Pacific Journal of Mathematics

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Volume 255 No. 1 January 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 255, No. 1, 2012

REMARKS ON THE CURVATURE BEHAVIOR AT THE FIRST SINGULAR TIME OF THE RICCI FLOW

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We study the curvature behavior at the first singular time of a solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \quad t \in [0, T),$$

on a smooth, compact n-dimensional Riemannian manifold M. If the flow has uniformly bounded scalar curvature and develops Type I singularities at T, we show that suitable blow-ups of the evolving metrics converge in the pointed Cheeger–Gromov sense to a Gaussian shrinker by using Perelman's $\mathcal{W}$-functional. If the flow has uniformly bounded scalar curvature and develops Type II singularities at T, we show that suitable scalings of the potential functions in Perelman's entropy functional converge to a positive constant on a complete, Ricci flat manifold. We also show that if the scalar curvature is uniformly bounded along the flow in certain integral sense then the flow either develops a Type II singularity at T or it can be smoothly extended past time T.

1. Introduction

The Ricci flow and previous results. Let M be a smooth, compact n-dimensional Riemannian manifold without boundary equipped with a smooth Riemannian metric g_0, where $n \geq 3$. Let $g(t)$, $0 \leq t < T$, be a one-parameter family of metrics on M. The Ricci flow equation on M with initial metric g_0

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)), \quad g(0) = g_0. \tag{1-1}$$

was introduced in the seminal paper [Hamilton 1982]. It is a weakly parabolic system of equations whose short-time existence was proved by Hamilton using the Nash–Moser implicit function theorem in the same paper and after that simplified by DeTurck [1983]. The goal in the analysis of (1-1) is to understand the long-time behavior of the flow and possible singularity formation or convergence of the flow in the cases when we do have a long-time existence. In general, the behavior of the flow can give insight into the topology of the underlying manifold. One of the

Sesum was partially supported by NSF grant 0905749.
MSC2010: primary 53C44; secondary 35K10.
Keywords: Ricci flow, scalar curvature, evolution.

great successes is the resolution of the Poincaré conjecture by Perelman. In order to discuss the long-time behavior we have to understand what happens at the singular time and also what the optimal conditions for having a smooth solution are.

Hamilton [1995b] showed that if the norm of Riemannian curvature $|\mathrm{Rm}|(g(t))$ stays uniformly bounded in time for all $t \in [0, T)$ with $T < \infty$, then we can extend the flow (1-1) smoothly past time T. In other words, either the flow exists forever or the norm of Riemannian curvature blows up in finite time. Wang [2008] and Ye [2008] extended this result, assuming certain integral bounds on the Riemannian curvature. Namely, if

$$\int_0^T \int_M |\mathrm{Rm}|^\alpha \, d\mathrm{vol}_{g(t)} \, dt \le C \quad \text{for some } \alpha \ge \frac{n+2}{2},$$

then the flow can be extended smoothly past time T. Throughout the paper, $d\mathrm{vol}_g$ denotes the Riemannian volume density on (M, g). On the other hand, Sesum [2005] improved Hamilton's extension result and showed if the norm of Ricci curvature is uniformly bounded over a finite time interval $[0, T)$, then we can extend the flow smoothly past time T. Wang [2008] improved this even further, showing that if Ricci curvature is uniformly bounded from below and if the space-time integral of the scalar curvature is bounded, say

$$\int_0^T \int_M |R|^\alpha \, d\mathrm{vol}_{g(t)} \, dt \le C \quad \text{for } \alpha \ge \frac{n+2}{2},$$

where R is the scalar curvature, then we can extend the flow smoothly past time T. The requirement on Ricci curvature in [Wang 2008] is rather restrictive. Ricci flow does not in general preserve nonnegative Ricci curvature in dimensions $n \ge 4$. See [Knopf 2006] for noncompact examples starting in dimension $n = 4$ and [Böhm and Wilking 2007] for compact examples starting in dimension $n = 12$. Recently, Maximo [2011] brought the result of [Böhm and Wilking 2007] down to dimension 4 by showing that nonnegative Ricci curvature is not preserved under Ricci flow for closed compact manifolds of dimensions 4 and above. Without assuming the boundedness from below of Ricci curvature, Ma and Cheng [2010] proved that the norm of Riemannian curvature can be controlled given integral bounds on the scalar curvature R and the Weyl tensor W from the orthogonal decomposition of the Riemannian curvature tensor. Their bounds are of the form

$$\int_0^T \int_M (|R|^\alpha + |W|^\alpha) \, d\mathrm{vol}_{g(t)} \, dt \le C \quad \text{for } \alpha \ge \frac{n+2}{2}.$$

This is not surprising since Knopf [2009] has shown that the trace-free Ricci tensor is controlled pointwise by the scalar curvature and the Weyl tensor without any additional hypotheses. Zhang [2010] proved that the scalar curvature controls the Kähler Ricci flow $\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} - g_{i\bar{j}}$ starting from any Kähler metric g_0.

Main results. The above results, in particular that of [Zhang 2010], support the belief that the scalar curvature should control the Ricci flow in the Riemannian setting as well. Enders, Müller and Topping [2010] justified this belief for Type I Ricci flow:

Theorem 1.1 [Enders et al. 2010]. *Let M be a smooth, compact n-dimensional Riemannian manifold equipped with a smooth Riemannian metric g_0 and $g(\cdot, t)$ be a solution to the Type I Ricci flow (1-1) on M. Assume there is a constant C so that $\sup_M |R(\cdot, t)| \le C$ for all $t \in [0, T)$ and $T < \infty$. Then we can extend the flow past time T.*

Their proof was based on a blow-up argument using Perelman's reduced distance and pseudolocality theorem.

Assume the flow (1-1) develops a singularity at $T < \infty$.

Definition 1.1. We say that (1-1) has a *Type I singularity at T* if there exists a constant $C > 0$ such that for all $t \in [0, T)$

$$\max_M |\mathrm{Rm}(\cdot, t)| \cdot (T - t) \le C. \tag{1-2}$$

Otherwise we say the flow develops *Type II singularity at T*. Moreover, the flow that satisfies (1-2) will be referred to as to the *Type I Ricci flow.*

In this paper, we also use a blow-up argument to study curvature behavior at the first singular time of the Ricci flow. We deal with both Type I and II singularities. Assume that the scalar curvature is uniformly bounded along the flow. If the flow develops Type I singularities at some finite time T then by using Perelman's entropy functional $\mathcal{W}$, we show that suitable blow-ups of the evolving metrics converge in the pointed Cheeger–Gromov sense to a Gaussian shrinker.

Theorem 1.2. *Let M be a smooth, compact n-dimensional Riemannian manifold ($n \ge 3$) and $g(\cdot, t)$ be a solution to the Ricci flow (1-1) on M. Assume there is a constant C so that $\sup_M |R(\cdot, t)| \le C$ for all $t \in [0, T)$ and $T < \infty$. Assume that at T we have a Type I singularity and the norm of the curvature operator blows up. Then by suitably rescaling the metrics, we get a Gaussian shrinker in the limit.*

A simple consequence of the proof of Theorem 1.2 is following result, which is also proved in [Naber 2010]. Instead of the reduced distance techniques used by Naber, we use Perelman's monotone functional $\mathcal{W}$.

Corollary 1.1. *Let M be a smooth, compact n-dimensional Riemannian manifold ($n \ge 3$) and $g(\cdot, t)$ be a solution to the Ricci flow (1-1) on M. If the flow has a Type I singularity at T, then a suitable rescaling of the solution converges to a gradient shrinking Ricci soliton.*

Naber [2010] proved that in the case of a Type I singularity, a suitable rescaling of the flow converges to gradient shrinking Ricci soliton. Enders, Müller and Topping [2010] recently showed that the limiting soliton represents a singularity model, that is, it is nonflat (see also [Cao and Zhang 2011]). The open question is whether using Perelman's $\mathcal{W}$-functional, one can produce in the limit a singularity model (*nonflat* gradient shrinking Ricci solitons). We prove some interesting estimates on the minimizers of Perelman's $\mathcal{W}$-functional which could be of independent interest.

On the other hand, if the flow develops Type II singularities at some finite time T, then we show that suitable scalings of the potential functions in Perelman's entropy functional converge to a positive constant on a complete, Ricci flat manifold which is the pointed Cheeger–Gromov limit of a suitably chosen sequence of blow-ups of the original evolving metrics.

Theorem 1.3. *Let M be a smooth, compact n-dimensional Riemannian manifold ($n \geq 3$) and $g(\cdot, t)$ be a solution to the Ricci flow (1-1) on M. Assume there is a constant C so that $\sup_M |R(\cdot, t)| \leq C$ for all $t \in [0, T)$ and $T < \infty$. Assume that at T we have a Type II singularity and the norm of the curvature operator blows up. Let ϕ_i be as in the proof of Theorem 1.2* (see, for example, (3-9)). *Then by suitably rescaling the metrics and ϕ_i, we get as a limit of ϕ_i a positive constant on a complete, Ricci flat manifold.*

We believe that Theorem 1.3 may play a role in proving the nonexistence of Type II singularities if the scalar curvature is uniformly bounded along the flow. We are still investigating this issue.

For a precise definition of ϕ_i, see Section 3.

There has been a striking analogy between the Ricci flow and the mean curvature flow for decades now. Around the same time Hamilton proved that the norm of the Riemannian curvature under the Ricci flow must blow up at a finite singular time, Huisken [1984] showed that the norm of the second fundamental form of an evolving hypersurface under the mean curvature flow must blow up at a finite singular time. The analogue of Wang's result holds for the mean curvature flow as well [Le and Sesum 2011], namely if the second fundamental form of an evolving hypersurface is uniformly bounded from below and if the mean curvature is bounded in a certain integral sense, then we can smoothly extend the flow. In the follow-up paper [Le and Sesum 2010] the authors show that given only the uniform bound on the mean curvature of the evolving hypersurface, the flow either develops a Type II singularity or can be smoothly extended. In the case the dimension of the evolving hypersurfaces is 2 they show that under some density assumptions one can smoothly extend the flow provided that the mean curvature is uniformly bounded. Finally, in contrast to the lower bound on the scalar curvature (2-3), at the first singular time of the mean curvature flow, the mean curvature can either tend to ∞ (as in the case of a round sphere) or $-\infty$ as in some examples of Type II singularities [Angenent and Velázquez 1997].

If we replace the pointwise scalar curvature bound in Theorem 1.1 with an integral bound, we can prove the following theorems.

Theorem 1.4. *If $g(\cdot, t)$ solves (1-1) and if*

$$\int_M |R|^\alpha(t)\, d\mathrm{vol}_{g(t)} \le C_\alpha \tag{1-3}$$

for all $t \in [0, T)$ where $\alpha > n/2$ and $T < \infty$, then either the flow develops a Type II singularity at T or the flow can be smoothly extended past time T.

Remark 1.1. The condition on α in Theorem 1.4 is optimal. Let (S^n, g_0) be the space form of constant sectional curvature 1. The Ricci flow on $M = S^n$ with initial metric g_0 has the solution $g(t) = (1-2(n-1)t)g_0$. Therefore $T = 1/(2(n-1))$ is the maximal existence time. Rewrite $g(t) = 2(n-1)(T-t)g_0$ to compute

$$\begin{aligned}\int_M |R|^\alpha(t)\, d\mathrm{vol}_{g(t)} &= \mathrm{vol}_{g(t)}(M)\Big(\frac{n}{2(T-t)}\Big)^\alpha\\ &= \mathrm{vol}_{g(0)}(M)(2(n-1)(T-t))^{n/2}\Big(\frac{n}{2(T-t)}\Big)^\alpha\\ &= \mathrm{vol}_{g(0)}(M)2^{n/2-\alpha}(n-1)^{n/2}n^\alpha\frac{1}{(T-t)^{\alpha-n/2}}.\end{aligned}$$

Hence $\int_M |R|^\alpha(t)\, d\mathrm{vol}_{g(t)}$ tends to ∞ as $t \to T$ if and only if $\alpha > n/2$.

Theorem 1.5. *If $g(\cdot, t)$ solves (1-1) and if we have the space-time integral bound*

$$\int_0^T \int_M |R|^\alpha(t)\, d\mathrm{vol}_{g(t)}\, dt \le C_\alpha \tag{1-4}$$

for $\alpha \ge (n+2)/2$, then the flow either develops a Type II singularity at T or can be smoothly extended past time T.

Remark 1.2. The condition on α in Theorem 1.5 is optimal. As in Remark 1.1 consider the Ricci flow on the round sphere. Following the computation in Remark 1.1 we get

$$\int_0^T \int_M |R|^\alpha\, d\mathrm{vol}_{g(t)}\, dt = \mathrm{vol}_{g(0)}(M)2^{n/2-\alpha}(n-1)^{n/2}n^\alpha \int_0^T \frac{1}{(T-t)^{\alpha-n/2}}\, dt,$$

and therefore the integral is ∞ if and only if $\alpha \ge (n+2)/2$.

For the mean curvature flow, a similar result to Theorem 1.5 has been obtained in [Le and Sesum 2010].

The rest of the paper is organized as follows. In Section 2 we give some necessary preliminaries. Section 3 is devoted to the statements and proofs of Theorems 1.2 and 1.3. In Section 4 we prove Theorems 1.4 and 1.5.

2. Preliminaries

In this section, we recall basic evolution equations during the Ricci flow and the definition of singularity formation. Then we recall Perelman's entropy functional $\mathcal{W}$ and in Lemma 2.1 prove one of its properties, nonpositivity of the μ-energy. The nonpositivity of the μ-energy turns out to be very crucial for the proof of Theorem 1.1.

Evolution equations and singularity formation. Consider the Ricci flow (1-1) on $[0, T)$. Then, the scalar curvature R and the volume form $\mathrm{vol}_{g(t)}$ evolve by

$$\frac{\partial}{\partial t} R = \Delta R + 2|\mathrm{Ric}|^2, \tag{2-1}$$

$$\frac{\partial}{\partial t} \mathrm{vol}_{g(t)} = -R \, \mathrm{vol}_{g(t)} \,. \tag{2-2}$$

Because $|\mathrm{Ric}|^2 \geq R^2/n$, the maximum principle applied to (2-1) yields

$$R(g(t)) \geq \frac{\min_M R(g(0))}{1 - (2 \min_M R(g(0))t)/n}. \tag{2-3}$$

If $T < +\infty$ and the norm of the Riemannian curvature $|\mathrm{Rm}|(g(t))$ becomes unbounded as t tends to T, we say the Ricci flow develops singularities as t tends to T and T is a singular time. It is well-known that the Ricci flow generally develops singularities.

If a solution $(M, g(t))$ to the Ricci flow develops singularities at $T < +\infty$, then according to [Hamilton 1995b], we say that it develops a *Type I singularity* if

$$\sup_{t \in [0,T)} (T - t) \max_M |\mathrm{Rm}(\,\cdot\,, t)| < +\infty,$$

and it develops a *Type II singularity* if

$$\sup_{t \in [0,T)} (T - t) \max_M |\mathrm{Rm}(\,\cdot\,, t)| = +\infty.$$

Clearly, the Ricci flow of a round sphere develops a Type I singularity in finite time. The existence of Type II singularities for the Ricci flow has been recently established in [Gu and Zhu 2008], proving the degenerate neckpinch conjecture of [Hamilton 1995b].

Finally, by the curvature gap estimate for Ricci flow solutions with a finite-time singularity (see, for example, [Chow et al. 2006, Lemma 8.7]), we have

$$\max_{x \in M} |\mathrm{Rm}(x, t)| \geq \frac{1}{8(T - t)}. \tag{2-4}$$

Perelman's entropy functional $\mathcal{W}$ and the μ-energy. Perelman [2002] introduced a very important functional, the entropy functional $\mathcal{W}$, for the study of the Ricci flow:

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M \big(\tau(R + |\nabla f|^2) + f - n\big)\, e^{-f}\, d\mathrm{vol}_g, \tag{2-5}$$

under the constraint $(4\pi\tau)^{-n/2} \int_M e^{-f}\, d\mathrm{vol}_g = 1$. The functional $\mathcal{W}$ is invariant under the parabolic scaling of the Ricci flow and invariant under diffeomorphism. Namely, for any positive number α and any diffeomorphism φ, we have $\mathcal{W}(\alpha\varphi^* g, \varphi^* f, \alpha\tau) = \mathcal{W}(g, f, \tau)$. Perelman showed that if $\dot\tau = -1$ and $f(\,\cdot\,, t)$ is a solution to the backwards heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \tag{2-6}$$

and if $g(\,\cdot\,, t)$ solves the Ricci flow (1-1) then

$$\frac{d}{dt}\mathcal{W}(g(t), f(t), \tau) = (2\tau)\cdot(4\pi\tau)^{-n/2} \int_M \left| R_{ij} + \nabla_i\nabla_j f - \frac{g_{ij}}{2\tau}\right|^2 e^{-f}\, d\mathrm{vol}_{g(t)} \geq 0.$$

The functional $\mathcal{W}$ is constant on metrics g with the property that

$$R_{ij} + \nabla_i\nabla_j f - \frac{g_{ij}}{2\tau} = 0$$

for a smooth function f. These metrics are called gradient shrinking Ricci solitons and appear often as singularity models, that is, limits of blown up solutions around finite-time singularities of the Ricci flow.

Let $g(t)$ be a solution to the Ricci flow (1-1) on $(-\infty, T)$. We call a triple $(M, g(t), f(t))$ on $(-\infty, T)$ with smooth functions $f : M \to \mathbb{R}$ a *gradient shrinking soliton in canonical form* if it satisfies

$$\mathrm{Ric}(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f(t) - \frac{1}{2(T-t)} g(t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} f(t) = |f(t)|^2_{g(t)}. \tag{2-7}$$

Perelman also defines the μ-energy

$$\mu(g, \tau) = \inf \mathcal{W}(g, f, \tau) \quad \text{over } \left\{ f \mid (4\pi\tau)^{-n/2} \textstyle\int_M e^{-f}\, d\mathrm{vol}_g = 1 \right\} \tag{2-8}$$

and shows that

$$\frac{d}{dt}\mu(g(\,\cdot\,, t), \tau) \geq (2\tau)\cdot(4\pi\tau)^{-n/2} \int_M \left| R_{ij} + \nabla_i\nabla_j - \frac{g_{ij}}{2\tau}\right|^2 e^{-f}\, d\mathrm{vol}_{g(t)} \geq 0, \tag{2-9}$$

where $f(\,\cdot\,, t)$ is the minimizer for $\mathcal{W}(g(\,\cdot\,, t), f, \tau)$ with the constraint on f as in (2-8). The μ-energy $\mu(g, \tau)$ corresponds to the best constant of a logarithmic Sobolev inequality. Adjusting some of Perelman's arguments to our situation we get the following lemma whose proof we include for the reader's convenience.

Lemma 2.1 (nonpositivity of the μ-energy). *If $g(t)$ is a solution to (1-1) for all $t \in [0, T)$, then $\mu(g(t), T-t) \le 0$ for all $t \in [0, T)$.*

Proof. We are assuming the Ricci flow exists for all $t \in [0, T)$. Fix $t \in [0, T)$. Define $\tilde g(s) = g(t+s)$ for $s \in [0, T-t)$. Pick any $\bar\tau < T-t$. Let $\tau_0 = \bar\tau - \varepsilon$ with $\varepsilon > 0$ small. Pick $p \in M$. We use normal coordinates about p on $(M, \tilde g(\tau_0))$ to define

$$f_1(x) = \begin{cases} |x|^2/4\varepsilon & \text{if } d_{\tilde g(\tau_0)}(x, x_0) < \rho_0, \\ \rho_0^2/4\varepsilon & \text{elsewhere,} \end{cases} \tag{2-10}$$

where $\rho_0 > 0$ is smaller than the injectivity radius. Note that $d\mathrm{vol}_{\tilde g(\tau_0)}(x) = 1 + O(|x|^2)$ near p. We compute

$$\begin{aligned}\int_M (4\pi\varepsilon)^{-n/2} e^{-f_1}\, d\mathrm{vol}_{\tilde g(\tau_0)} &= \int_{|x|\le\rho_0} (4\pi\varepsilon)^{-n/2} e^{-|x|^2/4\varepsilon}(1 + O(|x|^2))\, dx + O(\varepsilon^{-n/2} e^{-\rho_0^2/4\varepsilon}) \\ &= \int_{|y|\le\rho_0/\sqrt{\varepsilon}} (4\pi)^{-n/2} e^{-|y|^2/4}(1 + O(\varepsilon|y|^2))\, dy + O(\varepsilon^{-n/2} e^{-\rho_0^2/4\varepsilon}).\end{aligned}$$

The second term goes to zero as $\varepsilon \to 0$ while the first term converges to

$$\int_{\mathbb{R}^n} (4\pi)^{-n/2} e^{-|y|^2/4}\, dy = 1.$$

Writing the integral as e^C, then $C \to 0$ as $\varepsilon \to 0$. And $f = f_1 + C$ then satisfies the constraint $\int_M (4\pi\varepsilon)^{-n/2} e^{-f}\, d\mathrm{vol}_{\tilde g(\tau_0)} = 1$.

Solve Equation (2-6) backwards with initial value f at τ_0. Then

$$\begin{aligned}\mathcal{W}(\tilde g(\tau_0), f(\tau_0), \bar\tau - \tau_0) &= \int_{|x|\le\rho_0} \left(\varepsilon\left(\frac{|x|^2}{4\varepsilon^2} + R\right) + \frac{|x|^2}{4\varepsilon} + C - n\right)(4\pi\varepsilon)^{-n/2} e^{-|x|^2/4\varepsilon - C}(1 + O(|x|^2))\, dx \\ &\qquad + \int_{M - B(p,\rho_0)} \left(\frac{\rho_o^2}{4\varepsilon} + \varepsilon R + C - n\right)(4\pi\varepsilon)^{-n/2} e^{-r_0^2/4\varepsilon - C} \\ &= \mathrm{I} + \mathrm{II},\end{aligned}$$

where $\mathrm{I} = e^{-C} \int_{|x|\le\rho_0} (|x|^2/2\varepsilon - n)(4\pi\varepsilon)^{-n/2} e^{-|x|^2/4\varepsilon}(1 + O(|x|^2))\, dx$ and II contains all the remaining terms. It is obvious that $\mathrm{II} \to 0$ as $\varepsilon \to 0$ while

$$\begin{aligned}\mathrm{I} &= e^{-C} \int_{|y|\le\rho_0/\sqrt{\varepsilon}} \left(\frac{|y|^2}{2} - n\right)(4\pi)^{-n/2} e^{-|y|^2/4}(1 + O(\varepsilon|y|^2))\, dy \\ &\to \int_{\mathbb{R}^n} \left(\frac{|y|^2}{2} - n\right)(4\pi)^{-n/2} e^{-|y|^2/4}\, dy = 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

Therefore $\mathcal{W}(\tilde{g}(\tau_0), f(\tau_0), \bar{\tau}-\tau_0) \to 0$ as $\tau_0 \to \bar{\tau}$. By the monotonicity of μ along the flow, $\mu(g(t), \bar{\tau}) = \mu(\tilde{g}(0), \bar{\tau}) \le \mathcal{W}(\tilde{g}(0), f(0), \bar{\tau}) \le \mathcal{W}(\tilde{g}(\tau_0), f(\tau_0), \bar{\tau}-\tau_0)$. Letting $\tau_0 \to \bar{\tau}$, we get $\mu(g(t), \bar{\tau}) \le 0$. Since $\bar{\tau} < T - t$ is arbitrary,

$$\mu(g(t), T-t) \le 0. \qquad \square$$

3. Uniform bound on scalar curvature

In this section, we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. By our assumptions, there exists a sequence of times $t_i \to T$ so that $Q_i := \max_{M\times[0,t_i]} |\mathrm{Rm}|(x,t) \to \infty$ as $i \to \infty$. Assume that the maximum is achieved at $(p_i, t_i) \in M \times [0, t_i]$. Define a rescaled sequence of solutions

$$g_i(t) = Q_i \cdot g(t_i + t/Q_i). \tag{3-1}$$

We have that

$$|\mathrm{Rm}(g_i)| \le 1 \text{ on } M \times [-t_i Q_i, 0] \quad \text{and} \quad |\mathrm{Rm}(g_i)|(p_i, 0) = 1. \tag{3-2}$$

By Hamilton's compactness theorem [1995a] and Perelman's κ-noncollapsing theorem [2002] we can extract a pointed subsequence of solutions $(M, g_i(t), q_i)$, converging in the Cheeger–Gromov sense to a solution to (1-1), which we denote by $(M_\infty, g_\infty(t), q_\infty)$ for any sequence of points $q_i \in M$. In particular, if we take that sequence of points to be exactly $\{p_i\}$, we can guarantee the limiting metric is nonflat. The limiting metric has a sequence of nice properties: Since

$$|R(g_i(t))| = \frac{|R(g(t_i + t/Q_i))|}{Q_i} \le \frac{C}{Q_i} \to 0,$$

the limiting solution $(M_\infty, g_\infty(t))$ is scalar flat for each $t \in (-\infty, 0]$. Since it solves the Ricci flow (1-1) and $R_\infty := R(g_\infty)$ evolves by

$$\frac{\partial}{\partial t} R_\infty = \Delta R_\infty + 2|\operatorname{Ric}(g_\infty)|^2,$$

we have that $\operatorname{Ric}(g_\infty) \equiv 0$, that is, the limiting metric is Ricci flat. We will get a Gaussian shrinker by using Perelman's functional μ defined by (2-8). Recall that (see the computation in [Kleiner and Lott 2008])

$$\frac{d}{dt}\mu(g(t), \tau) \ge 2\tau \cdot (4\pi\tau)^{-n/2} \int_M \left|\operatorname{Ric} + \nabla\nabla f - \frac{g}{2\tau}\right|^2 e^{-f}\, d\mathrm{vol}_{g(t)},$$

where $f(\cdot, t)$ is the minimizer realizing $\mu(g(t), \tau)$, and $\tau = T - t$.

In this proof of Theorem 1.2, we take $s, v \in [-10, 0]$ with $s < v$. Then, by (3-2), $g_i(s)$ and $g_i(v)$ are defined for i sufficiently large. Then, by the invariant property of μ under the parabolic scaling of the Ricci flow, for $s < v \in [-10, 0]$ one has

$$\begin{aligned}
(3\text{-}3)\quad & \mu(g_i(v), Q_i(T-t_i)-v) - \mu(g_i(s), Q_i(T-t_i)-s) \\
&= \mu\Big(g\Big(t_i+\frac{v}{Q_i}\Big), T-t_i-\frac{v}{Q_i}\Big) - \mu\Big(g\Big(t_i+\frac{s}{Q_i}\Big), T-t_i-\frac{s}{Q_i}\Big) \\
&= \int_{t_i+s/Q_i}^{t_i+v/Q_i} \frac{d}{dt}\mu(g(t), T-t)\,dt \\
&\geq \int_{t_i+s/Q_i}^{t_i+v/Q_i} \int_M 2\tau(4\pi\tau)^{-n/2}\cdot \Big|\mathrm{Ric}+\nabla\nabla f-\frac{g}{2\tau}\Big|^2 e^{-f}\,d\mathrm{vol}_{g(t)}\,dt \\
&= 2\int_s^v \int_M \Big(m_i(r)(4\pi m_i(r))^{-n/2} \\
&\qquad\qquad \cdot \Big|\mathrm{Ric}(g_i(r))+\nabla\nabla f-\frac{g_i}{2m_i(r)}\Big|^2 e^{-f}\Big)\,d\mathrm{vol}_{g_i(r)}\,dr,
\end{aligned}$$

where, for simplicity, $m_i(r) = Q_i(T-t_i)-r$.

Since we are assuming the flow develops a Type I singularity at T, we have

$$(3\text{-}4)\qquad \lim_{i\to\infty} Q_i(T-t_i) = a < \infty.$$

Thus, by (2-4), one has for $r \in [-10, 0]$,

$$(3\text{-}5)\qquad \lim_{i\to\infty} m_i(r) = a-r > 0.$$

By Lemma 2.1 and by the monotonicity of $\mu(g(t), T-t)$ (see (2-9)),

$$(3\text{-}6)\qquad \mu(g(0), T) \leq \mu(g(t), T-t) \leq 0.$$

Estimate (3-6) implies that there exists a finite $\lim_{t\to T}\mu(g(t), T-t)$ which implies that the left-hand side of (3-3) tends to zero as $i\to\infty$. Letting $i\to\infty$ in (3-3) and using (3-5), we get

$$\begin{aligned}
(3\text{-}7)\quad \lim_{i\to\infty}\int_s^v\int_M \Big(&(a-r)(4\pi(a-r))^{-n/2} \\
&\times \Big|\mathrm{Ric}(g_i)+\nabla\nabla f-\frac{g_i}{2(a-r)}\Big|^2 e^{-f}\Big)\,d\mathrm{vol}_{g_i(r)}\,dr = 0.
\end{aligned}$$

We would like to say that we can extract a subsequence so that $f(\cdot, t_i+r/Q_i)$ converges smoothly to a smooth function $f_\infty(r)$ on $(M_\infty, g_\infty(r))$, which will then be a potential function for a limiting gradient shrinking Ricci soliton g_∞. In order to do that, we need some uniform estimates for $f(\cdot, t_i+r/Q_i)$. The equation satisfied by $f(t_i+r/Q_i)$ in (3-3) is

$$(3\text{-}8)\quad \Big(T-t_i-\frac{r}{Q_i}\Big)(2\Delta f-|\nabla f|^2+R)+f-n = \mu\Big(g\Big(t_i+\frac{r}{Q_i}\Big), T-t_i-\frac{r}{Q_i}\Big).$$

Let $f_i(\cdot, r) = f(\cdot, t_i + r/Q_i)$. Then

$$(Q_i(T-t_i)-r)(2\Delta_{g_i(r)} f_i(r) - |\nabla_{g_i(r)} f_i(r)|^2 + R(g_i(r))) + f_i(r) - n = \mu(g_i(r), Q_i(T-t_i)-r).$$

Define $\phi_i(\cdot, r) = e^{-f_i(\cdot, r)/2}$. This function $\phi_i(\cdot, r)$ satisfies a nice elliptic equation

$$(Q_i(T-t_i)-r)(-4\Delta_{g_i(r)} + R(g_i(r)))\phi_i = 2\phi_i \log\phi_i + (\mu(g_i(r), Q_i(T-t_i)-r)+n)\phi_i. \tag{3-9}$$

Recall that, in this proof of Theorem 1.2, we consider $r \in [-10, 0]$. We take the liberty of suppressing certain dependencies on r whenever no confusion may arise.

Our first estimates are uniform global $W^{1,2}$ estimates for $\phi_i(r)$:

Lemma 3.1. *There exists a uniform constant C so that for all $r \in [-10, 0]$ and all i, one has*

$$\int_M \phi_i^2(\cdot, r)\, d\mathrm{vol}_{g_i(r)} + \int_M |\nabla_{g_i(r)}\phi_i(\cdot, r)|^2\, d\mathrm{vol}_{g_i(r)} \le C(Q_i(T-t_i)-r)^{n/2} \le \widetilde{C}.$$

Proof. The function $\phi_i(r)$ satisfies the L^2-constraint

$$\int_M (4\pi m_i(r))^{-n/2}(\phi_i(r))^2\, d\mathrm{vol}_{g_i(r)} = 1$$

and is in fact smooth [Rothaus 1981]. Here, we have used $m_i(r) = Q_i(T-t_i)-r$.

To simplify, let $F_i(r) = \phi_i(r)/c_i(r)$, where $c_i(r) = (4\pi m_i(r))^{n/4}$. Then

$$\int_M (F_i(r))^2\, d\mathrm{vol}_{g_i(r)} = 1,$$

and the equation for $F_i(r)$ becomes

$$m_i(r)(-4\Delta_{g_i(r)} + R(g_i(r)))F_i(r) = 2F_i(r)\log F_i(r) + (\mu(g_i(r), m_i(r)) + n + 2\log c_i(r))F_i(r).$$

Introduce

$$\mu_i(r) = \mu(g_i(r), m_i(r)) + n + 2\log c_i(r).$$

Then

$$-\Delta_{g_i(r)} F_i = \frac{1}{2m_i(r)} F_i \log F_i + \Big(\frac{\mu_i(r)}{4m_i(r)} - \frac{1}{4}R(g_i(r))\Big)F_i.$$

Multiplying the above equation by $F_i(r)$ and integrating over M, we get

$$\int_M |\nabla_{g_i} F_i|^2\, d\mathrm{vol}_{g_i(r)} = \frac{1}{2m_i(r)}\int_M F_i^2 \log F_i\, d\mathrm{vol}_{g_i(r)} + \int_M \Big(\frac{\mu_i(r)}{4m_i(r)} - \frac{1}{4}R(g_i)\Big)F_i^2\, d\mathrm{vol}_{g_i(r)}. \tag{3-10}$$

Because $\int_M (F_i(r))^2\, d\mathrm{vol}_{g_i(r)} = 1$, by Jensen's inequality for the logarithm,

$$\int_M F_i^2 \log F_i \, d\mathrm{vol}_{g_i(r)} = \frac{n-2}{4} \int_M F_i^2 \log F_i^{4/(n-2)} \, d\mathrm{vol}_{g_i(r)} \le \frac{n-2}{4} \log \int_M F_i^{2+4/(n-2)} \, d\mathrm{vol}_{g_i(r)} = \frac{n-2}{4} \log \int_M F_i^{(2n)/(n-2)} \, d\mathrm{vol}_{g_i(r)} . \tag{3-11}$$

On the other hand, we recall the following Sobolev inequality (see also [Hebey 1999, Theorem 5.6]):

Theorem 3.1 [Hebey and Vaugon 1995]. *For any smooth, compact Riemannian n-manifold (M, g), where $n \ge 3$, such that*

$$|\mathrm{Rm}(g)| \le \Lambda_1, \quad |\nabla_g \mathrm{Rm}(g)| \le \Lambda_2, \quad \mathrm{inj}_{(M,g)} \ge \gamma,$$

there is a uniform constant $B(n, \Lambda_1, \Lambda_2, \gamma)$ so that for any $u \in W^{1,2}(M)$,

$$\left(\int_M |u|^{(2n)/(n-2)} \, d\mathrm{vol}_g \right)^{(n-2)/n} \le C(n) \int_M |\nabla u|^2 \, d\mathrm{vol}_g + B(n, \Lambda_1, \Lambda_2, \gamma) \int_M u^2 \, d\mathrm{vol}_g . \tag{3-12}$$

By Perelman's noncollapsing result, Theorem 3.1 applies to $(M, g_i(r))$ with uniform constants $\Lambda_1, \Lambda_2, \gamma$, independent of $r \in [-10, 0]$ and i. In particular, letting $u = F_i(r)$ in (3-12), we find that

$$\int_M (F_i(r))^{(2n)/(n-2)} \, d\mathrm{vol}_{g_i(r)} \le C(n) \left(\int_M |\nabla_{g_i(r)} F_i(r)|^2 \, d\mathrm{vol}_{g_i(r)} \right)^{n/(n-2)} + B(n, \Lambda_1, \Lambda_2, \gamma). \tag{3-13}$$

Combining (3-10), (3-11) and (3-13), we obtain

$$\int_M |\nabla_{g_i} F_i|^2 \, d\mathrm{vol}_{g_i(r)} \le \frac{n-2}{8m_i(r)} \log \int_M F_i^{(2n)/(n-2)} \, d\mathrm{vol}_{g_i(r)} + \int_M \left(\frac{\mu_i(r)}{4m_i(r)} - \frac{1}{4} R(g_i) \right) F_i^2 \, d\mathrm{vol}_{g_i(r)} \tag{3-14}$$

$$\le \frac{n-2}{8m_i(r)} \log\biggl(C(n) \biggl(\int_M |\nabla F_i|^2 \, d\mathrm{vol}_{g_i(r)} \biggr)^{n/(n-2)} + B(n, \Lambda_1, \Lambda_2, \gamma) \biggr) \\ + \int_M \biggl(\frac{\mu_i(r)}{4m_i(r)} - \frac{1}{4} R(g_i) \biggr) F_i^2 \, d\mathrm{vol}_{g_i(r)} .$$

Recall that $R(g_i(r))$ is uniformly bounded by the scaling and furthermore

$$\lim_{i\to\infty} Q_i(T-t_i) = a \in [\tfrac{1}{8}, \infty).$$

Thus, if $r \in [-10, 0]$, then Equation (3-14) gives a global uniform bound for $\int_M |\nabla_{g_i(r)} F_i(r)|^2 \, d\mathrm{vol}_{g_i(r)}$. Since $\phi_i(r) = c_i(r) F_i(r)$, we then have a global uniform bound for $\int_M |\nabla_{g_i(r)} \phi_i(r)|^2 \, d\mathrm{vol}_{g_i(r)}$. □

Now, elliptic L^p theory gives uniform $C^{1,\alpha}$ estimates for $\phi_i(r)$ on compact sets [Gilbarg and Trudinger 2001]. We need higher order derivative estimates on $\phi_i(r)$ to conclude that for a suitably chosen sequence of points q_i around which we take the limit, we have $f_\infty(r) = -2 \log \phi_\infty(r)$ for a smooth function $f_\infty(r)$ (where $f_\infty(r)$ is the limit of $f_i(r)$ and $\phi_\infty(r)$ is the limit of $\phi_i(r)$). For the higher order estimates, it is crucial to prove that $\{\phi_i(r)\}$ stay uniformly bounded from below on compact sets around q_i.

In (3-7), take $s = -10$ and $v = 0$. For each i, let $r_i \in [-10, 0]$ be such that

$$(a - r_i)(4\pi(a - r_i))^{-n/2} \biggl| \mathrm{Ric}(g_i(r_i)) + \nabla\nabla f\biggl(t_i + \frac{r_i}{Q_i}\biggr) - \frac{g_i}{2(a - r_i)} \biggr|^2 \\ \times e^{-f(t_i + r_i/Q_i)} \, d\mathrm{vol}_{g_i(r_i)} \\ \le (a - r)(4\pi(a - r))^{-n/2} \biggl| \mathrm{Ric}(g_i(r)) + \nabla\nabla f\biggl(t_i + \frac{r}{Q_i}\biggr) - \frac{g_i}{2(a - r)} \biggr|^2 \\ \times e^{-f(t_i + r/Q_i)} \, d\mathrm{vol}_{g_i(r)}$$

for all $r \in [-10, 0]$. Take $q_i \in M$ at which the maximum of $\phi_i(r_i)$ over M has been achieved and denote also by $(M_\infty, g_\infty(t), q)$ the smooth pointed Cheeger–Gromov limit of the rescaled sequence of metrics $(M, g_i(t), q_i)$, defined as above. Take any compact set $K \subset M_\infty$ containing q. Let $\psi_i : K_i \to K$ be the diffeomorphisms from the definition of Cheeger–Gromov convergence of (M, g_i, q_i) to (M_∞, g_∞, q) and $K_i \subset M$. Following the previous notation, consider the functions $F_i(r_i)$, $\phi_i(r_i)$ and for simplicity denote them by F_i and ϕ_i, respectively. Also denote the metric $g_i(r_i)$ by g_i.

Lemma 3.2. *For any $\alpha \in (0, 1)$, there is a uniform constant $C(\alpha)$ so that*

$$\|F_i\|_{C^{1,\alpha}(M)} \le C(\alpha). \tag{3-15}$$

Proof. The proof is via bootstrapping and rather standard for the equation satisfied by F_i:

$$-\Delta_{g_i} F_i = \frac{1}{2m_i(r)} F_i \log F_i + \left(\frac{\mu_i(r)}{4m_i(r)} - \frac{1}{4} R(g_i)\right) F_i. \tag{3-16}$$

The reason that bootstrapping works is simple. If F_i is uniformly bounded in $L^p(K_i)$, where $K_i \in M$ is a compact set, then $F_i \log F_i$ is uniformly bounded in $L^{p-\delta}(K_i)$ for any $\delta > 0$. Standard local parabolic estimates give (3-15), which is independent of a compact set since we have a uniform global $W^{1,2}$ bound on F_i. □

We now discuss how to get higher order derivatives estimates for F_i. Covariantly differentiating (3-16), commuting derivatives, and noting that

$$-\Delta_{g_i} \partial_l F_i = -\partial_l \Delta_{g_i} F_i - \mathrm{Ric}(g_i)_{lk} g_i^{kp} \partial_p F_i,$$

we get

$$\begin{aligned} -\Delta_{g_i} \partial_l F_i = \frac{1}{2m_i(r)} \partial_l F_i \log F_i + \left(\frac{2+\mu_i(r)}{4m_i(r)} - \frac{1}{4} R(g_i)\right) \partial_l F_i \\ - \frac{1}{4} \partial_l R(g_i) F_i - \mathrm{Ric}(g_i)_{lk} g_i^{kp} \partial_p F_i. \end{aligned} \tag{3-17}$$

The major obstacle in applying L^p theory to get uniform $C^{1,\alpha}$ estimates for $\partial_l F_i$ is the term $\partial_l F_i \log F_i$. This emanates from the potential smallness of $|F_i|$, though we have already found a nice uniform upper bound on it. Thus, to proceed further, we need to bound $|F_i|$ uniformly from below. Equivalently, we will prove in Lemma 3.3 that ϕ_i stays uniformly bounded from below on K_i.

As the first step, we bound $\phi_i(q_i)$ from below. This is simple. Applying the maximum principle to (3-8) gives $\min_M f_i \le C$, where $f_i = f_i(r_i)$ for a uniform constant C. This can be seen as follows. Define $\alpha_i = Q_i(T - t_i)$. At the minimum of f_i, we have

$$\frac{f_i - n}{\alpha_i - r_i} = \frac{\mu(g_i(r_i), \alpha_i - r_i)}{\alpha_i - r_i} - R(g_i(r_i)) - 2\Delta_{g_i(r_i)} f_i \le \frac{\mu(g_i(r_i), \alpha_i - r_i)}{\alpha_i - r_i} - R(g_i(r_i)).$$

Thus,

$$\begin{aligned} f_i &\le n + \mu(g_i(r_i), \alpha_i - r_i) - R(g_i(r_i))(\alpha_i - r_i) \\ &\le n + \mu(g_i(r_i), \alpha_i - r_i) + \frac{C}{Q_i}\big(Q_i(T - t_i) - r_i\big) \le C, \end{aligned} \tag{3-18}$$

where we have used the fact that $R(\cdot\,, t) \ge -C$ on M for all $t \in [0, T)$ (see (2-3)). This implies $\phi_i(q_i) \ge \delta > 0$ for all i, with a uniform constant δ.

Let $K \subset M_\infty$ and $K_i \subset M$ be compact sets as before. Also recall that $m_i(r_i) = Q_i(T - t_i) - r_i$.

Lemma 3.3. *For every compact set $K \subset M_\infty$ there exists a uniform constant $C(K)$ so that*

$$\phi_i \geq C(K) \text{ on } K_i \quad \text{for all } i.$$

Proof. Assume the lemma is not true and that there exist points $P_i \in K_i$ so that $\phi_i(P_i) \leq 1/i \to 0$ as $i \to \infty$. Assume $\psi_i(P_i)$ converge to a point $P \in K$. Then $\phi_\infty(P) = 0$. Take a smooth function $\eta \in C_0^\infty(M_\infty)$, compactly supported in $K \setminus \{P\}$. Then $\psi_i^* \eta \in C_0^\infty(M)$, compactly supported in $K_i \setminus \{P_i\}$. Multiplying (3-9) by $\psi_i^* \eta$, assuming $\lim_{i\to\infty} r_i = r_0$, and then integrating by parts, we get

$$\int_M \bigl(m_i(r_i) \cdot (4\nabla\phi_i \nabla(\psi_i^*\eta) + R_i \phi_i \psi_i^* \eta) - 2\phi_i \psi_i^* \eta \ln \phi_i \\ - n\phi_i \psi_i^* \eta - \mu(g_i, m_i(r_i))\phi_i \psi_i^* \eta\bigr)\, d\mathrm{vol}_{g_i(r_i)} = 0.$$

We now let $i \to \infty$ and observe that $\phi_i \to \phi_\infty$ $C^{1,\alpha}$ locally, that $\psi_i^* \eta \to \eta$ smoothly, that $\lim_{i\to\infty} R(g_i) = 0$, and that $a - r_0 := \lim_{i\to\infty} m_i(r_i) \equiv \lim_{i\to\infty}(Q_i(T - t_i) - r_i)$ is finite. Thus one finds that

$$\int_{M_\infty} \bigl(4(a-r_0)\nabla\phi_\infty \nabla\eta - 2\eta\phi_\infty \ln\phi_\infty - n\phi_\infty\eta - \mu(g_\infty, a-r_0)\eta\phi_\infty\bigr)\, d\mathrm{vol}_{g_\infty(r_0)} = 0.$$

Proceeding in the same manner as in [Rothaus 1981], we obtain $\phi_\infty \equiv 0$ in some small ball around P. Using the connectedness argument, $\phi_\infty \equiv 0$ everywhere in M_∞. That contradicts $\phi_\infty(q) \geq \delta > 0$. □

Having Lemma 3.3 and $C^{1,\alpha}$ uniform estimates on ϕ_i, we see that the right-hand side of (3-17) is uniformly bounded in $L^2(K_i)$. Because $\log F_i$ is uniformly bounded on K_i, we can bootstrap (3-17) to obtain $C^{1,\alpha}$ estimates for $|\nabla_{g_i} F_i|$. Hence, one has uniform $C^{2,\alpha}$ estimates for F_i on K_i. In terms of ϕ_i,

$$\|\phi_i\|_{C^{2,\alpha}(K_i)} \leq C(K,\alpha)(Q_i(T - t_i) - r_i)^{n/4}. \tag{3-19}$$

Differentiating (3-17) again gives all higher order derivative estimates on ϕ_i and therefore all higher order derivative estimates on $f_i = f_i(r_i) = -2\log\phi_i$. However, for our purpose, $C^{2,\alpha}$ estimates suffice.

Then, using (3-7), for $s = -10$ and $v = 0$,

$$\lim_{i\to\infty}\biggl(10(a - r_i)(4\pi(a - r_i))^{-n/2} \\ \times \int_M \biggl|\mathrm{Ric}(g_i(r_i)) + \nabla\nabla f_i - \frac{g_i(r_i)}{2(a - r_i)}\biggr|^2 e^{-f_i}\, d\mathrm{vol}_{g_i(r_i)}\biggr)$$

$$\leq \lim_{i\to\infty}\biggl(\int_{-10}^0 \int_M (a - r)(4\pi(a - r))^{-n/2} \\ \times \biggl|\mathrm{Ric}(g_i) + \nabla\nabla f_i - \frac{g_i}{2(a - r)}\biggr|^2 e^{-f_i}\, d\mathrm{vol}_{g_i(r)}\, dr\biggr) = 0.$$

By Lemma 3.3 and (3-7), applying the Arzelà–Ascoli theorem on f_i results in

$$\mathrm{Ric}_\infty + \nabla\nabla f_\infty - \frac{g_\infty}{2(a-r_0)} = 0.$$

Since $\mathrm{Ric}_\infty \equiv 0$, we get

$$g_\infty = 2(a-r_0)\nabla\nabla f_\infty,$$

and therefore M_∞ is isometric to a standard Euclidean space $\mathbb{R}^n$; see, for example, [Ni 2005, Proposition 1.1]. It is now easy to see that

$$f_\infty = \frac{|x|^2}{4(a-r_0)}, \tag{3-20}$$

that is, the limiting manifold $(\mathbb{R}^n, g_\infty, q_\infty)$ is a Gaussian shrinker. □

Proof of Theorem 1.3. We will use many estimates and arguments developed in the proof of Theorem 1.2. Assume the flow does develop a Type II singularity at T. Then we can pick a sequence of times $t_i \to T$ and points $p_i \in M$ as in [Hamilton 1995b] so that the rescaled sequence of solutions $(M, g_i(t) := Q_i g(t_i + t/Q_i), p_i)$, converges in a pointed Cheeger–Gromov sense to a Ricci flat, nonflat, complete, eternal solution $(M_\infty, g_\infty(t), p_\infty)$. Here $Q_i := \max_{M\times[0,t_i]} |\mathrm{Rm}|(x,t) \to \infty$ as $i \to \infty$. The reasons for getting Ricci flat metric are the same as in the proof of Theorem 1.2. Define

$$\alpha_i := (T - t_i)Q_i.$$

Since we are assuming a Type II singularity occurs at T, we may assume that for a chosen sequence t_i we have $\lim_{i\to\infty} \alpha_i = \infty$.

By Lemma 2.1 and the monotonicity of μ, we have $|\mu(g(t), T-t)| \le C$ for all $t \in [0,T)$. Let $f_i(\,\cdot\,, s)$ be a smooth minimizer realizing

$$\mu\Big(g\Big(t_i+\frac{s}{Q_i}\Big), T-t_i-\frac{s}{Q_i}\Big) = \mu(g_i(s), \alpha_i - s) = \inf \mathcal{W}\Big(g\Big(t_i+\frac{s}{Q_i}\Big), f, T-t_i-\frac{s}{Q_i}\Big)$$

over the set of all smooth functions f satisfying

$$\Big(4\pi\Big(T-t_i-\frac{s}{Q_i}\Big)\Big)^{-n/2}\int_M e^{-f}\, d\mathrm{vol}_{g(t_i+s/Q_i)} = 1.$$

Then $f_i = f_i(\,\cdot\,, s)$ satisfies

$$2\Delta_{g_i(s)} f_i - |\nabla_{g_i(s)} f_i|^2 + R_i + \frac{f_i - n}{\alpha_i - s} = \frac{\mu(g_i(s), \alpha_i - s)}{\alpha_i - s}. \tag{3-21}$$

In terms of $\phi_i(x,s) = e^{-f_i(x,s)/2}$ this is equivalent to

$$\begin{aligned} -4\Delta_{g_i(s)}\phi_i(s) + R(g_i(s))\phi_i(s) \\ = \frac{2\,\phi_i(s)\log\phi_i(s)}{\alpha_i - s} + \frac{(\mu(g_i(s), \alpha_i - s) + n)\phi_i(s)}{\alpha_i - s}, \end{aligned} \tag{3-22}$$

with

$$\int_M (\phi_i(s))^2 \, d\mathrm{vol}_{g_i(s)} = (4\pi(\alpha_i - s))^{n/2}. \tag{3-23}$$

In what follows, we fix $s = 0$. Define $\widetilde{\phi}_i(\,\cdot\,) := \phi_i(\,\cdot\,, 0)/\beta_i$, where

$$\beta_i := \max_M \left(\phi_i(x, 0) + |\nabla_{g_i(0)}\phi_i(x, 0)|\right). \tag{3-24}$$

This choice of β_i gives us uniform C^1 estimates for $\widetilde{\phi}_i$ on M. Thus, we can apply L^p theory to get uniform $C^{1,\alpha}$ estimates for $\widetilde{\phi}_i$ on compact sets around the points where the maxima in (3-24) are achieved. To be more precise, we proceed as follows.

Take $q_i \in M$ at which this maximum in (3-24) has been achieved and denote also by $(M_\infty, g_\infty(t), q)$ the smooth pointed Cheeger–Gromov limit of the rescaled sequence of metrics $(M, g_i(t), q_i)$, defined as above. Lemma 3.1, Theorem 3.1 and standard elliptic L^p estimates applied to (3-22) yield the estimates on β_i in terms of the $W^{1,2}$ norm of ϕ_i with respect to metric $g_i(0)$, that is, there exists a uniform constant C so that for all i, $\beta_i \le C\alpha_i^{n/4}$, which implies

$$\log \beta_i \le C_2 \log \alpha_i + C_2, \tag{3-25}$$

for some uniform constants C_1 and C_2. This can be proved the same way we obtained (3-19) in Theorem 1.2. After dividing (3-22) by β_i we get

$$-4\Delta_{g_i(0)}\widetilde{\phi}_i + R(g_i(0))\widetilde{\phi}_i = 2\widetilde{\phi}_i \cdot \frac{\log\widetilde{\phi}_i + \log\beta_i}{\alpha_i} + \frac{(\mu(g(t_i), T - t_i) + n)\widetilde{\phi}_i}{\alpha_i}. \tag{3-26}$$

Since $(M, g_i(t), q_i)$ converges to $(M_\infty, g_\infty(t), q)$ in the pointed Cheeger–Gromov sense, and $\|\widetilde{\phi}_i\|_{C^1(M, g_i(0))}$ is uniformly bounded, we can get uniform $C^{1,\alpha}$ estimates for $\widetilde{\phi}_i$ on compact sets around points q_i. By the Arzelà–Ascoli theorem, $\widetilde{\phi}_i$ converges uniformly in the C^1 norm on compact sets around points q_i to a smooth function $\widetilde{\phi}_\infty$. We will show in the next paragraph that $\widetilde{\phi}_\infty(\,\cdot\,)$ is a positive constant.

Indeed, if we apply the maximum principle to (3-21), similarly as in the proof of Theorem 1.2, we obtain $\min_M f_i(\,\cdot\,, 0) \le C$ for a uniform constant C. This implies $\log \beta_i \ge -C_1$ for a uniform constant C_1. In particular, there is a uniform constant $\delta > 0$ such that for all i, one has

$$\beta_i \ge \delta > 0. \tag{3-27}$$

This together with (3-25) and the $\lim_{i\to\infty} \alpha_i = \infty$ implies

$$\lim_{i\to\infty} \frac{\log \beta_i}{\alpha_i} = 0. \tag{3-28}$$

Multiplying (3-26) by any cut-off function $\eta_i = \psi_i^* \eta$ (where η is any cut-off function on M_∞ and ψ_i is a sequence of diffeomorphisms from the definition of Cheeger–Gromov convergence) and integrating by parts, we get

$$\begin{aligned} 4 \int_M & \nabla \widetilde{\phi}_i \nabla \eta_i \, d\mathrm{vol}_{g_i(0)} \\ &= - \int_M R(g_i(0)) \widetilde{\phi}_i \eta_i \, d\mathrm{vol}_{g_i(0)} \\ &\quad + 2 \int_M \eta_i \widetilde{\phi}_i \cdot \frac{\log \widetilde{\phi}_i + \log \beta_i}{\alpha_i} \, d\mathrm{vol}_{g_i(0)} - \frac{\mu(g(t_i), T - t_i) + n}{\alpha_i} \int_M \eta_i \widetilde{\phi}_i \, d\mathrm{vol}_{g_i(0)} . \end{aligned}$$

Let $i \to \infty$ in the previous identity. From (3-28) and the limits $\lim_{i\to\infty} \alpha_i = \infty$, $R(g_i(0)) \to 0$ uniformly on compact sets, and $\widetilde{\phi}_i \to \widetilde{\phi}_\infty$ in the C^1 sense, and using uniform bounds on $\mu(g(t), T - t)$, we obtain

$$\int_M \nabla \widetilde{\phi}_\infty \nabla \eta \, d\mathrm{vol}_{g_\infty(0)} = 0.$$

This means $\Delta \widetilde{\phi}_\infty = 0$ in the distributional sense. By Weyl's theorem, $\widetilde{\phi}_\infty$ is a harmonic function on M_∞. Since $(M_\infty, g_\infty(0))$ is a complete, Ricci flat manifold and $\phi_\infty \geq 0$, by the theorem of [Yau 1975], $\widetilde{\phi}_\infty = C_\infty$ is a constant function on M_∞. At the same time, from the definition of $\widetilde{\phi}_i$, we get for x in compact sets around points q_i,

$$1 = \lim_{i\to\infty} \big(\widetilde{\phi}_i(x) + |\nabla_{g_i(0)} \widetilde{\phi}_i(x)| \big) = \widetilde{\phi}_\infty(x) + |\nabla_{g_\infty(0)} \widetilde{\phi}_\infty(x)| \equiv C_\infty. \tag{3-29}$$

This implies, in particular $C_\infty \equiv 1 > 0$. □

4. Integral bounds on scalar curvature

In this section we will prove Theorem 1.4 and Theorem 1.5. Theorem 1.1 is a special case of Theorem 1.4 when $\alpha = \infty$ in the case with Type I singularities only. A crucial ingredient in our arguments is the following result.

Theorem 4.1 [Enders et al. 2010, Theorem 1.4]. *Let $g(t)$ be the solution to a Type I Ricci flow (1-1) on $[0, T)$ and suppose that the flow develops a Type I singularity at T. Then for every sequence $\lambda_j \to \infty$, the rescaled Ricci flows $(M, g_j(t))$ defined on $[-\lambda_j T, 0)$ by $g_j(t) := \lambda_j g(T + t/\lambda_j)$ subconverge in the Cheeger–Gromov sense to a normalized nontrivial gradient shrinking soliton in canonical form on $(-\infty, 0)$.*

Proof of Theorem 1.4. The proof is by contradiction. Assume the flow develops a Type I singularity at $p \in M$ at $T < \infty$. Consider any sequence $\lambda_j \to \infty$ and define $g_j(t) := \lambda_j g(T + t/\lambda_j)$ where $t \in [-\lambda_j T, 0)$. By Theorem 4.1, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ subconverge in the Cheeger–Gromov sense to a normalized nontrivial gradient shrinking soliton $(M_\infty, g_\infty(t), p_\infty)$ in canonical form on $(-\infty, 0)$. Under the condition (1-3), one has

$$\int_M |R(g_j(t))|^\alpha \, d\mathrm{vol}_{g_j(t)} = \frac{1}{\lambda_j^{\alpha-n/2}} \int_M \left| R(g(T+\frac{t}{\lambda_j})) \right|^\alpha d\mathrm{vol}_{g(T+t/\lambda_j)} \le \frac{C_\alpha}{\lambda_j^{\alpha-n/2}} \to 0.$$

Thus the limiting solution $(M_\infty, g_\infty(t), p_\infty)$ is scalar flat. Arguing as in the proof of Theorem 1.1, we see that M_∞ is isometric to a standard Euclidean space $\mathbb{R}^n$. However, this contradicts the nontriviality of M_∞. □

Proof of Theorem 1.5. By Hölder's inequality, it suffices to consider the case when $\alpha = (n+2)/2$. Then the integral bound is invariant under the usual parabolic scaling of the Ricci flow.

The proof is by contradiction. Assume the flow develops a Type I singularity at $p \in M$ at $T < \infty$. Consider any sequence $\lambda_j \to \infty$ and define $g_j(t) := \lambda_j g(T + t/\lambda_j)$ where $t \in [-\lambda_j T, 0)$. Then, by Theorem 4.1, the rescaled Ricci flows $(M, g_j(t), p)$ defined on $[-\lambda_j T, 0)$ subconverge in the Cheeger–Gromov sense to a normalized nontrivial gradient shrinking soliton $(M_\infty, g_\infty(t), p_\infty)$ in canonical form on $(-\infty, 0)$. Observe that

$$\int_{-1}^{0} \int_M |R(g_j(t)|^\alpha \, d\mathrm{vol}_{g_j(t)} \, dt = \int_{T-1/\lambda_j}^{T} \int_M |R(g(s))|^\alpha \, d\mathrm{vol}_{g(s)} \, ds.$$

Since $\int_0^T \int_M |R(g(t))|^\alpha \, d\mathrm{vol}_{g(t)} \, dt < \infty$, letting $j \to \infty$, we obtain

$$\int_{-1}^{0} \int_{M_\infty} |R(g_\infty(t)|^\alpha \, d\mathrm{vol}_{g_\infty(t)} \, dt \le \lim_{j\to\infty} \int_{T-1/\lambda_j}^{T} \int_M |R(g(s))|^\alpha \, d\mathrm{vol}_{g(s)} \, ds = 0,$$

which implies $R(g_\infty(t)) \equiv 0$ on M_∞ for $t \in [-1, 0]$. Thus the limiting solution $(M_\infty, g_\infty(t))$ is scalar flat. Arguing as in the proof of Theorem 1.1, we see that M_∞ is isometric to a standard Euclidean space $\mathbb{R}^n$. However, this contradicts the nontriviality of M_∞. □

Acknowledgement

The authors would like to thank John Lott for helpful conversations during the preparation of this paper.

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Received October 7, 2010. Revised February 23, 2011.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 255 No. 1 January 2012