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Let $d \geq 3$, $n_1 > 0$ and $n_2 > 0$ be integers. Let $\underline{e} = (e_1, \dots, e_r)$ and $\underline{q} = (q_1, \dots, q_s)$ be two partitions of d. Let X, X' and Y be smooth, connected, projective complex curves. In this paper we study coverings that decompose into a sequence

$$X \xrightarrow{\pi} X' \xrightarrow{f} Y,$$

where π is a degree-two coverings with n_1 branch points and branch locus D_π and f is a degree-d coverings with n_2 points of simple branching and two special points whose local monodromy is given by $\underline{e}$ and $\underline{q}$, respectively. Furthermore the covering f has monodromy group S_d and $f(D_\pi) \cap D_f = \varnothing$ where D_f denotes the branch locus of f. We prove that the corresponding Hurwitz spaces are irreducible under the hypothesis $n_2 - s - r \geq d + 1$.

Introduction

In this paper we study Hurwitz spaces that parametrize branched coverings with two special fibers whose monodromy group is a Weyl group of type B_d.

We notice that the irreducibility of Hurwitz spaces, parametrizing branched coverings of a smooth, connected, projective complex curve Y with monodromy group S_d and with at most two special fibers, has been well studied both when $Y \simeq \mathbb{P}^1$ and when Y has positive genus. The case of simple coverings was studied in [Berstein and Edmonds 1984; Hurwitz 1891], the case of coverings with one special fiber in addition to points of simple branching was studied in [Kanev 2004; Kluitmann 1988; Natanzon 1991; Vetro 2006] and the case of two special fibers in addition to points of simple branching was studied in [Vetro 2010; Wajnryb 1996].

S_d is the Weyl group of a root system of type A_{d-1} and so it is interesting to study coverings with monodromy group a Weyl group different by S_d. Furthermore coverings of this type are interesting, for example, because they appear in the study of spectral curves and of Prym–Tyurin varieties.

MSC2010: primary 14H30; secondary 14H10.

Keywords: Hurwitz spaces, special fibers, branched coverings, Weyl group of type B_d, monodromy, braid moves.

Hurwitz spaces parametrizing coverings of this type were studied in [Biggers and Fried 1986; Kanev 2006; Vetro 2007; 2008a; 2008b; 2009]. Biggers and Fried proved the irreducibility of Hurwitz spaces parametrizing coverings of $\mathbb{P}^1$ whose monodromy group is a Weyl group of type D_d and whose local monodromies are all reflections. Kanev extended the result to Hurwitz spaces of Galois coverings of $\mathbb{P}^1$ whose Galois group is an arbitrary Weyl group.

Let X and X' be smooth, connected, projective complex curves. We studied Hurwitz spaces of coverings that decompose into a sequence of coverings of type $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, where π is a degree-two covering and f is a degree $d \geq 3$ covering with one special fiber and with monodromy group S_d. We analyzed in [Vetro 2007; 2008a] the case that π is branched, and in [Vetro 2008b; 2009] the unramified case.

In this paper we continue the study of coverings of type $X \xrightarrow{\pi} X' \xrightarrow{f} Y$, with π a degree-two covering and f a degree-d covering. Let $\underline{e} = (e_1, \ldots, e_r)$ and $\underline{q} = (q_1, \ldots, q_s)$ be two partitions of d and let b_0 be a point of Y. In particular we study equivalence classes of pairs $[X \xrightarrow{\pi} X' \xrightarrow{f} Y, \phi]$ satisfying the following conditions:

- π is branched in n_1 points and has branch locus D_π, f is simply branched in n_2 points and has two special points with local monodromy given by $\underline{e}$ and $\underline{q}$, respectively;
- f has monodromy group S_d and $f(D_\pi) \cap D_f = \varnothing$, where D_f denotes the branch locus of f;
- $f \circ \pi$ is unramified in b_0 and $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \ldots, -1, 1, \ldots, d\}$ is a bijection.

We study the irreducibility of the corresponding Hurwitz spaces both when $Y \simeq \mathbb{P}^1$ and when Y has genus > 0. We prove that, in both the cases, these spaces are irreducible under the hypothesis $n_2 - s - r \geq d + 1$. This condition is necessary in [Vetro 2010] in order to prove the irreducibility of the Hurwitz spaces $H^o_{d,n_2,\underline{e},\underline{q}}(Y, b_0)$ that parametrize equivalence classes of pairs $[f, \varphi]$ where f is a coverings as above and $\varphi : f^{-1}(b_0) \to \{1, \ldots, d\}$ is a bijection. Here, we also use the results of [Vetro 2010].

Notation. Two degree-d branched coverings of Y, $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, are *equivalent* if there exists a biholomorphic map $p : X_1 \to X_2$ such that $f_2 \circ p = f_1$. Two sequences of coverings,

$$X_1 \xrightarrow{\pi_1} X_1' \xrightarrow{f_1} Y \quad \text{and} \quad X_2 \xrightarrow{\pi_2} X_2' \xrightarrow{f_2} Y,$$

are *equivalent* if there exist two biholomorphic maps $p : X_1 \to X_2$ and $p' : X_1' \to X_2'$ such that $p' \circ \pi_1 = \pi_2 \circ p$ and $f_2 \circ p' = f_1$. The equivalence class containing $f \circ \pi$ is denoted by $[f \circ \pi]$. The natural action of S_d on $\{1, \ldots, d\}$ is on the right.

1. Preliminaries

Throughout this section, d and n denote positive integers.

1.1. ***Weyl groups of type B_d.*** (Refer to [Bourbaki 1968; Carter 1972] for details.) Let $\{\varepsilon_1, \dots, \varepsilon_d\}$ be the standard base of $\mathbb{R}^d$ and let R be the root system

$$\{\pm\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j : 1 \le i, j \le d\}.$$

Let us denote by $W(B_d)$ the group generated by the reflections s_{ε_i}, with $1 \le i \le d$, and by the reflections $s_{\varepsilon_i - \varepsilon_j}$, with $1 \le i < j \le d$. We call $W(B_d)$ a Weyl group of type B_d.

We notice that the reflection $s_{\varepsilon_i-\varepsilon_j}$ exchanges ε_i with ε_j and $-\varepsilon_i$ with $-\varepsilon_j$, leaving fixed each ε_h with $h \neq i, j$. The reflection s_{ε_i} exchanges ε_i with $-\varepsilon_i$ and fixes all the ε_h with $h \neq i$. Thus if we identify $\{\pm\varepsilon_i : 1 \le i \le d\}$ with $\{\pm 1, \dots, \pm d\}$ by the map $\pm\varepsilon_i \to \pm i$, we can easily define an injective homomorphism from $W(B_d)$ into S_{2d} such that

$$s_{\varepsilon_i-\varepsilon_j} \to (i\ j)(-i\ -j), \quad s_{\varepsilon_i} \to (i\ -i), \quad s_{\varepsilon_i+\varepsilon_j} = s_{\varepsilon_i} s_{\varepsilon_j} s_{\varepsilon_i-\varepsilon_j} \to (i\ -j)(-i\ j).$$

Let $\mathbb{Z}_2^d$ be the set of the functions from $\{1, \dots, d\}$ into $\mathbb{Z}_2$ equipped with the sum operation. We will use $\bar{1}_j$ to denote the function in $\mathbb{Z}_2^d$ defined by

$$\bar{1}_j(j) = \bar{1} \quad \text{and} \quad \bar{1}_j(h) = \bar{0} \quad \text{for each } h \neq j$$

and we will write z_{ij} to denote the function in $\mathbb{Z}_2^d$ defined by

$$z_{ij}(i) = z_{ij}(j) = z \quad \text{and} \quad z_{ij}(h) = \bar{0} \quad \text{for each } h \neq i, j \text{ and } z \in \mathbb{Z}_2.$$

Let Ψ be the homomorphism from S_d into $\mathrm{Aut}(\mathbb{Z}_2^d)$ that assigns to $t \in S_d$ the element $\Psi(t) \in \mathrm{Aut}(\mathbb{Z}_2^d)$, where $[\Psi(t)\, a]\,(j) := a(j^t)$ for each $a \in \mathbb{Z}_2^d$.

Let $\mathbb{Z}_2^d \times^s S_d$ be the semidirect product of $\mathbb{Z}_2^d$ and S_d through the homomorphism Ψ. Given $(a'; t_1), (a''; t_2) \in \mathbb{Z}_2^d \times^s S_d$, we put

$$(a'; t_1) \cdot (a''; t_2) := (a' + \Psi(t_1)a''; t_1 t_2).$$

It is easy to check that the homomorphism from $W(B_d) \to \mathbb{Z}_2^d \times^s S_d$ defined by

$$s_{\varepsilon_i-\varepsilon_j} \to (0; (i\ j)), \quad s_{\varepsilon_i} \to (\bar{1}_i; \mathrm{id}), \quad s_{\varepsilon_i+\varepsilon_j} \to (\bar{1}_{ij}; (i\ j))$$

is an isomorphism. We will identify $W(B_d)$ with $\mathbb{Z}_2^d \times^s S_d$ via this isomorphism.

Definition 1. Let k be a positive integer. Let $(c; \xi)$ be an element of $W(B_d)$ such that ξ is a k-cycle of S_d and c is a function that sends to $\bar{0}$ all the indexes fixed by ξ. We call an such element a *positive k-cycle* if c is either zero or a function which sends to $\bar{1}$ an even number of indexes. We call it *negative k-cycle* if it is not positive.

We notice that two cycles $(c; \xi)$ and $(c'; \xi')$ in $W(B_d)$ are disjoint if ξ and ξ' are disjoint. Furthermore, all the elements in $W(B_d)$ can be expressed as a product of disjoint positive and negative cycles. The lengths of such disjoint cycles together with their signs determine the signed cycle type of the elements of $W(B_d)$. Two elements of $W(B_d)$ are conjugate if and only if they have the same signed cycle type [Carter 1972].

Braid group actions on Hurwitz systems. (Refer to [Birman 1969; Fadell and Neuwirth 1962; Graber et al. 2002; Hurwitz 1891; Kanev 2004; Scott 1970].) Let Y be a smooth, connected, projective complex curve of genus g and let $b_0 \in Y$. Let $(Y - b_0)^{(n)}$ be the n-fold symmetric product of $(Y - b_0)$ and let Δ be the codimension 1 locus of $(Y - b_0)^{(n)}$ consisting of non simple divisors. The generators of the braid group $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ were studied in [Birman 1969; Fadell and Neuwirth 1962; Scott 1970]. They are the elementary braids σ_i, with $1 \le i \le n-1$, and the braids $\rho_{jk},\ \tau_{jk}$, with $1 \le j \le n$ and $1 \le k \le g$.

Definition 2. Let G be a subgroup of S_h. An ordered sequence of elements of G

$$(\underline{t}; \underline{\lambda}, \underline{\mu}) := (t_1, \dots, t_n; \lambda_1, \mu_1, \dots, \lambda_g, \mu_g)$$

such that $t_i \ne \mathrm{id}$ for each i and $t_1 \cdots t_n = [\lambda_1, \mu_1] \cdots [\lambda_g, \mu_g]$ is called a *Hurwitz system with values in G*. The subgroup of G generated by $t_1, \dots, t_n, \lambda_1, \mu_1, \dots, \lambda_g, \mu_g$ is called the *monodromy group* of the Hurwitz system.

Remark 3. An ordered sequence $\underline{t} := (t_1, \dots, t_n)$ of elements of G, with $t_i \ne \mathrm{id}$ for each i, is a Hurwitz system if $t_1 \cdots t_n = \mathrm{id}$.

To each generator of $\pi_1((Y - b_0)^{(n)} - \Delta, D)$ one associates a pair of braid moves. We denote by σ_i' and $\sigma_i'' = (\sigma_i')^{-1}$ the moves associated with σ_i, and we call them elementary moves. Similarly, ρ_{jk}' and $\rho_{jk}'' = (\rho_{jk}')^{-1}$ denote the moves associated to ρ_{jk}, and likewise for τ_{jk}.

The moves σ_i' and σ_i'' fix all the λ_k, all the μ_k and all the t_h with $h \ne i, i+1$. The elementary move σ_i' transforms (t_i, t_{i+1}) into $(t_i t_{i+1} t_i^{-1}, t_i)$, while the move σ_i'' transforms (t_i, t_{i+1}) into $(t_{i+1}, t_{i+1}^{-1} t_i t_{i+1})$; see [Hurwitz 1891].

The braid moves ρ_{jk}' and ρ_{jk}'' fix all the λ_l, all the t_h with $h \ne j$ and all the μ_l with $l \ne k$. They modify t_j and μ_k. Analogously the braid moves τ_{jk}' and τ_{jk}'' modify t_j and λ_k, leaving unchanged μ_l for all l, λ_l with $l \ne k$ and t_h with $h \ne j$.

The braid moves $\rho_{jk}',\ \rho_{jk}'',\ \tau_{jk}'$ and τ_{jk}'' transform t_j to an element belonging to the same conjugate class (see Theorem 1.8, [Kanev 2004]).

By [Kanev 2004, Corollary 1.9], when $\lambda_1 = \cdots = \lambda_k = \mu_1 = \cdots = \mu_{k-1} = \mathrm{id}$, the braid move ρ_{1k}' transforms μ_k into $t_1^{-1} \mu_k$.

Analogously when $\lambda_1 = \cdots = \lambda_{k-1} = \mu_1 = \cdots = \mu_{k-1} = \mathrm{id}$, the braid move τ_{1k}'' transforms λ_k into $t_1^{-1} \lambda_k$.

Definition 4. Two Hurwitz systems with values in G are *braid-equivalent* if one is obtained from the other by a finite sequence of braid moves σ_i', ρ_{jk}', τ_{jk}', σ_i'', ρ_{jk}'', τ_{jk}'', where $1 \le i \le n-1$, $1 \le j \le n$ and $1 \le k \le g$. Two ordered sequences of elements of G, $(t_1, \dots, t_l)$ and $(t_1', \dots, t_l')$, are *braid-equivalent* if $(t_1', \dots, t_l')$ is obtained from $(t_1, \dots, t_l)$ by a finite sequence of braid moves of type σ_i', σ_i''. We denote braid equivalence by $\sim$.

2. The Hurwitz spaces $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ and $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$

Let X, X' and Y be smooth, connected, projective complex curves. Let $d \ge 3$, $n_1 > 0$ and $n_2 > 0$ be integers. Let $\underline{e} = (e_1, \dots, e_r)$ and $\underline{q} = (q_1, \dots, q_s)$ be two partitions of d with $e_1 \ge e_2 \ge \cdots \ge e_r \ge 1$ and $q_1 \ge q_2 \ge \cdots \ge q_s \ge 1$. Let b_0 be a point of Y and let g be the genus of Y. In this paper we study equivalence classes of pairs $[X \xrightarrow{\pi} X' \xrightarrow{f} Y, \phi]$ satisfying the following conditions:

(a) π is a degree-two coverings with n_1 branch points and branch locus D_π;

(b) f is a degree-d coverings with n_2 points of simple branching and two special points whose local monodromy has cycle type given by $\underline{e}$ and $\underline{q}$, respectively;

(c) the covering f has monodromy group S_d and $f(D_\pi) \cap D_f = \varnothing$ where D_f denotes the branch locus of f;

(d) $f \circ \pi$ is unramified in b_0 and $\phi : (f \circ \pi)^{-1}(b_0) \to \{-d, \dots, -1, 1, \dots, d\}$ is a bijection such that if $f^{-1}(b_0) = \{y_1, \dots, y_d\}$ then $\pi^{-1}(y_i) = \{\phi^{-1}(i), \phi^{-1}(-i)\}$ for each $i = 1, \dots, d$.

$H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ will denote the Hurwitz space that parametrizes equivalence classes of pairs $[f \circ \pi, \phi]$ satisfying conditions (a)–(d).

$H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$ will denote the Hurwitz space that parametrizes equivalence classes of coverings $f \circ \pi$ satisfying conditions (a)–(c).

Definition 5. A $(n_1, n_2, \underline{e}, \underline{q})$*-Hurwitz system* is a Hurwitz system with values in $\mathbb{Z}_2^d \times^s S_d$, $(t_1, \dots, t_{n_1+n_2+2}; \underline{\lambda}, \underline{\mu})$, such that n_1 of $t_1, \dots, t_{n_1+n_2+2}$ are of the form $(\underline{1}_*; \mathrm{id})$, n_2 are of the form $(z_{hk}; (hk))$, one is a product of r disjoint positive cycles whose lengths are given by the elements of the partition $\underline{e}$, and one is a product of s disjoint positive cycles whose lengths are given by the elements of the partition $\underline{q}$.

Let $D = f(D_\pi) \cup D_f$ and let $m : \pi_1(Y - D, b_0) \to S_{2d}$ be the monodromy homomorphism associated to $[f \circ \pi, \phi]$. Let $(\gamma_1, \dots, \gamma_{n_1+n_2+2}, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ be a standard generating system for $\pi_1(Y - D, b_0)$. The images under m of γ_1, $\dots, \gamma_{n_1+n_2+2}, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ determine an $(n_1, n_2, \underline{e}, \underline{q})$-Hurwitz system with monodromy group $W(B_d)$.

In the sequel we will denote by $A^o_{n_1,n_2,\underline{e},\underline{q},g}$ the set of all $(n_1, n_2, \underline{e}, \underline{q})$-Hurwitz systems with monodromy group $W(B_d)$. When $g = 0$ we will write $A^o_{n_1,n_2,\underline{e},\underline{q}}$ instead of $A^o_{n_1,n_2,\underline{e},\underline{q},g}$.

Let $\delta : H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0) \to (Y - b_0)^{(n_1+n_2+2)} - \Delta$ be the map that assigns to each pair $[f \circ \pi, \phi]$ the branch locus of $f \circ \pi$. By Riemann's existence theorem we can identify the fiber of δ over D with $A^o_{n_1,n_2,\underline{e},\underline{q},g}$. There is a unique topology on $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ such that δ is a topological covering map; see [Fulton 1969]. Therefore the braid group $\pi_1((Y - b_0)^{(n_1+n_2+2)} - \Delta, D)$ acts on $A^o_{n_1,n_2,\underline{e},\underline{q},g}$. If this action is transitive, $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ is connected and hence, since $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ is smooth, it is also irreducible.

Remark 6. The forgetful map $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0) \to H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$ defined by $[f \circ \pi, \phi] \to [f \circ \pi]$ is a morphism, whose image is a dense subset of $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$. This ensures that if $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ is irreducible also $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$ is irreducible.

3. The results

We denote by ϵ the following element in S_d having cycle type $\underline{e}$:

(1) $$(1\ 2\ \dots\ e_1)(e_1{+}1\ \dots\ e_1{+}e_2)\cdots((e_1{+}\cdots{+}e_{r-1}){+}1\ \dots\ d)\,.$$

We denote by ν the following element in S_d having cycle type $\underline{q}$:

(2) $$(1\ d\ d{-}1\ \dots\ d{-}q_1{+}2)(d{-}q_1{+}1\ \dots\ d{-}(q_1{+}q_2){+}2)\\ \cdots\ (d{-}(q_1{+}\cdots{+}q_{s-1}){+}1\ \dots 2).$$

Lemma 7. *Let $(t_1, \dots, t_i, t_{i+1}, \dots, t_l)$ be a sequence of permutations in S_d where t_i and t_{i+1} are two equal transpositions of S_d. Then we can move to the right and to the left the pair (t_i, t_{i+1}) leaving unchanged the other permutations of the sequence.*

Proof. Applying the elementary moves $\sigma''_{i-1}, \sigma''_i$ we obtain

$$(t_{i-1}, t_i, t_{i+1}) \sim (t_i, t_i^{-1}t_{i-1}t_i, t_{i+1}) \sim (t_i, t_{i+1}, t_{i-1});$$

applying the moves σ'_{i+1}, σ'_i we have

$$(t_i, t_{i+1}, t_{i+2}) \sim (t_i, t_{i+1}t_{i+2}t_{i+1}^{-1}, t_{i+1}) \sim (t_{i+2}, t_i, t_{i+1}).$$

Hence using sequences of elementary moves of type either $\sigma''_{j-1}, \sigma''_j$ or σ'_{j+1}, σ'_j we can move respectively on the left and on the right the pair (t_i, t_{i+1}), leaving unchanged the other permutations of the sequence. □

Lemma 8. *Let $(t_1, \dots, t_l, \tau, \tau)$ be a sequence of permutations of S_d, with τ a transposition. Let H be the subgroup of S_d generated by $t_1, \dots, t_l$. Then, for each $h \in H$, one has*

$$(t_1, \dots, t_l, \tau, \tau) \sim (t_1, \dots, t_l, h^{-1}\tau h, h^{-1}\tau h).$$

Proof. Let $h \in H$, then $h = h_1 h_2 \cdots h_k$ where h_i or h_i^{-1}, with $i = 1, \dots, k$, belonging to $\{t_1, \dots, t_l\}$. If h_1 is equal to t_j for some $j \in \{1, \dots, l\}$, we use Lemma 7 to bring the pair (τ, τ) to the left of t_j and then we act by the moves $\sigma''_{j+1}, \sigma''_j$ in order to replace (τ, τ, t_j) with $(t_j, t_j^{-1}\tau t_j, t_j^{-1}\tau t_j)$.

On the contrary, if h_1 is equal to t_j^{-1} for some $j \in \{1, \dots, l\}$, we use Lemma 7 to shift the pair (τ, τ) on the right of t_j and then we apply σ'_j, σ'_{j+1}. In this way we replace (t_j, τ, τ) with $(t_j \tau t_j^{-1}, t_j \tau t_j^{-1}, t_j)$.

For h_2 we reason as above but we bring the pair $(h_1^{-1}\tau h_1, h_1^{-1}\tau h_1)$ to the left or to the right of t_n depending on whether h_2 is equal to t_n or to t_n^{-1}.

Following this line for each h_i, with $i = 3, \dots, k$, we obtain the claim. □

Proposition 9 [Vetro 2010, Proposition 2]. *Let $\underline{t} = (t_1, \dots, t_{n_2+2})$ be a Hurwitz system of permutations of S_d with monodromy group S_d such that one of $t_1, \dots, t_{n_2+2}$ has cycle type $\underline{e}$, one has cycle type $\underline{q}$ and the other n_2 permutations in $t_1, \dots, t_{n_2+2}$ are transpositions. If $n_2 - s - r \geq d + 1$, $\underline{t}$ is braid-equivalent to the Hurwitz system*

$(\epsilon, \tilde{t}_2, \dots, \tilde{t}_{n_2+2-s}, \nu)$ *if* $s = 1$,

$(\epsilon, \tilde{t}_2, \dots, \tilde{t}_{n_2+2-s}, \nu, (1\ d - q_1 + 1), \dots, (1\ d - (q_1 + \cdots + q_{s-1}) + 1))$ *if* $s > 1$,

where ϵ and ν are the permutations defined in (1) *and* (2), *and where the sequence $(\tilde{t}_2, \dots, \tilde{t}_{n_2+2-s})$ is equal to*

$((1\,2), \dots, (1\,2))$ *if* $r = 1$,

$((1\ e_1 + 1), \dots, (1\ (e_1 + \cdots + e_{r-1}) + 1), (1\,2), \dots, (1\,2))$ *if* $r > 1$

with the transposition $(1\,2)$ *appearing an even number of times.*

Remark 10. Seeing that $d \geq 3$, the hypothesis $n_2 - s - r \geq d + 1$ ensures that in the sequence $(\tilde{t}_2, \dots, \tilde{t}_{n_2+2-s})$ there are more than 3 transpositions (12).

3.1. ***Irreducibility of $H_{W(B_d), n_1, n_2, \underline{e}, \underline{q}}(\mathbb{P}^1, b_0)$ and $H_{W(B_d), n_1, n_2, \underline{e}, \underline{q}}(\mathbb{P}^1)$.*** We next show that, if $n_2 - s - r \geq d + 1$, the braid group $\pi_1((\mathbb{P}^1 - b_0)^{(n_1+n_2+2)} - \Delta, D)$ acts transitively on $A^o_{n_1, n_2, \underline{e}, \underline{q}}$. To prove this we show that each $(n_1, n_2, \underline{e}, \underline{q})$-Hurwitz system in $A^o_{n_1, n_2, \underline{e}, \underline{q}}$ is braid-equivalent to a given normal form.

Proposition 11. *If $n_2 - s - r \geq d + 1$, each Hurwitz system in $A^o_{n_1, n_2, \underline{e}, \underline{q}}$ is braid-equivalent to a Hurwitz system of the form*

$\big(\tilde{t}_1, \dots, \tilde{t}_{n_2+2-s}, (0; \nu), (\bar{1}_1; \mathrm{id}), \dots, (\bar{1}_1; \mathrm{id})\big)$ *if* $s = 1$,

$$\Big(\tilde{t}_1, \dots, \tilde{t}_{n_2+2-s}, (0; \nu), (0; (1\ d - q_1 + 1)), \dots, \Big(0; \Big(1\ d - \sum_{h=1}^{s-1} q_h + 1\Big)\Big), (\bar{1}_1; \mathrm{id}), \dots, (\bar{1}_1; \mathrm{id})\Big) \quad \textit{if } s > 1,$$

where $(\bar{1}_1; \mathrm{id})$ *appears n_1 times and where $(\tilde{t}_1, \dots, \tilde{t}_{n_2+2-s})$ is the sequence*

$\big((0; \epsilon), (0; (1\ 2)), \dots, (0; (1\ 2))\big)$ *if* $r = 1$,

$$\big((0;\epsilon),(0;(1e_1+1)),\dots,\big(0;\big(1\sum_{i=1}^{r-1}e_i+1\big)\big),(0;(12)),\dots,(0;(12))\big)\ \textit{if}\ r>1,$$

with (0; (1 2)) *appearing an even number of times.*

Proof. Step 1. Let $\underline{t}\in A^{o}_{n_1,n_2,\underline{e},\underline{q}}$. We prove first that $\underline{t}$ is braid-equivalent to a Hurwitz system of either the form

$$\big(\dots,(0;\nu),(\bar{1}_1;\mathrm{id}),\dots,(\bar{1}_1;\mathrm{id})\big)$$

or the form

$$\big(\dots,(0;\nu),(0;(1\,d-q_1+1)),\dots,\big(0;\big(1\,d-\sum_{h=1}^{s-1}q_h+1\big)\big),(\bar{1}_1;\mathrm{id}),\dots,(\bar{1}_1;\mathrm{id})\big),$$

depending on whether $s=1$ or $s>1$, where $(\bar{1}_1;\mathrm{id})$ appears n_1 times.

Acting by elementary moves σ'_j we shift on the right the elements of the form $(\bar{1}_*;\mathrm{id})$ obtaining that $\underline{t}$ is braid-equivalent to

$$\big(\hat{t}_1,\dots,\hat{t}_{n_2+2},(\bar{1}_h;\mathrm{id}),\dots,(\bar{1}_k;\mathrm{id})\big),$$

where $\hat{t}_i=(*;t'_i)$. We notice that $(t'_1,\dots,t'_{n_2+2})$ is a Hurwitz system of permutations of S_d with monodromy group S_d such that one of $t'_1,\dots,t'_{n_2+2}$ has cycle type given by $\underline{e}$, one has cycle type given by $\underline{q}$ and the other n_2 permutations are transpositions. Since $n_2-s-r\geq d+1$, by Proposition 9, the system $(t'_1,\dots,t'_{n_2+2})$ is braid-equivalent to either

$$\big(\epsilon,\dots,(1\,2),\dots,(1\,2),(1\,2),(1\,2),\nu\big)$$

or

$$\big(\epsilon,\dots,(1\,2),\dots,(1\,2),(1\,2),(1\,2),\nu,(1\,d-q_1+1),\dots,\big(1\,d-\sum_{h=1}^{s-1}q_h+1\big)\big)$$

depending on whether $s=1$ or $s>1$.

We notice that from

$$\epsilon\cdots(1\,2)\cdots(1\,2)(1\,2)(1\,2)=(12\dots d)$$

it follows that the group generated by the permutations $\epsilon,\dots,(1\,2)$ is all of S_d. Hence, by Lemma 8, the sequence $(\epsilon,\dots,(1\,2),\dots,(1\,2),(1\,2),(1\,2))$ is braid-equivalent to a sequence of the form $(\epsilon,\dots,(1\,2),\dots,(1\,2),\tau,\tau)$, where τ is an arbitrary transposition of S_d.

This ensures that $\underline{t}$ is braid-equivalent to a system of type either

$$\big(\bar{t}_1,\dots,\bar{t}_{n_2+2-s},(b;\nu),(\bar{1}_h;\mathrm{id}),\dots\big)$$

or

$$\big(\bar{t}_1, \dots, \bar{t}_{n_2+2-s}, (b;\nu), (z^1_{1\,d-q_1+1}; (1\ d-q_1+1)), \dots, \\ \big(z^{s-1}_{1\,d-\sum_{h=1}^{s-1} q_h+1}; \big(1\ d-\sum_{h=1}^{s-1} q_h+1\big)\big), (\bar{1}_h;\mathrm{id}), \dots\big),$$

depending on whether $s = 1$ or $s > 1$, where $\bar{t}_i = (*; t''_i)$ and

$$(t''_1, \dots, t''_{n_2+2-s}) = (\epsilon, \dots, (12), \dots, (1\,2), \tau, \tau).$$

Furthermore we can affirm that our system is braid-equivalent to either

$$\big(\bar{t}_1, \dots, \check{t}_{n_2+2-s}, (\bar{1}_u;\mathrm{id}), (b;\nu), (\bar{1}_*;\mathrm{id}), \dots\big)$$

or

$$\big(\bar{t}_1, \dots, \check{t}_{n_2+2-s}, (\bar{1}_u;\mathrm{id}), (b;\nu), \dots, \\ \big(z^{s-1}_{1\,d-\sum_{h=1}^{s-1} q_h+1}; \big(1\ d-\sum_{h=1}^{s-1} q_h+1\big)\big), (\bar{1}_*;\mathrm{id}), \dots\big),$$

depending on whether $s = 1$ or $s > 1$, where u is an arbitrary index in $\{1, \dots, d\}$ and $\check{t}_{n_2+2-s} = (\star'; \tau)$.

In fact, acting by elementary moves of the form σ'_j we can bring to the left of $(b;\nu)$ one element of type $(\bar{1}_*;\mathrm{id})$. We choose $\tau = (u\ *)$ and then we act by σ'_{n_2+2-s} two times to replace $((\star;\tau), (\bar{1}_*;\mathrm{id}))$ by $((\star';\tau), (\bar{1}_u;\mathrm{id}))$.

Now we analyze separately the cases $s = 1$ and $s > 1$.

<u>*Case $s = 1$*</u>. Let $i_1,\ i_2, \dots, i_l$ be the indexes that b sends to $\bar{1}$. We suppose that $i_1 > i_2 > \cdots > i_{l-1} > i_l$. Since our system is braid-equivalent to

$$\big(\bar{t}_1, \dots, \bar{t}_{n_2}, \check{t}_{n_2+1}, (\bar{1}_{i_l};\mathrm{id}), (b;\nu), (\bar{1}_*;\mathrm{id}), \dots\big),$$

acting two times by the move σ'_{n_2+2} we can replace the pair $((\bar{1}_{i_l};\mathrm{id}), (b;\nu))$, with $((\bar{1}_{i_{l+1}};\mathrm{id}), (\hat{b};\nu))$ where $\hat{b}$ is a function that sends to $\bar{1}$ the indexes $i_1, i_2, \dots, i_{l-1}$, $i_l + 1$, where $i_l + 1$ is the index that precedes i_l in ν. Observe that if there are h indexes among i_{l-1} and i_l, it is sufficient to use the move σ'_{n_2+2} another $2h$ times, to replace the pair $((\bar{1}_{i_{l+1}};\mathrm{id}), (\hat{b};\nu))$ with $((\bar{1}_{i_{l-1}};\mathrm{id}), (\check{b};\nu))$ where $\check{b}$ is a function that sends to $\bar{1}$ the indexes $i_1, i_2, \dots, i_{l-2}$.

Since b is a function that sends to $\bar{1}$ an even number of indexes (see Definition 1), following this line we can replace the pair $((\bar{1}_*;\mathrm{id}), (\check{b};\nu))$ with $((\bar{1}_\star;\mathrm{id}), (0;\nu))$. Now, we use σ''_{n_2+2} to shift $(0;\nu)$ to the place $n_2 + 2$.

We notice that if all the elements of the form $(\bar{1}_*;\mathrm{id})$ in our system are equal to $(\bar{1}_1;\mathrm{id})$ we have the claim. Otherwise we place the elements $(\bar{1}_1;\mathrm{id})$ to the last places and then we act by σ'_{n_2+2} to bring one element of type $(\bar{1}_*;\mathrm{id})$ to the left of

$(0;\nu)$. By Lemma 8 and by using σ'_{n_2+1} two times, we can replace our system by a system of type

$$\big((*;\epsilon),\dots,(*;(1\,2)),(*;\tau'),(*;\tau'),(\bar{1}_2;\mathrm{id}),(0;\nu),(\bar{1}_*;\mathrm{id}),\dots\big).$$

Thus, acting by the elementary move σ''_{n_2+2}, we can replace the pair $((\bar{1}_2;\mathrm{id}),(0;\nu))$ with $((0;\nu),(\bar{1}_1;\mathrm{id}))$. Now, acting with elementary moves of type σ'_j, we bring $(\bar{1}_1;\mathrm{id})$ next to the other elements $(\bar{1}_1;\mathrm{id})$.

Reasoning in this way for each $(\bar{1}_*;\mathrm{id})$ such that $*\neq 1$ we obtain the claim.

Case $s>1$. Our system is braid-equivalent to a system of the form

$$\Big(\dots,\bar{t}_{n_2+1-s},\check{t}_{n_2+2-s},(\bar{1}_1;\mathrm{id}),(b;\nu),(z^1_{1\,d-q_1+1};(1\;d-q_1+1)),\dots,\\ \big(z^{s-1}_{1\,d-\sum_{h=1}^{s-1}q_h+1};\big(1\;d-\sum_{h=1}^{s-1}q_h+1\big)\big),(\bar{1}_*;\mathrm{id}),\dots\Big),$$

so if $z^{s-1}=\bar{1}$ we can use the moves $\sigma'_{n_2+3-s},\sigma'_{n_2+4-s},\dots,\sigma'_{n_2+1},\sigma'_{n_2+2}$ in order to replace it by

$$\Big(\dots,\check{t}_{n_2+2-s},(b';\nu),(\hat{z}^1_{1\,d-q_1+1};(1\;d-q_1+1)),\dots,\\ \big(0;\big(1\;d-\sum_{h=1}^{s-1}q_h+1\big)\big),(\bar{1}_1;\mathrm{id}),\dots\Big).$$

Since this system is braid-equivalent to a system of type

$$\Big((*;\epsilon),\dots,(*;(1\,2)),(*;\tau'),(*;\tau'),(\bar{1}_1;\mathrm{id}),(b';\nu),\\ (\hat{z}^1_{1\,d-q_1+1};(1\;d-q_1+1)),\dots,\big(0;\big(1\;d-\sum_{h=1}^{s-1}q_h+1\big)\big),\dots\Big),$$

we can reason as above for all the elements

$$(*;(1\;d-q_1+1)),\quad\dots,\quad\big(*;(1\;d-\sum_{h=1}^{s-2}q_h+1)\big)$$

such that $*$ is a function different from 0. In this way, after at most $s-2$ steps, we transform our system into

$$\Big(\dots,(\bar{1}_1;\mathrm{id}),(\hat{b};\nu),(0;(1\;d-q_1+1)),\dots,\big(0;\big(1\;d-\sum_{h=1}^{s-1}q_h+1\big)\big),\dots\Big).$$

Now if $\hat{b}\neq 0$, it is sufficient to proceed as in the case $s=1$ in order to obtain the system

$$\Big((*;\epsilon),\dots,(*;(1\,2)),(*;\tau),(*;\tau),(\bar{1}_*;\mathrm{id}),(0;\nu),\\ (0;(1\;d-q_1+1)),\dots,\big(0;\big(1\;d-\sum_{h=1}^{s-1}q_h+1\big)\big),\dots\Big).$$

Using elementary moves σ'_j, we move to the left of $(0; \nu)$ all the elements of type $(\bar{1}_*; \mathrm{id})$, so we replace our system with

$$\big(\dots, (*; \tau), (*; \tau), (\bar{1}_{h_1}; \mathrm{id}), \dots, (\bar{1}_{h_{n_1}}; \mathrm{id}), (0; \nu), \\ (0; (1\ d - q_1 + 1)), \dots, \big(0; \big(1\ d - \sum_{h=1}^{s-1} q_h + 1\big)\big)\big).$$

By Lemma 8 we can choose $\tau = (1\ h_1)$. We apply σ'_{n_2+2-s} two times in order to replace $(\bar{1}_{h_1}; \mathrm{id})$ with $(\bar{1}_1; \mathrm{id})$. Now we use elementary moves σ'_j to bring $(\bar{1}_1; \mathrm{id})$ next to $(0; \nu)$. We repeat this reasoning for all $(\bar{1}_{h_i}; \mathrm{id})$ such that $h_i \neq 1$. Since by the Hurwitz formula n_1 is even, we obtain the claim using the sequence of moves $\sigma'_{n_2+n_1+2-s}, \sigma'_{n_2+n_1+1-s}, \dots, \sigma'_{n_2+3-s}, \sigma'_{n_2+n_1+3-s}, \sigma'_{n_2+n_1+2-s}, \dots, \sigma'_{n_2+4-s}, \dots, \sigma'_{n_2+n_1+1}, \dots, \sigma'_{n_2+2}$.

Step 2. By Step 1 and by Lemma 8, $\underline{t}$ is braid-equivalent to either

$$\big((a; \epsilon), (z^1_{12}; (1\,2)), \dots, (z^l_{12}; (1\,2)), (0; \nu), \dots, (\bar{1}_1; \mathrm{id})\big)$$

or

$$\big((a; \epsilon), (v^1_{1e_1+1}; (1e_1+1)), \dots, \big(v^{r-1}_{1\sum_{i=1}^{r-1} e_i+1}; \big(1 \sum_{i=1}^{r-1} e_i + 1\big)\big), \\ (z^1_{12}; (1\,2)), \dots, (z^l_{12}; (1\,2)), (0; \nu), \dots, (\bar{1}_1; \mathrm{id})\big),$$

depending on whether $r = 1$ or $r > 1$. We analyze separately the two cases.

<u>*Case* $r = 1$</u>. From

$$(a; \epsilon)(z^1_{12}; (1\,2)) \cdots (z^l_{12}; (1\,2))(0; \nu) \cdots (\bar{1}_1; \mathrm{id}) = (0; \mathrm{id})$$

it follows that

$$a + z^1_{1\,d} + \cdots + z^l_{1\,d} + \bar{1}_1 + \cdots + \bar{1}_1 = 0.$$

Since in our system there are n_1 elements of type $(\bar{1}_1; \mathrm{id})$ and n_1 is even, by the Hurwitz formula we can affirm that a is either 0 or $\bar{1}_{1\,d}$ depending on whether the number of z^i equal to $\bar{1}$ is even or odd. Acting by moves of type σ'_j we move the elements of the form $(0; (1\,2))$ to the left of $(0; \nu)$. Successively, acting by sequences of moves of type $\sigma''_j, \sigma''_{j+1}$, we shift a pair of type $((\bar{1}_1; \mathrm{id}), (\bar{1}_1; \mathrm{id}))$ to the right of the elements $(\bar{1}_{12}; (1\,2))$.

If the function a is equal to 0 and the elements of type $(\bar{1}_{12}; (1\,2))$ are in the places $r+1, \dots, h$, it is sufficient to use the sequence of moves $\sigma''_h, \sigma''_{h-1}, \dots, \sigma''_{r+1}, \sigma''_{r+1}, \dots, \sigma''_h$ to obtain the system

$$\big((0; \epsilon), (0; (1\,2)), \dots, (0; (1\,2)), \\ (\bar{1}_1; \mathrm{id}), (\bar{1}_1; \mathrm{id}), (0; (1\,2)), \dots, (0; (1\,2)), (0; \nu), \dots\big).$$

The claim follows by using the sequence of moves $\sigma'_{h+2}, \sigma'_{h+1}, \dots, \sigma'_{n_2+3}, \sigma'_{n_2+2}$.

On the contrary, if $a = \bar{1}_{1\,d}$ and the elements of type $(\bar{1}_{12}; (1\,2))$ are in the places $r+1, \dots, h$, we use the sequence of moves $\sigma''_h, \sigma''_{h-1}, \dots, \sigma''_{r+2}, \sigma'_{r+1}$ to bring our system to the form

$$\big((\bar{1}_{1\,d}; \epsilon), (\bar{1}_2; \text{id}), (\bar{1}_{12}; (1\,2)), (0; (1\,2)), \dots, (0; (1\,2)),\\ (\bar{1}_1; \text{id}), (0; (1\,2)), \dots, (0; \nu), \dots\big).$$

We use σ'_1 to replace the pair $((\bar{1}_{1\,d}; \epsilon), (\bar{1}_2; \text{id}))$ with $((\bar{1}_1; \text{id}), (\bar{1}_{1\,d}; \epsilon))$ and then we apply the moves σ'_1, σ'_2 to replace $((\bar{1}_1; \text{id}), (\bar{1}_{1\,d}; \epsilon), (\bar{1}_{12}; (1\,2)))$ by

$$\big((0; \epsilon), (0; (1\,2)), (\bar{1}_1; \text{id})\big).$$

Now we obtain the claim acting by the sequence of elementary moves $\sigma''_{r+2}, \sigma''_{r+3}, \dots, \sigma''_h, \sigma'_{h+2}, \sigma'_{h+1}, \dots, \sigma'_{n_2+3}, \sigma'_{n_2+2}$.

Case $r > 1$. Seeing that

$$(a; \epsilon)(v^1_{1e_1+1}; (1e_1+1)) \cdots (z^1_{12}; (1\,2)) \cdots (0; \nu) \cdots (\bar{1}_1; \text{id}) = (0; \text{id}),$$

one has

$$a + v^1_{e_1(e_1+e_2)} + v^2_{(e_1+e_2)(e_1+e_2+e_3)} + \cdots + v^{r-1}_{(e_1+\cdots+e_{r-1})d} + z^1_{1\,d} + \cdots + \bar{1}_1 + \cdots + \bar{1}_1 = 0.$$

Since a is a function that sends to $\bar{1}$ at most an even number of indexes moved by every disjoint cycle of which is product ϵ, the equality above ensures that a is either 0 or $\bar{1}_{1e_1}$.

If $a = 0$, we have $v^1 = v^2 = \cdots = v^{r-1} = 0$. Furthermore there is an even number of z^i equal to $\bar{1}$. So in order to obtain the claim, it is sufficient to act as in the case $r = 1$ and $a = 0$.

On the contrary, if $a = \bar{1}_{1e_1}$ we have $v^1 = v^2 = \cdots = v^{r-1} = \bar{1}$; furthermore, there is an odd number of z^i equal to $\bar{1}$. Then we act as in the case $r = 1$ and $a = \bar{1}_{1\,d}$ to replace our system with the braid-equivalent system

$$\Big((\bar{1}_{1e_1}; \epsilon), \dots, \big(\bar{1}_{1\sum_{i=1}^{r-1} e_i+1}; \big(1 \sum_{i=1}^{r-1} e_i + 1\big)\big), (\bar{1}_2; \text{id}), (\bar{1}_{12}; (1\,2)),\\ (0; (1\,2)), \dots, (0; (1\,2)), (\bar{1}_1; \text{id}), (0; (1\,2)), \dots, (0; \nu), \dots\Big).$$

Using the moves $\sigma'_r, \sigma'_{r-1}, \dots, \sigma'_2, \sigma'_1$ we transform the sequence

$$\Big((\bar{1}_{1e_1}; \epsilon), \dots, \big(\bar{1}_{1\sum_{i=1}^{r-1} e_i+1}; (1 \sum_{i=1}^{r-1} e_i + 1)\big), (\bar{1}_2; \text{id}), (\bar{1}_{12}; (1\,2))\Big)$$

into

$$\Big((\bar{1}_1; \text{id}), (\bar{1}_{1e_1}; \epsilon), \dots, \big(\bar{1}_{1\sum_{i=1}^{r-1} e_i+1}; \big(1 \sum_{i=1}^{r-1} e_i + 1\big)\big), (\bar{1}_{12}; (1\,2))\Big).$$

Now in order to obtain the claim it is sufficient to act by the sequence of moves $\sigma'_1, \dots, \sigma'_r, \sigma'_{r+1}, \sigma''_{r+2}, \dots, \sigma''_h, \sigma'_{h+2}, \sigma'_{h+1}, \dots, \sigma'_{n_2+3}, \sigma'_{n_2+2}$. □

The following result is a direct consequence of Proposition 11.

Theorem 12. *If* $n_2 - s - r \geq d+1$, *the Hurwitz space* $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(\mathbb{P}^1, b_0)$ *is irreducible.*

Combining Theorem 12 and Remark 6, we derive the following result.

Corollary 13. *If* $n_2 - s - r \geq d+1$, *the Hurwitz space* $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(\mathbb{P}^1)$ *is irreducible.*

3.2. ***Irreducibility of*** $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ ***and*** $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y)$**.** Let Y be a smooth, connected, projective complex curve of genus $g \geq 1$.

Theorem 14. *If* $n_2 - s - r \geq d+1$, *the Hurwitz space* $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ *is irreducible.*

Proof. To prove the irreducibility of $H_{W(B_d),n_1,n_2,\underline{e},\underline{q}}(Y, b_0)$ it is sufficient to show that each $(n_1, n_2, \underline{e}, \underline{q})$-Hurwitz system in $A^o_{n_1,n_2,\underline{e},\underline{q},g}$ is braid-equivalent to a system of the form

$$(\hat{\underline{t}}; (0; \mathrm{id}), \dots, (0; \mathrm{id})).$$

In fact, $\hat{\underline{t}} \in A^o_{n_1,n_2,\underline{e},\underline{q}}$ and so the theorem follows by Proposition 11.

Let $(\underline{t}; \underline{\lambda}, \underline{\mu}) \in A^o_{n_1,n_2,\underline{e},\underline{q},g}$. Acting by elementary moves of type σ'_j we shift to the right the elements of the form $(\bar{1}_*; \mathrm{id})$ transforming our system into

$$\big(\tilde{t}_1, \dots, \tilde{t}_{n_2+2}, (\bar{1}_*; \mathrm{id}), \dots, (\bar{1}_*; \mathrm{id}); \lambda_1, \mu_1, \dots, \lambda_g, \mu_g\big),$$

where $\tilde{t}_i = (*; t'_i)$, $\lambda_k = (*; \lambda'_k)$ and $\mu_k = (*; \mu'_k)$.

We notice that $(t'_1, \dots, t'_{n_2+2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$ is the Hurwitz system of a covering of Y of degree $d \geq 3$, with monodromy group S_d and with n_2+2 branch points, n_2 of which are points of simple branching, one is a special point whose local monodromy is given by $\underline{e}$ and one is a special point whose local monodromy is given by $\underline{q}$.

Since $n_2 - s - r \geq d+1$, the Hurwitz space $H^o_{d,n_2,\underline{e},\underline{q}}(Y, b_0)$ is irreducible (see [Vetro 2010], Theorem 2) and then the Hurwitz system

$$(t'_1, \dots, t'_{n_2+2}; \lambda'_1, \mu'_1, \dots, \lambda'_g, \mu'_g)$$

is braid-equivalent to one of the form

$$(t''_1, \dots, t''_{n_2+2}; \mathrm{id}, \mathrm{id}, \dots, \mathrm{id}, \mathrm{id}).$$

Hence it follows that $(\underline{t}; \underline{\lambda}, \underline{\mu})$ is braid-equivalent to a system of type

$$(\bar{t}_1, \dots, \bar{t}_{n_2+2}, (\bar{1}_*; \mathrm{id}), \dots; (a_1; \mathrm{id}), (b_1; \mathrm{id}), \dots, (a_g; \mathrm{id}), (b_g; \mathrm{id})).$$

We notice that if $a_h = 0$ and $b_k = 0$ for each $1 \leq h, k \leq g$ the theorem follows by Proposition 11. So let $a_1 \neq 0$ and i be one of the indexes that a_1 sends to $\bar{1}$.

Since it is not restrictive to suppose that among the element of type $(\bar{1}_*; \mathrm{id})$ in our system there is $(\bar{1}_i; \mathrm{id})$ (see Step 1, Proposition 11), acting by elementary moves of type σ''_j we can transform our system into

$$\big((\bar{1}_i; \mathrm{id}), \dots; (a_1; \mathrm{id}), (b_1; \mathrm{id}), \dots, (a_g; \mathrm{id}), (b_g; \mathrm{id})\big).$$

Now we use the move τ''_{11} to replace $(a_1; \mathrm{id})$ with $(\bar{1}_i; \mathrm{id})(a_1; \mathrm{id})$, where $\bar{1}_i + a_1$ is a function that sends i to $\bar{0}$.

So reasoning for all the indexes that a_1 sends to $\bar{1}$, after a finite number of steps, we obtain a new Hurwitz system with $(0; \mathrm{id})$ at the place $(n_2 + n_1 + 3)$.

On the contrary, if $a_1 = 0$, $b_1 \neq 0$ and b_1 sends i to $\bar{1}$, we at first use elementary moves of type σ''_j to bring to the first place $(\bar{1}_i; \mathrm{id})$ and then we act by the braid move ρ'_{11} in order to transform $(b_1; \mathrm{id})$ into $(\bar{1}_i; \mathrm{id})(b_1; \mathrm{id})$ where the function $\bar{1}_i + b_1$ sends i to $\bar{0}$. Following this line for all the indexes that b_1 sent to $\bar{1}$, we can replace $(\bar{1}_i + b_1; \mathrm{id})$ by $(0; \mathrm{id})$.

We notice that if $a_k \neq 0$ and $a_l = b_l = 0$, for each $l \leq k - 1$, in order to obtain the claim one can reason in the same way but this time applying the braid move τ'_{1k}. Analogously if $b_k \neq 0$, $a_l = b_l = 0$, for each $l \leq k - 1$, and $a_k = 0$ one can apply the braid move ρ'_{1k} to transform $(b_k; \mathrm{id})$ into $(0; \mathrm{id})$. □

From Theorem 14 and Remark 6 we deduce the following result.

Corollary 15. *If $n_2 - s - r \geq d + 1$, the Hurwitz space $H_{W(B_d), n_1, n_2, \underline{e}, \underline{q}}(Y)$ is irreducible.*

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