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We study immersed stable capillary surfaces with planar boundary in the absence of gravity. We assume that the surface approaches the boundary from one side. If the boundary of the capillary surface is embedded in a plane, we prove that the only immersed weakly stable capillary surface is the spherical cap.

Introduction

In this paper we study capillary surfaces with planar boundary in the absence of gravity. A comprehensive treatment of the theory of capillary surfaces can be found in [Finn 1986]. The problem we address arises from the related physical problem concerning a homogeneous liquid drop in contact with a smooth rigid boundary surface Σ. We call the free surface of the drop Ω and the angle of contact γ, and the wetted part of Σ we call Σ'. The liquid drop occupies a connected region in space, T, with a prescribed volume. The contact angle γ is measured relative to the interior of the liquid bounded by Ω and Σ. The problem is to describe the possible shapes of Ω if the liquid drop is in equilibrium.

There are three energies associated with this configuration. The first is the free surface energy, which is proportional to the area of Ω, with coefficient equal to the surface tension. The second is the wetting energy, which is a multiple of the area of Σ'. The third is the gravitational energy. Here we assume that there is no gravity acting, so the gravitational energy does not contribute. In order for the drop to be in equilibrium, it must be a critical point for the potential energy functional E. From this discussion we obtain a formula for E, that is, $E = \sigma \operatorname{Area}(\Omega) - \sigma\tau \operatorname{Area}(\Sigma')$, where σ is the surface tension and τ is the capillary constant. The constant τ is a physical quantity that is predetermined and, in equilibrium, equal to $\cos\gamma$. The wetting ability and the surface tension of the liquid are the two physical phenomena that cause the drop to become stationary. The above configuration is said to be in a stationary state if the first variation of E is zero for any volume-preserving perturbation. It is weakly stable (resp. stable) if it is stationary and the second

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variation of E is nonnegative (resp. positive) for any nontrivial volume-preserving perturbation.

In this paper we study the stability problem when the fixed boundary surface Σ is plane. We denote the first variation of E by ∂E. If ∂E is zero subject to a volume constraint, one finds that the angle of contact γ must be constant along $\partial\Omega \subset \Sigma$, the mean curvature of Ω must be constant, and $\tau = \cos\gamma$. In the case when gravity is present, the mean curvature of Ω is proportional to the height. This discussion naturally leads to the study of constant mean curvature surfaces with boundary. If Ω forms a constant angle with Σ along $\partial\Omega$, one can ask what the possible shapes of Ω are. This question is hard to resolve [Earp et al. 1991]. There are a few known examples for planar or spherical Σ, including spherical caps, right cylinders and Delaunay surfaces. We generalize the problem and assume Ω to be immersed; that is, Ω could have self-intersections, which further complicates the discussion. For this reason we put an additional restriction on Ω and study the same problem. The physical discussion above leads us to consider the case when the second variation of the potential energy, $\partial^2 E$, is nonnegative, that is, Ω is weakly stable. Assuming stability, in the case of Σ being a plane we can say much more.

It is known that if Ω is bounded and embedded and sits on one side of the boundary plane, then it is a spherical cap [Wente 1980]. We assume only that $\partial\Omega$ is embedded and that the surface Ω comes close to the boundary from above, allowing the immersed Ω to be below Σ away from $\partial\Omega$. Our main theorem shows that the only possible stable configuration of this type is the spherical cap. The spherical cap is weakly stable, as shown in [Wente 1966]. For the proof of the main theorem we consider three cases. The first is when Ω is of disk type with genus zero. The proof of this case comes from a result of Nitsche [1985] (see also [Finn and McCuan 2000]) and does not assume stability. The second case is when Ω is of genus zero, but not of disk type. For this case we use an argument involving a Killing field, as suggested in [Ros and Souam 1997]. The third case is when the genus of Ω is positive. For this case we construct a perturbation that depends on the mean curvature of Ω and on the contact angle γ. A similar perturbation is used in [Barbosa and do Carmo 1984] to show that the round spheres are the only immersed stable constant mean curvature hypersurfaces in $\mathbb{R}^n$. The normal component of the constructed perturbation makes the second variation of the energy negative, while preserving the volume; therefore, Ω cannot be weakly stable.

1. Preliminaries

In this section we define capillarity and stability in terms of the energy and the volume for a given configuration. Throughout this paper, let Ω be an oriented compact surface immersed in $\mathbb{R}^3$ with nonempty planar boundary in the xy-plane.

Let it be given by a $C^{2,\alpha}$-immersion $x(u, v) : D \to \mathbb{R}^3$, with $x(D) = \Omega$ and $x(\partial D) = \partial\Omega$. Also, assume that $\partial\Omega$ is a finite collection of nonintersecting simple closed curves; that is, $\partial\Omega$ is embedded in $\mathbb{R}^3$. We denote the boundary $\partial\Omega$ by Γ, and the regions in $\mathbb{R}^2$ bounded by Γ we name Σ'. The boundary Γ is also oriented and it is assumed that Ω comes from above near the boundary. We denote the areas on the surface and the wetted area by $|\Omega|$ and $|\Sigma'|$, respectively. We denote the angle of contact between the surface Ω and the wetted region Σ' by γ. The surface area of Ω is given by

$$|\Omega| = \iint_D dS,$$

where dS is the surface element on $\Omega = x(D)$. We assume that Ω is extendable in a neighborhood of Γ, so we can compute tangent vectors, normal vectors, etc.

Definition. An immersed surface is called capillary if it has constant mean curvature and makes constant contact angle with the walls along its boundary.

Now we define our main object of interest: the energy.

Definition. The energy function of the above configuration, after dividing by the surface tension σ, is given by

$$E = |\Omega| - \tau|\Sigma'|,$$

with $-1 < \tau < 1$ being some predetermined constant.

Our main goal in the problems we consider is to minimize the energy subject to the natural constraints that arise. To do this we should look among all the nearby surfaces that are admissible. We get them if we apply a perturbation to the original surface. Thus we need to define what an admissible variation is; see for example [Ros and Souam 1997].

Definition. Admissible variation of x is a differentiable map $\Phi : (-\epsilon, \epsilon) \times D \to \mathbb{R}^3$ such that $\Phi_t(p) = \Phi(t, p)$ with $p \in D$ is an immersion and $\Phi_0 = x$.

As we assumed before, the surface Ω can be extended across its boundary. That will allow us to keep the boundary planar after applying an admissible variation.

Definition. The volume functional $V : (-\epsilon, \epsilon) \to \mathbb{R}$ is defined by

$$V(t) = \tfrac{1}{3} \iint_D (\Phi_t \cdot \xi_t)\, dS_t,$$

where ξ_t and dS_t are the unit outward normal and the surface element on $\Phi_t(D)$.

The corresponding variational field is $Y(p) = (\partial\Phi/\partial t)(p)\big|_{t=0}$, and we denote its normal part by ϕ. Now we need to write the first and the second variation formulae related to ϕ. We need to set the first variation equal to 0, subject to the volume constraint $V'(0) = 0$, and investigate the second variation. We also have a volume constraint, because the variation ϕ must preserve the volume. This means that we should introduce a Lagrange multiplier λ and compute the first and second variations for the expression $E+\lambda V$. For proofs of the first and second variation formulae subject to a volume constraint, one may check [Wente 1966] and [Ros and Souam 1997], respectively.

Theorem 1.1 (first variation formula). *Let $d\sigma$ be the line element on boundary Γ, and let dS be the surface element on Ω. The first variation formula for the energy of x in the direction of ϕ, subject to a volume constraint, implies that*

$$\partial(E)[\phi] \equiv \frac{d}{dt}E(t)\Big|_{t=0} = -2\iint_D H\phi\,dS + \oint_{\partial D}(-\tau\csc\gamma+\cot\gamma)\phi\,d\sigma, \tag{1}$$

$$\partial(V)[\phi] \equiv \frac{d}{dt}V(t)\Big|_{t=0} = \iint_D \phi\,dS \equiv 0. \tag{2}$$

Formula (2) represents the rate of change of the volume at time $t=0$, so if we want constant volume it must be zero. It follows from (1) and (2) that H and γ must be constants in order to have an extremal of E, subject to the volume being stationary. This follows from the observation that the constant τ must equal $\cos\gamma$ in order for the boundary integral to equal zero. This means that Ω must be a capillary surface.

Definition. A capillary surface is called *weakly stable* if the second variation is nonnegative for all admissible perturbations with normal components $\phi \neq 0$, and *stable* if the second variation is positive for all admissible perturbations.

The next theorem gives a formula for the second variation. This is our main object of interest. We assume that the configuration is weakly stable and choose a special ϕ, manipulating the formula to get a contradiction unless we have a spherical cap.

Theorem 1.2 (second variation formula). *With the notation above, the formula for the second variation of E is*

$$\partial^2(E)[\phi] \equiv \frac{d^2}{dt^2}E(t)\Big|_{t=0} = \iint_D \big[|\nabla\phi|^2-(k_1^2+k_2^2)\phi^2\big]\,dS + \oint_{\partial D} p\phi^2\,d\sigma, \tag{3}$$

where $\nabla\phi$ is the surface gradient of ϕ, k_1 and k_2 are the principal curvatures, and $p = K_\Omega\cot\gamma + K_\Sigma\csc\gamma$. Here K_Ω and K_Σ are the signed normal curvatures of Ω and Σ with respect to the boundary. Of course, condition (2) should be fulfilled.

In our case Σ is planar, so $K_\Sigma = 0$; and if we take a vertical slice and consider the profile curve, K_Ω will be its curvature. If the profile curve bends toward the boundary, the sign of that normal curvature is taken to be positive. In the proof of our main theorem, we use Green's identities to write this formula more concisely.

2. The main theorem

In this section we show that the only immersed weakly stable capillary surface with boundary embedded in a plane is the spherical cap. (Recall that there is no gravitational action involved.) To do this we consider three cases.

The first case is when Ω is an immersed disc type surface. It has been solved by Nitsche [1985] and Finn and McCuan [2000]. The first author proves a theorem that states that an immersed disk type surface in a ball that makes constant angle with the boundary sphere is a flat disk or a round spherical cap. He proves the theorem for a right angle but points out that the idea works for any angle. This is proved again in [Ros and Souam 1997]. Finn and McCuan have similar results for such surfaces with planar boundary. Notice that the stability condition is unnecessary for this case.

The second case is the general genus-zero case. Here the surface Ω has genus zero, but there could be possibly more than one boundary curve. We assume that the planar boundary $\Gamma = \partial\Omega$ is embedded, but the surface itself could be immersed. Let Γ belong to the plane $z = 0$. We adapt the method used in [Ros and Souam 1997] to our purposes. As before, Ω is given by mapping $x : D \to \Omega$. Let $p_0 \in \Omega$ be a point such that the Euclidean distance to the plane containing Γ is maximal. Obviously there is at least one point with that property. Let ξ be the unit normal to the surface. From our setup it follows that $\xi(p_0)$ is parallel to the z-axis. Later in this section, we see that the mean curvature of Ω must be negative, so the vector $\xi(p_0)$ points out in the positive z-direction. Denote by X the Killing field induced by rotations around the line directed by $\xi(p_0) = \boldsymbol{k}$; that is, $X = p \wedge \xi(p_0)$, where p is a point in $\mathbb{R}^3$ and $\wedge$ is the usual wedge product in $\mathbb{R}^3$. Consider the function $\phi(p) = \langle X(x(p)), \xi(p)\rangle$. Because of rotational invariance around $\xi(p)$, it follows that $\phi(p)$ is a Jacobi field on the surface. Using the notation from the previous sections, one has

$$\Delta\phi + (k_1^2 + k_2^2)\phi = 0,$$

with $\phi_\nu + p\phi = 0$ on Γ. Also $\phi(p_0) = 0$ and $\nabla\phi(p_0) = 0$. Therefore, the second variation of energy in the direction of ϕ is zero, and the volume constraint holds. As in [Ros and Souam 1997], with the Gauss–Bonnet theorem one can show that there are at least three nodal regions of ϕ; that is, $\Omega - \phi^{-1}(0)$ has at least three connected components. Let Ω_i, $i = 1, 2, 3$ be the nodal regions of ϕ, and let ϕ_i be equal to ϕ on Ω_i and zero elsewhere. Now construct $\widetilde{\phi} = \sum_{i=1}^{3} c_i\phi_i$ with c_1, c_2, c_3 constants. Then one can adjust the constants using the volume constraint to get a smaller number for the second variation, making it negative. Hence there are no surfaces of the assumed type with two or more connected boundary components. Thus we can conclude that the spherical caps are the only immersed weakly stable

CMC-surfaces with planar embedded boundary having genus $g = 0$ and constant contact angle along the boundary with the plane Σ.

The third and final case is when the genus is positive. Here we state the main theorem of this paper.

Main Theorem. *No weakly stable capillary surface with planar boundary exists that is immersed in $\mathbb{R}^3$ and has genus $g > 0$.*

We split the proof into several lemmas. Again we assume that the boundary of Ω is embedded; that is, it consists of a finite number of simple closed curves. Also we assume that the surface can be extended across its boundary. Thus we ensure that the boundary stays planar after a normal perturbation. Also we assume that Ω comes from above to Σ. We construct a special normal perturbation for which the second variation is negative and the volume is preserved. First we need to rewrite (3) using Green's first identity. We also assume that our mappings are $C^{2,\alpha}$ (in fact capillary surfaces are analytic by standard regularity theory), so we can compute derivatives at the boundary and extend the surface around the boundary Γ. The variation that we use does not necessarily keep the boundary planar. That is why we extend the surface across the boundary, so that after the perturbation, the new surface has planar boundary; that is, $\partial\Phi_t(D)$ belongs to the plane Σ. This is how it is done in [Wente 1966].

Applying Green's first identity to the second variation formula from Section 1, one gets

$$\iint_D |\nabla\phi|^2 - (k_1^2+k_2^2)\phi^2\,dS = \iint_D \phi[-\Delta\phi - (k_1^2+k_2^2)\phi]\,dS + \oint_{\partial D} \phi\phi_\nu\,d\sigma, \tag{4}$$

and for $\partial^2 E$ one obtains

$$\partial^2 E = \iint_D (-L\phi)\phi\,dS + \oint_{\partial D} (\phi_\nu + p\phi)\phi\,d\sigma, \tag{5}$$

where $L\phi = \Delta\phi + (k_1^2+k_2^2)\phi$, $\partial V \equiv \iint_D \phi ds = 0$, and $p = K_\Omega \cot\gamma + K_\Sigma \csc\gamma$. The operator L is called the *Jacobi operator*.

Now we write the perturbation used to prove the main theorem. Let Φ be the perturbation that sends $x \to x + t\xi + Htx + ct\boldsymbol{k} + O(t^2)$. Here t lies in $[-\epsilon, \epsilon]$, $\boldsymbol{k} = (0, 0, 1)$ is the unit vertical vector, c is a constant, ξ is the outward unit normal on the surface, and H is the mean curvature of the surface. The normal part of Φ is $\phi = \xi \cdot (\partial\Phi/\partial t)(p)\big|_{t=0}$, where $p \in \Omega$. When we compute this quantity, we get $\phi = 1 + H(x\cdot\xi) + c(\boldsymbol{k}\cdot\xi)$, with c to be determined from (2).

Lemma 2.1. *Condition (2) implies that $c = -\cos\gamma$; that is,*

$$\phi = 1 + H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi),$$

in order to keep the volume fixed.

Proof. One needs to adjust c in $\phi = 1 + H(x\cdot\xi) + c(\boldsymbol{k}\cdot\xi)$ to get the integral of ϕ over the surface Ω to be zero.

$$\begin{aligned} 0 = \iint_D \phi\, dS &= \iint_D (1 + H(x\cdot\xi) + c(\boldsymbol{k}\cdot\xi))\, dS \\ &= |\Omega| + H \iint_D (x\cdot\xi)\, dS + c\iint_D (\boldsymbol{k}\cdot\xi)\, dS, \end{aligned}$$

where $|\Omega|$ is the area of the surface Ω. The quantity $\iint_D (\boldsymbol{k}\cdot\xi)\, dS$ is easily computed by the Divergence theorem. We know for the embedded case that

$$\iint_D (\boldsymbol{k}\cdot\xi)\, dS + \iint_{\Sigma'} (\boldsymbol{k}\cdot\xi)\, dS = \iiint_T \operatorname{div} \boldsymbol{k}\, dV = 0,$$

since $\boldsymbol{k}$ is a constant vector. Here Σ' is the wetted part bounded by Γ, T is the solid bounded by Ω and Σ, ξ is unit outward normal to $\partial T = \Omega \cup \Sigma'$, and dV is the volume element in $\mathbb{R}^3$. On Σ' the unit vector $\boldsymbol{k}$ is equal to $-\xi$, so

$$\iint_D (\boldsymbol{k}\cdot\xi)\, dS = -\iint_{\Sigma'} (\boldsymbol{k}\cdot\xi)\, dS = \iint_{\Sigma'} dS = |\Sigma'|.$$

For the immersed case there is not actually a solid T, but one can still apply the divergence theorem. In this case $\Omega\cup\Sigma'$ separates $\mathbb{R}^3$ into a finite number of connected regions, with one of them unbounded. On the bounded regions one can use the divergence theorem, and the calculation is the same as in the embedded case, since $\operatorname{div}\boldsymbol{k} = 0$ everywhere on $\mathbb{R}^3$. One can also apply Stokes's theorem to obtain the same result. Thus

$$\iint_D (\boldsymbol{k}\cdot\xi)\, dS = |\Sigma'|.$$

Next we compute $\iint_D H(x\cdot\xi)\, dS$. Assume conformal coordinates. It is well-known that in conformal coordinates one has $\Delta x = 2H\xi$ [Oprea 2007]; therefore

$$\iint_D H(x\cdot\xi)\, dS = \tfrac{1}{2}\iint_D (x\cdot\Delta x)\, dS = -\tfrac{1}{2}\iint_D |\nabla x|^2\, dS + \tfrac{1}{2}\oint_{\partial D} (x\cdot x_\nu)\, d\sigma.$$

Here Δx and ∇x are the vector surface Laplacian and the vector surface gradient of x. In conformal coordinates, the square of the surface gradient of x is

$$|\nabla x|^2 = \frac{1}{E}((x_u\cdot x_u) + (x_v\cdot x_v)) = \frac{1}{E}(E+E) = 2,$$

so

$$-\tfrac{1}{2}\iint_D |\nabla x|^2\, dS = -\tfrac{1}{2}\iint_D 2\, dS = -|\Omega|.$$

Also, if $\boldsymbol{n}$ is the unit normal of Γ in Σ, we have $x_\nu = (\cos\gamma)\boldsymbol{n} - (\sin\gamma)\boldsymbol{k}$. Therefore $(x \cdot x_\nu) = \cos\gamma (x \cdot \boldsymbol{n})$. It follows that

$$\frac{1}{2}\oint_{\partial D}(x \cdot x_\nu)\, d\sigma = \frac{\cos\gamma}{2}\oint_{\partial D}(x \cdot \boldsymbol{n})\, d\sigma = \cos\gamma|\Sigma'|. \tag{6}$$

The proof of the last equality in (6) can be seen in [Marinov 2010]. Combining the above results, we get

$$0 = |\Omega| - |\Omega| + \frac{1}{2}\oint_{\partial D}(x \cdot x_\nu)\, d\sigma + c|\Sigma'| = \cos\gamma|\Sigma'| + c|\Sigma'|.$$

This implies that $c = -\cos\gamma$, and therefore $\phi = 1 + H(x \cdot \xi) - \cos\gamma(\boldsymbol{k} \cdot \xi)$ and $\iint_D \phi\, dS = 0$. □

For this particular ϕ, the boundary term in the second variation happens to be zero.

Lemma 2.2. *For $\phi = 1 + H(x \cdot \xi) - \cos\gamma(\boldsymbol{k} \cdot \xi)$, we have $\phi_\nu + p\phi = 0$; that is,*

$$\partial^2 E = \iint_D (-L\phi)\phi\, dS.$$

Proof. One useful fact is that Γ is a line of curvature for both the plane Σ and the surface Ω, by the Terquem–Joachimsthal theorem [Spivak 1979]. On Γ, we have

$$\begin{aligned}\phi_\nu + p\phi &= (1 + H(x \cdot \xi) - \cos\gamma(\boldsymbol{k} \cdot \xi))_\nu + K_\Omega(\cot\gamma)\phi \\ &= H\frac{\partial}{\partial\nu}(x \cdot \xi) - \cos\gamma\frac{\partial}{\partial\nu}(\boldsymbol{k} \cdot \xi) + K_\Omega(\cot\gamma)\phi.\end{aligned}$$

Now we compute the normal derivative, taking into account that $(x_\nu \cdot \xi) = 0$ and $\boldsymbol{k}$ is a constant vector. Since Γ is a line of curvature, we get $(\partial/\partial\nu)(k \cdot \xi) = -K_\Omega(\boldsymbol{k} \cdot x_\nu)$ and $(\partial/\partial\nu)(x \cdot \xi) = -K_\Omega(x \cdot x_\nu)$. Substituting in the boundary expression from above, we have

$$\phi_\nu + p\phi = -K_\Omega H(x \cdot x_\nu) + (\cos\gamma)K_\Omega(\boldsymbol{k} \cdot x_\nu) + K_\Omega(\cot\gamma)\phi. \tag{7}$$

We use some more relations to rewrite this expression. Here $\boldsymbol{n}$ is the unit normal vector of Γ in the plane Σ.

$$(x \cdot x_\nu) = \cos\gamma(x \cdot \boldsymbol{n}), \qquad (x \cdot \xi) = \sin\gamma(x \cdot \boldsymbol{n}), \qquad (k \cdot x_\nu) = -\sin\gamma.$$

Using this and the fact that $\boldsymbol{k} \cdot \xi = \cos\gamma$ on Γ, we obtain from (7)

$$\begin{aligned}&K_\Omega\big[-H(\cos\gamma)(x \cdot \boldsymbol{n}) - \cos\gamma\sin\gamma + \cot\gamma\big(1 + H(x \cdot \xi) - \cos\gamma(\boldsymbol{k} \cdot \xi)\big)\big] \\ &\quad = K_\Omega\big(-H(\cos\gamma)(x\cdot\boldsymbol{n}) - \cos\gamma\sin\gamma + (\cot\gamma)H\sin\gamma(x\cdot\boldsymbol{n}) + \cot\gamma - \cot\gamma\cos^2\gamma\big) \\ &\quad = K_\Omega(-\cos\gamma\sin\gamma + \cot\gamma - \cot\gamma\cos^2\gamma) = 0.\end{aligned}$$

This shows that $\phi_\nu + p\phi \equiv 0$ on Γ, and thus in the second variation formula (3), the boundary term is zero and the formula becomes

$$\partial^2 E = \iint_D (-L\phi)\phi\, dS. \tag{8}$$ □

Next is to rewrite (8).

Lemma 2.3.

$$\begin{aligned}\partial^2 E &= \iint_D (-L\phi)\phi\, dS \\ &= -\iint_D \frac{(k_1-k_2)^2}{2}\, dS - \oint_{\partial D} K_\Omega (\cos\gamma)[H(x\cdot \boldsymbol{n}) + \sin\gamma]\, d\sigma.\end{aligned} \tag{9}$$

Proof. For $\phi = 1 + H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi)$, we have

$$(L\phi)\phi = (L\phi)(1 + H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi)) = (L\phi) + (L\phi)(H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi)),$$

so

$$\iint_D (-L\phi)\phi\, dS = -\iint_D (L\phi)\, dS - \iint_D (L\phi)(H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi))\, dS.$$

Now let's compute $L\phi$:

$$L\phi = L1 + L(H(x\cdot\xi)) - (\cos\gamma)L(\boldsymbol{k}\cdot\xi) = k_1^2 + k_2^2 + HL(x\cdot\xi).$$

Here $L1 = \Delta 1 + (k_1^2 + k_2^2)1 = k_1^2 + k_2^2$ and $L(\boldsymbol{k}\cdot\xi) = 0$, and it follows that $L(x\cdot\xi) = -2H$ [Barbosa and do Carmo 1984]. Taking this into account, we have

$$L\phi = k_1^2 + k_2^2 - 2H^2 = k_1^2 + k_2^2 - \frac{(k_1+k_2)^2}{2} = \frac{(k_1-k_2)^2}{2}.$$

Getting back to the integral of $(-L\phi)\phi$, we obtain

$$\begin{aligned}&\iint_D (-L\phi)\phi\, dS \\ &\qquad = -\iint_D \frac{(k_1-k_2)^2}{2}\, dS - \iint_D \frac{(k_1-k_2)^2}{2}(H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi))\, dS.\end{aligned}$$

We set $\psi = H(x\cdot\xi) - \cos\gamma(\boldsymbol{k}\cdot\xi)$; thus we need to compute $\iint_D (L\phi)\psi\, dS$. Green's second identity implies that

$$\iint_D (L\phi)\psi\, dS = \iint_D (L\psi)\phi\, dS + \oint_{\partial D} (\phi_\nu\psi - \psi_\nu\phi)\, d\sigma.$$

We know from the previous calculations that $L\psi = -2H^2$, so

$$\iint_D (L\phi)\psi\, dS = -2H^2 \iint_D \phi\, dS + \oint_{\partial D} (\phi_\nu\psi - \psi_\nu\phi)\, d\sigma; \tag{10}$$

but we know that $\iint_D \phi\, dS = \partial V = 0$. Using $\psi = \phi - 1$, this reduces (10) to

$$\iint_D (L\phi)\psi\, dS = \oint_D (\phi_\nu(\phi-1) - (\phi-1)_\nu\phi)\, d\sigma = -\oint_{\partial D} \phi_\nu\, d\sigma.$$

On Γ, we know from (7) that $\phi_\nu = -K_\Omega H(x\cdot x_\nu) + (\cos\gamma)K_\Omega(\boldsymbol{k}\cdot x_\nu)$. We also know that on the boundary, $(x\cdot x_\nu) = \cos\gamma(x\cdot\boldsymbol{n})$ and $(k\cdot x_\nu) = -\sin\gamma$; therefore

$$\phi_\nu = -K_\Omega H(\cos\gamma)(x\cdot\boldsymbol{n}) - (\cos\gamma)K_\Omega\sin\gamma$$

and

$$\iint_D (L\phi)\psi\, dS = \oint_{\partial D} K_\Omega[H(\cos\gamma)(x\cdot\boldsymbol{n}) + (\cos\gamma)K_\Omega\sin\gamma]\, d\sigma.$$

Now substituting $(L\phi)\psi$ back into the formula

$$-\iint_D (L\phi)\phi\, dS = -\iint_D \frac{(k_1-k_2)^2}{2}\, dS - \iint_D (L\phi)\psi\, dS,$$

we get for the left-hand side the value

$$-\iint_D \frac{(k_1-k_2)^2}{2}\, dS - \oint_{\partial D}[K_\Omega H(\cos\gamma)(x\cdot\boldsymbol{n}) + (\cos\gamma)K_\Omega\sin\gamma]\, d\sigma.$$

Thus we have (9), which was the statement of the lemma. □

Lemma 2.4. *Let Σ' be the region bounded by Γ, let $|\Sigma'|$ be its area, and let $|\Gamma|$ be the length of the boundary. The boundary may consist of several curves, so Σ' may not be connected. Let d be the number of boundary curves, that is, the number of components of Γ. Then*

$$\oint_{\partial D}(x\cdot\boldsymbol{n})\, d\sigma = 2|\Sigma'|, \tag{11}$$

$$\oint_{\partial D} k_\Gamma(x\cdot\boldsymbol{n})\, d\sigma = -|\Gamma|, \tag{12}$$

$$\left|\oint_{\partial D} k_\Gamma\, d\sigma\right| \le 2\pi d, \tag{13}$$

$$\sin\gamma|\Gamma| = -2H|\Sigma'|. \tag{14}$$

Proof. The proof of this lemma is given in [Marinov 2010]. Here we only prove formula (14), which is a version of the *balancing formula*, of which a general statement and proof can be found in [Earp et al. 1991]. Note that (14) implies that H is negative.

Choose conformal coordinates. In the proof of Lemma 2.1 we saw that

$$\iint_D (\boldsymbol{k}\cdot\xi)\, dS = |\Sigma'|$$

and that the surface Laplacian in conformal coordinates is $\Delta x = 2H\xi$ [Oprea 2007]. We use this and Green's first identity to get

$$\iint_D (\boldsymbol{k}\cdot\xi)\,dS = \frac{1}{2H}\iint_D (\boldsymbol{k}\cdot\Delta x)\,dS = \frac{1}{2H}\oint_{\partial D} (\boldsymbol{k}\cdot x_\nu)\,d\sigma.$$

This equality holds since $\boldsymbol{k}$ is a constant vector, and therefore its surface gradient is zero. From a previous discussion of the result that the boundary term in $\partial^2 E$ is zero, we know that $(\boldsymbol{k}\cdot x_\nu) = -\sin\gamma$ on Γ. Combining this fact with the above expressions for the integral of $(\boldsymbol{k}\cdot\xi)$ over Ω, we arrive at

$$|\Sigma'| = \iint_D (\boldsymbol{k}\cdot\xi)\,dS = \frac{1}{2H}\oint_{\partial D} (\boldsymbol{k}\cdot x_\nu)\,d\sigma = -\frac{1}{2H}\sin\gamma\,|\Gamma|.$$

Taking the first and last expressions above and multiplying by $-2H$, we get

$$-2H|\Sigma'| = \sin\gamma|\Gamma|,$$

which was the result to prove. This is an indication that the mean curvature H of Ω is negative for the immersed case, since all other quantities in (14) are positive. □

To continue, we rewrite (9). From Meusnier's theorem and Euler's theorem [Struik 1988], we know that $2H = K_\Omega + k_2 = K_\Omega + (\sin\gamma)k_\Gamma$ on Γ, where k_Γ is the curvature of the boundary. We have

$$K_\Omega = 2H - k_\Gamma \sin\gamma,$$

and therefore (9) becomes

$$\begin{aligned}\partial^2 E &= -\iint_D \frac{(k_1-k_2)^2}{2}\,dS - \oint_{\partial D} \cos\gamma\,(2H - k_\Gamma\sin\gamma)(H(x\cdot\boldsymbol{n}) + \sin\gamma)\,d\sigma\\ &= -\iint_D \frac{(k_1-k_2)^2}{2}\,dS\\ &\quad - \oint_{\partial D} \cos\gamma\,\big(2H^2(x\cdot\boldsymbol{n}) + 2H\sin\gamma - \sin\gamma\,k_\Gamma H(x\cdot\boldsymbol{n}) - \sin^2\gamma\,k_\Gamma\big)\,d\sigma.\end{aligned}$$

Lemma 2.5. *The following estimate holds for the second variation of energy*:

$$\partial^2 E < 4\pi(2-2g) - 2\pi d\big(2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma\big),$$

where g is the genus of Ω and d is the number of boundary components of Γ.

Proof. Using (11) and (12), we get

$$\begin{aligned}\partial^2 E = &-\iint_D \frac{(k_1-k_2)^2}{2}\,dS\\ &+ \cos\gamma\left(-4H^2|\Sigma'| - 2H|\Gamma|\sin\gamma - \sin\gamma\,H|\Gamma| + \sin^2\gamma\oint_{\partial D} k_\Gamma\,d\sigma\right),\end{aligned}$$

and if we use (14) we arrive at

$$\partial^2 E = -\iint_D \frac{(k_1-k_2)^2}{2}\,dS + \cos\gamma\left(2H\sin\gamma\,|\Gamma| - 2H\sin\gamma\,|\Gamma| - H\sin\gamma\,|\Gamma| + \sin^2\gamma \oint_{\partial D} k_\Gamma\,d\sigma\right).$$

After the obvious cancellation, we obtain

$$\partial^2 E = -\iint_D \frac{(k_1-k_2)^2}{2}\,dS + \cos\gamma\left(-\sin\gamma\,H|\Gamma| + \sin^2\gamma \oint_{\partial D} k_\Gamma\,d\sigma\right), \tag{15}$$

and by using (14) again, we get

$$\partial^2 E = -\iint_D \frac{(k_1-k_2)^2}{2}\,dS + \cos\gamma\left(2H^2|\Sigma'| + \sin^2\gamma \oint_{\partial D} k_\Gamma\,d\sigma\right). \tag{16}$$

We can easily see that this last expression is zero if Ω is the standard spherical cap. On a spherical cap all points are umbilical; that is, $k_1 = k_2$ everywhere, so the first integral is zero. Also we can see that $H^2|\Sigma'| = \pi\sin^2\gamma$ no matter what the scaling, and for a spherical cap we have $\oint_{\partial D} k_\Gamma\,d\sigma = -2\pi$, since we chose to work with the outward unit normal, and for us $k_\Gamma \le 0$. This means that the second expression is also zero, so the whole variation is zero. Thus, on a spherical cap this particular variation does not change the geometry.

One can express the first integral in this formula in another way:

$$\frac{(k_1-k_2)^2}{2} = \frac{k_1^2+k_2^2-2k_1k_2}{2} = \frac{(k_1+k_2)^2-4k_1k_2}{2} = 2H^2 - 2K.$$

This is the integrand in the Willmore energy. Continuing with the second variation and using the previous formula, we get

$$\partial^2 E = -2\iint_D H^2\,dS + 2\iint_D K\,dS + \cos\gamma\left(2H^2|\Sigma'| + \sin^2\gamma \oint_{\partial D} k_\Gamma\,d\sigma\right).$$

Using the Gauss–Bonnet formula we obtain for the right-hand side the value

$$-2\iint_D H^2\,dS + 4\pi\chi(\Omega) - 2\oint_{\partial D} k_g\,d\sigma + \cos\gamma\left(2H^2|\Sigma'| + \sin^2\gamma \oint_{\partial D} k_\Gamma\,d\sigma\right).$$

Again we use Meusnier's theorem and Euler's theorem to get $k_g = \pm(\cos\gamma)k_\Gamma$ on each Γ_i. From (13) it follows that

$$\left|\oint_{\partial D} k_g\,d\sigma\right| \le |\cos\gamma| 2\pi d.$$

Also, if Ω has d boundary curves and genus g, then $\chi(\Omega) = 2 - 2g - d$. This follows from the fact that one can attach flat discs to the surface at the boundary to make it closed, and reattaching the disks will decrease $\chi(\Omega)$ exactly with d. Taking all this into account, we have

$$\begin{aligned}
\partial^2 E \\
&= -2H^2\left(\iint_D dS - \cos\gamma\,|\Sigma'|\right) + 4\pi\chi(\Omega) - 2\oint_{\partial D} k_g\,d\sigma + \cos\gamma\sin^2\gamma\oint_{\partial D} k_\Gamma\,d\sigma \\
&\le -2H^2\big(|\Omega| - \cos\gamma\,|\Sigma'|\big) + 4\pi(2-2g-d) + 4\pi d|\cos\gamma| + |\cos\gamma|\sin^2\gamma\cdot 2\pi d \\
&= -2H^2\big(|\Omega| - \cos\gamma\,|\Sigma'|\big) + 4\pi(2-2g) - 2\pi d\big(2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma\big).
\end{aligned}$$

The first term is always negative because $|\Omega| > |\Sigma'|$, since Σ' is a planar surface spanning Γ, so

$$\partial^2 E < 4\pi(2-2g) - 2\pi d\left(2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma\right). \qquad \square$$

The last lemma basically proves the main theorem of this paper. Through simple calculus, we can easily see that the expression

$$2 - 2|\cos\gamma| - |\cos\gamma|\sin^2\gamma$$

is nonnegative for any angle $\gamma \in (0, \pi)$. The computation is shown in [Marinov 2010]. This observation implies that

$$\partial^2 E < 4\pi(2-2g),$$

and if the genus is positive, the second variation of energy is negative. Therefore Ω cannot be weakly stable, which was the statement of the main theorem. This discussion fully resolves the case for an immersed stable capillary surface with planar embedded boundary.

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