zero, we call ψ a *Calabi product* Lagrangian immersion of ψ_2 (or ψ_1) and a point.

Using the arguments of Bolton et al. [2009], Calabi products were characterized in Li and Wang [2011]. In particular we recall:

Theorem 4.12 [Li and Wang 2011, Theorem 1.6]. *Let $\psi : M \to \mathbb{CP}^n(4)$ be a Lagrangian immersion. Suppose that M admits orthogonal distributions $\mathfrak{D}_1$ (of dimension 1, spanned by a unit vector E_1) and $\mathfrak{D}_2$ (of dimension $n-1$, spanned by $\{E_2, \dots, E_n\}$), and that there exist local functions $\lambda_1,\ \lambda_2$ such that $\lambda_1 \ne 2\lambda_2$ and*

$$h(E_1, E_1) = \lambda_1 J E_1 \quad \textit{and} \quad h(E_1, E_i) = \lambda_2 J E_i \quad \textit{for } i = 2, \dots, n. \tag{4-20}$$

Then M has parallel second fundamental form if and only if ψ is locally a Calabi product Lagrangian immersion of a point and an $(n-1)$-dimensional Lagrangian immersion $\psi_1 : M_1 \to \mathbb{CP}^{n-1}(4)$ which has parallel second fundamental form.

Theorem 4.13 [Li and Wang 2011, Theorem 4.6]. *Let $\psi : M \to \mathbb{CP}^n(4)$ be a Lagrangian immersion. Suppose that M admits three mutually orthogonal distributions $\mathfrak{D}_1$ (spanned by a unit vector E_1), $\mathfrak{D}_2$, and $\mathfrak{D}_3$ of dimension 1, n_1 and n_2 respectively, with $1+n_1+n_2=n$, and that there are three real constants λ_1, λ_2 and λ_3 that satisfy $2\lambda_3 \neq \lambda_1 \neq 2\lambda_2 \neq 2\lambda_3$ such that for all $E_i \in \mathfrak{D}_2,\ E_\alpha \in \mathfrak{D}_3$,*

$$h(E_1,E_1)=\lambda_1 JE_1, \qquad h(E_1,E_i)=\lambda_2 JE_i,$$
$$h(E_1,E_\alpha)=\lambda_3 JE_\alpha, \qquad h(E_i,E_\alpha)=0. \tag{4-21}$$

Then M has parallel second fundamental form if and only if ψ is locally a Calabi product Lagrangian immersion of two lower-dimensional Lagrangian submanifolds ψ_i $(i=1,2)$ with parallel second fundamental form.

5. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0=1$

In this section, we consider the case $\mathfrak{C}_m$ for $2\le m\le n-1$ with $k_0=1$. In view of Proposition 4.9 this implies that $\dim(\operatorname{Im} L)=1$.

Theorem 5.1. *Let $M\subset \mathbb{CP}^n(4)$ be a Lagrangian submanifold in a complex space form with constant holomorphic sectional curvature 4. Suppose that M is not totally geodesic and has parallel second fundamental form, that $k_0=1$ and that $1\le \dim \mathfrak{D}_2 = m-1\le n-2$. Then M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form or the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form.*

Proof. In view of Lemma 4.2 and Lemma 4.6 we see that there exists a unit vector $w_1\in \operatorname{Im} L\subset \mathfrak{D}_3$ such that

$$L(v_1,v_2)=\sqrt{\frac{\lambda_1\eta}{2}}\langle v_1,v_2\rangle w_1 =: \rho\langle v_1,v_2\rangle w_1, \tag{5-1}$$

for all $v_1,\ v_2\in\mathfrak{D}_2$.

By (4-8) we get

$$h(w_1,w_1)=\mu Je_1+(2\rho+\mu\eta/\rho)Jw_1. \tag{5-2}$$

By (3-5) we get the operator $A_{Jw_1}:\mathfrak{D}_2\to\mathfrak{D}_2$ is well defined and self adjoint. From the definition of L, we get for orthonormal vectors $\{v_1,\dots,v_{m-1}\}$ belonging to $\mathfrak{D}_2$ that

$$h(e_1,v_j)=\tfrac{1}{2}\lambda_1 Jv_j, \quad h(w_1,v_j)=\rho Jv_j \quad \text{and} \quad h(v_j,v_k)=\left(\tfrac{1}{2}\lambda_1 Je_1+\rho Jw_1\right)\delta_{jk}$$

for $1\le j,k\le m-1$.

From $\dim(\operatorname{Im} L)=1$, we have $\mathfrak{D}_{31}=\{w_1\}$. Denote $\tilde n = n-m-1$, then $\dim(\mathfrak{D}_{32})=\tilde n$. We choose $\{\tilde w_1,\dots,\tilde w_{\tilde n}\}$ to be an orthonormal basis of $\mathfrak{D}_{32}$. Then by Lemma 3.4 and Lemma 3.5 we have

$$h(w_1,\tilde{w}_r)=\frac{\mu\eta}{\rho}J\tilde{w}_r,\quad 1\le r\le\tilde{n}. \tag{5-3}$$

Now we define $T=\alpha e_1+\beta w_1$ and $T^*=-\beta e_1+\alpha w_1$, where

$$\alpha=\frac{\rho}{\sqrt{\rho^2+\eta^2}}\quad\text{and}\quad\beta=\frac{\eta}{\sqrt{\rho^2+\eta^2}}. \tag{5-4}$$

Then $\{T,\,T^*,\,v_{i|1\le i\le m-1},\,\tilde{w}_{r|1\le r\le\tilde{n}}\}$ forms an orthonormal basis of T_pM. By (5-2), we easily obtain

$$h(T,T)=\eta_1 JT,\quad h(T,u)=\eta_2 Ju\quad\text{and}\quad h(T,\tilde{w}_r)=\eta_3 J\tilde{w}_r \tag{5-5}$$

for $1\le r\le\tilde{n}$, where η_1, η_2 and η_3 are defined by

$$\eta_1=\alpha\big(\tfrac12\lambda_1+\eta\big)+\mu/\alpha,\quad\eta_2=\alpha\big(\tfrac12\lambda_1+\eta\big)\quad\text{and}\quad\eta_3=\mu/\alpha, \tag{5-6}$$

which satisfy the relations $\eta_2\ne\eta_3$, $2\eta_2\ne\eta_1\ne 2\eta_3$ and

$$\eta_1=\eta_2+\eta_3\quad\text{and}\quad\eta_2\eta_3=\mu\big(\eta+\tfrac12\lambda_1\big)=-1, \tag{5-7}$$

and $u\in\{T^*,v_1,\dots,v_{m-1}\}$.

From (5-5), we have

$$\begin{cases} T(\eta_1)=\langle(\nabla h)(T,T,T),\,JT\rangle\,, \\ u(\eta_1)=\langle(\nabla h)(u,T,T),\,JT\rangle & \text{for } u\in\{T^*,v_1,\dots,v_{m-1}\}, \\ \tilde{w}_r(\eta_1)=\langle(\nabla h)(\tilde{w}_r,T,T),\,JT\rangle & \text{for } 1\le r\le\tilde{n}, \end{cases} \tag{5-8}$$

Since M has parallel second fundamental form, (5-8) implies that η_1 is constant on M. By a similar argument, we can prove that η_2 and η_3 are also constant on M.

By the Gauss equation (2-2) and Equation (5-5), we have

$$R^\perp(u,\tilde{w}_r)\,h(T,T)=\eta_1(\eta_3-\eta_2)JA_{Ju}\tilde{w}_r, \tag{5-9}$$

while on the other hand, from (2-4), we have

$$R^\perp(u,\tilde{w}_r)\,h(T,T)=2(\eta_3-\eta_2)\,h(T,A_{Ju}\tilde{w}_r). \tag{5-10}$$

Since $\eta_3-\eta_2\ne 0$, (5-9) and (5-10) imply that

$$h(T,A_{Ju}\tilde{w}_r)=\tfrac12\eta_1 JA_{Ju}\tilde{w}_r, \tag{5-11}$$

so from (2-1), (5-5) and (5-11) we deduce that $h(u,\tilde{w}_r)=JA_{Ju}\tilde{w}_r=0$.

Now we apply Theorem 4.13 (see also [Li and Wang 2011, Theorem 4.6])—or, if $\tilde{n}=0$, Theorem 4.12 (see also [ibid., Theorem 1.6])—to conclude that M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form or the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form. □

6. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0 \ge 2$ and $\mathfrak{p} = 0$

Theorem 6.1. *Let $M \subset \mathbb{CP}^n(4)$ be a Lagrangian submanifold in a complex space form with constant holomorphic sectional curvature 4. Suppose that M is not totally geodesic and that M has parallel second fundamental form. Suppose also that $1 \le \dim \mathfrak{D}_2 = m-1 \le n-2$, and that k_0 and $\mathfrak{p}$ defined in Section 4 satisfy $k_0 \ge 2$ and $\mathfrak{p} = 0$. Then $n \ge \frac{1}{2}m(m+1)-1$. Moreover, if $n = \frac{1}{2}m(m+1)$, then M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, and if $n \ge \frac{1}{2}m(m+1)+1$, then M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form.*

For the proof we need some observations and a lemma. Suppose M is not totally geodesic. In the present situation, the decomposition (4-6) reduces to $\mathfrak{D}_2 = \{v_1\}\oplus\cdots\oplus\{v_{k_0}\}$. Then $\dim \mathfrak{D}_2 = k_0 = m-1$ and $\{v_1, \dots, v_{k_0}\}$ forms an orthonormal basis of $\mathfrak{D}_2$.

According to Lemma 3.7 and the fact that for $j \ne l$, $v_j \in V_{v_l}(\tau)$, we have

$$\langle L(v_j, v_l), L(v_j, v_l)\rangle = \tau, \quad j \ne l, \tag{6-1}$$

$$\langle L(v_j, v_{l_1}), L(v_j, v_{l_2})\rangle = 0, \quad j, l_1, l_2 \text{ distinct}, \tag{6-2}$$

$$\langle L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})\rangle = 0, \quad j_1, j_2, j_3, j_4 \text{ distinct}. \tag{6-3}$$

It follows that $\left\{\frac{1}{\sqrt{\tau}}L(v_j, v_l)\right\}_{1\le j<l\le k_0}$ consists of $\frac{1}{2}k_0(k_0-1) = \frac{1}{2}(m-1)(m-2)$ orthonormal vectors. For $\{L(v_j, v_j)\}_{1\le j\le k_0}$, we note that

$$\langle L(v_j, v_j), L(v_j, v_j)\rangle = \tfrac{1}{2}\lambda_1\eta, \quad 1 \le j \le k_0, \tag{6-4}$$

$$\langle L(v_j, v_j), L(v_l, v_l)\rangle = \tfrac{1}{2}\mu\eta, \quad 1 \le j \ne l \le k_0, \tag{6-5}$$

$$\langle L(v_j, v_j), L(v_j, v_l)\rangle = 0, \quad 1 \le j \ne l \le k_0, \tag{6-6}$$

$$\langle L(v_j, v_j), L(v_{l_1}, v_{l_2})\rangle = 0, \quad 1 \le j, l_1, l_2 \text{ distinct and } \le k_0. \tag{6-7}$$

Then $\{L_j := L(v_1, v_1)+\cdots+L(v_j, v_j) - jL(v_{j+1}, v_{j+1}) \mid 1 \le j \le k_0-1\}$ is a set of k_0-1 mutually orthogonal vectors which are all orthogonal to $L(v_j, v_l)$, $j \ne l$. Moreover, we easily have $\langle L_j, L_j\rangle = 2j(j+1)\tau \ne 0$. Hence

$$\begin{aligned} w_j &= \frac{1}{\sqrt{2j(j+1)\tau}}L_j, \quad 1 \le j \le k_0-1 = m-2,\\ w_{kl} &= \frac{1}{\sqrt{\tau}}L(v_k, v_l), \qquad 1 \le k < l \le k_0 = m-1, \end{aligned} \tag{6-8}$$

are $\frac{1}{2}(m-1)(m-2)+(m-2)$ orthonormal vectors in $\mathrm{Im}(L) \subset \mathfrak{D}_3$.

Finally, it is easily known that $\operatorname{Tr} L = L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})$ is orthogonal to the above $\frac{1}{2}(m-1)(m-2) + (m-2)$ vectors and satisfies

$$\langle \operatorname{Tr} L, \operatorname{Tr} L\rangle = \tfrac{1}{2} k_0 \eta (\lambda_1 + (k_0 - 1)\mu) =: \rho^2, \tag{6-9}$$

where $\rho \geq 0$. These results imply that

$$\begin{aligned} n &= 1 + \dim \mathfrak{D}_2 + \dim \mathfrak{D}_3 \\ &\geq 1 + (m-1) + \tfrac{1}{2}(m-1)(m-2) + (m-2) = \tfrac{1}{2} m(m+1) - 1. \end{aligned} \tag{6-10}$$

Lemma 6.2. *We have that* $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{2} m(m+1) - 1$.

Proof. Suppose $\operatorname{Tr} L = 0$, we can first prove that $\mathfrak{D}_3 = \operatorname{Im}(L)$. If not, we can choose a vector $w \in \mathfrak{D}_3$ which is orthogonal to $\operatorname{Im}(L)$, then by (3-7) we get

$$0 = h(\operatorname{Tr} L, w) = (m-1)\mu\eta J w,$$

hence we get $w = 0$ which is a contraction. So we have

$$n = 1 + \dim \mathfrak{D}_2 + \dim \mathfrak{D}_3 = 1 + (m-1) + \tfrac{1}{2}(m+1)(m-2) = \tfrac{1}{2} m(m+1) - 1.$$

On the other hand, suppose that $n = \frac{1}{2} m(m+1) - 1$. By Equation (6-10) we get $\dim \mathfrak{D}_3 = \frac{1}{2}(m-1)(m-2) + (m-2)$ hence $\operatorname{Tr} L = 0$. □

Proof of Theorem 6.1. We need to consider two cases:

(i) $n = \frac{1}{2} m(m+1)$.

(ii) $n \geq \frac{1}{2} m(m+1) + 1$.

We define a unit vector $t = \frac{1}{\rho} \operatorname{Tr} L$.

In case (i), the previous results and particularly (6-9) show that

$$\{t,\ w_{kl \mid 1 \leq k < l \leq m-1},\ w_{j \mid 1 \leq j \leq m-2}\}$$

is an orthonormal basis of $\operatorname{Im}(L) = \mathfrak{D}_3$. By direct calculations with application of Lemma 3.6, Lemma 4.8 and (6-1)–(6-8), we have:

Lemma 6.3. *Under the above assumptions, we have*

$$\begin{gathered} h(t, e_1) = \mu J t, \quad h(t, u) = \frac{\rho}{k_0} J u, \quad h(t, w) = \frac{2\rho}{k_0} J w, \\ h(t, t) = \mu J e_1 + \Big(\frac{2\rho}{k_0} + \frac{k_0 \mu \eta}{\rho}\Big) J t, \end{gathered} \tag{6-11}$$

where $u = v_i$ *for* $1 \leq i \leq k_0 = m-1$, *and* w *stands for either* w_j *or* w_{kl}, *with* $1 \leq j \leq k_0 - 1 = m-2$ *and* $1 \leq k < l \leq k_0 = m-1$.

Put $T = \alpha e_1 + \beta t$ and $T^* = -\beta e_1 + \alpha t$, where

$$\alpha = \frac{\rho}{\sqrt{\rho^2 + k_0^2\eta^2}} \quad \text{and} \quad \beta = \frac{k_0\eta}{\sqrt{\rho^2 + k_0^2\eta^2}}. \tag{6-12}$$

Then $\{T,\ T^*,\ v_{i|1\le i\le m-1},\ w_{j|1\le j\le m-2},\ w_{kl|1\le k<l\le m-1}\}$ is an orthonormal basis of T_pM. By Lemma 6.3 we easily obtain:

Lemma 6.4. *Under the above assumptions, we have*

$$h(T,T) = \eta_1 JT \quad \textit{and} \quad h(T,u) = \eta_2 Ju, \tag{6-13}$$

where η_1 and η_2 are defined by

$$\eta_1 = \alpha\big(\tfrac{1}{2}\lambda_1 + \eta\big) + \mu/\alpha \quad \textit{and} \quad \eta_2 = \alpha\big(\tfrac{1}{2}\lambda_1 + \eta\big), \tag{6-14}$$

which satisfy the relation

$$\eta_1\eta_2 - \eta_2^2 = \mu(\tfrac{1}{2}\lambda_1 + \eta) = -1, \tag{6-15}$$

where u stands for one of T^*, v_i, w_j, w_{kl} and $1 \le i, k, l \le m-1,\ 1 \le j \le m-2$.

We note that $\eta_1 \ne 2\eta_2$. Otherwise, by definition we have $\mu/\alpha = \alpha(\frac{1}{2}\lambda_1 + \eta)$, then by using the definition of α, ρ and the fact that $\eta \ne 0$ for case $\{\mathfrak{C}_m\}$ we get

$$\lambda_1 + 2\eta = \lambda_1 + \sqrt{\lambda_1^2 + 4} = 0,$$

which cannot happen.

Based on the conclusions of Lemma 6.4, we can apply Theorem 4.12 (see also Theorem 1.6 in [Li and Wang 2011]) to conclude that in case (i) M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form.

In case (ii), we proceed in the same way. We still have that

$$\{t,\ w_{kl|1\le k<l\le m-1},\ w_{j|1\le j\le m-2}\}$$

is an orthonormal basis of $\mathrm{Im}(L)$. But now we no longer have that $\mathrm{Im}(L)$ coincides with $\mathfrak{D}_3$. Denote $\tilde{n} = n - \frac{1}{2}m(m+1)$ and choose $\tilde{w}_1, \ldots, \tilde{w}_{\tilde{n}}$ in the orthogonal complement of $\mathrm{Im}(L)$ in $\mathfrak{D}_3$ such that

$$\{t,\ w_{kl|1\le k<l\le m-1}, w_{j|1\le j\le m-2},\ \tilde{w}_{r|1\le r\le \tilde{n}}\}$$

is an orthonormal basis of $\mathfrak{D}_3$. Then, besides (6-11), we further use (3-7) to get

$$h(t,\tilde{w}_r) = \frac{k_0\mu\eta}{\rho} J\tilde{w}_r, \quad 1 \le r \le \tilde{n}. \tag{6-16}$$

Now we define T and T^* as in case (i). Similarly to Lemma 6.4, we can easily show:

Lemma 6.5. *For case (ii), we have*

$$h(T,T)=\eta_1 JT,\quad h(T,u)=\eta_2 Ju \quad \text{and} \quad h(T,\tilde{w}_r)=\eta_3 J\tilde{w}_r, \tag{6-17}$$

for $1\le r\le \tilde{n}$. Here η_1 and η_2 are defined by (6-14) and $\eta_3=\mu/\alpha$. These satisfy the relations $\eta_2\ne\eta_3$, $2\eta_2\ne\eta_1\ne 2\eta_3$,

$$\eta_1=\eta_2+\eta_3 \quad \text{and} \quad \eta_2\eta_3=\mu(\eta+\tfrac{1}{2}\lambda_1)=-1, \tag{6-18}$$

where u is one of T^*, v_i, w_j, w_{kl} and $1\le i,k,l\le m-1$, $1\le j\le m-2$.

Based on the conclusions of Lemma 6.5, after a similar argument as in the proof of Theorem 5.1, we can apply Theorem 4.13 (see also [Li and Wang 2011, Theorem 4.6]) to conclude that in case (ii) M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form. This completes the proof of Theorem 6.1. □

7. Case $\{\mathfrak{C}_m\}_{2\le m\le n-1}$ with $k_0\ge 2$ and $\mathfrak{p}=1$

Theorem 7.1. *Let $M\subset \mathbb{CP}^n(4)$ be a Lagrangian submanifold in a complex space form with constant holomorphic sectional curvature 4. Suppose that M is not totally geodesic and has parallel second fundamental form. Suppose also that $1\le \dim\mathfrak{D}_2=m-1\le n-2$, and k_0 and $\mathfrak{p}$ defined in Section 4 satisfy $k_0\ge 2$ and $\mathfrak{p}=1$. Then $n\ge \frac{1}{4}(m+1)^2-1$. Moreover, if $n=\frac{1}{4}(m+1)^2$, then M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, and if $n\ge\frac{1}{4}(m+1)^2+1$, then M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form.*

Lemma 7.2. *Suppose $\dim\mathfrak{D}_2=m-1\ge 1$, $k_0\ge 2$ and $\mathfrak{p}=1$. Then from the decomposition (4-6) there exist unit vectors $u_j\in V_{v_j}(0)$, $1\le j\le k_0=\frac{1}{2}(m-1)$, such that the orthonormal basis $\{v_1,u_1,\dots,v_{k_0},u_{k_0}\}$ of $\mathfrak{D}_2$ satisfies the relations*

$$L(v_j,u_l)=-L(u_j,v_l) \quad \text{and} \quad L(v_j,v_l)=L(u_j,u_l) \tag{7-1}$$

for $1\le j,l\le k_0$.

Proof. We have the decomposition (4-6) with $\dim V_{v_j}(0)=1$, $1\le j\le k_0$. Let $V_{v_2}(0)=\{u_2\}$, here u_2 is a unit vector.

According to Lemma 4.7, for each $j\ne 2$, we have a unique unit vector u_j in $V_{v_j}(0)$ satisfying

$$L(v_j,-u_2)=L(u_j,v_2) \quad \text{and} \quad L(u_j,u_2)=L(v_j,v_2) \tag{7-2}$$

for $1\le j\le k_0$, $j\ne 2$. The lemma now follows from the following claim. □

Claim 7.3. $L(v_j,u_l)=-L(u_j,v_l)$ and $L(v_j,v_l)=L(u_j,u_l)$ for $1\le j,\ l\le k_0$, $j,l\ne 2$.

Proof. For $j=l$, the fact that $u_j \in V_{v_j}(0)$ implies $L(v_j,u_j)=0$. It follows that $L(u_j,u_j)=L(v_j,v_j)$.

Next, for $k_0 \geq 3$, we fix $j,l \neq 2$ such that $j \neq l$. By Lemma 4.7, there exists a unique unit vector in $V_{v_j}(0)$, denoted $u_j(l)$, such that

$$L(v_j,u_l)=-L(u_j(l),v_l). \tag{7-3}$$

Since both unit vectors u_j and $u_j(l)$ are in $V_{v_j}(0)$ and $\dim V_{v_j}(0)=1$, we have $u_j(l)=\epsilon u_j$ with $\epsilon=\pm 1$, which implies that $u_j(l)-\epsilon u_j=0$ and

$$L(v_j,u_l)=-\epsilon L(u_j,v_l) \quad \text{and} \quad L(v_j,v_l)=\epsilon L(u_j,u_l). \tag{7-4}$$

By using (7-2) and Lemma 4.8, we find that

$$\begin{aligned} &h(L(u_j,u_l),L(v_2,u_j))=\tau JL(u_l,v_2)=-\tau JL(v_l,u_2) \quad \text{and} \\ &h(L(v_j,v_l),L(v_2,u_j))=h(L(v_j,v_l),-L(v_j,u_2))=-\tau JL(v_l,u_2), \end{aligned}$$

which imply

$$0=h(L(v_j,v_l)-\epsilon L(u_j,u_l),L(v_2,u_j))=-\tau(1-\epsilon)JL(v_l,u_2). \tag{7-5}$$

Combining equations (7-4) and (7-5) we get $\epsilon=1$, which completes the proof of the claim. $\square$

Remark 7.4. For $\mathfrak{p}=1$ we have $\dim \mathfrak{D}_2=2k_0$. Denote

$$V_j=\{v_j\}\oplus V_{v_j}(0)=\{v_j\}\oplus\{u_j\}, \quad 1\leq j\leq k_0.$$

For each $1\leq j\leq k_0$, we define a linear map $J_0: V_j \to V_j$ by setting

$$J_0 v_j = u_j \quad \text{and} \quad J_0 u_j=-v_j.$$

Then $J_0:\mathfrak{D}_2\to\mathfrak{D}_2$ is an almost complex structure and Lemma 7.2 shows that it satisfies the relations

$$L(J_0u,v)=-L(u,J_0v) \quad \text{and} \quad L(J_0u,J_0v)=L(u,v) \tag{7-6}$$

for all $u,v\in\mathfrak{D}_2$.

Let $\{v_1,u_1,\dots,v_{k_0},u_{k_0}\}$ be the orthonormal basis of $\mathfrak{D}_2$ from Lemma 7.2. Combining Lemma 4.4 with the fact that $u_j,v_j\in V_{v_l}(\tau)=V_{u_l}(\tau)$ for $j\neq l$, we have

$$\langle L(v_j,u_l),L(v_j,u_l)\rangle=\langle L(v_j,v_l),L(v_j,v_l)\rangle=\tau, \tag{7-7}$$

for $j\neq l$. Next we get

$$\begin{aligned} \langle L(u_j,v_{l_1}),L(u_j,v_{l_2})\rangle &= \langle L(v_j,u_{l_1}),L(v_j,u_{l_2})\rangle \\ &= \langle L(v_j,v_{l_1}),L(v_j,v_{l_2})\rangle = 0, \end{aligned} \tag{7-8}$$

for j, l_1, l_2 distinct. Then

$$\langle L(v_{j_1}, v_{j_2}), L(v_{j_3}, v_{j_4})\rangle = 0, \qquad j_1, j_2, j_3, j_4 \text{ distinct}, \tag{7-9}$$

$$\langle L(v_j, v_l), L(v_{j_1}, u_{l_1})\rangle = 0, \qquad j \neq l \text{ and } j_1 \neq l_1. \tag{7-10}$$

Thus

$$\left\{\tfrac{1}{\sqrt{\tau}} L(v_j, v_l)\right\}_{1\le j<l\le k_0} \cup \left\{\tfrac{1}{\sqrt{\tau}} L(v_j, u_l)\right\}_{1\le j<l\le k_0}$$

consists of $k_0(k_0-1) = \frac{1}{4}(m-1)(m-3)$ orthonormal vectors. For the subset $\{L(v_j, v_j) = L(u_j, u_j)\}_{1\le j\le k_0}$, we note that

$$\langle L(v_j, v_j), L(v_j, v_j)\rangle = \lambda_1\eta/2, \tag{7-11}$$

$$\langle L(v_j, v_j), L(v_l, v_l)\rangle = \mu\eta/2, \tag{7-12}$$

$$\langle L(v_j, v_j), L(v_j, v_l)\rangle = \langle L(v_j, v_j), L(v_j, u_l)\rangle = 0, \tag{7-13}$$

$$\langle L(v_j, v_j), L(v_{l_1}, v_{l_2})\rangle = \langle L(v_j, v_j), L(v_{l_1}, u_{l_2})\rangle = 0, \tag{7-14}$$

where $1 \le j \neq l \le k_0$ and $1 \le j, l_1, l_2$ distinct $\le k_0$.

As in the previous section, we see that

$$\{L_j := L(v_1, v_1) + \cdots + L(v_j, v_j) - jL(v_{j+1}, v_{j+1}) \mid 1 \le j \le k_0 - 1\}$$

are $k_0 - 1 = \frac{1}{2}(m-3)$ mutually orthogonal vectors which are orthogonal to all $L(v_j, v_l)$ and $L(v_j, u_l)$, $j \neq l$. We also easily have $\langle L_j, L_j\rangle = 2j(j+1)\tau \neq 0$. Hence

$$\begin{cases} w_j = \frac{1}{\sqrt{2j(j+1)\tau}} L_j, & 1 \le j \le k_0 - 1 = \frac{1}{2}(m-3), \\ w_{kl} = \frac{1}{\sqrt{\tau}} L(v_k, v_l), & 1 \le k < l \le k_0 = \frac{1}{2}(m-1), \\ w'_{kl} = \frac{1}{\sqrt{\tau}} L(v_k, u_l), & 1 \le k < l \le k_0 = \frac{1}{2}(m-1), \end{cases} \tag{7-15}$$

are $\frac{1}{4}(m+1)(m-3)$ orthonormal vectors in $\operatorname{Im}(L) \subset \mathfrak{D}_3$.

Finally, it is easily verified that $\frac{1}{2}\operatorname{Tr} L = L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})$ is orthogonal to the above $(m+1)(m-3)/4$ vectors and satisfies

$$\tfrac{1}{4}\langle \operatorname{Tr} L, \operatorname{Tr} L\rangle = \tfrac{1}{2}k_0\eta(\lambda_1 + (k_0-1)\mu) =: \rho^2, \quad \rho \ge 0. \tag{7-16}$$

Similarly as in the previous section we get that

Lemma 7.5. *We have* $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{4}(m+1)^2 - 1$.

Proof of Theorem 7.1. We define a unit vector $t = \frac{1}{2\rho}\operatorname{Tr} L$. Again we need to consider two cases.

(i) $n = \frac{1}{4}(m+1)^2$. The previous results show that the set $\{t, w_{kl}, w'_{kl}, w_j\}$, where we have $1 \le k < l \le \frac{1}{2}(m-1)$ and $1 \le j \le \frac{1}{2}(m-3)$, is an orthonormal basis of $\mathrm{Im}(L) = \mathfrak{D}_3$. By direct calculations applying Lemma 3.6, Lemma 4.8 and (7-7)-(7-14) we obtain again the expressions of (6-11) for $u = v_i, u_i$ and $w = w_j, w_{kl}, w'_{kl}$ with $1 \le i, k, l \le \frac{1}{2}(m-1)$ and $1 \le j \le \frac{1}{2}(m-3)$. Proceeding then in the same way as before, we can again apply Theorem 4.12 (see also [Li and Wang 2011, Theorem 1.6]) to conclude that in this case, M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form.

(ii) $n \ge \frac{1}{4}(m+1)^2 + 1$. Here we see that $\{t, w_{kl}, w'_{kl}, w_j\}$, where j, k, l are as before, is still an orthonormal basis of $\mathrm{Im}(L)$. But now $\mathrm{Im}(L) \subsetneq \mathfrak{D}_3$. Introduce the notation

$$\tilde{n} = n - \tfrac{1}{4}(m+1)^2 \ge 1$$

and choose $w'_1, \ldots, w'_{\tilde{n}}$ in the orthogonal complement of $\mathrm{Im}(L)$ in $\mathfrak{D}_3$, such that

$$\{t,\ w_{kl}, w'_{kl}, w_j, w'_r\}$$

where j, k, l are as before and $1 \le r \le \tilde{n}$, is an orthonormal basis of $\mathfrak{D}_3$. Then (3-7) gives that

$$h(t, w'_r) = \frac{k_0 \mu \eta}{2\rho} J w'_r, \quad 1 \le r \le \tilde{n}, \tag{7-17}$$

and we can again proceed exactly as in the previous section to conclude that in this case, M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form. □

8. Case $\mathfrak{C}_m$ ($2 \le m \le n-1$) with $k_0 \ge 2$ and $\mathfrak{p} = 3$

Theorem 8.1. *Let $M \subset \mathbb{CP}^n(4)$ be a Lagrangian submanifold in a complex space form with constant holomorphic sectional curvature* 4. *Suppose that M is not totally geodesic and that it has parallel second fundamental form. Suppose also that $1 \le \dim \mathfrak{D}_2 = m-1 \le n-2$, and k_0 and $\mathfrak{p}$ defined in Section 4 satisfy $k_0 \ge 2$ and $\mathfrak{p} = 3$. Then $n \ge \frac{1}{8}(m-1)(m+5)$. If $n = \frac{1}{8}(m-1)(m+5)+1$, then M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, and if $n \ge \frac{1}{8}(m-1)(m+5)+2$, then M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form.*

Lemma 8.2. *Suppose* $\dim \mathfrak{D}_2 = m-1 \ge 1$, $k_0 \ge 2$ *and* $\mathfrak{p} = 3$. *Then from the decomposition (4-6) there exist unit orthogonal vectors*

$$x_j, y_j, z_j \in V_{v_j}(0), \quad 1 \le j \le k_0 = \tfrac{1}{4}(m-1),$$

such that the orthonormal basis $\{v_1, x_1, y_1, z_1, \dots, v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$ *of* $\mathscr{D}_2$ *satisfies*

$$
\begin{aligned}
L(x_j, x_l) &= L(y_j, y_l) = L(z_j, z_l) = L(v_j, v_l),\\
L(v_j, x_l) &= -L(x_j, v_l) = -L(y_j, z_l) = L(y_l, z_j),\\
L(v_j, y_l) &= -L(y_j, v_l) = -L(z_j, x_l) = L(x_j, z_l),\\
L(v_j, z_l) &= -L(z_j, v_l) = -L(x_j, y_l) = L(x_l, y_j),
\end{aligned}
\tag{8-1}
$$

for all $1 \le j, l \le k_0$.

Proof. We use the decomposition (4-6) with $\dim V_{v_j}(0) = 3$ for $1 \le j \le k_0$.

Denote $V_j = \{v_j\} \oplus V_{v_j}(0)$. First we choose arbitrary orthonormal vectors $x_1, y_1 \in V_{v_1}(0)$, next by using Lemma 4.6 and Lemma 4.7 we can first find unit vectors $x_j, y_j \in V_{v_j}(0)$, $j \ge 2$ such that

$$
\begin{aligned}
&L(x_j, x_1) = L(y_j, y_1) = L(v_j, v_1), && L(v_j, x_1) = -L(x_j, v_1),\\
&L(v_j, y_1) = -L(y_j, v_1), && L(x_1, y_j) = -L(x_j, y_1).
\end{aligned}
\tag{8-2}
$$

Next we choose z_j, z_1^j such that $L(v_j, z_1^j) = -L(z_j, v_1) = -L(x_j, y_1)$. By using the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
L(x_j, x_1) &= L(y_j, y_1) = L(z_j, z_1^j) = L(v_j, v_1),\\
L(v_j, x_1) &= -L(x_j, v_1) = -L(y_j, z_1^j) = L(y_1, z_j),\\
L(v_j, y_1) &= -L(y_j, v_1) = -L(z_j, x_1) = L(x_j, z_1^j),\\
L(v_j, z_1^j) &= -L(z_j, v_1) = -L(x_j, y_1) = L(x_1, y_j).
\end{aligned}
\tag{8-3}
$$

Claim 8.3. For all $j \ge 2$, the families $\{x_1, y_1, z_1^j\}$ and $\{x_j, y_j, z_j\}$ of (8-3) are orthonormal bases of $V_{v_1}(0)$ and $V_{v_j}(0)$, respectively.

Proof of claim. In fact, from (8-3) we have

$$
\begin{aligned}
\tau\langle z_1^j, x_1\rangle &= \langle L(v_j, z_1^j), L(v_j, x_1)\rangle = \langle L(x_j, -y_1), L(x_j, -v_1)\rangle = \tau\langle y_1, v_1\rangle = 0,\\
\tau\langle z_1^j, y_1\rangle &= \langle L(v_j, z_1^j), L(v_j, y_1)\rangle = \langle L(y_j, -x_1), L(y_j, -v_1)\rangle = \tau\langle x_1, v_1\rangle = 0,
\end{aligned}
$$

hence we get $\{x_1, y_1, z_1^j\}$ is an orthonormal basis of $V_{v_1}(0)$.

For $j \ge 2$, from (8-3) we have

$$
\tau\langle x_j, y_j\rangle = \langle L(v_1, x_j), L(v_1, y_j)\rangle = \langle L(v_j, -x_1), L(v_j, -y_1)\rangle = \tau\langle x_1, y_1\rangle = 0,
$$

similarly, we get $\langle x_j, z_j\rangle = \langle x_1, z_1^j\rangle = 0$ and $\langle y_j, z_j\rangle = \langle y_1, z_1^j\rangle = 0$. This completes the proof. □

Claim 8.4. The vectors z_1^j and z_1^l of (8-3) are equal for all $2 \le j, l \le k_0$. If we denote this common value by z_1, then (8-1) holds.

Proof of claim. By Claim 8.3, we know that for $j \neq l$, $j, l \geq 2$ we have $z_1^j = \varepsilon_{jl} z_1^l$ with $\varepsilon_{jl} = \pm 1$. From Lemma 4.8 and (8-3) we get

$$\begin{aligned} \varepsilon_{jl} \tau JL(v_j, v_l) &= h(L(v_j, z_1^j), L(v_l, z_1^l)) \\ &= h(L(x_j, y_1), L(x_l, y_1)) = \tau JL(x_j, x_l). \end{aligned} \tag{8-4}$$

Similarly, we get

$$\begin{aligned} \varepsilon_{jl} L(v_j, v_l) &= L(y_j, y_l) = L(z_j, z_l), \\ \varepsilon_{jl} L(x_j, x_l) &= L(y_j, y_l) = L(z_j, z_l) = L(v_j, v_l). \end{aligned} \tag{8-5}$$

From (8-4) and (8-5) we get $\varepsilon_{jl} = 1$ and

$$L(v_j, v_l) = L(x_j, x_l) = L(y_j, y_l) = L(z_j, z_l), \quad j \neq l, \, j, l \geq 2. \tag{8-6}$$

Let $z_1 = z_1^2 = \cdots = z_1^{k_0}$, then by (8-3) and Lemma 4.8 we have

$$\begin{aligned} \tau JL(x_j, y_l) &= h(L(y_1, x_j), L(y_1, y_l)) \\ &= h(L(v_1, z_j), L(v_1, v_l)) = \tau JL(z_j, v_l). \end{aligned} \tag{8-7}$$

From (8-6) and (8-7), and by using Lemma 4.6 and Lemma 4.7 we get that (8-1) holds. □

Combining the above claims completes the proof of the lemma. □

Remark 8.5. Having fixed the orthonormal basis of $\mathfrak{D}_2$ satisfying (8-1), we can now define three almost complex structures $J_1, J_2, J_3 : \mathfrak{D}_2 \to \mathfrak{D}_2$ such that for all $1 \leq j \leq k_0$,

$$\begin{aligned} J_1 v_j &= x_j, & J_2 v_j &= y_j, & J_3 v_j &= z_j, \\ J_1 x_j &= -v_j, & J_2 y_j &= -v_j, & J_3 z_j &= -v_j, \end{aligned} \tag{8-8}$$

and furthermore J_1, J_2 and J_3 satisfy

$$J_1 \circ J_1 = J_2 \circ J_2 = J_3 \circ J_3 = -\,\mathrm{id} \quad \text{and} \quad J_1 J_2 = -J_2 J_1 = J_3. \tag{8-9}$$

Then we define a quaternionic structure $\{J_1, J_2, J_3\}$ on $\mathfrak{D}_2$. It is important to remark that (8-1) is equivalent to the relations

$$L(J_s u, v) = -L(u, J_s v) \quad \text{and} \quad L(J_s u, J_s v) = L(u, v) \tag{8-10}$$

for all $s = 1, 2, 3$ and $u, v \in \mathfrak{D}_2$.

We have $m - 1 = 4k_0$ and $k_0 \geq 2$. Let $\{v_1, x_1, y_1, z_1, \ldots, v_{k_0}, x_{k_0}, y_{k_0}, z_{k_0}\}$ be an orthonormal basis of $\mathfrak{D}_2$ as constructed in Lemma 8.2. Applying Lemma 4.4 and the fact that for $j \neq l$, $v_j, x_j, y_j, z_j \in V_{v_l}(\tau) = V_{x_l}(\tau) = V_{y_l}(\tau) = V_{z_l}(\tau)$, we easily show that

$$\text{(8-11)}\quad \langle L(v_j,x_l),L(v_j,x_l)\rangle=\langle L(v_j,y_l),L(v_j,y_l)\rangle \\ =\langle L(v_j,z_l),L(v_j,z_l)\rangle=\langle L(v_j,v_l),L(v_j,v_l)\rangle=\tau,$$

for $j\neq l$. We also get

$$\text{(8-12)}\quad \langle L(x_j,v_{l_1}),L(x_j,v_{l_2})\rangle=\langle L(v_j,x_{l_1}),L(v_j,x_{l_2})\rangle=\langle L(y_j,v_{l_1}),L(y_j,v_{l_2})\rangle \\ =\langle L(v_j,y_{l_1}),L(v_j,y_{l_2})\rangle=\langle L(z_j,v_{l_1}),L(z_j,v_{l_2})\rangle \\ =\langle L(v_j,z_{l_1}),L(v_j,z_{l_2})\rangle=\langle L(v_j,v_{l_1}),L(v_j,v_{l_2})\rangle \\ =0,$$

for j, l_1, l_2 distinct. Next we get

$$\text{(8-13)}\quad \langle L(v_{j_1},v_{j_2}),L(v_{j_3},v_{j_4})\rangle=\langle L(v_{j_1},x_{j_2}),L(v_{j_3},x_{j_4})\rangle=\langle L(v_{j_1},y_{j_2}),L(v_{j_3},y_{j_4})\rangle \\ =\langle L(v_{j_1},z_{j_2}),L(v_{j_3},z_{j_4})\rangle=0,$$

for j_1, j_2, j_3, j_4 distinct, and then

$$\text{(8-14)}\quad \langle L(v_j, v_l), L(v_{j_1}, x_{l_1})\rangle=\langle L(v_j, v_l), L(v_{j_1}, y_{l_1})\rangle \\ =\langle L(v_j, v_l), L(v_{j_1}, z_{l_1})\rangle=0,$$

for $j\neq l$ and $j_1\neq l_1$.

For $\{L(v_j, v_j)=L(x_j, x_j)=L(y_j, y_j)=L(z_j, z_j)\}_{1\le j\le k_0}$, we note that

$$\text{(8-15)}\quad \langle L(v_j, v_j), L(v_j, v_j)\rangle=\tfrac{1}{2}\lambda_1\eta,$$

$$\text{(8-16)}\quad \langle L(v_j, v_j), L(v_l, v_l)\rangle=\frac{n+1}{4(n-i)}\lambda_1^2-2\tau=\tfrac{1}{2}\mu\eta,$$

$$\text{(8-17)}\quad \langle L(v_j, v_j), L(v_j, v_l)\rangle=\langle L(v_j, v_j), L(v_j, u_l)\rangle=0,$$

$$\text{(8-18)}\quad \langle L(v_j, v_j), L(v_{l_1}, v_{l_2})\rangle=\langle L(v_j, v_j), L(v_{l_1}, u_{l_2})\rangle=0,$$

for $1\le j, l, l_1, l_2\le k_0$ distinct. Similarly to the previous section, we deduce that

$$\{L_j := L(v_1, v_1)+\cdots+L(v_j, v_j)-jL(v_{j+1}, v_{j+1}) \mid 1\le j\le k_0-1\}$$

are $k_0-1=\frac{1}{4}(m-5)$ mutually orthogonal vectors which are orthogonal to all of the vectors $L(v_j, v_l)$, $L(v_j, x_l)$, $L(v_j, y_l)$, and $L(v_j, z_l)$, where $j\neq l$. Also, we have $\langle L_j, L_j\rangle=2j(j+1)\tau\neq 0$. Hence the vectors

$$w_j=\frac{1}{\sqrt{2j(j+1)\tau}}L_j,\quad w_{kl}=\frac{1}{\sqrt{\tau}}L(v_k, v_l),$$
$$w_{kl}^1=\frac{1}{\sqrt{\tau}}L(v_k, x_l),\quad w_{kl}^2=\frac{1}{\sqrt{\tau}}L(v_k, y_l),\quad w_{kl}^3=\frac{1}{\sqrt{\tau}}L(v_k, z_l),$$

where $1\le j\le k_0=\frac{1}{4}(m-1)$ and $1\le k<l\le k_0$, comprise $2k_0(k_0-1)+k_0-1=\frac{1}{8}(m+1)(m-5)$ orthonormal vectors in $\mathrm{Im}(L)\subset\mathfrak{D}_3$.

Finally, from Lemma 8.2, (8-15) and (8-16) it is easily known that the vector

$$\operatorname{Tr} L = 4\big(L(v_1, v_1) + \cdots + L(v_{k_0}, v_{k_0})\big)$$

is orthogonal to the above $\frac{1}{8}(m+1)(m-5)$ vectors and satisfies

$$\frac{1}{16}\langle \operatorname{Tr} L, \operatorname{Tr} L\rangle = \frac{1}{2}k_0\eta(\lambda_1 + (k_0-1)\mu) =: \rho^2, \quad \rho \geq 0. \tag{8-19}$$

The above results imply that

$$n = 1 + \dim \mathfrak{D}_2 + \dim \mathfrak{D}_3 \geq 1 + (m-1) + \tfrac{1}{8}(m+1)(m-5) = \tfrac{1}{8}(m-1)(m+5).$$

Lemma 8.6. *We have* $\operatorname{Tr} L = 0$ *if and only if* $n = \frac{1}{8}(m-1)(m+5)$.

Proof of Theorem 8.1. We need to consider two cases:

(i) $n = \frac{1}{8}(m-1)(m+5) + 1$.

(ii) $n \geq \frac{1}{8}(m-1)(m+5) + 2$.

In case (i), we have that

$$\{t,\ w_j\,|_{1\leq j\leq (i-5)/4},\ w_{kl},\ w_{kl}^1,\ w_{kl}^2,\ w_{kl}^3\,|_{1\leq k<l\leq (i-1)/4}\}$$

is an orthonormal basis of $\operatorname{Im}(L) = \mathfrak{D}_3$. In case (ii), in order to have an orthonormal basis we still need to add an orthonormal basis of $\mathfrak{D}_{32}$.

As in the previous sections, we get that (6-13) in case (i) and (6-17) in case (ii) are satisfied. Consequently we deduce that in case (i), M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, and in case (ii), we deduce that M is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form. This completes the proof. □

9. Case $\{\mathfrak{C}_m\}_{2\leq m\leq n-1}$ with $k_0 \geq 2$ and $\mathfrak{p} = 7$

Theorem 9.1. *Let* $M \subset \mathbb{CP}^n(4)$ *be a Lagrangian submanifold in a complex space form with constant holomorphic sectional curvature* 4. *Suppose that* M *is not totally geodesic and has parallel second fundamental form. Suppose also that* $1 \leq \dim \mathfrak{D}_2 = m-1 \leq n-2$ *and* k_0 *and* $\mathfrak{p}$ *defined in Section 4 satisfy* $k_0 \geq 2$ *and* $\mathfrak{p} = 7$. *Then* $k_0 = 2$ *and* $m = 17$, *which implies that* $n \geq 26$. *Moreover, if* $n = 27$ *we have that* M *is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, and if* $n \geq 28$, *then* M *is locally the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form.*

Lemma 9.2. *Suppose* $\dim \mathfrak{D}_2 = m-1 \geq 1$, $k_0 \geq 2$ *and* $\mathfrak{p} = 7$. *Then from the decomposition* (4-6), *if* $k_0 \geq 2$, *we can choose an orthonormal basis* $\{x_j\}_{1\leq j\leq 7}$

for $V_{v_1}(0)$ *and an orthonormal basis* $\{y_j\}_{1\le j\le 7}$ *for* $V_{v_2}(0)$ *so that by identifying* $e_j(v_1)=x_j$ *and* $e_j(v_2)=y_j$, *we have the relations*

$$L(e_j(v_1),e_l(v_2))=-L(v_1,e_je_l(v_2))=-L(e_le_j(v_1),v_2), \tag{9-1}$$

for $1\le j,l\le 7$, *where the product is defined by the following multiplication table*:

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-\mathrm{id}$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-\mathrm{id}$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-\mathrm{id}$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-\mathrm{id}$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-\mathrm{id}$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-\mathrm{id}$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-\mathrm{id}$

Proof. Let $k_0\ge 2$ and suppose we have the decomposition (4-6) with $\dim V_{v_j}(0)=7$ $(1\le j\le k_0)$.

Denote $V_j=\{v_j\}\oplus V_{v_j}(0)$. First we choose arbitrary orthonormal vectors $x_1,x_2\in V_{v_1}(0)$. Next we can use Lemma 4.6 and Lemma 4.7 to consecutively find unit vectors $y_1,y_2\in V_{v_2}(0)$, $x_3\in V_{v_1}(0)$ and $y_3\in V_{v_2}(0)$ satisfying

$$L(y_1,v_1)=-L(x_1,v_2),\qquad L(y_2,v_1)=-L(x_2,v_2), \tag{9-2}$$

$$L(y_1,x_2)=-L(v_2,x_3),\qquad L(y_3,v_1)=-L(x_3,v_2). \tag{9-3}$$

Now we pick an arbitrary unit vector $x_4\in V_{v_1}(0)$ so that it is orthogonal to all x_1,x_2 and x_3. Then we can take unit vectors $x_5,x_6,x_7\in V_{v_1}(0)$ and unit vectors $y_4,y_5,y_6,y_7\in V_{v_2}(0)$ inductively such that the following hold:

$$L(x_4,y_1)=-L(y_4,x_1)=-L(v_2,x_5)=L(v_1,y_5), \tag{9-4}$$

$$L(x_4,y_2)=-L(v_2,x_6)=L(v_1,y_6),\quad L(x_4,y_3)=-L(v_2,x_7)=L(v_1,y_7). \tag{9-5}$$

From the previous equations, together with the isotropy conditions and the Cauchy–Schwarz inequality, it immediately follows that $L(x_i,y_i)=L(v_1,v_2)$, for $i=1,\dots 7$. Applying once more the same properties it also follows that $L(x_i,y_j)=-L(x_j,y_i)$ and $L(x_i,v_2)=-L(y_i,v_1)$.

From (9-3) and (9-4) it additionally follows that

$$\begin{aligned}
L(y_1,x_3)&=L(x_2,v_2), & L(x_4,v_2)&=L(x_5,y_1),\\
L(x_4,y_5)&=-L(v_1,y_1), & L(x_4,v_2)&=L(x_6,y_2),\\
L(x_4,y_6)&=-L(v_1,y_2), & L(x_4,v_2)&=L(x_7,y_3),\\
L(x_4,y_7)&=-L(v_1,y_3). & &
\end{aligned}$$

Hence $L(x_4, v_2) = L(x_5, y_1) = L(x_6, y_2) = L(x_7, y_3)$. Repeating now the same procedure on the newly found identities shows that L has the desired form.

Finally note that the fact that $\{v_1, x_1, \dots, x_7\}$ and $\{v_2, y_1, \dots, y_7\}$ are orthonormal can be seen as follows. First, we have

$$\tau\langle x_1, x_3\rangle = \langle L(v_2, x_3), L(v_2, x_1)\rangle = \langle L(x_1, y_2), L(v_2, x_1)\rangle = \tau\langle v_2, y_2\rangle = 0.$$

The other equations are obtained similarly. □

Lemma 9.3. *Suppose* $\dim \mathcal{D}_2 = m-1 \geq 1$ *and* $\mathfrak{p} = 7$. *If* $k_0 \geq 2$ *in the decomposition* (4-6), *then in fact* $k_0 = 2$.

Proof. Suppose on the contrary that $k_0 \geq 3$. To choose a basis for $V_{v_3}(0)$, we follow the same ideas as in Lemma 9.2 for $V_{v_1}(0)$ and $V_{v_2}(0)$. Let x_1, x_2, x_3 be given as in Lemma 9.2, then we have unique unit vectors $z_1, z_2 \in V_{v_3}(0)$ and $\tilde{x}_3 \in V_{v_1}(0)$ that satisfy

$$L(z_1, v_1) = -L(x_1, v_3), \quad L(z_2, v_1) = -L(x_2, v_3) \quad \text{and} \quad L(z_1, x_2) = -L(v_3, \tilde{x}_3).$$

Now we pick an arbitrary unit vector $x_4 \in V_{v_1}(0)$ so that it is orthogonal to x_1, x_2, x_3 and $\tilde{x}_3$. Then we can choose unit vectors $\tilde{x}_5, \tilde{x}_6, \tilde{x}_7 \in V_{v_1}(0)$ and vectors $z_3, z_4, z_5, z_6, z_7 \in V_{v_3}(0)$ inductively by the following conditions:

$$\begin{aligned}
&L(z_3, v_1) = -L(\tilde{x}_3, v_3),\\
&L(x_4, z_2) = -L(v_3, \tilde{x}_6) = L(v_1, z_6),\\
&L(x_4, z_3) = -L(v_3, \tilde{x}_7) = L(v_1, z_7),\\
&L(x_4, z_1) = -L(z_4, x_1) = -L(v_3, \tilde{x}_5) = L(v_1, z_5).
\end{aligned}$$

Then, similarly to the proof of Lemma 9.2, we get that $\{z_1, z_2, z_3, z_4, z_5, z_6, z_7\}$ forms an orthonormal basis of $V_{v_3}(0)$ together with the relations between inner products of L:

$$L(e_j(v_1), e_l(v_3)) = -L(v_1, e_j e_l(v_3)) = -L(e_l e_j(v_1), v_3), \quad 1 \leq j, l \leq 7, \tag{9-6}$$

where $e_j e_l$ denotes a product defined by the multiplication table in Lemma 9.2.

We have two orthonormal bases of $V_{v_1}(0)$, namely $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $\{x_1, x_2, \tilde{x}_3, x_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7\}$. We first show that $\tilde{x}_i = x_i$ for $i = 3, 5, 6, 7$:

By Lemma 4.8 and the relations between the inner products of L, we get

$$\tau L(y_1, z_1) = h(L(y_1, x_2), L(z_1, x_2)) = h(L(v_1, y_3), L(v_1, z_3)) = \tau L(y_3, z_3).$$

Similarly, we get $L(v_2, v_3) = L(y_j, z_j)$ for $j = 1, \dots, 7$.

Since $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ and $\{x_1, x_2, \tilde{x}_3, x_4, \tilde{x}_5, \tilde{x}_6, \tilde{x}_7\}$ are two orthonormal bases for $V_{v_1}(0)$, we may assume that $x_3 = b_3\tilde{x}_3 + b_5\tilde{x}_5 + b_6\tilde{x}_6 + b_7\tilde{x}_7$. Then by Lemma 4.8 and the relations between the inner products of L, we get

$$\begin{aligned}\tau L(y_2,z_2) &= h(L(v_1,y_2),L(v_1,z_2))\\ &= -h(L(x_3,y_1),L(v_1,z_2))\\ &= b_3h(L(\tilde{x}_3,y_1),L(\tilde{x}_3,z_1))+b_5h(L(\tilde{x}_5,y_1),L(\tilde{x}_5,z_7))\\ &\qquad -b_6h(L(\tilde{x}_6,y_1),L(\tilde{x}_6,z_4))-b_7h(L(\tilde{x}_7,y_1),L(\tilde{x}_7,z_5))\\ &= b_3\tau L(y_1,z_1)+b_5\tau L(y_1,z_7)-b_6\tau L(y_1,z_4)-b_7\tau L(y_1,z_5).\end{aligned}$$

By the relations between the inner products of L, we get $L(y_1,z_1)=L(y_2,z_2)$, and that $L(y_1,z_4)$, $L(y_1,z_5)$ and $L(y_1,z_7)$ are orthogonal to each other. Hence we get $b_3=1,\ b_5=b_6=b_7=0$ and $x_3=\tilde{x}_3$. By a similar argument, we can prove that $\tilde{x}_i=x_i$ for $i=5,6,7$.

In order to complete the proof of Lemma 9.3, we will first use (9-1) and (9-6) to show that we have also similar relations between the spaces $V_2=\{v_2\}\oplus V_{v_2}(0)$ and $V_3=\{v_3\}\oplus V_{v_3}(0)$, that is,

(9-7) $$L(e_j(v_2),e_l(v_3))=-L(v_2,e_je_l(v_3))=-L(e_le_j(v_2),v_3),\qquad 1\le j,l\le 7,$$

where e_je_l denotes a product defined by the multiplication table in Lemma 9.2.

For $j=l$, by Lemma 4.8, (9-1) and (9-6) we have

$$\begin{aligned}\tau JL(e_j(v_2),e_j(v_3)) &= h(L(e_j(v_2),e_k(v_1)),L(e_j(v_3),e_k(v_1)))\\ &= h(-L(e_je_k(v_1),v_2),-L(e_je_k(v_1),v_3))=\tau JL(v_2,v_3).\end{aligned}$$

For $j\ne l$, from the table in Lemma 9.2 we have that there exists a unique k such that $e_le_j=\epsilon e_k,\ e_je_k=\epsilon e_l,\ e_ke_l=\epsilon e_j$, where ϵ is 1 or -1. Then by Lemma 4.8, (9-1) and (9-6) we have

$$\begin{aligned}\tau JL(e_j(v_2),e_l(v_3)) &= h(L(e_j(v_2),v_1),L(e_l(v_3),v_1))\\ &= h(L(-\epsilon e_le_k(v_2),v_1),L(e_l(v_3),v_1))\\ &= \epsilon h(L(e_l(v_1),e_k(v_2)),-L(e_l(v_1),v_3))\\ &= -\epsilon\tau JL(e_k(v_2),v_3)=-\tau L(e_le_j(v_2),v_3)\end{aligned}$$

and

$$\begin{aligned}\tau JL(v_2,e_je_l(v_3)) &= h\big(L(v_2,e_k(v_1)),L(e_je_l(v_3),e_k(v_1))\big)\\ &= h\big(L(v_2,\epsilon e_le_j(v_1)),L(-\epsilon e_k(v_3),e_k(v_1)))\big)\\ &= h\big(L(v_1,-\epsilon e_le_j(v_2)),L(-\epsilon v_3,v_1))\big)=\tau JL(e_le_j(v_2),v_3).\end{aligned}$$

From (9-1), (9-6), (9-7) and Lemma 4.8 we have

(9-8) $$h(L(v_1,y_6)+L(x_1,y_7),L(x_2,v_3))=0.$$

On the other hand, we have

$$h(L(v_1, y_6), L(x_2, v_3)) = h(L(v_1, y_6), -L(v_1, z_2)) = -\tau J L(y_6, z_2),$$
$$h(L(x_1, y_7), L(x_2, v_3)) = h(L(x_1, y_7), -L(x_1, z_3)) = -\tau J L(y_7, z_3).$$

These together with (9-8) give that

$$L(y_6, z_2) + L(y_7, z_3) = 0. \tag{9-9}$$

From (9-7) we have $L(y_6, z_2) = L(y_7, z_3)$. We also have that

$$\langle L(y_6, z_2), L(y_6, z_2)\rangle = \tau,$$

so we get a contradiction with (9-9). This completes the proof. □

Proof of Theorem 9.1. By Lemma 9.3, we have $k_0 = 2$, $m = 8k_0 + 1 = 17$ and $\dim \mathfrak{D}_2 = m - 1 = 16$.

Let $\{v_1, v_2, x_j, y_j \mid 1 \le j \le 7\}$ be the orthonormal basis of $\mathfrak{D}_2$ as constructed in Lemma 9.2, whose elements satisfy (9-1). Define $L_1 = L(v_1, v_1) - L(v_2, v_2)$, then direct calculation shows that

$$\langle L_1, L_1\rangle = 4\tau \neq 0. \tag{9-10}$$

We now easily see that the nine vectors

$$w_0 = \frac{1}{2\sqrt{\tau}} L_1, \quad w_1 = \frac{1}{\sqrt{\tau}} L(v_1, v_2) \quad \text{and} \quad w_{j+1} = \frac{1}{\sqrt{\tau}} L(v_1, y_j), \quad 1 \le j \le 7,$$

in $\operatorname{Im}(L) \subset \mathfrak{D}_3$ are orthonormal one to another.

Note that $\operatorname{Tr} L = 8(L(v_1, v_1) + L(v_2, v_2))$ is orthogonal to the above nine vectors. Using (3-3) and (3-4), the vector $\operatorname{Tr} L$ obviously satisfies

$$\tfrac{1}{64}\langle \operatorname{Tr} L, \operatorname{Tr} L\rangle = \tfrac{1}{2} k_0 \eta(\lambda_1 + (k_0 - 1)\mu) = \eta(\lambda_1 + \mu) =: \rho^2, \quad \rho \ge 0.$$

Then we have the conclusion

$$n = 1 + \dim \mathfrak{D}_2 + \dim \mathfrak{D}_3 \ge 1 + 16 + 9 = 26,$$

and as proved in previous sections we see that $n = 26$ if and only if $\operatorname{Tr} L = 0$.

When $n = 27$ or $n \ge 28$, we can still define a unit vector $t = \frac{1}{8\rho} \operatorname{Tr} L$. As before we get the same expressions as in Lemma 6.3, 6.4 and 6.5 which allows us to conclude that M is locally the Calabi product of a point with a lower-dimensional Lagrangian submanifold with parallel second fundamental form, or the Calabi product of two lower-dimensional Lagrangian submanifolds with parallel second fundamental form. □

10. The remaining cases

In this section we will complete the proof of the classification theorem. Let $k = k_0 + 1$, we will show that if M is neither totally geodesic nor can be decomposed as a Calabi product then one of the following applies:

(i) $n = \frac{1}{2}k(k+1) - 1$, $k \geq 3$, and M is congruent with $\mathrm{SU}(k)/\mathrm{SO}(k)$.

(ii) $n = k^2 - 1$, $k \geq 3$, and M is congruent with $\mathrm{SU}(k)$.

(iii) $n = 2k^2 - k - 1$, $k \geq 3$, and M is congruent with $\mathrm{SU}(2k)/\mathrm{Sp}(k)$.

(iv) $n = 26$ and M is congruent with $\mathrm{E}_6/\mathrm{F}_4$.

From Naitoh [1981b; 1983a; 1983b] we see that there indeed exist parallel immersions of the above spaces of the previously mentioned dimensions into the complex projective space.

From the previous remaining sections, each of the resulting cases corresponds to one of the cases $\mathfrak{p} = 0, 1, 3, 7$ with $\mathfrak{D}_{32} = \{0\}$ (from Lemma 6.2, Lemma 7.5, Lemma 8.6 and the arguments in Section 9) and $\operatorname{Tr} L$ vanishing. Note that in each of the above cases, the vanishing of $\operatorname{Tr} L$ allows to determine λ_1 explicitly. We also have in each of the cases a basis and we can compute the components of the second fundamental form from Lemmas 3.2, 3.4 and 4.8. For example in the case of $\mathfrak{p} = 0$, this basis is spanned by

$$\{e_1,\ v_1, \dots, v_{k_0},\ L(v_j, v_j)|_{1\leq j\leq k_0-1},\ L(v_j, v_k) \mid 1 \leq j < k \leq k_0\}.$$

As M is parallel we can extend this basis using parallel translation thus obtaining the same expression of the second fundamental form at every point. Applying then the lemma of Cartan, as the previously mentioned spaces are also parallel and therefore must admit a similar basis, shows that M is isometric with one of the previously mentioned spaces. Finally applying the uniqueness result for Lagrangian immersions shows also that the immersion of M is congruent to one of Naitoh's examples.

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Received January 10, 2011. Revised May 31, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 255 No. 1 January 2012