*Pacific Journal of Mathematics*

# ULTRA-DISCRETIZATION OF THE $D_4^{(3)}$-GEOMETRIC CRYSTAL TO THE $G_2^{(1)}$-PERFECT CRYSTALS

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Volume 255 No. 1 January 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 255, No. 1, 2012

# ULTRA-DISCRETIZATION OF THE $D_4^{(3)}$-GEOMETRIC CRYSTAL TO THE $G_2^{(1)}$-PERFECT CRYSTALS

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**Let $\mathfrak{g}$ be an affine Lie algebra and $\mathfrak{g}^L$ its Langlands dual. It was conjectured by Kashiwara, Nakashima, and Okado that $\mathfrak{g}$ has a positive geometric crystal whose ultra-discretization is isomorphic to the limit of certain coherent family of perfect crystals for $\mathfrak{g}^L$. We prove that the ultra-discretization of the positive geometric crystal for $\mathfrak{g} = D_4^{(3)}$ given by Igarashi and Nakashima is isomorphic to the limit of the coherent family of perfect crystals for $\mathfrak{g}^L = G_2^{(1)}$ constructed by Misra, Mohamad, and Okado.**

## 1. Introduction

Let $A = (a_{ij})_{i,j\in I}$, where $I = \{0, 1, \dots, n\}$, be an affine Cartan matrix and let $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be a given Cartan datum. Let $\mathfrak{g} = \mathfrak{g}(A)$ denote the associated affine Lie algebra [Kac 1990] and $U_q(\mathfrak{g})$ denote the corresponding quantum affine algebra. Let $P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta$ denote the affine weight lattice and $P^\vee = \mathbb{Z}\alpha_0^\vee \oplus \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee \oplus \mathbb{Z}d$ the dual affine weight lattice. For a dominant weight $\lambda \in P^+ = \{\mu \in P \mid \mu(h_i) \geq 0 \text{ for all } i \in I\}$ of level $l = \lambda(\mathbf{c})$ (where $\mathbf{c}$ is the canonical central element), Kashiwara [1990] defined the crystal base $(L(\lambda), B(\lambda))$ for the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$. The crystal $B(\lambda)$ is the $q = 0$ limit of the canonical basis [Lusztig 1990] or the global crystal basis [Kashiwara 1991]. It has many interesting combinatorial properties. To give an explicit realization of $B(\lambda)$, the notions of affine crystal and perfect crystal were introduced in [Kang et al. 1992a]. It is shown there that the affine crystal $B(\lambda)$ for the level $l \in \mathbb{Z}_{>0}$ integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ can be realized as the semi-infinite tensor product $\cdots \otimes B_l \otimes B_l \otimes B_l$, where $B_l$ is a perfect crystal of level $l$. This is known as the path realization.

Kang et al. [1994] remarked that one needs a coherent family of perfect crystals $\{B_l\}_{l\geq 1}$ in order to give a path realization of the Verma module $M(\lambda)$ (or $U_q^-(\mathfrak{g})$). In particular, the crystal $B(\infty)$ of $U_q^-(\mathfrak{g})$ can be realized as the semi-infinite tensor

Misra was supported in part by NSA Grant H98230-08-1-0080. Nakashima was supported in part by JSPS Grants in Aid for Scientific Research #22540031.
*MSC2010:* primary 17B37, 17B67; secondary 22E65, 14M15.
*Keywords:* geometric crystals, perfect crystals, ultra-discretization.

product $\cdots \otimes B_\infty \otimes B_\infty \otimes B_\infty$ where $B_\infty$, is the limit of the coherent family of perfect crystals $\{B_l\}_{l\geq 1}$.

At least one coherent family $\{B_l\}_{l\geq 1}$ of perfect crystals and its limit is known for $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}, D_4^{(3)}, G_2^{(1)}$. (See [Kang et al. 1992b; 1994; Yamane 1998; Kashiwara et al. 2007; Misra et al. 2010].)

A perfect crystal is indeed a crystal for certain finite-dimensional modules of the quantum affine algebra $U_q(\mathfrak{g})$ named after Kirillov and Reshetikhin [1987], and known as KR-modules for short. KR-modules are parametrized by two integers, $i \in I \setminus \{0\}$ and $l > 0$. Let $\{\varpi_i\}_{i\in I\setminus\{0\}}$ be the set of level 0 fundamental weights [Kashiwara 2002]. Hatayama et al. [1999; 2002] conjectured that any KR-module $W(l\varpi_i)$ admits a crystal base $B^{i,l}$ in the sense of Kashiwara and that $B^{i,l}$ is perfect if $l$ is a multiple of $c_i^\vee := \max(1, 2/(\alpha_i, \alpha_i))$. This conjecture has been proved for quantum affine algebras $U_q(\mathfrak{g})$ of classical types [Okado and Schilling 2008; Fourier et al. 2009; 2010]. When $\{B^{i,l}\}_{l\geq 1}$ is a coherent family of perfect crystals we denote its limit by $B_\infty(\varpi_i)$, or just $B_\infty$ if there is no confusion.

The notion of geometric crystals is a geometric analog to Kashiwara's crystal [Kashiwara 1990]. It was defined in [Berenstein and Kazhdan 2000] for reductive algebraic groups and extended to general Kac–Moody groups in [Nakashima 2005a]. For a given Cartan datum $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$, a geometric crystal is defined as a quadruple $\mathcal{V}(\mathfrak{g}) = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, where $X$ is an algebraic variety, $e_i : \mathbb{C}^\times \times X \to X$ are rational $\mathbb{C}^\times$-actions and $\gamma_i, \varepsilon_i : X \to \mathbb{C}$ $(i \in I)$ are rational functions satisfying certain conditions (see Definition 2.1). Geometric crystals have many properties similar to algebraic crystals. For instance, the product of two geometric crystals admits the structure of a geometric crystal if they are induced from unipotent crystals [Berenstein and Kazhdan 2000]. A geometric crystal is said to be a positive geometric crystal if it admits a positive structure (see Definition 2.5). A remarkable relation between positive geometric crystals and algebraic crystals is the ultra-discretization functor $\mathcal{UD}$ between them (page 123). Applying this functor, positive rational functions are transferred to piecewise linear functions by the simple correspondence:

$$x \times y \mapsto x + y, \qquad \frac{x}{y} \mapsto x - y, \qquad x + y \mapsto \max(x, y).$$

Let $G$ denote the affine Kac–Moody group associated with the affine Lie algebra $\mathfrak{g}$. Let $B^\pm$ be fixed Borel subgroups and $T$ the maximal torus of $G$ such that $B^+ \cap B^- = T$. Set $y_i(c) := \exp(cf_i)$, and let $\alpha_i^\vee(c) \in T$ be the image of $c \in \mathbb{C}^\times$ under the group morphism $\mathbb{C}^\times \to T$ induced by the simple coroot $\alpha_i^\vee$. We set $Y_i(c) := y_i(c^{-1})\,\alpha_i^\vee(c) = \alpha_i^\vee(c)\,y_i(c)$. Let $W$ and $\widetilde{W}$ be the Weyl group and extended Weyl group associated with $\mathfrak{g}$. The Schubert cell

$$X_w := BwB/B,$$

where $w = s_{i_1} \cdots s_{i_k} \in W$, is birationally isomorphic to the variety

$$B_\iota^- := \{Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mid x_1, \dots, x_k \in \mathbb{C}^\times\} \subset B^-,$$

and $X_w$ has a natural geometric crystal structure, where $\iota = i_1, \dots, i_k$ is a reduced word for $w$. [Berenstein and Kazhdan 2000; Nakashima 2005a].

Let $W(\varpi_i)$ be the KR-module (also called the fundamental representation) of $U_q(\mathfrak{g})$ with $\varpi_i$ as an extremal weight (see [Kashiwara 2002]). Denote its specialization at $q = 1$ by the same symbol, $W(\varpi_i)$. It is a finite-dimensional $\mathfrak{g}$-module (not necessarily irreducible). Let $\mathbb{P}(\varpi_i)$ be the projective space $(W(\varpi_i) \setminus \{0\})/\mathbb{C}^\times$. For any $i \in I$ the translation $t(c_i^\vee \varpi_i)$ belongs to $\widetilde{W}$ (see [Kashiwara et al. 2008]). For a subset $J$ of $I$, let us denote by $\mathfrak{g}_J$ the subalgebra of $\mathfrak{g}$ generated by $\{e_i, f_i\}_{i \in J}$. For an integral weight $\mu$, define $I(\mu) := \{j \in I \mid \langle \alpha_j^\vee, \mu \rangle \geq 0\}$.

**Conjecture 1.1** [Kashiwara et al. 2008]. *For any $i \in I \setminus \{0\}$ there exist a unique variety $X$ endowed with a positive $\mathfrak{g}$-geometric crystal structure and a rational mapping $\pi : X \to \mathbb{P}(\varpi_i)$ satisfying the following properties*:

(i) *For an arbitrary extremal vector $u \in W(\varpi_i)_\mu$, writing the translation $t(c_i^\vee \mu)$ as $\tau w \in \widetilde{W}$ with a Dynkin diagram automorphism $\tau$ and $w = s_{i_1} \cdots s_{i_k}$, there exists a birational mapping $\xi : B^-_{i_1,\dots,i_k} \to X$ such that $\xi$ is a morphism of $\mathfrak{g}_{I(\mu)}$-geometric crystals and that the composition $\pi \circ \xi : B^-_{i_1,\dots,i_k} \to \mathbb{P}(\varpi_i)$ coincides with $Y_{i_1}(x_1) \cdots Y_{i_k}(x_k) \mapsto Y_{i_1}(x_1) \cdots Y_{i_k}(x_k)\bar{u}$, where $\bar{u}$ is the line including $u$.*

(ii) *The ultra-discretization* (*Section 2*) *of $X$ is isomorphic to the crystal $B_\infty = B_\infty(\varpi_i)$ of the Langlands dual $\mathfrak{g}^L$.*

In [Kashiwara et al. 2008], it was shown that this conjecture is true for $i = 1$ and $\mathfrak{g} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$. In [Nakashima 2007], a positive geometric crystal for $\mathfrak{g} = G_2^{(1)}$ and $i = 1$ was constructed and it was shown in [Nakashima 2010] that the ultra-discretization of this positive geometric crystal is isomorphic to the limit of the coherent family of perfect crystals for $\mathfrak{g}^L = D_4^{(3)}$ given in [Kashiwara et al. 2007].

More recently, two of the authors have constructed a positive geometric crystal for $\mathfrak{g} = D_4^{(3)}$, $i = 1$ in [Igarashi and Nakashima 2010]. In this paper we describe the structure of the crystal obtained by the ultra-discretization of the geometric crystal $\mathcal{V}(\mathfrak{g})$ constructed in [Igarashi and Nakashima 2010] and then prove that it is isomorphic to the limit $B_\infty$ of the coherent family of perfect crystals for its Langlands dual $\mathfrak{g}^L = G_2^{(1)}$ constructed in [Misra et al. 2010]. This proves Conjecture 4.5 in [Igarashi and Nakashima 2010].

This paper is organized as follows. In Section 2, we recall necessary definitions and facts about geometric crystals. In Section 3, we review needed facts about affine crystals and perfect crystals. We recall from [Misra et al. 2010] the coherent family of perfect crystals for $\mathfrak{g} = G_2^{(1)}$ and its limit in Section 4. In Section 5, we review the positive geometric crystal $\mathcal{V}(\mathfrak{g})$ for $\mathfrak{g} = D_4^{(3)}$ constructed in [Igarashi and Nakashima 2010]. In Section 6, we state and prove our main result, Theorem 6.1.

## 2. Geometric crystals

In this section, we review Kac–Moody groups and geometric crystals following [Peterson and Kac 1983; Kumar 2002; Berenstein and Kazhdan 2000].

***Kac–Moody algebras and Kac–Moody groups.*** Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$ with a finite index set $I$. Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be the associated root data, where $\mathfrak{t}$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i\in I} \subset \mathfrak{t}^*$ and $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{t}$ are linearly independent satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$.

The Kac–Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with $A$ is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{t}$, the Chevalley generators $e_i$ and $f_i$ $(i \in I)$ with the usual defining relations [Kac and Peterson 1983; Peterson and Kac 1983]. There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{t}^*} \mathfrak{g}_\alpha$. Denote the set of roots by

$$\Delta := \{\alpha \in \mathfrak{t}^* \mid \alpha \neq 0,\ \ \mathfrak{g}_\alpha \neq (0)\}.$$

Set $Q = \sum_i \mathbb{Z}\alpha_i$, $Q_+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$, $Q^\vee := \sum_i \mathbb{Z}\alpha_i^\vee$ and $\Delta_+ := \Delta \cap Q_+$. An element of $\Delta_+$ is called a *positive root*. Let $P \subset \mathfrak{t}^*$ be a weight lattice such that $\mathbb{C} \otimes P = \mathfrak{t}^*$, whose element is called a weight.

Define simple reflections $s_i \in \mathrm{Aut}(\mathfrak{t})$ $(i \in I)$ by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$; they generate the Weyl group $W$, which acts on $\mathfrak{t}^*$ by

$$s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i.$$

Set $\Delta^{\mathrm{re}} := \{w(\alpha_i) \mid w \in W,\ \ i \in I\}$, whose elements are called real roots.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and $G$ the Kac–Moody group associated with $\mathfrak{g}'$ [Peterson and Kac 1983]. Let $U_\alpha := \exp \mathfrak{g}_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$ be a one-parameter subgroup of $G$. The group $G$ is generated by $U_\alpha$ $(\alpha \in \Delta^{\mathrm{re}})$. Let $U^\pm$ be the subgroup generated by $U_{\pm\alpha}$ $(\alpha \in \Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+)$, *i.e.*, $U^\pm := \langle U_{\pm\alpha} | \alpha \in \Delta_+^{\mathrm{re}}\rangle$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$\phi_i\left(\begin{pmatrix} c & 0 \\ 0 & c^{-1}\end{pmatrix}\right) = c^{\alpha_i^\vee}, \quad \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\right) = \exp(te_i), \quad \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1\end{pmatrix}\right) = \exp(tf_i),$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := c^{\alpha_i^\vee}$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \phi_i(\{\mathrm{diag}(c, c^{-1}) | c \in \mathbb{C}^\vee\})$ and $N_i := N_{G_i}(T_i)$. Let $T$ (resp. $N$) be the subgroup of $G$ with the Lie algebra $\mathfrak{t}$ (resp. generated by the $N_i$'s), which is called a *maximal torus* in $G$, and let $B^\pm = U^\pm T$ be the Borel subgroup of $G$. We have the isomorphism $\phi : W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element

$\bar{s}_i := x_i(-1)y_i(1)x_i(-1) = \phi_i\big(\big(\begin{smallmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{smallmatrix}\big)\big)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

***Geometric crystals.*** Let $X$ be an ind-variety, $\gamma_i : X \to \mathbb{C}$ and $\varepsilon_i : X \to \mathbb{C}$ $(i \in I)$ rational functions on $X$, and $e_i : \mathbb{C}^\times \times X \to X$ $((c, x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action.

**Definition 2.1.** A quadruple $(X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ is a $G$ (or $\mathfrak{g}$)-*geometric crystal* if it satisfies these conditions:

(i) $\{1\} \times X \subset \mathrm{dom}(e_i)$ for any $i \in I$.

(ii) $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$.

(iii) The $e_i$ satisfy

$$\begin{aligned}
&e_i^{c_1}e_j^{c_2} = e_j^{c_2}e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0,\\
&e_i^{c_1}e_j^{c_1c_2}e_i^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1,\\
&e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1} && \text{if } a_{ij} = -2,\ a_{ji} = -1,\\
&e_i^{c_1}e_j^{c_1^3c_2}e_i^{c_1^2c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2} = e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^2c_2}e_j^{c_1^3c_2}e_i^{c_1} && \text{if } a_{ij} = -3,\ a_{ji} = -1.
\end{aligned}$$

(iv) $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$.

Condition (iv) is slightly modified from the one in [Igarashi and Nakashima 2010; Nakashima 2007; 2010].

Let $W$ be the Weyl group associated with $\mathfrak{g}$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \ldots, i_l) \in I^l \mid w = s_{i_1}s_{i_2}\cdots s_{i_l}\},$$

where $l$ is the length of $w$. Then $R(w)$ is the set of reduced words of $w$. For a word $\mathbf{i} = (i_1, \ldots, i_l) \in R(w)$ $(w \in W)$, set $\alpha^{(j)} := s_{i_l}\cdots s_{i_{j+1}}(\alpha_{i_j})$ $(1 \le j \le l)$ and

$$e_{\mathbf{i}} : T \times X \to X, \quad (t, x) \mapsto e_{\mathbf{i}}^t(x) := e_{i_1}^{\alpha^{(1)}(t)}e_{i_2}^{\alpha^{(2)}(t)}\cdots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

Condition (iii) above amounts to saying that $e_{\mathbf{i}} = e_{\mathbf{i}'}$ for any $w \in W$ and $\mathbf{i}, \mathbf{i}' \in R(w)$.

***Geometric crystal on Schubert cell.*** Let $w \in W$ be a Weyl group element and take a reduced expression $w = s_{i_1}\cdots s_{i_l}$. Let $X := G/B$ be the flag variety, which is an ind-variety and $X_w \subset X$ the Schubert cell associated with $w$, which has a natural geometric crystal structure [Berenstein and Kazhdan 2000; Nakashima 2005a]. For $\mathbf{i} := (i_1, \ldots, i_k)$, set

$$B_{\mathbf{i}}^- := \{Y_{\mathbf{i}}(c_1, \ldots, c_k) := Y_{i_1}(c_1)\cdots Y_{i_l}(c_k) \mid c_1\cdots, c_k \in \mathbb{C}^\times\} \subset B^-, \tag{2-1}$$

where $Y_i(c) := y_i(\frac{1}{c})\alpha_i^\vee(c)$. This has a geometric crystal structure [Nakashima 2005a] isomorphic to $X_w$. The explicit forms of the action $e_i^c$, the rational function $\varepsilon_i$ and $\gamma_i$ on $B_{\mathbf{i}}^-$ are given by

$$e_i^c(Y_{\mathbf{i}}(c_1, \ldots, c_k)) = Y_{\mathbf{i}}(\mathscr{C}_1, \ldots, \mathscr{C}_k)),$$

where

$$\mathscr{C}_j := c_j \cdot \frac{\displaystyle\sum_{1\le m\le j, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j<m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}{\displaystyle\sum_{1\le m< j, i_m=i} \frac{c}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m} + \sum_{j\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}}, \tag{2-2}$$

$$\varepsilon_i(Y_{\mathbf{i}}(c_1, \dots, c_k)) = \sum_{1\le m\le k, i_m=i} \frac{1}{c_1^{a_{i_1,i}} \cdots c_{m-1}^{a_{i_{m-1},i}} c_m}, \tag{2-3}$$

$$\gamma_i(Y_{\mathbf{i}}(c_1, \dots, c_k)) = c_1^{a_{i_1,i}} \cdots c_k^{a_{i_k,i}}. \tag{2-4}$$

***Positive structure, ultra-discretizations and tropicalizations.*** The setting is the same as in [Kashiwara et al. 2008]. Let $T = (\mathbb{C}^\times)^l$ be an algebraic torus over $\mathbb{C}$, with character lattice $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l$ and cocharacter lattice $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l$. Set $R := \mathbb{C}(c)$ and define

$$v : R \setminus \{0\} \to \mathbb{Z}, \quad f(c) \mapsto \deg f(c),$$

where deg is the degree of poles at $c = \infty$. Note that for $f_1, f_2 \in R \setminus \{0\}$, we have

$$v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2). \tag{2-5}$$

A nonzero rational function on an algebraic torus $T$ is called *positive* if it can be written as $g/h$ where $g$ and $h$ are a positive linear combination of characters of $T$.

**Definition 2.2.** Let $f : T \to T'$ be a rational morphism between two algebraic tori $T$ and $T'$. We say that $f$ is *positive* if $\eta \circ f$ is positive for any character $\eta : T' \to \mathbb{C}$.

Denote by $\mathrm{Mor}^+(T, T')$ the set of positive rational morphisms from $T$ to $T'$.

**Lemma 2.3** [Berenstein and Kazhdan 2000]. *For any $f \in \mathrm{Mor}^+(T_1, T_2)$ and any $g \in \mathrm{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is well-defined and lies in $\mathrm{Mor}^+(T_1, T_3)$.*

By Lemma 2.3, we can define a category $\mathscr{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

Let $f : T \to T'$ be a positive rational morphism of algebraic tori $T$ and $T'$. We define a map $\widehat{f} : X_*(T) \to X_*(T')$ by

$$\langle \eta, \widehat{f}(\xi) \rangle = v(\eta \circ f \circ \xi),$$

where $\eta \in X^*(T')$ and $\xi \in X_*(T)$.

**Lemma 2.4** [Berenstein and Kazhdan 2000]. *For any algebraic tori $T_1$, $T_2$, $T_3$, and positive rational morphisms $f \in \mathrm{Mor}^+(T_1, T_2)$, $g \in \mathrm{Mor}^+(T_2, T_3)$, we have $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.*

Let $\mathfrak{Set}$ denote the category of sets and set maps. By the lemma, we obtain a functor

$$\begin{array}{rccc} \mathcal{UD}: & \mathcal{T}_+ & \to & \mathfrak{Set} \\ & T & \mapsto & X_*(T) \\ & (f:T\to T') & \mapsto & (\widehat{f}:X_*(T)\to X_*(T'))). \end{array}$$

**Definition 2.5** [Berenstein and Kazhdan 2000]. Let

$$\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$$

be a geometric crystal, $T'$ an algebraic torus and $\theta:T'\to X$ a birational isomorphism. The isomorphism $\theta$ is called *positive structure* on $\chi$ if

(i) for any $i\in I$ the rational functions $\gamma_i\circ\theta:T'\to\mathbb{C}$ and $\varepsilon_i\circ\theta:T'\to\mathbb{C}$ are positive, and

(ii) for any $i\in I$, the rational morphism $e_{i,\theta}:\mathbb{C}^\times\times T'\to T'$ defined by $e_{i,\theta}(c,t):=\theta^{-1}\circ e_i^c\circ\theta(t)$ is positive.

Let $\theta:T\to X$ be a positive structure on a geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta}:\mathbb{C}^\times\times T'\to T'$ and $\gamma_i,\varepsilon_i\circ\theta:T'\to\mathbb{C}$ (the notations are as above), we obtain

$$\begin{aligned} \tilde{e}_i&:=\mathcal{UD}(e_{i,\theta}):\mathbb{Z}\times X_*(T)\to X_*(T),\\ \mathrm{wt}_i&:=\mathcal{UD}(\gamma_i\circ\theta):X_*(T')\to\mathbb{Z},\\ \varepsilon_i&:=\mathcal{UD}(\varepsilon_i\circ\theta):X_*(T')\to\mathbb{Z}. \end{aligned}$$

Now, for given positive structure $\theta:T'\to X$ on a geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$, we associate the quadruple $(X_*(T'),\{\tilde{e}_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ with a free pre-crystal structure (see [Berenstein and Kazhdan 2000, 2.2]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$.

**Theorem 2.6** [Berenstein and Kazhdan 2000; Nakashima 2005a]. *For any geometric crystal $\chi=(X,\{e_i\}_{i\in I},\{\gamma_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ and positive structure $\theta:T'\to X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi)=(X_*(T'),\{\tilde{e}_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I})$ is a crystal* (see [Berenstein and Kazhdan 2000, 2.2]).

Now, let $\mathcal{GC}^+$ be the category whose objects are triplets $(\chi,T',\theta)$, where $\chi=(X,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ is a geometric crystal and $\theta:T'\to X$ is a positive structure on $\chi$, and whose morphisms $f:(\chi_1,T_1',\theta_1)\to(\chi_2,T_2',\theta_2)$ are given by morphisms $\varphi:X_1\to X_2$ ($\chi_i=(X_i,\dots)$) such that

$$f:=\theta_2^{-1}\circ\varphi\circ\theta_1:T_1'\to T_2',$$

is a positive rational morphism. Let $\mathcal{CR}$ be the category of crystals. Theorem 2.6 yields:

**Corollary 2.7.** *The map $\mathcal{UD} = \mathcal{UD}_{\theta,T'}$ defined above is a functor*

$$\begin{array}{llcl} \mathcal{UD}: & \mathcal{GC}^+ & \to & \mathcal{CR} \\ & (\chi, T', \theta) & \mapsto & X_*(T'), \\ & (f : (\chi_1, T_1', \theta_1) \to (\chi_2, T_2', \theta_2)) & \mapsto & (\widehat{f} : X_*(T_1') \to X_*(T_2')). \end{array}$$

We call the functor $\mathcal{UD}$ *"ultra-discretization"* as in [Nakashima 2005a; 2005b] instead of "tropicalization" as in [Berenstein and Kazhdan 2000]. And for a crystal $B$, if there exists a geometric crystal $\chi$ and a positive structure $\theta : T' \to X$ on $\chi$ such that $\mathcal{UD}(\chi, T', \theta) \cong B$ as crystals, we call an object $(\chi, T', \theta)$ in $\mathcal{GC}^+$ a *tropicalization* of $B$, where it is not known that this correspondence is a functor.

## 3. Limit of perfect crystals

We review limit of perfect crystals following [Kang et al. 1994]. (See also [Kang et al. 1992a; 1992b].)

***Crystals.*** First we review the theory of crystals, which is the notion obtained by abstracting the combinatorial properties of crystal bases. Let $(A, \{\alpha_i\}_{i\in I}, \{\alpha_i^\vee\}_{i\in I})$ be a Cartan data.

**Definition 3.1.** A *crystal* $B$ is a set endowed with maps

$$\begin{aligned} &\mathrm{wt} : B \to P, \\ &\varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \qquad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \qquad \text{for } i \in I, \\ &\tilde{e}_i : B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \quad \text{for } i \in I, \\ &\tilde{e}_i(0) = \tilde{f}_i(0) = 0. \end{aligned}$$

satisfying the following axioms, for all $b, b_1, b_2 \in B$:

$$\begin{aligned} &\varphi_i(b) = \varepsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle, \\ &\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \quad \text{if } \tilde{e}_i b \in B, \\ &\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \quad \text{if } \tilde{f}_i b \in B, \\ &\tilde{e}_i b_2 = b_1 \quad \Longleftrightarrow \tilde{f}_i b_1 = b_2, \\ &\varepsilon_i(b) = -\infty \implies \tilde{e}_i b = \tilde{f}_i b = 0. \end{aligned}$$

The following tensor product structure is one of the most crucial properties of crystals.

**Theorem 3.2.** *Let $B_1$ and $B_2$ be crystals, and set*

$$B_1 \otimes B_2 := \{b_1 \otimes b_2;\ b_j \in B_j \ (j = 1, 2)\}.$$

(i) *$B_1 \otimes B_2$ is a crystal.*

(ii) *For $b_1 \in B_1$ and $b_2 \in B_2$, we have*

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$
$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \\ \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2). \end{cases}$$

**Definition 3.3.** Let $B_1$ and $B_2$ be crystals. A *strict morphism* of crystals

$$\psi : B_1 \to B_2$$

is a map $\psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ such that $\psi(0) = 0$, $\psi(B_1) \subset B_2$, $\psi$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$, and

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b) \text{ for any } b \in B_1.$$

A bijective strict morphism is called an *isomorphism of crystals*.

**Example 3.4.** If $(L, B)$ is a crystal base, then $B$ is a crystal. Hence, for the crystal base $(L(\infty), B(\infty))$ of the nilpotent subalgebra $U_q^-(\mathfrak{g})$ of the quantum algebra $U_q(\mathfrak{g})$, $B(\infty)$ is a crystal.

**Example 3.5.** For $\lambda \in P$, set $T_\lambda := \{t_\lambda\}$. We define a crystal structure on $T_\lambda$ by

$$\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0, \quad \varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty, \quad \mathrm{wt}(t_\lambda) = \lambda.$$

**Definition 3.6.** For a crystal $B$, a colored oriented graph structure is associated with $B$ by

$$b_1 \xrightarrow{i} b_2 \iff \tilde{f}_i b_1 = b_2.$$

We call this graph the *crystal graph* of $B$.

***Affine weights.*** Let $\mathfrak{g}$ be an affine Lie algebra. The sets $\mathfrak{t}$, $\{\alpha_i\}_{i\in I}$ and $\{\alpha_i^\vee\}_{i\in I}$ be as in Section 2. We take $\dim \mathfrak{t} = \#I + 1$. Let $\delta \in Q_+$ be the unique element satisfying $\{\lambda \in Q \mid \langle \alpha_i^\vee, \lambda\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\delta$ and $\mathbf{c} \in \mathfrak{g}$ be the canonical central element satisfying $\{h \in Q^\vee \mid \langle h, \alpha_i\rangle = 0 \text{ for any } i \in I\} = \mathbb{Z}\mathbf{c}$. We write, as in [Kac 1990, 6.1],

$$\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee, \qquad \delta = \sum_i a_i \alpha_i.$$

Let ( , ) be the nondegenerate $W$-invariant symmetric bilinear form on $\mathfrak{t}^*$ normalized by $(\delta, \lambda) = \langle \mathbf{c}, \lambda\rangle$ for $\lambda \in \mathfrak{t}^*$. Let us set $\mathfrak{t}^*_{\mathrm{cl}} := \mathfrak{t}^*/\mathbb{C}\delta$ and let $\mathrm{cl} : \mathfrak{t}^* \to \mathfrak{t}^*_{\mathrm{cl}}$ be the canonical projection. Here we have $\mathfrak{t}^*_{\mathrm{cl}} \cong \bigoplus_i (\mathbb{C}\alpha_i^\vee)^*$. Set $\mathfrak{t}^*_0 := \{\lambda \in \mathfrak{t}^* \mid \langle \mathbf{c}, \lambda\rangle = 0\}$, $(\mathfrak{t}^*_{\mathrm{cl}})_0 := \mathrm{cl}(\mathfrak{t}^*_0)$. Since $(\delta, \delta) = 0$, we have a positive definite symmetric form on $\mathfrak{t}^*_{\mathrm{cl}}$ induced by the one on $\mathfrak{t}^*$. Let $\Lambda_i \in \mathfrak{t}^*_{\mathrm{cl}}$ $(i \in I)$ be a classical weight such that $\langle \alpha_i^\vee, \Lambda_j\rangle = \delta_{i,j}$, which is called a fundamental weight. We choose $P$ so that $P_{\mathrm{cl}} := \mathrm{cl}(P)$ coincides with $\bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ and we call $P_{\mathrm{cl}}$ a *classical weight lattice*.

***Perfect crystals and their limits.*** Let $\mathfrak{g}$ be an affine Lie algebra, let $P_{\rm cl}$ be a classical weight lattice as above and set $(P_{\rm cl})_l^+ := \{\lambda \in P_{\rm cl} \mid \langle \mathbf{c}, \lambda \rangle = l,\ \langle \alpha_i^\vee, \lambda \rangle \geq 0\}$ $(l \in \mathbb{Z}_{>0})$.

**Definition 3.7.** A crystal $B$ is a *perfect crystal* of level $l$ if the following conditions are satisfied:

(i) $B \otimes B$ is connected as a crystal graph.

(ii) There exists $\lambda_0 \in P_{\rm cl}$ such that

$$\mathrm{wt}(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \mathrm{cl}(\alpha_i), \qquad \# B_{\lambda_0} = 1.$$

(iii) There exists a finite-dimensional $U_q'(\mathfrak{g})$-module $V$ with a crystal pseudobase $B_{\rm ps}$ such that $B \cong B_{\rm ps}/\pm 1$.

(iv) For any $b \in B$, we have $\langle \mathbf{c}, \varepsilon(b) \rangle \geq l$.

(v) The maps $\varepsilon, \varphi : B^{\min} := \{b \in B \mid \langle \mathbf{c}, \varepsilon(b) \rangle = l\} \longrightarrow (P_{\rm cl}^+)_l$ are bijective, where $\varepsilon(b) := \sum_i \varepsilon_i(b) \Lambda_i$ and $\varphi(b) := \sum_i \varphi_i(b) \Lambda_i$.

Let $\{B_l\}_{l \geq 1}$ be a family of perfect crystals of level $l$ and set $J := \{(l, b) \mid l > 0,\ b \in B_l^{\min}\}$.

**Definition 3.8.** A crystal $B_\infty$ with an element $b_\infty$ is called a *limit of* $\{B_l\}_{l \geq 1}$ if

(i) $\mathrm{wt}(b_\infty) = \varepsilon(b_\infty) = \varphi(b_\infty) = 0$;

(ii) for any $(l, b) \in J$, there exists an embedding of crystals

$$f_{(l,b)} : T_{\varepsilon(b)} \otimes B_l \otimes T_{-\varphi(b)} \hookrightarrow B_\infty, \quad t_{\varepsilon(b)} \otimes b \otimes t_{-\varphi(b)} \mapsto b_\infty;$$

(iii) $B_\infty = \bigcup_{(l,b) \in J} \mathrm{Im} f_{(l,b)}$.

As for the crystal $T_\lambda$, see Example 3.5. If a limit exists for a family $\{B_l\}$, we say that $\{B_l\}$ is a *coherent family* of perfect crystals.

Here is one of the most important properties of limit of perfect crystals.

**Proposition 3.9.** *For the crystal $B(\infty)$ as in Example 3.4, we have an isomorphism of crystals*

$$B(\infty) \otimes B_\infty \xrightarrow{\sim} B(\infty).$$

## 4. Perfect crystals of type $G_2^{(1)}$

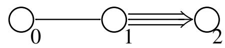
In this section, we review the family of perfect crystals of type $G_2^{(1)}$ and its limit [Misra et al. 2010].

We fix the data for $G_2^{(1)}$. Let $\{\alpha_0, \alpha_1, \alpha_2\}$, $\{\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee\}$ and $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of simple roots, simple coroots and fundamental weights, respectively. The

Cartan matrix $A = (a_{ij})_{i,j=0,1,2}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix},$$

and its Dynkin diagram is as follows:



The standard null root $\delta$ and the canonical central element $\mathbf{c}$ are given by

$$\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 \quad \text{and} \quad \mathbf{c} = \alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta$, $\alpha_1 = -\Lambda_0 + 2\Lambda_1 - 3\Lambda_2$, and $\alpha_2 = -\Lambda_1 + 2\Lambda_2$.

For a positive integer $l$ we introduce $G_2^{(1)}$-crystals $B_l$ and $B_\infty$ as

$$B_l = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}_{\geq 0}/3)^6 \;\middle|\; \begin{array}{l} 3b_3 \equiv 3\bar{b}_3 \pmod 2, \\ \sum_{i=1,2}(b_i + \bar{b}_i) + \frac{1}{2}(b_3 + \bar{b}_3) \leq l \\ b_1, \bar{b}_1, b_2 - b_3, \bar{b}_3 - \bar{b}_2 \in \mathbb{Z} \end{array} \right\},$$

$$B_\infty = \left\{ b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in (\mathbb{Z}/3)^6 \;\middle|\; \begin{array}{l} 3b_3 \equiv 3\bar{b}_3 \pmod 2, \\ b_1, \bar{b}_1, b_2 - b_3, \bar{b}_3 - \bar{b}_2 \in \mathbb{Z} \end{array} \right\}.$$

Now we describe the explicit crystal structures of $B_l$ and $B_\infty$. Indeed, most of them coincide with each other except for $\varepsilon_0$ and $\varphi_0$. In the rest of this section, we use the following convention: $(x)_+ = \max(x, 0)$. For $b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1)$ we define

$$s(b) = b_1 + b_2 + \tfrac{1}{2}(b_3 + \bar{b}_3) + \bar{b}_2 + \bar{b}_1, \tag{4-1}$$

and

$$z_1 = \bar{b}_1 - b_1, \quad z_2 = \bar{b}_2 - \bar{b}_3, \quad z_3 = b_3 - b_2, \quad z_4 = \tfrac{1}{2}(\bar{b}_3 - b_3). \tag{4-2}$$

Now we define conditions and $(F_1)$–$(F_6)$ as follows:

$$\begin{cases} (F_1) & z_1+z_2+z_3+3z_4 \leq 0,\ z_1+z_2+3z_4 \leq 0,\ z_1+z_2 \leq 0,\ z_1 \leq 0, \\ (F_2) & z_1+z_2+z_3+3z_4 \leq 0,\ z_2+3z_4 \leq 0,\ z_2 \leq 0,\ z_1 > 0, \\ (F_3) & z_1+z_3+3z_4 \leq 0,\ z_3+3z_4 \leq 0,\ z_4 \leq 0,\ z_2 > 0,\ z_1+z_2 > 0, \\ (F_4) & z_1+z_2+3z_4 > 0,\ z_2+3z_4 > 0,\ z_4 > 0,\ z_3 \leq 0,\ z_1+z_3 \leq 0, \\ (F_5) & z_1+z_2+z_3+3z_4 > 0,\ z_3+3z_4 > 0,\ z_3 > 0,\ z_1 \leq 0, \\ (F_6) & z_1+z_2+z_3+3z_4 > 0,\ z_1+z_3+3z_4 > 0,\ z_1+z_3 > 0,\ z_1 > 0. \end{cases} \tag{4-3}$$

Conditions $(E_i)$, for $1 \leq i \leq 6$, are defined from $(F_i)$ by replacing $>$ with $\geq$ and $\leq$ with $<$.

We also define

(4-4) $A = (0, z_1, z_1+z_2, z_1+z_2+3z_4, z_1+z_2+z_3+3z_4, 2z_1+z_2+z_3+3z_4)$.

Then for $b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in B_l$ or $B_\infty$, the values of $\tilde{e}_i b$, $\tilde{f}_i b$, $\varepsilon_i(b)$, and $\varphi_i(b)$, for $i = 0, 1, 2$, are as follows:

$$\tilde{e}_0 b = \begin{cases} (b_1-1, \dots) & \text{if } (E_1), \\ (\dots, b_3-1, \bar{b}_3-1, \dots, \bar{b}_1+1) & \text{if } (E_2), \\ (\dots, b_2-\frac{2}{3}, b_3-\frac{2}{3}, \bar{b}_3+\frac{4}{3}, \bar{b}_2+\frac{1}{3}, \dots) & \text{if } (E_3) \text{ and } z_4 = -\frac{1}{3}, \\ (\dots, b_2-\frac{1}{3}, b_3-\frac{4}{3}, \bar{b}_3+\frac{2}{3}, \bar{b}_2+\frac{2}{3}, \dots) & \text{if } (E_3) \text{ and } z_4 = -\frac{2}{3}, \\ (\dots, b_3-2, \dots, \bar{b}_2+1, \dots) & \text{if } (E_3) \text{ and } z_4 \neq -\frac{1}{3}, -\frac{2}{3}, \\ (\dots, b_2-1, \dots, \bar{b}_3+2, \dots) & \text{if } (E_4), \\ (b_1-1, \dots, b_3+1, \bar{b}_3+1, \dots) & \text{if } (E_5), \\ (\dots, \bar{b}_1+1) & \text{if } (E_6), \end{cases}$$

$$\tilde{f}_0 b = \begin{cases} (b_1+1, \dots) & \text{if } (F_1), \\ (\dots, b_3+1, \bar{b}_3+1, \dots, \bar{b}_1-1) & \text{if } (F_2), \\ (\dots, b_3+2, \dots, \bar{b}_2-1, \dots) & \text{if } (F_3), \\ (\dots, b_2+\frac{1}{3}, b_3+\frac{4}{3}, \bar{b}_3-\frac{2}{3}, \bar{b}_2-\frac{2}{3}, \dots) & \text{if } (F_4) \text{ and } z_4 = \frac{1}{3}, \\ (\dots, b_2+\frac{2}{3}, b_3+\frac{2}{3}, \bar{b}_3-\frac{4}{3}, \bar{b}_2-\frac{1}{3}, \dots) & \text{if } (F_4) \text{ and } z_4 = \frac{2}{3}, \\ (\dots, b_2+1, \dots, \bar{b}_3-2, \dots) & \text{if } (F_4) \text{ and } z_4 \neq \frac{1}{3}, \frac{2}{3}, \\ (b_1+1, \dots, b_3-1, \bar{b}_3-1, \dots) & \text{if } (F_5), \\ (\dots, \bar{b}_1-1) & \text{if } (F_6), \end{cases}$$

$$\tilde{e}_1 b = \begin{cases} (\dots, \bar{b}_2+1, \bar{b}_1-1) & \text{if } \bar{b}_2-\bar{b}_3 \geq (b_2-b_3)_+, \\ (\dots, b_3+1, \bar{b}_3-1, \dots) & \text{if } \bar{b}_2-\bar{b}_3 < 0 \leq b_3-b_2, \\ (b_1+1, b_2-1, \dots) & \text{if } (\bar{b}_2-\bar{b}_3)_+ < b_2-b_3, \end{cases}$$

$$\tilde{f}_1 b = \begin{cases} (b_1-1, b_2+1, \dots) & \text{if } (\bar{b}_2-\bar{b}_3)_+ \leq b_2-b_3, \\ (\dots, b_3-1, \bar{b}_3+1, \dots) & \text{if } \bar{b}_2-\bar{b}_3 \leq 0 < b_3-b_2, \\ (\dots, \bar{b}_2-1, \bar{b}_1+1) & \text{if } \bar{b}_2-\bar{b}_3 > (b_2-b_3)_+, \end{cases}$$

$$\tilde{e}_2 b = \begin{cases} (\dots, \bar{b}_3+\frac{2}{3}, \bar{b}_2-\frac{1}{3}, \dots) & \text{if } \bar{b}_3 \geq b_3, \\ (\dots, b_2+\frac{1}{3}, b_3-\frac{2}{3}, \dots) & \text{if } \bar{b}_3 < b_3, \end{cases}$$

$$\tilde{f}_2 b = \begin{cases} (\dots, b_2-\frac{1}{3}, b_3+\frac{2}{3}, \dots) & \text{if } \bar{b}_3 \leq b_3, \\ (\dots, \bar{b}_3-\frac{2}{3}, \bar{b}_2+\frac{1}{3}, \dots) & \text{if } \bar{b}_3 > b_3, \end{cases}$$

$$\varepsilon_1(b) = \bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+, \quad \varphi_1(b) = b_1 + (b_3 - b_2 + (\bar{b}_2 - \bar{b}_3)_+)_+,$$
$$\varepsilon_2(b) = 3\bar{b}_2 + \tfrac{3}{2}(b_3 - \bar{b}_3)_+, \qquad \varphi_2(b) = 3b_2 + \tfrac{3}{2}(\bar{b}_3 - b_3)_+,$$
$$\varepsilon_0(b) = \begin{cases} l - s(b) + \max\ A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_l, \\ -s(b) + \max\ A - (2z_1 + z_2 + z_3 + 3z_4) & b \in B_\infty, \end{cases}$$
$$\varphi_0(b) = \begin{cases} l - s(b) + \max\ A & b \in B_l, \\ -s(b) + \max\ A & b \in B_\infty. \end{cases}$$

For $b \in B_l$ if $\tilde{e}_i b$ or $\tilde{f}_i b$ does not belong to $B_l$, namely, if $b_j$ or $\bar{b}_j$ for some $j$ becomes negative or $s(b)$ exceeds $l$, we understand it to be 0.

**Theorem 4.1** [Misra et al. 2010]. (i) *The $G_2^{(1)}$-crystal $B_l$ is a perfect crystal of level $l$.*

(ii) *The family of the perfect crystals $\{B_l\}_{l\geq 1}$ forms a coherent family and the crystal $B_\infty$ is its limit with the vector $b_\infty = (0, 0, 0, 0, 0, 0)$.*

As was shown in [Misra et al. 2010], the minimal elements are given by

$$(B_l)_{\min} = \{(\alpha, \beta, \beta, \beta, \beta, \alpha) \mid \alpha \in \mathbb{Z}_{\geq 0}, \beta \in (\mathbb{Z}_{\geq 0})/3, 2\alpha + 3\beta \leq l\}.$$

Set $J = \{(l, b) \mid l \in \mathbb{Z}_{\geq 1}, b \in (B_l)_{\min}\}$ and let the maps $\varepsilon, \varphi : (B_l)_{\min} \to (P_{\mathrm{cl}}^+)_l$ be as in Definition 3.7. Then we have $\mathrm{wt}\, b_\infty = 0$ and

$$\varepsilon_i(b_\infty) = \varphi_i(b_\infty) = 0 \quad \text{for } i = 0, 1, 2.$$

For $(l, b_0) \in J$, since $\varepsilon(b_0) = \varphi(b_0)$, one can set $\lambda = \varepsilon(b_0) = \varphi(b_0)$. For

$$b = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1) \in B_l$$

we define a map

$$f_{(l,b_0)} : T_\lambda \otimes B_l \otimes B_{-\lambda} \to B_\infty$$

by

$$f_{(l,b_0)}(t_\lambda \otimes b \otimes t_{-\lambda}) = b' = (\nu_1, \nu_2, \nu_3, \bar{\nu}_3, \bar{\nu}_2, \bar{\nu}_1)$$

where $b_0 = (\alpha, \beta, \beta, \beta, \beta, \alpha)$, and

$$\nu_1 = b_1 - \alpha, \quad \bar{\nu}_1 = \bar{b}_1 - \alpha,$$
$$\nu_2 = b_2 - \beta, \quad \bar{\nu}_2 = \bar{b}_2 - \beta,$$
$$\nu_3 = b_3 - \beta, \quad \bar{\nu}_3 = \bar{b}_3 - \beta.$$

Finally, we obtain

$$B_\infty = \bigcup_{(l,b)\in J} \mathrm{Im}\, f_{(l,b)}.$$

## 5. Affine geometric crystal $\mathcal{V}_1(D_4^{(3)})$

***Fundamental representation $W(\varpi_1)$ for $D_4^{(3)}$.*** Let $\mathbf{c} = \sum_i a_i^\vee \alpha_i^\vee$ be the canonical central element in an affine Lie algebra $\mathfrak{g}$ (see [Kac 1990, 6.1]), $\{\Lambda_i \mid i \in I\}$ the set of fundamental weights as in the previous section and $\varpi_1 := \Lambda_1 - a_1^\vee \Lambda_0$ the fundamental weight (of level 0). Let $W(\varpi_1)$ be the fundamental representation of $U_q'(\mathfrak{g})$ associated with $\varpi_1$ [Kashiwara 2002].

By [Kashiwara 2002, Theorem 5.17], $W(\varpi_1)$ is a finite-dimensional irreducible integrable $U_q'(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $\mathfrak{g}$-module $W(\varpi_1)$, which we call a fundamental representation of $\mathfrak{g}$ and use the same notation as above.

We shall present the explicit form of $W(\varpi_1)$ for $\mathfrak{g} = D_4^{(3)}$.

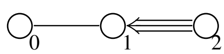
***$W(\varpi_1)$ for $D_4^{(3)}$.*** The Cartan matrix $A = (a_{i,j})_{i,j=0,1,2}$ of type $D_4^{(3)}$ is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then the simple roots are

$$\alpha_0 = 2\Lambda_0 - \Lambda_1 + \delta, \quad \alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2, \quad \alpha_2 = -3\Lambda_1 + 2\Lambda_2,$$

and the Dynkin diagram is this:




The $D_4^{(3)}$-module $W(\varpi_1)$ is an 8-dimensional module with the basis

$$\{v_1, v_2, v_3, v_0, \varnothing, v_{\bar{3}}, v_{\bar{2}}, v_{\bar{1}}\}.$$

The explicit form of $W(\varpi_1)$ is given in [Kashiwara et al. 2007].

$$\mathrm{wt}(v_1) = \Lambda_1 - 2\Lambda_0, \quad \mathrm{wt}(v_2) = -\Lambda_0 - \Lambda_1 + \Lambda_2, \quad \mathrm{wt}(v_3) = -\Lambda_0 + 2\Lambda_1 - \Lambda_2,$$
$$\mathrm{wt}(v_{\bar{i}}) = -\mathrm{wt}(v_i) \ (i = 1, 2, 3), \quad \mathrm{wt}(v_0) = \mathrm{wt}(\varnothing) = 0.$$

The actions of $e_i$ and $f_i$ on these basis vectors are given as follows:

$$f_0(v_0, v_{\bar{3}}, v_{\bar{2}}, v_{\bar{1}}, \varnothing) = \left(v_1, v_2, v_3, \varnothing + \tfrac{1}{2}v_0, \tfrac{3}{2}v_1\right),$$
$$f_1(v_1, v_3, v_0, v_{\bar{2}}) = (v_2, v_0, 2v_{\bar{3}}, v_{\bar{1}}), \quad f_2(v_2, v_{\bar{3}}) = (v_3, v_{\bar{2}}),$$
$$e_0(v_1, v_2, v_3, v_0, \varnothing) = \left(\varnothing + \tfrac{1}{2}v_0, v_{\bar{3}}, v_{\bar{2}}, v_{\bar{1}}, \tfrac{3}{2}v_{\bar{1}}\right),$$
$$e_1(v_2, v_0, v_{\bar{3}}, v_{\bar{1}}) = (v_1, 2v_3, v_0, v_{\bar{2}}), \quad e_2(v_3, v_{\bar{2}}) = (v_2, v_{\bar{3}}),$$

where we give nontrivial actions only.

***Construction of the affine geometric crystal $\mathcal{V}_1(D_4^{(3)})$ in $W(\varpi_1)$.*** In this section, we follow [Igarashi and Nakashima 2010]. For $\xi \in (\mathfrak{t}^*_{\mathrm{cl}})_0$, let $t(\xi)$ be the translation as in [Kashiwara 2002, Section 4] and $\widetilde{\varpi}_i$ as in [Kashiwara 2005]; indeed, $\widetilde{\varpi}_i := \max(1, 2/(\alpha_i, \alpha_i))\varpi_i$. Then we have

$$t(\widetilde{\varpi}_1) = s_0 s_1 s_2 s_1 s_2 s_1 =: w_1,$$
$$t(\mathrm{wt}(v_{\bar{2}})) = s_2 s_1 s_2 s_1 s_0 s_1 =: w_2.$$

Associated with these Weyl group elements $w_1$ and $w_2$, we define algebraic varieties $\mathcal{V}_1 = \mathcal{V}_1(D_4^{(3)})$ and $\mathcal{V}_2 = \mathcal{V}_2(D_4^{(3)}) \subset W(\varpi_1)$ respectively:

$$\mathcal{V}_1 := \{V_1(x) := Y_0(x_0)Y_1(x_1)Y_2(x_2)Y_1(x_3)Y_2(x_4)Y_1(x_5)v_1 \mid x_i \in \mathbb{C}^\times,\ 0 \le i \le 5\},$$
$$\mathcal{V}_2 := \{V_2(y) := Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)v_{\bar{2}} \mid y_i \in \mathbb{C}^\times,\ 0 \le i \le 5\}.$$

Owing to the explicit forms of $f_i$'s on $W(\varpi_1)$ as above, we have $f_0^3 = 0$, $f_1^3 = 0$ and $f_2^2 = 0$ and then

$$Y_i(c) = \Big(1 + \frac{f_i}{c} + \frac{f_i^2}{2c^2}\Big)\alpha_i^\vee(c)\ (i = 0, 1), \quad Y_2(c) = \Big(1 + \frac{f_2}{c}\Big)\alpha_2^\vee(c).$$

We get explicit forms of $V_1(x) \in \mathcal{V}_1$ and $V_2(y) \in \mathcal{V}_2$ as in [Nakashima 2007]:

$$V_1(x) = \sum_{1 \le i \le 3} \big(X_i v_i + X_{\bar{i}} v_{\bar{i}}\big) + X_0 v_0 + X_\varnothing \varnothing,$$
$$V_2(y) = \sum_{1 \le i \le 3} \big(Y_i v_i + Y_{\bar{i}} v_{\bar{i}}\big) + Y_0 v_0 + Y_\varnothing \varnothing.$$

where the rational functions $X_i$'s and $Y_i$'s are all positive in $(x_0, \dots, x_5)$ and $(y_0, \dots, y_5)$ respectively (as for their explicit forms, see [Igarashi and Nakashima 2010]) and for any $x$ there exist a unique rational function $a(x)$ and $y$ such that $V_2(y) = a(x)V_1(x)$. Using this result, we get the positive birational isomorphism $\overline{\sigma} : \mathcal{V}_1 \to \mathcal{V}_2$ $(V_1(x) \mapsto V_2(y))$ and we know that its inverse $\overline{\sigma}^{-1}$ is also positive. The actions of $\bar{e}_0^c$ on $V_2(y)$ (respectively $\overline{\gamma}_0(V_2(y))$ and $\overline{\varepsilon}_0(V_2(y))$) are induced from the ones on $Y_2(y_2)Y_1(y_1)Y_2(y_4)Y_1(y_3)Y_0(y_0)Y_1(y_5)$ as an element of the geometric crystal $\mathcal{V}_2$. We define the action $e_0^c$ on $V_1(x)$ by

$$e_0^c(V_1(x)) := \overline{\sigma}^{-1} \circ \bar{e}_0^c \circ \overline{\sigma}(V_1(x)). \tag{5-1}$$

We also define $\gamma_0(V_1(x))$ and $\varepsilon_0(V_1(x))$ by

$$\gamma_0(V_1(x)) := \overline{\gamma}_0(\overline{\sigma}(V_1(x))), \qquad \varepsilon_0(V_1(x)) := \overline{\varepsilon}_0(\overline{\sigma}(V_1(x))). \tag{5-2}$$

**Theorem 5.1** [Igarashi and Nakashima 2010]. *Together with* (5-1), (5-2) *on* $\mathcal{V}_1$, *we obtain a positive affine geometric crystal* $\chi := (\mathcal{V}_1, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$

($I = \{0, 1, 2\}$), *whose explicit form is as follows*: *first we have* $e_i^c$, $\gamma_i$, *and* $\varepsilon_i$, *for* $i = 1, 2$, *from* (2-2), (2-3), *and* (2-4):

$$e_1^c(V_1(x)) = V_1(x_0, \mathscr{C}_1 x_1, x_2, \mathscr{C}_3 x_3, x_4, \mathscr{C}_5 x_5),$$
$$e_2^c(V_1(x)) = V_1(x_0, x_1, \mathscr{C}_2 x_2, x_3, \mathscr{C}_4 x_4, x_5),$$

*where*

$$\mathscr{C}_1 = \frac{\dfrac{c\,x_0}{x_1} + \dfrac{x_0\,x_2}{{x_1}^2\,x_3} + \dfrac{x_0\,x_2\,x_4}{{x_1}^2\,{x_3}^2\,x_5}}{\dfrac{x_0}{x_1} + \dfrac{x_0\,x_2}{{x_1}^2\,x_3} + \dfrac{x_0\,x_2\,x_4}{{x_1}^2\,{x_3}^2\,x_5}}, \quad \mathscr{C}_3 = \frac{\dfrac{c\,x_0}{x_1} + \dfrac{c\,x_0\,x_2}{{x_1}^2\,x_3} + \dfrac{x_0\,x_2\,x_4}{{x_1}^2\,{x_3}^2\,x_5}}{\dfrac{c\,x_0}{x_1} + \dfrac{x_0\,x_2}{{x_1}^2\,x_3} + \dfrac{x_0\,x_2\,x_4}{{x_1}^2\,{x_3}^2\,x_5}},$$

$$\mathscr{C}_5 = \frac{c\left(\dfrac{x_0}{x_1} + \dfrac{x_0 x_2}{x_1^2 x_3} + \dfrac{x_0 x_2 x_4}{x_1^2 x_3^2 x_5}\right)}{\dfrac{c x_0}{x_1} + \dfrac{c x_0 x_2}{x_1^2 x_3} + \dfrac{x_0 x_2 x_4}{x_1^2 x_3^2 x_5}}, \quad \mathscr{C}_2 = \frac{\dfrac{c{x_1}^3}{x_2} + \dfrac{{x_1}^3{x_3}^3}{{x_2}^2 x_4}}{\dfrac{{x_1}^3}{x_2} + \dfrac{{x_1}^3\,{x_3}^3}{{x_2}^2\,x_4}}, \quad \mathscr{C}_4 = \frac{c\left(\dfrac{{x_1}^3}{x_2} + \dfrac{{x_1}^3{x_3}^3}{{x_2}^2 x_4}\right)}{\dfrac{c\,{x_1}^3}{x_2} + \dfrac{{x_1}^3\,{x_3}^3}{{x_2}^2\,x_4}},$$

$$\varepsilon_1(V_1(x)) = \frac{x_0}{x_1} + \frac{x_0\,x_2}{{x_1}^2\,x_3} + \frac{x_0\,x_2\,x_4}{{x_1}^2\,{x_3}^2\,x_5}, \quad \varepsilon_2(V_1(x)) = \frac{{x_1}^3}{x_2} + \frac{{x_1}^3\,{x_3}^3}{{x_2}^2\,x_4},$$
$$\gamma_1(V_1(x)) = \frac{x_1^2 x_3^2 x_5^2}{x_0 x_2 x_4}, \qquad \gamma_2(V_1(x)) = \frac{x_2^2 x_4^2}{x_1^3 x_3^3 x_5^3}.$$

*We also have* $e_0^c$, $\varepsilon_0$ *and* $\gamma_0$ *on* $V_1(x)$:

$$e_0^c(V_1(x)) = V_1\left(\frac{D}{c\cdot E}x_0, \frac{F}{c\cdot E}x_1, \frac{G}{c^3\cdot E^3}x_2, \frac{D\cdot H}{c^2\cdot E\cdot F}x_3, \frac{D^3}{c^3\cdot G}x_4, \frac{D}{c\cdot H}x_5\right),$$
$$\varepsilon_0(V_1(x)) = \frac{E}{{x_0}^3\,x_2\,x_3}, \qquad \gamma_0(V_1(x)) = \frac{x_0^2}{x_1 x_3 x_5},$$

*where*

$$\begin{aligned}
D &= c^2\,{x_0}^2\,x_2\,x_3 + x_1\,x_2\,{x_3}^2\,x_5 + c\,x_0\big(x_1\,{x_3}^3 + x_2({x_3}^2 + x_1\,x_4 + x_1\,x_3\,x_5)\big),\\
E &= {x_0}^2\,x_2\,x_3 + x_1\,x_2\,{x_3}^2\,x_5 + x_0\left(x_1\,{x_3}^3 + x_2({x_3}^2 + x_1\,x_4 + x_1\,x_3\,x_5)\right),\\
F &= x_2\,{x_3}^2\,(x_0 + x_1\,x_5) + c\,x_0\left(x_0\,x_2\,x_3 + x_1({x_3}^3 + x_2\,x_4 + x_2\,x_3\,x_5)\right),\\
G &= c^3\,{x_0}^6\,{x_2}^3\,{x_3}^3 + 3\,c^2\,{x_0}^5\,{x_2}^3\,{x_3}^4 + 3\,c^2\,{x_0}^5\,x_1\,{x_2}^2\,{x_3}^5 + 3\,c\,{x_0}^4\,{x_2}^3\,{x_3}^5\\
&\quad + 6\,c\,{x_0}^4\,x_1\,{x_2}^2\,{x_3}^6 + {x_0}^3\,{x_2}^3\,{x_3}^6 + 3\,c\,{x_0}^4\,{x_1}^2\,x_2\,{x_3}^7 + 3\,{x_0}^3\,x_1\,{x_2}^2\,{x_3}^7\\
&\quad + 3\,{x_0}^3\,{x_1}^2\,x_2\,{x_3}^8 + {x_0}^3\,{x_1}^3\,{x_3}^9 + 3\,c^3\,{x_0}^5\,x_1\,{x_2}^3\,{x_3}^2\,x_4 + 6\,c^2\,{x_0}^4\,x_1\,{x_2}^3\,{x_3}^3\,x_4\\
&\quad + 3\,c\,{x_0}^4\,{x_1}^2\,{x_2}^2\,{x_3}^4\,x_4 + 3\,c^3\,{x_0}^4\,{x_1}^2\,{x_2}^2\,{x_3}^4\,x_4 + 3\,c\,{x_0}^3\,x_1\,{x_2}^3\,{x_3}^4\,x_4\\
&\quad + 3\,{x_0}^3\,{x_1}^2\,{x_2}^2\,{x_3}^5\,x_4 + 3\,c^2\,{x_0}^3\,{x_1}^2\,{x_2}^2\,{x_3}^5\,x_4 + 2\,{x_0}^3\,{x_1}^3\,x_2\,{x_3}^6\,x_4\\
&\quad + c^3\,{x_0}^3\,{x_1}^3\,x_2\,{x_3}^6\,x_4 + 3\,c^3\,{x_0}^4\,{x_1}^2\,{x_2}^3\,x_3\,{x_4}^2 + 3\,c^2\,{x_0}^3\,{x_1}^2\,{x_2}^3\,{x_3}^2\,{x_4}^2\\
&\quad + {x_0}^3\,{x_1}^3\,{x_2}^2\,{x_3}^3\,{x_4}^2 + 2\,c^3\,{x_0}^3\,{x_1}^3\,{x_2}^2\,{x_3}^3\,{x_4}^2 + c^3\,{x_0}^3\,{x_1}^3\,{x_2}^3\,{x_4}^3
\end{aligned}$$

$$+3c^3x_0^5x_1x_2^3x_3^3x_5+9c^2x_0^4x_1x_2^3x_3^4x_5+6c^2x_0^4x_1^2x_2^2x_3^5x_5$$
$$+9cx_0^3x_1x_2^3x_3^5x_5+12cx_0^3x_1^2x_2^2x_3^6x_5+3x_0^2x_1x_2^3x_3^6x_5$$
$$+3cx_0^3x_1^3x_2x_3^7x_5+6x_0^2x_1^2x_2^2x_3^7x_5+3x_0^2x_1^3x_2x_3^8x_5$$
$$+6c^3x_0^4x_1^2x_2^3x_3^2x_4x_5+12c^2x_0^3x_1^2x_2^3x_3^3x_4x_5+3cx_0^3x_1^3x_2^2x_3^4x_4x_5$$
$$+3c^3x_0^3x_1^3x_2^2x_3^4x_4x_5+6cx_0^2x_1^2x_2^3x_3^4x_4x_5+3x_0^2x_1^3x_2^2x_3^5x_4x_5$$
$$+3c^2x_0^2x_1^3x_2^2x_3^5x_4x_5+3c^3x_0^3x_1^3x_2^3x_3x_4^2x_5+3c^2x_0^2x_1^3x_2^3x_3^2x_4^2x_5$$
$$+3c^3x_0^4x_1^2x_2^3x_3^3x_5^2+9c^2x_0^3x_1^2x_2^3x_3^4x_5^2+3c^2x_0^3x_1^3x_2^2x_3^5x_5^2$$
$$+9cx_0^2x_1^2x_2^3x_3^5x_5^2+6cx_0^2x_1^3x_2^2x_3^6x_5^2+3x_0x_1^2x_2^3x_3^6x_5^2$$
$$+3x_0x_1^3x_2^2x_3^7x_5^2+3c^3x_0^3x_1^3x_2^3x_3^2x_4x_5^2+6c^2x_0^2x_1^3x_2^3x_3^3x_4x_5^2$$
$$+3cx_0x_1^3x_2^3x_3^4x_4x_5^2+c^3x_0^3x_1^3x_2^3x_3^3x_5^3+3c^2x_0^2x_1^3x_2^3x_3^4x_5^3$$
$$+3cx_0x_1^3x_2^3x_3^5x_5^3+x_1^3x_2^3x_3^6x_5^3,$$

$$H=cx_0^2x_2x_3+x_0x_2x_3^2+x_0x_1x_3^3+x_0x_1x_2x_4+cx_0x_1x_2x_3x_5+x_1x_2x_3^2x_5.$$

## 6. Ultra-discretization

We denote the positive structure on $\chi$ as in the previous section by $\theta : T' := (\mathbb{C}^\times)^6 \to \mathcal{V}_1$ ($x \mapsto V_1(x)$). Then by Corollary 2.7 we obtain the ultra-discretization $\mathcal{UD}(\chi, T', \theta)$, which is a Kashiwara's crystal. Now we show that the conjecture in [Igarashi and Nakashima 2010] is correct.

**Theorem 6.1.** *The crystal $\mathcal{UD}(\chi, T', \theta)$ as above is isomorphic to the crystal $B_\infty$ of type $G_2^{(1)}$ as in Section 4.*

To show this, we display the explicit crystal structure on $\mathcal{X} := \mathcal{UD}(\chi, T', \theta)$. Note that $\mathcal{UD}(\chi) = \mathbb{Z}^6$ as a set. Here as for variables in $\mathcal{X}$, we use the same notations $c, x_0, x_1, \dots, x_5$ as for $\chi$.

For $x = (x_0, x_1, \dots, x_5) \in \mathcal{X}$, it follows from the results in the previous section that the functions $\mathrm{wt}_i$ and $\varepsilon_i$ ($i = 0, 1, 2$) are given as

$$\mathrm{wt}_0(x) = 2x_0 - x_1 - x_3 - x_5,\quad \mathrm{wt}_1(x) = 2(x_1+x_3+x_5) - x_0 - x_2 - x_4,$$
$$\mathrm{wt}_2(x) = 2(x_2+x_4) - 3(x_1 - x_3 - x_5).$$

Set

$$\text{(6-1)}\quad \begin{aligned} &\alpha:=2x_0+x_2+x_3, && \beta:=x_1+x_2+2x_3+x_5, && \gamma:=x_0+x_1+3x_3,\\ &\delta:=x_0+x_2+2x_3, && \epsilon:=x_0+x_1+x_2+x_4, && \phi:=x_0+x_1+x_2+x_3+x_5. \end{aligned}$$

Then we have

$$\text{(6-2)}\quad \begin{aligned} \varepsilon_0(x) &= \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi) - (3x_0+x_2+x_3),\\ \varepsilon_1(x) &= \max(x_0-x_1,\, x_0+x_2-2x_1-x_3,\, x_0+x_2+x_4-2x_1-2x_3-x_5),\\ \varepsilon_2(x) &= \max(3x_1-x_2,\, 3x_1+3x_3-2x_2-x_4). \end{aligned}$$

Indeed, from the explicit form of $G$ in the previous section we have $\mathcal{UD}(G)|_{c=-1}$ $= \max(-3+3\alpha,\ -2+2\alpha+\delta,\ -2+2\alpha+\gamma,\ -1+\alpha+2\delta,\ -1+\alpha+\gamma+\delta,\ 3\delta,$ $-1+\alpha+2\gamma,\ \gamma+2\delta,\ 2\gamma+\delta,\ 3\gamma,\ -3+2\alpha+\epsilon,\ -2+\alpha+\delta+\epsilon,\ -1+\alpha+\gamma+\epsilon,$ $-1+2\delta+\epsilon, \gamma+\delta+\epsilon, 2\gamma+\epsilon, -3+\alpha+2\epsilon, -2+\delta+2\epsilon, \gamma+2\epsilon, -3+3\epsilon, -3+2\alpha+\phi,$ $-2+\alpha+\delta+\phi,\ -2+\alpha+\gamma+\phi,\ -1+2\delta+\phi,\ -1+\gamma+\delta+\phi,\ \beta+2\delta,\ -1+2\gamma+\phi,$ $\beta+\gamma+\delta,\ \beta+2\gamma,\ -3+\alpha+\epsilon+\phi,\ -2+\delta+\epsilon+\phi,\ -1+\gamma+\epsilon+\phi,\ -1+\beta+\delta+\epsilon,$ $\beta+\gamma+\epsilon,\ -3+2\epsilon+\phi,\ -2+\beta+2\epsilon,\ -3+\alpha+2\phi,\ -2+\delta+2\phi,\ -2+\gamma+2\phi,$ $-1+\alpha+2\beta, -1+\beta+\gamma+\phi, 2\beta+\delta, 2\beta+\gamma, -3+\epsilon+2\phi, -2+\beta+\epsilon+\phi, -1+2\beta+\epsilon,$ $-3+3\phi,\ -2+\beta+2\phi,\ -1+2\beta+\phi,\ 3\beta)$.

We simplify this by using the following lemma:

**Lemma 6.2.** *For $m_1, \dots, m_k \in \mathbb{R}$ and $t_1, \dots, t_k \in \mathbb{R}_{\geq 0}$ such that $t_1 + \cdots + t_k = 1$, we have*
$$\max\left(m_1, \dots, m_k, \sum_{i=1}^{k} t_i m_i\right) = \max(m_1, \dots, m_k).$$

Since we have

$$-2+2\alpha+\delta = \frac{2(-3+3\alpha)+3\delta}{3}, \qquad -2+2\alpha+\gamma = \frac{2(-3+3\alpha)+3\gamma}{3},$$
$$-1+\alpha+2\delta = \frac{2\cdot 3\delta+(-3+3\alpha)}{3}, \quad -1+\alpha+\gamma+\delta = \frac{(-3+3\alpha)+3\gamma+3\delta}{3},$$
$$-1+\alpha+2\gamma = \frac{(-3+3\alpha)+2\cdot 3\gamma}{3}, \qquad \gamma+2\delta = \frac{2\cdot 3\delta+3\gamma}{3},$$

and so on, we deduce using the lemma that

$$\begin{aligned}\mathcal{UD}(G)|_{c=-1} = \max(&-3+3\alpha, 3\beta, 3\gamma, 3\delta, -3+3\epsilon, -3+3\phi, -1+\alpha+\gamma+\epsilon,\\ &\gamma+\delta+\epsilon, \gamma+2\epsilon, 2\gamma+\epsilon, -1+\gamma+\epsilon+\phi, \beta+\gamma+\epsilon).\end{aligned}$$

Next, we describe the actions of $\tilde{f}_i$ $(i = 0, 1, 2)$. Set $\Xi_j := \mathcal{UD}(\mathcal{C}_j)|_{c=-1}$, for $j = 1, \dots, 5$. Then we have

$$\begin{aligned}
\Xi_1 &= \max(-1+x_0-x_1, x_0+x_2-2x_1-x_3, x_0+x_2+x_4-2x_1-2x_3-x_5)\\
&\qquad - \max(x_0-x_1, x_0+x_2-2x_1-x_3, x_0+x_2+x_4-2x_1-2x_3-x_5),\\
\Xi_3 &= \max(-1+x_0-x_1, -1+x_0+x_2-2x_1-x_3, x_0+x_2+x_4-2x_1-2x_3-x_5)\\
&\qquad - \max(-1+x_0-x_1, x_0+x_2-2x_1-x_3, x_0+x_2+x_4-2x_1-2x_3-x_5),\\
\Xi_5 &= \max(-1+x_0-x_1, -1+x_0+x_2-2x_1-x_3, -1+x_0+x_2+x_4-2x_1-2x_3-x_5)\\
&\qquad - \max(-1+x_0-x_1, -1+x_0+x_2-2x_1-x_3, x_0+x_2+x_4-2x_1-2x_3-x_5),\\
\Xi_2 &= \max(-1+3x_1-x_2, 3x_1+3x_3-2x_2-x_4)\\
&\qquad - \max(3x_1-x_2, 3x_1+3x_3-2x_2-x_4),\\
\Xi_4 &= \max(-1+3x_1-x_2, -1+3x_1+3x_3-2x_2-x_4)\\
&\qquad - \max(-1+3x_1-x_2, 3x_1+3x_3-2x_2-x_4).
\end{aligned}$$

Therefore, for $x \in \mathcal{X}$ we have

$$\tilde{f}_1(x) = (x_0,\, x_1+\Xi_1,\, x_2,\, x_3+\Xi_3,\, x_4,\, x_5+\Xi_5),$$
$$\tilde{f}_2(x) = (x_0,\, x_1,\, x_2+\Xi_2,\, x_3,\, x_4+\Xi_4,\, x_5).$$

We obtain the action $\tilde{e}_i$ $(i = 1, 2)$ by setting $c = 1$ in $\mathcal{UD}(\mathcal{C}_i)$.

Finally, we describe the action of $\tilde{f}_0$. Set

$$\begin{aligned}
\Psi_0 &:= \max(-2+\alpha,\, \beta,\, -1+\gamma,\, -1+\delta,\, -1+\epsilon,\, -1+\phi) - \max(\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, \phi)+1,\\
\Psi_1 &:= \max(-1+\alpha,\, \beta,\, -1+\gamma,\, \delta,\, -1+\epsilon,\, -1+\phi) - \max(\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, \phi)+1,\\
\Psi_2 &:= \max(-3+3\alpha,\, 3\beta,\, 3\gamma,\, 3\delta,\, -3+3\epsilon,\, -3+3\phi,\, -1+\alpha+\gamma+\epsilon,\, \gamma+\delta+\epsilon,\\
&\qquad \gamma+2\epsilon,\, 2\gamma+\epsilon,\, -1+\gamma+\epsilon+\phi,\, \beta+\gamma+\epsilon) - 3\max(\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, \phi)+3,\\
\Psi_3 &:= \max(-2+\alpha,\, \beta,\, -1+\gamma,\, -1+\delta,\, -1+\epsilon,\, -1+\phi)\\
&\qquad +\max(-1+\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, -1+\phi) - \max(\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, \phi)\\
&\qquad -\max(-1+\alpha,\, \beta,\, -1+\gamma,\, \delta,\, -1+\epsilon,\, -1+\phi)+2,\\
\Psi_4 &:= 3\max(-2+\alpha,\, \beta,\, -1+\gamma,\, -1+\delta,\, -1+\epsilon,\, -1+\phi)\\
&\qquad -\max(-3+3\alpha,\, 3\beta,\, 3\gamma,\, 3\delta,\, -3+3\epsilon,\, -3+3\phi,\, -1+\alpha+\gamma+\epsilon,\, \gamma+\delta+\epsilon,\\
&\qquad \gamma+2\epsilon,\, 2\gamma+\epsilon,\, -1+\gamma+\epsilon+\phi,\, \beta+\gamma+\epsilon)+3,\\
\Psi_5 &:= \max(-2+\alpha,\, \beta,\, -1+\gamma,\, -1+\delta,\, -1+\epsilon,\, -1+\phi)\\
&\qquad - \max(1+\alpha,\, \beta,\, \gamma,\, \delta,\, \epsilon,\, -1+\phi)+1,
\end{aligned}$$

where $\alpha, \beta, \dots, \phi$ are as in (6-1). Therefore, by the explicit form of $e_0^c$ as in the previous section, we have

$$\tilde{f}_0(x) = (x_0 + \Psi_0,\, x_1 + \Psi_1,\, x_2 + \Psi_2,\, x_3 + \Psi_3,\, x_4 + \Psi_4,\, x_5 + \Psi_5). \tag{6-3}$$

We have the explicit form of $\tilde{e}_0$ by setting $c = 1$ in $\mathcal{UD}(\mathcal{C}_i)$.

*Proof of Theorem 6.1.* Define the map

$$\begin{array}{cccc}\Omega: & \mathcal{X} & \to & B_\infty\\ & (x_0, \dots, x_5) & \mapsto & (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1),\end{array}$$

by

$$b_1 = x_5,\ b_2 = \tfrac{1}{3}x_4 - x_5,\ b_3 = x_3 - \tfrac{2}{3}x_4,\ \bar{b}_3 = \tfrac{2}{3}x_2 - x_3,\ \bar{b}_2 = x_1 - \tfrac{1}{3}x_2,\ \bar{b}_1 = x_0 - x_1,$$

and $\Omega^{-1}$ is given by

$$x_0 = b_1 + b_2 + \tfrac{1}{2}(b_3 + \bar{b}_3) + \bar{b}_2 + \bar{b}_1, \quad x_1 = b_1 + b_2 + \tfrac{1}{2}(b_3 + \bar{b}_3) + \bar{b}_2,$$
$$x_2 = 3b_1 + 3b_2 + \tfrac{3}{2}(b_3 + \bar{b}_3),\ x_3 = 2b_1 + 2b_2 + b_3,\ x_4 = 3b_1 + 3b_2,\ x_5 = b_1,$$

which means that $\Omega$ is bijective. Note that $\frac{3}{2}(b_3 + \bar{b}_3) \in \mathbb{Z}$ by the definition of $B_\infty$ as on page 127. We shall show that $\Omega$ is commutative with actions of $\tilde{f}_i$ and preserves the functions $\mathrm{wt}_i$ and $\varepsilon_i$, that is,

$$\tilde{f}_i(\Omega(x)) = \Omega(\tilde{f}_i x), \quad \mathrm{wt}_i(\Omega(x)) = \mathrm{wt}_i(x), \quad \varepsilon_i(\Omega(x)) = \varepsilon_i(x) \quad (i = 0, 1, 2),$$

Indeed, the commutativity $\tilde{e}_i(\Omega(x)) = \Omega(\tilde{e}_i x)$ is shown by a similar way. First, let us check $\mathrm{wt}_i$:

Set $b = \Omega(x)$ and let $(z_1, z_2, z_3, z_4)$ be as in (4-2). By the explicit forms of $\mathrm{wt}_i$ on $\mathscr{X}$ and $B_\infty$, we have

$$\begin{aligned}
\mathrm{wt}_0(\Omega(x)) &= \varphi_0(\Omega(x)) - \varepsilon_0(\Omega(x)) = 2z_1 + z_2 + z_3 + 3z_4 \\
&= 2(\bar{b}_1 - b_1) + (\bar{b}_2 - \bar{b}_3) + (b_3 - b_2) + \tfrac{3}{2}(\bar{b}_3 - b_3) \\
&= 2(\bar{b}_1 - b_1) + \bar{b}_2 - b_2 + \tfrac{1}{2}(\bar{b}_3 - b_3) = 2x_0 - x_1 - x_3 - x_5 = \mathrm{wt}_0(x), \\
\mathrm{wt}_1(\Omega(x)) &= \varphi_1(\Omega(x)) - \varepsilon_1(\Omega(x)) \\
&= b_1 + (b_3 - b_2 + (\bar{b}_2 - \bar{b}_3)_+)_+ - (\bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+) \\
&= b_1 - \bar{b}_1 - b_2 + \bar{b}_2 + b_3 - \bar{b}_3 = 2(x_1 + x_3 + x_5) - x_0 - x_2 - x_4 \\
&= \mathrm{wt}_1(x), \\
\mathrm{wt}_2(\Omega(x)) &= \varphi_2(\Omega(x)) - \varepsilon_2(\Omega(x)) \\
&= 3b_2 + \tfrac{3}{2}(\bar{b}_3 - b_3)_+ - 3\bar{b}_2 - \tfrac{3}{2}(b_3 - \bar{b}_3)_+ \\
&= 3b_2 - 3\bar{b}_2 + \tfrac{3}{2}(\bar{b}_3 - b_3) = 2(x_2 + x_4) - 3(x_1 + x_3 + x_5) = \mathrm{wt}_2(x).
\end{aligned}$$

Next, we check $\varepsilon_i$:

$$\begin{aligned}
\varepsilon_1(\Omega(x)) &= \bar{b}_1 + (\bar{b}_3 - \bar{b}_2 + (b_2 - b_3)_+)_+ \\
&= \max(\bar{b}_1, \bar{b}_1 + \bar{b}_3 - \bar{b}_2, \bar{b}_1 + \bar{b}_3 - \bar{b}_2 + b_2 - b_3) \\
&= \max(x_0 - x_1, x_0 - 2x_1 + x_2 - x_3, x_0 - 2x_1 + x_2 - 2x_3 + x_4 - x_5) = \varepsilon_1(x), \\
\varepsilon_2(\Omega(x)) &= 3\bar{b}_2 + \tfrac{3}{2}(b_3 - \bar{b}_3)_+ = \max\bigl(3\bar{b}_2, 3\bar{b}_2 + \tfrac{3}{2}(b_3 - \bar{b}_3)\bigr) \\
&= \max(3x_1 - x_2, 3x_1 - 2x_2 + 3x_3 - x_4) = \varepsilon_2(x).
\end{aligned}$$

Now let us see $\varepsilon_0$:

$$\begin{aligned}
&\varepsilon_0(\Omega(x)) \\
&\quad = -s(b) + \max A - (2z_1 + z_2 + z_3 + 3z_4) \\
&\quad = -x_0 + \max(0, z_1, z_1 + z_2, z_1 + z_2 + 3z_4, \\
&\qquad\qquad z_1 + z_2 + z_3 + 3z_4, 2z_1 + z_2 + z_3 + 3z_4) - (\alpha - \beta) \\
&\quad = -x_0 + \max(-2x_0 + x_1 + x_3 + x_5, -x_0 + x_3, -x_0 + x_1 - x_2 + 2x_3, \\
&\qquad\qquad -x_0 + x_1 - x_3 + x_4, -x_0 + x_1 + x_5, 0) \\
&\quad = -(3x_0 + x_2 + x_3) + \max(x_1 + x_2 + 2x_3 + x_5, x_0 + x_2 + 2x_3, x_0 + x_1 + 3x_3, \\
&\qquad\qquad x_0 + x_1 + x_2 + x_4, x_0 + x_1 + x_2 + x_3 + x_5, 2x_0 + x_2 + x_3) \\
&\quad = -(3x_0 + x_2 + x_3) + \max(\beta, \delta, \gamma, \epsilon, \phi, \alpha).
\end{aligned}$$

On the other hand, we have

$$\varepsilon_0(x) = -(3x_0 + x_2 + x_3) + \max(\alpha, \beta, \gamma, \delta, \epsilon, \phi).$$

which shows $\varepsilon_0(\Omega(x)) = \varepsilon_0(x)$.

Let us show $\tilde{f}_i(\Omega(x)) = \Omega(\tilde{f}_i(x))$ $(x \in \mathscr{X},\ i = 0, 1, 2)$. As for $\tilde{f}_1$, set

$$A = x_0 - x_1, \quad B = x_0 + x_2 - 2x_1 - x_3, \quad C = x_0 + x_2 + x_4 - 2x_1 - 2x_3 - x_5.$$

Then we obtain

$$\begin{aligned}
\Xi_1 &= \max(A-1, B, C) - \max(A, B, C),\\
\Xi_3 &= \max(A-1, B-1, C) - \max(A-1, B, C),\\
\Xi_5 &= \max(A-1, B-1, C-1) - \max(A-1, B-1, C).
\end{aligned}$$

Therefore, we have

$$\begin{array}{llll}
\Xi_1 = -1, & \Xi_3 = 0, & \Xi_5 = 0, & \text{if } A > B, C,\\
\Xi_1 = 0, & \Xi_3 = -1, & \Xi_5 = 0, & \text{if } A \le B > C,\\
\Xi_1 = 0, & \Xi_3 = 0, & \Xi_5 = -1, & \text{if } A, B \le C,
\end{array}$$

which implies

$$\tilde{f}_1(x) = \begin{cases} (x_0, x_1 - 1, x_2, \dots, x_5) & \text{if } A > B, C,\\ (x_0, \dots, x_3 - 1, x_4, x_5) & \text{if } A \le B > C,\\ (x_0, \dots, x_4, x_5 - 1) & \text{if } A, B \le C. \end{cases}$$

Since $A = \bar{b}_1$, $B = \bar{b}_1 + \bar{b}_3 - \bar{b}_2$ and $C = \bar{b}_1 + \bar{b}_3 - \bar{b}_2 + b_2 - b_3$, we get $(b = \Omega(x))$

$$\Omega(\tilde{f}_1(x)) = \begin{cases} (\dots, \bar{b}_2 - 1, \bar{b}_1 + 1) & \text{if } \bar{b}_2 - \bar{b}_3 > (b_2 - b_3)_+,\\ (\dots, b_3 - 1, \bar{b}_3 + 1, \dots) & \text{if } \bar{b}_2 - \bar{b}_3 \le 0 < b_3 - b_2,\\ (b_1 - 1, b_2 + 1, \dots) & \text{if } (\bar{b}_2 - \bar{b}_3)_+ \le b_2 - b_3, \end{cases}$$

which is the same as the action of $\tilde{f}_1$ on $b = \Omega(x)$ as on page 128. Hence, we have $\Omega(\tilde{f}_1(x)) = \tilde{f}_1(\Omega(x))$.

Let us see $\Omega(\tilde{f}_2(x)) = \tilde{f}_2(\Omega(x))$. Set

$$L = 3x_1 - x_2, \quad M = 3x_1 + 3x_3 - 2x_2 - x_4.$$

Then $\Xi_2 = \max(-1 + L, M) - \max(L, M)$ and $\Xi_4 = \max(-1 + L, -1 + M) - \max(-1 + L, M)$. Thus, one has

$$\begin{array}{lll}
\Xi_2 = -1, & \Xi_4 = 0 & \text{if } L > M,\\
\Xi_2 = 0, & \Xi_4 = -1 & \text{if } L \le M,
\end{array}$$

which means

$$\tilde{f}_2(x) = \begin{cases} (x_0, x_1, x_2 - 1, x_3, x_4, x_5) & \text{if } L > M, \\ (x_0, x_1, x_2, x_3, x_4 - 1, x_5) & \text{if } L \le M. \end{cases}$$

Since $L - M = x_2 - 3x_3 + x_4 = \frac{3}{2}(\bar{b}_3 - b_3)$, one gets

$$\Omega(\tilde{f}_2(x)) = \begin{cases} (\dots, \bar{b}_3 - \frac{2}{3}, \bar{b}_2 + \frac{1}{3}, \dots) & \text{if } \bar{b}_3 > b_3, \\ (\dots, b_2 - \frac{1}{3}, b_3 + \frac{2}{3}, \dots) & \text{if } \bar{b}_3 \le b_3, \end{cases}$$

where $b = \Omega(x)$. This action coincides with the one of $\tilde{f}_2$ on $b \in B_\infty$ on page 128. Therefore, we get $\Omega(\tilde{f}_2(x)) = \tilde{f}_2(\Omega(x))$.

Finally, we shall check $\tilde{f}_0(\Omega(x)) = \Omega(\tilde{f}_0(x))$. For the purpose, we shall estimate the values $\Psi_0, \dots, \Psi_5$ explicitly.

First, the following cases are investigated:

$$\begin{aligned}
&(\mathrm{f}_1) && \beta \ge \gamma, \delta, \epsilon, \phi, \ \ \phi \ge \alpha, \ \ \delta \ge \alpha. \\
&(\mathrm{f}_2) && \beta < \delta \ge \alpha, \gamma, \epsilon, \ \ \alpha > \phi, \ \ \beta \ge \phi. \\
&(\mathrm{f}_3) && \beta, \delta < \gamma \ge \alpha, \epsilon, \phi. \\
&(\mathrm{f}_4) && \beta, \delta < \epsilon \ge \alpha, \phi, \ \ \epsilon = \gamma + 1. \\
&(\mathrm{f}'_4) && \beta, \delta < \epsilon \ge \alpha, \phi, \ \ \epsilon = \gamma + 2. \\
&(\mathrm{f}''_4) && \beta, \delta < \epsilon \ge \alpha, \phi, \ \ \epsilon > \gamma + 2. \\
&(\mathrm{f}_5) && \beta, \gamma, \epsilon < \phi \ge \alpha, \ \ \alpha > \delta, \ \ \beta \ge \delta. \\
&(\mathrm{f}_6) && \alpha > \gamma, \delta, \epsilon, \phi, \ \ \delta, \phi > \beta.
\end{aligned}$$

It is easy to see that each of these conditions are equivalent to the conditions $(F_1)$–$(F_6)$ in (4-3); more precisely, we have $(\mathrm{f}_i) \Longleftrightarrow (F_i)$ $(i = 1, 2, 3, 5, 6)$, $(\mathrm{f}_4) \Longleftrightarrow (F_4)$ and $z_4 = \frac{1}{3}$, $(\mathrm{f}'_4) \Longleftrightarrow (F_4)$ and $z_4 = \frac{2}{3}$ and $(\mathrm{f}''_4) \Longleftrightarrow (F_4)$ and $z_4 \ne \frac{1}{3}, \frac{2}{3}$, and that $(\mathrm{f}_1)$–$(\mathrm{f}_6)$ cover all cases and they have no intersection.

Let us show $(\mathrm{f}_1) \Longleftrightarrow (F_1)$: the condition $(\mathrm{f}_1)$ means $\beta - \gamma = -(z_1 + z_2) \ge 0$, $\beta - \delta = -z_1 \ge 0$, $\beta - \epsilon = -(z_1 + z_2 + 3z_4) \ge 0$ and $\beta - \phi = -(z_1 + z_2 + z_3 + 3z_4) \ge 0$, which is equivalent to the condition $z_1 + z_2 \le 0$, $z_1 \le 0$, $z_1 + z_2 + 3z_4 \le 0$ and $z_1 + z_2 + z_3 + 3z_4 \le 0$. (Note that $\phi - \alpha = \beta - \delta$, $\delta - \alpha = \beta - \phi$.) This is just the condition $(F_1)$. Other cases $i = 2, 3, 5, 6$ are shown similarly. Next, let us see the cases $(\mathrm{f}_4)$, $(\mathrm{f}'_4)$ and $(\mathrm{f}''_4)$. Indeed,

$$\epsilon - \gamma = x_2 - 3x_3 + x_4 = \frac{3}{2}(\bar{b}_3 - b_3) = 3z_4.$$

Thus, we can easily get that $(\mathrm{f}_4) \Longleftrightarrow (F_4)$ and $z_4 = \frac{1}{3}$, $(\mathrm{f}'_4) \Longleftrightarrow (F_4)$ and $z_4 = \frac{2}{3}$ and $(\mathrm{f}''_4) \Longleftrightarrow (F_4)$ and $z_4 \ne \frac{1}{3}, \frac{2}{3}$.

Under the condition $(\mathrm{f}_1) \Longleftrightarrow (F_1)$, we have

$$\Psi_0 = \Psi_1 = \Psi_5 = 1,\, \Psi_2 = \Psi_4 = 3, \quad \Psi_3 = 2,$$

which means $\tilde{f}_0(x) = (x_0+1,\, x_1+1,\, x_2+3,\, x_3+2,\, x_4+3,\, x_5+1)$. Thus, we have

$$\Omega(\tilde{f}_0(x)) = (b_1+1, b_2, \ldots, \bar{b}_1),$$

which coincides with the action of $\tilde{f}_0$ under $(F_1)$ given on page 128. Similarly, we have

$$\begin{aligned}(\mathrm{f}_2) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 1, 3, 1, 0, 0)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1+1, x_2+3, x_3+1, x_4, x_5)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2, b_3+1, \bar{b}_3+1, \bar{b}_2, \bar{b}_1-1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_2)$ on the same page;

$$\begin{aligned}(\mathrm{f}_3) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 3, 2, 0, 0)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1, x_2+3, x_3+2, x_4, x_5)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2, b_3+2, \bar{b}_3, \bar{b}_2-1, \bar{b}_1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_3)$;

$$\begin{aligned}(\mathrm{f}_4) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 2, 2, 1, 0)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1, x_2+2, x_3+2, x_4+1, x_5)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2+\tfrac{1}{3}, b_3+\tfrac{4}{3}, \bar{b}_3-\tfrac{2}{3}, \bar{b}_2-\tfrac{2}{3}, \bar{b}_1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_4)$ and $z_4 = \frac{1}{3}$;

$$\begin{aligned}(\mathrm{f}'_4) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 1, 2, 2, 0)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1, x_2+1, x_3+2, x_4+2, x_5)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2+\tfrac{2}{3}, b_3+\tfrac{2}{3}, \bar{b}_3-\tfrac{4}{3}, \bar{b}_2-\tfrac{1}{3}, \bar{b}_1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_4)$ and $z_4 = \frac{2}{3}$;

$$\begin{aligned}(\mathrm{f}''_4) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, 2, 3, 0)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1, x_2, x_3+2, x_4+3, x_5)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1, b_2+1, b_3, \bar{b}_3-2, \bar{b}_2, \bar{b}_1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_4)$ and $z_4 \neq \frac{1}{3}, \frac{2}{3}$;

$$\begin{aligned}(\mathrm{f}_5) &\Rightarrow (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (0, 0, 0, 1, 3, 1)\\ &\Rightarrow \tilde{f}_0(x) = (x_0, x_1, x_2, x_3+1, x_4+3, x_5+1)\\ &\Rightarrow \Omega(\tilde{f}_0(x)) = (b_1+1, b_2, b_3-1, \bar{b}_3-1, \bar{b}_2, \bar{b}_1),\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_5)$. Finally,

$$\begin{aligned}
(\mathrm{f}_6)\ &\Rightarrow\ (\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5) = (-1, 0, 0, 0, 0, 0)\\
&\Rightarrow\ \tilde{f}_0(x) = (x_0 - 1, x_1, x_2, x_3, x_4, x_5)\\
&\Rightarrow\ \Omega(\tilde{f}_0(x)) = (b_1, b_2, b_3, \bar{b}_3, \bar{b}_2, \bar{b}_1 - 1),
\end{aligned}$$

which coincides with the action of $\tilde{f}_0$ under $(F_6)$, still on page 128. Now, we have $\Omega(\tilde{f}_0(x)) = \tilde{f}_0(\Omega(x))$. The proof of Theorem 6.1 has been completed. □

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Received October 5, 2010.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

# PACIFIC JOURNAL OF MATHEMATICS

Volume 255 No. 1 January 2012





0030-8730(201201)255:1;1-D