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Let (M,π) be a Poisson manifold. A Poisson submanifold $P \subset M$ gives rise to a Lie algebroid $A_P \to P$. Formal deformations of π around P are controlled by certain cohomology groups associated to A_P. Assuming that these groups vanish, we prove that π is formally rigid around P; that is, any other Poisson structure on M, with the same first-order jet along P, is formally Poisson diffeomorphic to π. When P is a symplectic leaf, we find a list of criteria that are sufficient for these cohomological obstructions to vanish. In particular, we obtain a formal version of the normal form theorem for Poisson manifolds around symplectic leaves.

1. Introduction

A *Poisson bracket* on a manifold M is a Poisson algebra structure on the space of smooth functions on M, that is, a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the derivation property

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad \text{for all } f, g, h \in C^\infty(M). \tag{1}$$

Equivalently, it can be given by a bivector $\pi \in \mathfrak{X}^2(M)$ that satisfies $[\pi,\pi] = 0$. The two definitions are related by the formula

$$\langle \pi, df \wedge dg\rangle = \{f, g\} \quad \text{for all } f, g \in C^\infty(M).$$

An immersed submanifold $\iota : P \to M$ is called a *Poisson submanifold* of M if π is tangent to P. This ensures that $\pi_{|P}$ is a Poisson structure on P for which the restriction map

$$\iota^* : C^\infty(M) \to C^\infty(P)$$

is a Lie algebra homomorphism. We regard the Poisson algebra $(C^\infty(P), \{\cdot,\cdot\})$ as the 0th-order approximation of the Poisson structure on M. If P is embedded,

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then P is a Poisson submanifold if and only if its vanishing ideal

$$I(P) = \{f \in C^\infty(M) \mid \iota^*(f) = 0\}$$

is an ideal in the Lie algebra $(C^\infty(M), \{\cdot\,,\cdot\})$. Assuming that P is also closed[1], we have a canonical identification of Poisson algebras

$$(C^\infty(P), \{\cdot\,,\cdot\}) = (C^\infty(M)/I(P), \{\cdot\,,\cdot\}).$$

This gives a recipe for constructing higher-order approximations. For example, the first-order approximation fits into an exact sequence of Poisson algebras

$$0 \to (I(P)/I^2(P), \{\cdot\,,\cdot\}) \to (C^\infty(M)/I^2(P), \{\cdot\,,\cdot\}) \to (C^\infty(P), \{\cdot\,,\cdot\}) \to 0. \tag{2}$$

The Poisson algebra structures in this sequence depend only on $j^1_{|P}\pi$, the first jet of π along P. A better way to describe (2) is using the language of Lie algebroids. As explained in Section 2, the extension (2) gives rise to a Lie algebroid structure A_P on $T^*_P M$ that fits into a short exact sequence of Lie algebroids

$$0 \to TP^\circ \to A_P \to T^*P \to 0, \tag{3}$$

where $TP^\circ \subset T^*_P M = A_P$ is the annihilator of TP and T^*P is the cotangent Lie algebroid of $(P, \{\cdot\,,\cdot\})$. In particular, we obtain a representation of A_P on TP°, and thus also on its symmetric powers $\mathcal{S}^k(TP^\circ)$.

We study formal rigidity of Poisson structures around Poisson submanifolds. In general, deformation and rigidity problems in Poisson geometry are controlled by the Poisson cohomology groups $H^\bullet_\pi(M)$, which are the cohomology of the complex of multivector fields $(\mathfrak{X}^\bullet(M), d_\pi)$, where

$$d_\pi := [\pi, \cdot\,].$$

For a Poisson submanifold P, this d_π induces a differential on $\mathfrak{X}^\bullet(M)_{|P}$, the complex of multivector fields along M. The corresponding cohomology, denoted by $H^\bullet_\pi(M, P)$, is called the Poisson cohomology relative to P [Ginzburg and Lu 1992]. The formal rigidity of Poisson structures around Poisson submanifolds is controlled by a version of this cohomology with coefficients. Lie algebroids provide the right setting to make this precise; that is, the relative Poisson cohomology groups can be computed as the cohomology of the Lie algebroid A_P

$$H^\bullet_\pi(M, P) = H^\bullet(A_P),$$

[1] Since we study local properties of (M, π) around P, only the condition that P is embedded is essential; closeness can be achieved by replacing M with a tubular neighborhood of P.

and the cohomology groups of A_P with coefficients in $\mathcal{S}^k(TP^\circ)$, which we denote by $H^\bullet(A_P;\mathcal{S}^k(TP^\circ))$, control formal rigidity (see Section 2 for the definition of Lie algebroid cohomology).

Our main result is the following:

Theorem 1.1. *Let π_1 and π_2 be two Poisson structures on M, such that $P \subset M$ is an embedded Poisson submanifold for both, and such that they have the same first-order jet along P. If their common algebroid A_P has the property that*

$$H^2(A_P;\mathcal{S}^k(TP^\circ)) = 0 \quad \text{for all } k \geq 2,$$

then the two structures are formally Poisson diffeomorphic. More precisely, there exists a diffeomorphism

$$\psi : \mathcal{U} \to \mathcal{V},$$

with $d\psi_{|T_PM} = \mathrm{id}_{T_PM}$, where $\mathcal{U}$ and $\mathcal{V}$ are open neighborhoods of P, such that $\pi_{1|\mathcal{U}}$ and $\psi^*(\pi_{2|\mathcal{V}})$ have the same infinite jet along P:

$$j^\infty_{|P}(\pi_{1|\mathcal{U}}) = j^\infty_{|P}(\psi^*(\pi_{2|\mathcal{V}})).$$

Applying Theorem 1.1 to the linear Poisson structure on the dual of a compact, semisimple Lie algebra, we obtain the following result.

Corollary 1.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra of compact type and consider π_{lin} the linear Poisson structure on $\mathfrak{g}^*$. Let $\mathbb{S}(\mathfrak{g}) \subset \mathfrak{g}^*$ be the sphere in $\mathfrak{g}^*$ centered at 0, of radius 1 with respect to some invariant inner product. Then $\mathbb{S}(\mathfrak{g})$ is a Poisson submanifold, and any Poisson structure π_1 defined in some open neighborhood of $\mathbb{S}(\mathfrak{g})$, such that*

$$j^1_{|\mathbb{S}(\mathfrak{g})}(\pi_{\mathrm{lin}}) = j^1_{|\mathbb{S}(\mathfrak{g})}(\pi_1),$$

is formally Poisson diffeomorphic to π_{lin}.

The *symplectic leaves* of (M,π) are Poisson submanifolds of a special type. Recall that a Poisson manifold carries a canonical singular foliation whose leaves are the maximal integral submanifolds of the distribution $\pi^\sharp(T^*M)$. Such a leaf S has a natural symplectic structure given by $\omega_S := \pi_{|S}^{-1}$. If $(S,\omega_S) \subset (M,\pi)$ is an embedded symplectic leaf, then the Lie algebroid extension (3) — which encodes only the first-order jet π along S — can be used to construct a second Poisson structure π^1_S, called the *first-order approximation* of π around S, defined on some open neighborhood of S and having the same first jet as π along S.

In [Crainic and Mărcuţ 2010] we obtained a normal form theorem for Poisson structures around symplectic leaves: we proved that, under some assumptions on the first jet of π along S, the Poisson structures π and π^1_S are Poisson diffeomorphic around S. Our goal is to give a formal version of this result, which we state below in its most general form (observe that it is a direct consequence of Theorem 1.1).

Theorem 1.3. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf. If the cohomology groups*

$$H^2(A_S, \mathcal{S}^k(TS^\circ))$$

vanish for all $k \geq 2$, then π is formally Poisson diffeomorphic to its first-order approximation around S.

In many cases we show that these cohomological obstructions vanish, and we obtain the following corollaries.

Corollary 1.4. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf. Assume that the Poisson homotopy cover of S is a smooth principal bundle with vanishing second de Rham cohomology group, and that its structure group G satisfies*

$$H^2_{\mathrm{diff}}(G, \mathcal{S}^k(\mathfrak{g})) = 0 \quad \text{for all } k \geq 2,$$

where $\mathfrak{g}$ is the Lie algebra of G, and $H^\bullet_{\mathrm{diff}}(G, \mathcal{S}^k(\mathfrak{g}))$ denotes the differentiable cohomology of G with coefficients in the k-th symmetric power of the adjoint representation. Then π is formally Poisson diffeomorphic to its first-order approximation around S.

Since the differentiable cohomology of compact groups vanishes, we obtain the following immediate corollary.

Corollary 1.5. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf. If the Poisson homotopy cover of S is a smooth principal bundle with vanishing second de Rham cohomology group and compact structure group, then π is formally Poisson diffeomorphic to its first-order approximation around S.*

The next consequence is bit more technical:

Corollary 1.6. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf whose isotropy Lie algebra is reductive. If the abelianization algebroid*

$$A_S^{\mathrm{ab}} := A_S/[TS^\circ, TS^\circ]$$

is integrable by a simply connected principal bundle with compact structure group and vanishing second de Rham cohomology group, then π is formally Poisson diffeomorphic to its first-order approximation around S.

Corollary 1.7. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf through $x \in M$. If the isotropy Lie algebra at x is semisimple, $\pi_1(S, x)$ is finite, and $\pi_2(S, x)$ is torsion, then π is formally Poisson diffeomorphic to its first-order approximation around S.*

Some related results. The first-order approximation of a Poisson manifold (M, π) around a one-point leaf x (a zero of π) is the linear Lie–Poisson structure on $\mathfrak{g}_x^*$, the dual of the isotropy Lie algebra at x. Formal linearization in this setup was proven by Weinstein [1983] for semisimple $\mathfrak{g}_x$. This case is also covered by our Corollary 1.7. Under the stronger assumption that $\mathfrak{g}_x$ is semisimple of compact type, Conn [1985] proved that a neighborhood of x is in fact Poisson diffeomorphic to an open neighborhood of 0 in the local model $\mathfrak{g}_x^*$.

Vorobjev [2001] constructed the first-order approximation around arbitrary symplectic leaves (see [Crainic and Mărcuţ 2010] for a more geometrical approach).

A weaker version of our Theorem 1.1 — of which we became aware only at the end of this research — was stated by Itskov et al. [1998]. They work around compact symplectic leaves instead of embedded Poisson submanifolds, proving that for each k, there exists a diffeomorphism that identifies the Poisson structures up to order k [Itskov et al. 1998, Theorem 7.1]. Compactness of the leaf is nevertheless too strong an assumption for formal equivalence. For example, they conclude in their Corollary 7.4 that hypotheses similar to those in our Corollary 1.7 imply the vanishing of the cohomology groups $H^2(A_S, \mathcal{S}^k(TS^\circ))$, but also remark that compactness of the leaf is incompatible with these assumptions (it forces S to be a point).

To prove Theorem 1.1, we reduce it to a result on the equivalence of Maurer–Cartan elements in complete graded Lie algebras, which we prove in the Appendix. The same criteria for equivalence of Maurer–Cartan elements, but in the context of differential graded algebras, can be found in [Abad et al. 2010, Appendix A].

To prove the vanishing of the cohomological obstructions, and the corollaries listed above, we use techniques such as Whitehead's Lemma for semisimple Lie algebras and spectral sequences for Lie algebroids, but also the more powerful techniques developed in [Crainic 2003], such as the Van Est map and vanishing of cohomology of proper groupoids.

Theorem [Crainic and Mărcuţ 2010, main result]. *Let (M, π) be a Poisson manifold and $S \subset M$ an embedded symplectic leaf*; *π is Poisson diffeomorphic to its first-order approximation around S if the following conditions are satisfied*:

- *the Poisson homotopy cover P of S is smooth*;
- $H^2_{\mathrm{dR}}(P) = 0$;
- *the structure group of P is compact*;
- *S is compact.*

The first three conditions are the hypotheses of Corollary 1.5. So, giving up on compactness of the leaf, we still conclude that π and its first-order approximation are *formally* Poisson diffeomorphic. Nevertheless, the conditions of Corollary 1.5

are too strong in the formal setting; they force the semisimple part of the isotropy Lie algebra to be compact. Thus we consider the more technical Corollary 1.6 to be the correct analog in the formal category of the normal form theorem in [Crainic and Mărcuţ 2010]. In fact, Corollary 1.5 is a consequence of Corollary 1.6; it is precisely the case when the semisimple part of the isotropy Lie algebra is compact.

2. The first-order data

We recall some definitions; for more on Lie algebroids, see [Mackenzie 1987].

Definitions 2.1. A *Lie algebroid* over a manifold B is a vector bundle $\mathcal{A} \to B$ endowed with a Lie bracket $[\cdot,\cdot]$ on its space of sections $\Gamma(\mathcal{A})$ and a vector bundle map $\rho : \mathcal{A} \to TB$, called the *anchor*, which satisfy the Leibniz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + L_{\rho(\alpha)}(f)\beta \quad \text{for all } f \in C^\infty(B),\ \alpha, \beta \in \Gamma(\mathcal{A}).$$

A *representation* of $\mathcal{A}$ is a vector bundle $E \to B$ endowed with a bilinear map

$$\nabla : \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E),$$

satisfying

$$\nabla_{f\alpha}(s) = f\nabla_\alpha(s), \quad \nabla_\alpha(fs) = f\nabla_\alpha(s) + L_{\rho(\alpha)}(f)s,$$

and the flatness condition

$$\nabla_\alpha\nabla_\beta(s) - \nabla_\beta\nabla_\alpha(s) = \nabla_{[\alpha,\beta]}(s).$$

The *cohomology of a Lie algebroid* $(\mathcal{A}, [\cdot,\cdot], \rho)$ *with coefficients in a representation* (E, ∇) is defined by the complex $\Omega^\bullet(\mathcal{A}, E) := \Gamma(\Lambda^\bullet\mathcal{A}^* \otimes E)$ with differential given by the classical Koszul formula:

$$\begin{aligned} d_\nabla\omega(\alpha_0, \dots, \alpha_q) = &\sum_i (-1)^i \nabla_{\alpha_i}\big(\omega(\alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_q)\big) \\ &+ \sum_{i<j} (-1)^{i+j}\omega\big([\alpha_i, \alpha_j], \dots, \widehat{\alpha}_i, \dots, \widehat{\alpha}_j, \dots, \alpha_q\big). \end{aligned}$$

The corresponding cohomology groups are denoted by $H^\bullet(\mathcal{A}, E)$.

To a Poisson manifold (M, π) one can associate a Lie algebroid structure on the cotangent bundle T^*M, with anchor given by π viewed as a bundle map

$$\pi^\sharp : T^*M \to TM$$

and bracket uniquely determined by

$$[df, dg] := d\{f, g\} \quad \text{for all } f, g \in C^\infty(M);$$

see [Vaisman 1994] for details.

Let $P \subset M$ be an embedded Poisson submanifold. Since π is tangent to P, it is easy to see that the algebroid structure can be restricted to P, in the sense that there is a unique Lie algebroid structure on $A_P := T^*_P M$ with anchor $\pi^\sharp_{|P}$ and bracket such that the restriction map $\Gamma(T^*M) \to \Gamma(A_P)$ is a Lie algebra homomorphism. The dual of the inclusion $TP \subset T_P M$ gives a map $A_P \to T^*P$ that is a Lie algebroid homomorphism, where T^*P is the cotangent Lie algebroid of $(P, \pi_{|P})$. This way we obtain the extension of Lie algebroids from the introduction:

$$0 \to (TP^\circ, [\cdot\,,\cdot]) \to (A_P, [\cdot\,,\cdot]) \to (T^*P, [\cdot\,,\cdot]) \to 0. \tag{4}$$

This short exact sequence implies that TP° is an ideal in $(A_P, [\cdot\,,\cdot])$; therefore

$$\nabla : \Gamma(A_P) \times \Gamma(TP^\circ) \to \Gamma(TP^\circ), \quad \nabla_\alpha(\eta) := [\alpha, \eta]$$

defines a representation of A_P on TP°, and thus on its symmetric powers $\mathcal{S}^k(TP^\circ)$. The resulting cohomology groups are the obstructions appearing in Theorems 1.1 and 1.3. The Lie algebroid structures on A_P and the sequence (4) depend only on the first jet of π along P (that is, the brackets and anchors can be expressed in terms of $\pi_{|P}$ and the first-order derivatives of π restricted to P).

Remark 2.2. We regard the Lie algebroid A_P as the first-order approximation of the Poisson bracket at P. To justify this interpretation, fix a Poisson structure π_P on P, where $P \subset M$ is a closed embedded submanifold. Then there is a one-to-one correspondence between Poisson algebra structures on the commutative algebra $C^\infty(M)/I^2(P)$, which fit into the short exact sequence

$$\begin{aligned} 0 \to (I(P)/I^2(P), \{\cdot\,,\cdot\}) \\ \to (C^\infty(M)/I^2(P), \{\cdot\,,\cdot\}) \to (C^\infty(P), \{\cdot\,,\cdot\}) \to 0, \end{aligned} \tag{5}$$

and Lie algebroid structures on $A_P := T^*_P M$, which fit into a sequence of the form

$$0 \to (TP^\circ, [\cdot\,,\cdot]) \to (A_P, [\cdot\,,\cdot]) \to (T^*P, [\cdot\,,\cdot]) \to 0. \tag{6}$$

The exterior derivative induces a map

$$d : C^\infty(M)/I^2(P) \to \Gamma(A_P),$$

and the correspondence between the brackets is uniquely determined by the fact that this is a Lie algebra homomorphism.

Example 2.3. Consider $P := \mathbb{R}^2$ as the submanifold $\{z = 0\} \subset M := \mathbb{R}^3$. We construct a first-order extension of the trivial Poisson structure on P to M, that is, a Poisson algebra structure on the commutative algebra

$$C^\infty(M)/I^2(P) = C^\infty(M)/(z^2) = C^\infty(P) \oplus zC^\infty(P)$$

with the property that $\{f, g\} \in (z)$, for all $f, g \in C^\infty(M)/(z^2)$. Explicitly, define

$$\{f, g\} = z\Big(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} + x\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - x\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\Big) \mod (z^2).$$

A straightforward computation yields that $\{\cdot, \cdot\}$ satisfies the Jacobi identity, and therefore we have an extension of Poisson algebras

$$0 \to zC^\infty(P) \to C^\infty(P) \oplus zC^\infty(P) \to C^\infty(P) \to 0,$$

where the Poisson bracket on P is zero. The total space of the corresponding Lie algebroid A_P is $\mathbb{R}^3 \times P \to P$. The bracket is given on the global frame $dx_{|P}$, $dy_{|P}$, $dz_{|P}$ by

$$[dx_{|P}, dy_{|P}] = dz_{|P}, \quad [dy_{|P}, dz_{|P}] = 0, \quad [dx_{|P}, dz_{|P}] = x dz_{|P},$$

and extended bilinearly to all sections, since the anchor is zero.

Nevertheless, there is no Poisson structure on M (nor on any open neighborhood of P) that has this Poisson algebra as its first-order approximation. Assume, to the contrary, that on some open neighborhood $\mathcal{U}$ of P such a Poisson structure exists. Then it must have the form

$$\{x, y\} = z + z^2 h, \quad \{y, z\} = z^2 k, \quad \{x, z\} = xz + z^2 l,$$

for some smooth functions h, k, l defined on $\mathcal{U}$. Computing the Jacobiator of x, y, and z, we obtain

$$J = \{x, \{y, z\}\} + \{z, \{x, y\}\} + \{y, \{z, x\}\} = z^2\big((2-x)k(x, y, 0) + 1\big) + z^3 a(x, y, z),$$

where a is a smooth function. In particular, we see that J cannot vanish, since

$$\frac{\partial^2 J}{\partial z^2}(2, y, 0) = 2 \neq 0.$$

This example shows that not everything that looks like the first jet of a Poisson structure around P (that is, an extension of the form (6) or (5)) comes from an actual Poisson structure.

On the other hand, if P is a symplectic manifold, the situation changes for the better; every "first jet" of a Poisson structure can be extended to a Poisson structure around P. More precisely, consider (S, ω_S) a symplectic manifold, with $S \subset M$ embedded, and an algebroid structure on $A_S := T^*_S M$ that fits into the exact sequence

$$0 \to TS^\circ \to A_S \to T^*S \to 0.$$

Then, using a tubular neighborhood $\mathscr{E} : T_S M / TS \to M$, one can construct a Poisson structure $\pi^1_S = \pi^1_S(A_S, \omega_S, \mathscr{E})$ on some open neighborhood of S, from which we recover the first-order data: it has (S, ω_S) as a symplectic leaf, and the algebroid structure induced on $T^*_S M$ is A_S. This Poisson structure was first constructed by Vorobjev [2001]; we also recommend [Crainic and Mărcuţ 2010] for some different approaches. Applied to different tubular neighborhoods, this construction produces Poisson structures which, when restricted to small enough neighborhoods of S, are Poisson diffeomorphic [Vorobjev 2001]. So the isomorphism class of the germ around S of π^1_S doesn't depend on $\mathscr{E}$.

We can view the whole story from a different perspective; start with a Poisson structure π on M, for which (S, ω_S) is an embedded symplectic leaf, and denote as usual by A_S the Lie algebroid on $T^*_S M$. For $\mathscr{E}$ a tubular neighborhood of S, we call $\pi^1_S = \pi^1_S(A_S, \omega_S, \mathscr{E})$ the *first-order approximation* of π around S. The first-order approximation is defined on some open neighborhood of S in M, and it plays the role of a local normal form for π around S.

3. The formal equivalence theorem

The algebra of formal vector fields. Take the graded Lie algebra $(\mathfrak{X}^\bullet(M), [\cdot,\cdot])$ of multivector fields on M, with the Nijenhuis–Schouten bracket and $\deg(W) = k-1$ for $W \in \mathfrak{X}^k(M)$. For a closed, embedded submanifold $P \subset M$, denote by $\mathfrak{X}^\bullet_P(M)$ the following subalgebra of multivector fields tangent to P:

$$\mathfrak{X}^\bullet_P(M) := \{u \in \mathfrak{X}^\bullet(M) \mid u_{|P} \in \mathfrak{X}^\bullet(P)\}.$$

The vanishing ideal $I(P) \subset C^\infty(M)$ of P induces a filtration $\mathscr{F}$ on $\mathfrak{X}^\bullet_P(M)$:

$$\mathfrak{X}^\bullet_P(M) \supset \mathscr{F}^\bullet_0 \supset \mathscr{F}^\bullet_1 \supset \dots \mathscr{F}^\bullet_k \supset \mathscr{F}^\bullet_{k+1} \supset \dots,$$

$$\mathscr{F}^\bullet_k = I^{k+1}(P)\mathfrak{X}^\bullet(M), \quad k \geq 0.$$

It is readily checked that

$$[\mathscr{F}_k, \mathscr{F}_l] \subset \mathscr{F}_{k+l}, \quad [\mathfrak{X}^\bullet_P(M), \mathscr{F}_k] \subset \mathscr{F}_k. \tag{7}$$

Let $\hat{\mathfrak{X}}^\bullet_P(M)$ be the completion of $\mathfrak{X}^\bullet_P(M)$ with respect to the filtration $\mathscr{F}$, defined by the projective limit

$$\hat{\mathfrak{X}}^\bullet_P(M) := \varprojlim \mathfrak{X}^\bullet_P(M)/\mathscr{F}^\bullet_k.$$

By (7), it follows that $\hat{\mathfrak{X}}^\bullet_P(M)$ inherits a graded Lie algebra structure, such that, for $k \geq 0$, the natural maps

$$j^k_{|P} : \hat{\mathfrak{X}}^\bullet_P(M) \to \mathfrak{X}^\bullet_P(M)/\mathscr{F}^\bullet_k$$

are Lie algebra homomorphisms. The algebra $(\hat{\mathfrak{X}}^\bullet_P(M), [\cdot,\cdot])$ is called the *algebra of formal multivector fields along P*. Consider also the homomorphism

$$j^\infty_{|P} : \mathfrak{X}^\bullet_P(M) \to \hat{\mathfrak{X}}^\bullet_P(M).$$

From a version of Borel's Theorem (see, for example, [Moerdijk and Reyes 1991]) about the existence of a smooth section with a specified infinite jet along a submanifold, it follows that $j^{\infty}_{|P}$ is surjective. Observe that $\hat{\mathfrak{X}}^{\bullet}_{P}(M)$ inherits a filtration $\hat{\mathcal{F}}$ from $\mathfrak{X}^{\bullet}_{P}(M)$, given by

$$\hat{\mathcal{F}}^{\bullet}_{k} = j^{\infty}_{|P}\mathcal{F}^{\bullet}_{k},$$

that satisfies the corresponding equations (7).

The adjoint action of an element $X \in \hat{\mathcal{F}}^{1}_{1}$

$$\mathrm{ad}_X : \hat{\mathfrak{X}}^{\bullet}_{P}(M) \to \hat{\mathfrak{X}}^{\bullet}_{P}(M), \quad \mathrm{ad}_X(Y) := [X, Y]$$

increases the degree of the filtration by 1. Therefore the partial sums

$$\sum_{i=0}^{n} \frac{\mathrm{ad}^{i}_{X}}{i!}(Y)$$

are constant modulo $\hat{\mathcal{F}}_k$ for $n \geq k$ and all $Y \in \hat{\mathfrak{X}}^{\bullet}_{P}(M)$. This and the completeness of the filtration on $\hat{\mathcal{F}}$ show that the exponential of ad_X

$$e^{\mathrm{ad}_X} : \hat{\mathfrak{X}}^{\bullet}_{P}(M) \to \hat{\mathfrak{X}}^{\bullet}_{P}(M), \quad e^{\mathrm{ad}_X}(Y) := \sum_{n\geq 0} \frac{\mathrm{ad}^{n}_{X}}{n!}(Y)$$

is well-defined. It is readily checked that e^{ad_X} is a graded Lie algebra isomorphism with inverse $e^{-\mathrm{ad}_X}$ and that it preserves the filtration. We need the following geometric interpretation of these isomorphisms.

Lemma 3.1. *For every $X \in \hat{\mathcal{F}}^{1}_{1}$, there exists $\psi : M \to M$ a diffeomorphism of M, with $\psi_{|P} = \mathrm{id}_P$ and $d\psi_{|P} = \mathrm{id}_{T_PM}$, such that for every $W \in \mathfrak{X}^{\bullet}_{P}(M)$, we have*

$$j^{\infty}_{|P}(\psi^{*}(W)) = e^{\mathrm{ad}_X}(j^{\infty}_{|P}(W)).$$

Proof. By Borel's Theorem, there is a vector field V on M such that $X = j^{\infty}_{|P}(V)$. We claim that V can be chosen to be complete. Let g be a complete metric on M and let $\phi : M \to [0, 1]$ be a smooth function that satisfies $\phi = 1$ on the set $\{x \mid g_x(V_x, V_x) \leq \frac{1}{2}\}$ and $\phi = 0$ on the set $\{x \mid g_x(V_x, V_x) \geq 1\}$. Since $V_{|P} = 0$, it follows that ϕV has the same germ as V around P, and therefore $j^{\infty}_{|P}(\phi V) = X$. On the other hand, since ϕV is bounded, it is complete, so replace V by ϕV.

We show that $\psi := \mathrm{Fl}_V$, the flow of V at time 1, satisfies all requirements. Since $j^{1}_{|P}(V) = 0$, it is clear that $\psi_{|P} = \mathrm{id}_P$ and $d\psi_{|P} = \mathrm{id}_{T_PM}$.

Consider $W \in \mathfrak{X}^{\bullet}_{P}(M)$, and denote by $W_s := \mathrm{Fl}^{*}_{sV}(W)$ the pullback of W by the flow of V at time s. Since W_s satisfies the differential equation $dW_s/ds = [V, W_s]$, a simple computation gives

$$\frac{d}{ds}\Big(\sum_{i=0}^{k}\frac{(-s)^i\,\mathrm{ad}_V^i}{i!}(W_s)\Big)=\frac{(-s)^k\,\mathrm{ad}_V^{k+1}}{k!}(W_s).$$

This shows that the sum

$$\sum_{i=0}^{k}\frac{(-s)^i\,\mathrm{ad}_V^i}{i!}(W_s)$$

modulo $\mathcal{F}_{k+1}$ is independent of s, and therefore

$$W-\sum_{i=0}^{k}\frac{(-1)^i\,\mathrm{ad}_V^i}{i!}(\psi^*(W))\in\mathcal{F}_{k+1}.$$

Applying $j^\infty_{|P}$ to this equation yields

$$j^\infty_{|P}(W)-\sum_{i=0}^{k}\frac{(-1)^i\,\mathrm{ad}_X^i}{i!}j^\infty_{|P}(\psi^*(W))\in\hat{\mathcal{F}}_{k+1},$$

and hence we conclude

$$j^\infty_{|P}(W)=e^{-\mathrm{ad}_X}j^\infty_{|P}(\psi^*(W)).\qquad\square$$

The cohomology of the restricted algebroid. Let (M,π) be a Poisson manifold and $P\subset M$ a closed, embedded Poisson submanifold. The cohomologies we are considering are all versions of the Poisson cohomology $H^\bullet_\pi(M)$, computed by the complex $\mathfrak{X}^\bullet(M)$ of multivector fields on M and differential $d_\pi=[\pi,\cdot\,]$. Since P is a Poisson submanifold, we have that $[\pi, I(P)\mathfrak{X}^\bullet(M)]\subset I(P)\mathfrak{X}^\bullet(M)$, and more generally, it follows that $I^k(P)\mathfrak{X}^\bullet(M)$ forms a subcomplex. Taking consecutive quotients, we obtain the complexes

$$\big(I^k(P)\mathfrak{X}^\bullet(M)/I^{k+1}(P)\mathfrak{X}^\bullet(M),\,d^k_\pi\big),$$

with differential d^k_π induced by $[\pi,\cdot\,]$. For $k=0$, we obtain the Poisson cohomology relative to P. Observe that the differential on these complexes depends only on the first jet of π along P, and therefore, following the philosophy of Section 2, it can be described only in terms of the algebroid A_P.

Proposition 3.2. *The following two complexes are isomorphic*:

$$\big(I^k(P)\mathfrak{X}^\bullet(M)/I^{k+1}(P)\mathfrak{X}^\bullet(M),\,d^k_\pi\big)\cong\big(\Omega^\bullet(A_P,\mathcal{S}^k(TP^\circ)),\,d_{\nabla^k}\big)\quad\textit{for all } k\geq 0.$$

Proof. Since the space of sections of TP° is spanned by differentials of elements in $I(P)$, it is easy to see that the map given by

$$\tau_k: I^k(P)\mathfrak{X}^\bullet(M)\to\Omega^\bullet(A_P,\mathcal{S}^k(TP^\circ))=\Gamma(\Lambda^\bullet(T_PM)\otimes\mathcal{S}^k(TP^\circ)),$$

$$\tau_k(f_1 \dots f_k W) = W_{|P} \otimes df_{1|P} \odot \cdots \odot df_{k|P},$$

where $f_1, \dots, f_k \in I(P)$ and $W \in \mathfrak{X}^\bullet(M)$, is well-defined and surjective. Also, its kernel is precisely $I^{k+1}(P)\mathfrak{X}^\bullet(M)$. Hence, it remains to prove that

$$\tau_k([\pi, W]) = d_{\nabla^k}(\tau_k(W)) \quad \text{for all } W \in I^k(P)\mathfrak{X}^\bullet(M). \tag{8}$$

Recall that the algebroid A_P has anchor $\rho = \pi^\sharp_{|P}$ and bracket determined by

$$[d\phi_{|P}, d\psi_{|P}]_P := d\{\phi, \psi\}_{|P} \quad \text{for all } \phi, \psi \in C^\infty(M).$$

Also, for $k = 0$, we have that ∇^0 is given by

$$\nabla^0 : \Gamma(A_P) \times C^\infty(P) \to C^\infty(P), \quad \nabla^0_\eta(h) = L_{\rho(\eta)}(h).$$

Since both differentials d_π and d_{∇^k} act by derivations and ∇^k is obtained by extending ∇^1 by derivations, it suffices to prove (8) for $\phi \in C^\infty(M)$ and $X \in \mathfrak{X}^1(M)$ (with $k = 0$), and for $f \in I(P)$ (with $k = 1$).

Let $\phi \in C^\infty(M)$ and $\eta \in \Gamma(A_P)$. Since π is tangent to P, we obtain

$$\tau_0([\pi, \phi])(\eta) = [\pi, \phi]_{|P}(\eta) = d\phi_{|P}(\pi^\sharp_{|P}(\eta)) = L_{\rho(\eta)}(\tau_0(\phi)) = d_{\nabla^0}(\tau_0(\phi))(\eta).$$

Let $X \in \mathfrak{X}^1(M)$ and $\phi, \psi \in C^\infty(M)$, and define $\eta := d\phi_{|P}$ for $\theta := d\psi_{|P} \in \Gamma(A_P)$. Then

$$\begin{aligned}
\tau_0([\pi, X])(\eta, \theta) &= [\pi, X]_{|P}(d\phi_{|P}, d\psi_{|P}) \\
&= \big(\{X(\phi), \psi\} + \{\phi, X(\psi)\} - X(\{\phi, \psi\})\big)_{|P} \\
&= \pi^\sharp_{|P}(d\phi_{|P})(X_{|P}(d\psi_{|P})) \\
&\qquad - \pi^\sharp_{|P}(d\psi_{|P})(X_{|P}(d\phi_{|P})) - X_{|P}(d\{\phi, \psi\}_{|P}) \\
&= L_{\rho(\eta)}(\tau_0(X)(\theta)) - L_{\rho(\theta)}(\tau_0(X)(\eta)) - \tau_0(X)([\eta, \theta]_P) \\
&= d_{\nabla^0}(\tau_0(X))(\eta, \theta),
\end{aligned}$$

and thus (8) holds for X.

Consider now $f \in I(P)$ and $\eta := d\phi_{|P} \in \Gamma(A_P)$, with $\phi \in C^\infty(M)$. The formula defining τ_k implies that for every $W \in I^k(P)\mathfrak{X}^\bullet(M)$, we have

$$\tau_k(i_{d\phi}(W)) = i_{d\phi_{|P}}\tau_k(W).$$

Using this, the following computation finishes the proof:

$$\begin{aligned}
\tau_1([\pi, f])(\eta) &= \tau_1([\pi, f](d\phi)) = \tau_1(\{\phi, f\}) = d\{\phi, f\}_{|P} \\
&= [\eta, df_{|P}]_P = \nabla^1_\eta(\tau(f)) = d_{\nabla^1}(\tau(f))(\eta).
\end{aligned}$$

□

Proof of Theorem 1.1. By replacing M with a tubular neighborhood of P, we can assume that P is closed in M. Write

$$\gamma := j^\infty_{|P}\pi_1, \quad \gamma' := j^\infty_{|P}\pi_2 \in \hat{\mathfrak{X}}^2_P(M).$$

By Proposition 3.2, we can recast the hypothesis as

$$[\gamma,\gamma]=0, \quad [\gamma',\gamma']=0, \quad \gamma-\gamma' \in \hat{\mathscr{F}}_1, \quad H^2(\hat{\mathscr{F}}^\bullet_k/\hat{\mathscr{F}}^\bullet_{k+1}, d_\gamma)=0,$$

for all $k \geq 1$, where $d_\gamma := \mathrm{ad}_\gamma$. All these conditions are expressed in terms of the graded Lie algebra $\mathscr{L}^\bullet := \hat{\mathfrak{X}}^{\bullet+1}_P(M)$, with a complete filtration $\hat{\mathscr{F}}$. Theorem A.5 in the Appendix shows that there exists a formal vector field $X \in \hat{\mathscr{F}}^1_1$ such that $\gamma = e^{\mathrm{ad}_X}(\gamma')$. By Lemma 3.1, there exists a diffeomorphism ψ of M, such that $j^\infty_{|P}(\psi^*(W)) = e^{\mathrm{ad}_X} j^\infty_{|P}(W)$, for all $W \in \mathfrak{X}^\bullet_P(M)$. This concludes the proof, since

$$j^\infty_{|P}(\psi^*(\pi_2)) = e^{\mathrm{ad}_X} j^\infty_{|P}(\pi_2) = e^{\mathrm{ad}_X}(\gamma') = \gamma = j^\infty_{|P}(\pi_1).$$

Existence of Poisson structures with a specified infinite jet. This proof can be applied to obtain a result on existence of Poisson bivectors with a specified infinite jet. Let S be a closed embedded submanifold of M. An element $\hat{\pi} \in \hat{\mathfrak{X}}^2_S(M)$, satisfying $[\hat{\pi},\hat{\pi}]=0$, is called a formal Poisson bivector. Observe that

$$\hat{\pi}_{|S} := \hat{\pi} \mod \hat{\mathscr{F}}_0 \in \mathfrak{X}^2(S)$$

is a Poisson structure on S. We call S a symplectic leaf on $\hat{\pi}$ if $\hat{\pi}_{|S}$ is nondegenerate. Assuming that S is a symplectic leaf of $\hat{\pi}$, by the discussion in Section 2, the first jet of $\hat{\pi}$,

$$j^1_{|S}(\hat{\pi}) = \hat{\pi} \mod \hat{\mathscr{F}}_1,$$

determines a Lie algebroid A_S on $T^*_S M$, and thus can be used to construct a Poisson bivector π^1_S on some open neighborhood $\mathscr{U}$ of S, whose first jet coincides with that of $\hat{\pi}$. If the cohomology groups

$$H^2(A_S; \mathscr{S}^k(TS^\circ))$$

vanish for all $k \geq 2$, then by the proof of Theorem 1.1, there exists a formal vector field $X \in \hat{\mathscr{F}}^1_1$ such that $e^{\mathrm{ad}_X}(j^\infty_{|S}\pi^1_S) = \hat{\pi}$. By Lemma 3.1, we find a diffeomorphism $\psi : \mathscr{U} \to \mathscr{U}$ such that

$$j^\infty_{|S}(\psi^*(\pi^1_S)) = e^{\mathrm{ad}_X}(j^\infty_{|S}\pi^1_S) = \hat{\pi}.$$

Thus $\pi := \psi^*(\pi^1_S)$ gives a Poisson structure defined on an open neighborhood of S whose infinite jet is $\hat{\pi}$. Hence we have proved the following statement.

Corollary 3.3. *Let $\hat{\pi} \in \hat{\mathfrak{X}}^2_S(M)$ be a formal Poisson structure for which S is a symplectic leaf. If for any $k \geq 2$, the algebroid A_S induced by $j^1_{|S}\hat{\pi}$ satisfies*

$$H^2(A_S; \mathcal{S}^k(TS^\circ)) = 0,$$

then there exists a Poisson structure π defined on some open neighborhood of S such that $\hat{\pi} = j^\infty_{|S}\pi$.

4. Proofs of the criteria

Here we explain and prove the corollaries from the Introduction.

Integration of Lie algebroids and differentiable cohomology. We recall some properties of Lie groupoids and Lie algebroids; see [Mackenzie 1987; Moerdijk and Mrčun 2003] for the general theory. A Lie groupoid over a manifold B is denoted by $\mathcal{G}$, the source and target maps by $s, t : \mathcal{G} \to B$, and the unit map by $u : B \to \mathcal{G}$. To a Lie groupoid $\mathcal{G}$ one can associate a Lie algebroid $A(\mathcal{G}) \to B$, which is the infinitesimal counterpart of $\mathcal{G}$. A Lie algebroid $\mathcal{A}$ is called *integrable* if $\mathcal{A} \cong A(\mathcal{G})$ for some Lie groupoid $\mathcal{G}$. The relation between Lie algebroids and Lie groupoids is similar to that between Lie algebras and Lie groups, the most significant difference being that not every Lie algebroid is integrable.

Recall that a *transitive Lie algebroid* is a Lie algebroid $\mathcal{A} \to B$ with surjective anchor. For example, if $S \subset M$ is a symplectic leaf of a Poisson manifold (M, π), then the Lie algebroid A_S is transitive. A Lie groupoid $\mathcal{G}$ is called *transitive* if the map $(s, t) : \mathcal{G} \to M \times M$ is a surjective submersion. The Lie algebroid of a transitive Lie groupoid is transitive. Conversely, if the base B of a transitive Lie algebroid $\mathcal{A}$ is connected, and $\mathcal{A}$ is integrable, then any Lie groupoid $\mathcal{G}$ integrating it is transitive. Every transitive Lie groupoid is a *gauge groupoid*; that is, it is of the form $P \times_G P$, where G is a Lie group and $p : P \to B$ is a principal G-bundle. For P one can take any s-fiber $s^{-1}(x)$ of $\mathcal{G}$ for $x \in B$, and $G := s^{-1}(x) \cap t^{-1}(x)$. We can recover $\mathcal{A}$ from P as follows: as a bundle $\mathcal{A} = TP/G$, the Lie bracket is induced by the identification

$$\Gamma(\mathcal{A}) = \mathfrak{X}(P)^G,$$

and the anchor is given by dp. We will also say, about a principle G-bundle P for which $A \approxeq TP/G$, that it *integrates* A. As for Lie algebras, if a transitive Lie algebroid with connected base is integrable, then, up to isomorphism, there exists a unique 1-connected principal bundle integrating it [Mackenzie 1987].

Let $S \subset M$ be a symplectic leaf of a Poisson manifold (M, π), and assume that the transitive algebroid A_S is integrable. The connected and simply connected principal bundle $P \to S$ for which $P \times_G P$ integrates A_S is called *the Poisson homotopy cover* of S. We say that P is smooth if A_S is integrable; this terminology is justified by the fact that P exists also in the nonintegrable case as a topological principal bundle over S [Crainic and Fernandes 2003].

Let $\mathcal{A}$ be a transitive Lie algebroid with connected base space B, and denote by $\mathfrak{g} \subset \mathcal{A}$ the kernel of the anchor. On each fiber of $\mathfrak{g}$, the Lie bracket restricts to a Lie algebra structure $(\mathfrak{g}_x, [\cdot, \cdot]_x)$, and this Lie algebra is called *the isotropy Lie algebra at* x. In the integrable case, when $\mathcal{A} = A(\mathcal{G})$, the isotropy Lie algebra coincides with the Lie algebra of the *isotropy group* $G_x := s^{-1}(x) \cap t^{-1}(x)$. In the case of a symplectic leaf $S \subset M$ of a Poisson manifold, the kernel of the anchor of the Lie algebroid A_S is given by $\mathfrak{g} := TS^\circ$.

A Lie groupoid $\mathcal{G}$ is called *proper* if $(s, t) : \mathcal{G} \to B \times B$ is a proper map.

A *representation* of a Lie groupoid $\mathcal{G}$ over B is a vector bundle $E \to B$ and a smooth linear action $g : E_x \to E_y$ for every arrow $g : x \to y$ satisfying the obvious identities. A representation E of $\mathcal{G}$ can be differentiated to a representation of its Lie algebroid $A(\mathcal{G})$ on the same vector bundle E. If the s-fibers of $\mathcal{G}$ are connected and simply connected, then every representation of $A(\mathcal{G})$ comes from a representation of $\mathcal{G}$ [Crainic and Fernandes 2003, Proposition 2.2], and in our applications this is usually the case.

The *differentiable cohomology* of a Lie groupoid $\mathcal{G}$ with coefficients in a representation $E \to B$ is computed by the complex $\mathcal{C}^p_{\text{diff}}(\mathcal{G}; E)$ of smooth maps $c : \mathcal{G}^{(p)} \to E$, where

$$\mathcal{G}^{(p)} := \{(g_1, \dots, g_p) \in \mathcal{G}^p \mid s(g_i) = t(g_{i+1}), i = 1, \dots, p-1\}$$

with $c(g_1, \dots, g_p) \in E_{t(g_1)}$, and with differential given by

$$\begin{aligned} dc(g_1, \dots, g_{p+1}) &= g_1 c(g_2, \dots, g_{p+1}) \\ &\quad + \sum_{i=1}^{p} (-1)^i c(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1}) + (-1)^{p+1} c(g_1, \dots, g_p). \end{aligned}$$

The resulting cohomology groups are denoted $H^\bullet_{\text{diff}}(\mathcal{G}, E)$. For more details on this subject, see [Haefliger 1979].

In the following proposition we list some results from [Crainic 2003] that are needed in the proofs of the corollaries from the Introduction.

Proposition 4.1. *Let $\mathcal{G}$ be a Lie groupoid over B with Lie algebroid $\mathcal{A}$, and let $E \to B$ be a representation of $\mathcal{G}$.*

(1) *If the s-fibers of $\mathcal{G}$ are cohomologically 2-connected, then*

$$H^2(\mathcal{A}; E) \cong H^2_{\text{diff}}(\mathcal{G}; E).$$

(2) *If $\mathcal{G}$ is proper, then $H^2_{\text{diff}}(\mathcal{G}; E) = 0$.*

(3) *If $\mathcal{G}$ is transitive, then $H^2_{\text{diff}}(\mathcal{G}; E) \cong H^2_{\text{diff}}(\mathcal{G}_x; E_x)$, where $x \in B$ and $\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x)$.*

Proof. (1) is a particular case of [Crainic 2003, Theorem 4], and (2) follows from [Crainic 2003, Proposition 1]. Since $\mathcal{G}$ is transitive, it is Morita equivalent to $\mathcal{G}_x$ [Moerdijk and Mrčun 2003]; by [Crainic 2003, Theorem 1], a Morita equivalence induces an isomorphism in cohomology, and this proves (3). □

Proof of Corollary 1.2. Recall that the cotangent Lie algebroid of $(\mathfrak{g}^*, \pi_{\text{lin}})$ is isomorphic to the action Lie algebroid $\mathfrak{g} \ltimes \mathfrak{g}^* \to \mathfrak{g}^*$ for the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$, and that it is integrable by the action groupoid $G \ltimes \mathfrak{g}^*$, where G denotes the compact, connected and simply connected Lie group of $\mathfrak{g}$. Also, the symplectic leaves of $(\mathfrak{g}^*, \pi_{\text{lin}})$ are the orbits of the action of G. So, because $\mathbb{S}(\mathfrak{g})$ is G-invariant, it is a union of symplectic leaves, and therefore a Poisson submanifold. The algebroid $A_{\mathbb{S}(\mathfrak{g})}$ is isomorphic to the action algebroid $\mathfrak{g} \ltimes \mathbb{S}(\mathfrak{g})$, and therefore it is integrable by the action groupoid $G \ltimes \mathbb{S}(\mathfrak{g})$. Since G is simply connected, it follows that $H^2_{dR}(G) = 0$ [Duistermaat and Kolk 2000, Theorem 1.14.2]. On the other hand, all s-fibers of $G \ltimes \mathbb{S}(\mathfrak{g})$ are diffeomorphic to G, and so the assumptions of Proposition 4.1(1) are satisfied, and therefore, for any representation $E \to \mathbb{S}(\mathfrak{g})$ of $G \ltimes \mathbb{S}(\mathfrak{g})$, we have

$$H^2(\mathfrak{g} \ltimes \mathbb{S}(\mathfrak{g});\, E) \cong H^2_{\text{diff}}(G \ltimes \mathbb{S}(\mathfrak{g});\, E).$$

Since $G \ltimes \mathbb{S}(\mathfrak{g})$ is compact, it is proper, and hence by Proposition 4.1(2), we have $H^2_{\text{diff}}(G \ltimes \mathbb{S}(\mathfrak{g});\, E) = 0$ for every representation E. Now the corollary follows from Theorem 1.1. □

Proof of Corollary 1.4. Denote by P the Poisson homotopy cover of S with structure group G. By hypothesis, P is smooth, simply connected and with vanishing second de Rham cohomology group. Let $\mathcal{G} := P \times_G P$ be the gauge groupoid of P. Since every s-fiber of $\mathcal{G}$ is diffeomorphic to P, $\mathcal{G}$ satisfies the assumptions of Proposition 4.1(1), and therefore

$$H^2(A_S;\, \mathcal{S}^k(TS^\circ)) \cong H^2_{\text{diff}}(\mathcal{G};\, \mathcal{S}^k(TS^\circ)).$$

Since $\mathcal{G}$ is transitive, by Proposition 4.1(3), we have

$$H^2_{\text{diff}}(\mathcal{G};\, \mathcal{S}^k(TS^\circ)) \cong H^2_{\text{diff}}(G;\, \mathcal{S}^k(T_xS^\circ)).$$

Since $T_xS^\circ \cong \mathfrak{g}$ as G representations (both integrate the adjoint representation of $\mathfrak{g}$), the proof follows from Theorem 1.3. □

Proof of Corollary 1.5. This follows from Corollary 1.4, because the differentiable cohomology of compact Lie groups vanishes, by Proposition 4.1(2). □

Proof of Corollary 1.6. Let $x \in S$, and denote by $\mathfrak{g}_x := T_xS^\circ$ the isotropy Lie algebra of the transitive algebroid A_S. By hypothesis, $\mathfrak{g}_x$ is reductive; that is, it splits as a direct product of a semisimple Lie algebra and its center $\mathfrak{g}_x = \mathfrak{s}_x \oplus \mathfrak{z}_x$, where $\mathfrak{s}_x = [\mathfrak{g}_x, \mathfrak{g}_x]$ and $\mathfrak{z}_x = Z(\mathfrak{g}_x)$ is the center of $\mathfrak{g}_x$. Since $\mathfrak{g} = TS^\circ$ is a Lie algebra bundle, it follows that this splitting is in fact global:

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g}) = \mathfrak{s} \oplus \mathfrak{z}.$$

Since $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of A_S, we obtain a short exact sequence of algebroids

$$0 \to \mathfrak{s} \to A_S \to A_S^{\mathrm{ab}} \to 0,$$

with $A_S^{\mathrm{ab}} = A_S/[\mathfrak{g}, \mathfrak{g}]$. Similar to the spectral sequence for Lie algebra extensions [Hochschild and Serre 1953], there is a spectral sequence for extensions of Lie algebroids [Mackenzie 1987, Theorem 5.5 and the remark following it], which in our case converges to $H^\bullet(A_S; \mathcal{S}^k(\mathfrak{g}))$, with

$$E_2^{p,q} = H^p(A_S^{\mathrm{ab}}; H^q(\mathfrak{s}; \mathcal{S}^k(\mathfrak{g}))) \Rightarrow H^{p+q}(A_S; \mathcal{S}^k(\mathfrak{g})).$$

Since $\mathfrak{s}$ is in the kernel of the anchor, $H^q(\mathfrak{s}; \mathcal{S}^k(\mathfrak{g}))$ is indeed a vector bundle, with fiber $H^q(\mathfrak{s}; \mathcal{S}^k(\mathfrak{g}))_x = H^q(\mathfrak{s}_x; \mathcal{S}^k(\mathfrak{g}_x))$, and it inherits a representation of A_S^{ab}. Since $\mathfrak{s}_x$ is semisimple, by the Whitehead Lemma we have that $H^1(\mathfrak{s}_x; \mathcal{S}^k(\mathfrak{g}_x)) = 0$ and $H^2(\mathfrak{s}_x; \mathcal{S}^k(\mathfrak{g}_x)) = 0$. Therefore,

$$H^2(A_S; \mathcal{S}^k(\mathfrak{g})) \cong H^2(A_S^{\mathrm{ab}}; \mathcal{S}^k(\mathfrak{g})^{\mathfrak{s}}), \tag{9}$$

where $\mathcal{S}^k(\mathfrak{g}_x)^{\mathfrak{s}_x}$ is the $\mathfrak{s}_x$-invariant part of $\mathcal{S}^k(\mathfrak{g}_x)$. By hypothesis, A_S^{ab} is integrable by a principle bundle P^{ab} that is simply connected and that has vanishing second de Rham cohomology and compact structure group T. Therefore, by (9) and by applying Proposition 4.1(1), (2) and (3), we obtain that

$$\begin{aligned} H^2(A_S; \mathcal{S}^k(\mathfrak{g})) &\cong H^2(A_S^{\mathrm{ab}}; \mathcal{S}^k(\mathfrak{g})^{\mathfrak{s}}) \cong H^2_{\mathrm{diff}}(P^{\mathrm{ab}} \times_T P^{\mathrm{ab}}; \mathcal{S}^k(\mathfrak{g})^{\mathfrak{s}}) \\ &\cong H^2_{\mathrm{diff}}(T; \mathcal{S}^k(\mathfrak{g}_x)^{\mathfrak{s}_x}) = 0. \end{aligned}$$

Theorem 1.3 finishes the proof. □

Proof of Corollary 1.7. Assume that $\mathfrak{g}_x$ is semisimple, $\pi_1(S, x)$ is finite, and $\pi_2(S, x)$ is a torsion group. With the notation above, we have $A_S^{\mathrm{ab}} \cong TS$. Also, TS is integrable, and the simply connected principal bundle integrating it is $\widetilde{S}$, the universal cover of S. Finiteness of $\pi_1(S)$ is equivalent to compactness of the structure group of $\widetilde{S}$. By the Hurewicz theorem, we have $H_2(\widetilde{S}, \mathbb{Z}) \cong \pi_2(\widetilde{S})$, and since $\pi_2(\widetilde{S}) = \pi_2(S)$ is torsion, we have $H^2_{\mathrm{dR}}(\widetilde{S}) = 0$. So the result follows from Corollary 1.6. □

Appendix: Equivalence of MC-elements in complete GLAs

Here we discuss some general facts about graded Lie algebras endowed with a complete filtration, with the aim of proving a criterion for equivalence of Maurer–Cartan elements (Theorem A.5), which is used in the proof of Theorem 1.1. Some of the constructions given here can be also found in [Bursztyn et al. 2009, Appendix B.1] in the more general setting of differential graded Lie algebras with a complete filtration. In fact, all our constructions can be adapted to this setup, including in particular Theorem A.5. The analog of Theorem A.5 in the case of differential graded associative algebras is in [Abad et al. 2010, Appendix A].

Definitions A.1. (1) A *graded Lie algebra* $(\mathscr{L}^\bullet, [\cdot,\cdot])$ (or GLA) consists of a $\mathbb{Z}$-graded vector space $\mathscr{L}^\bullet$ endowed with a graded bracket $[\cdot,\cdot] : \mathscr{L}^p \times \mathscr{L}^q \to \mathscr{L}^{p+q}$, which is graded commutative and satisfies the graded Jacobi identity:

$$[X, Y] = -(-1)^{|X||Y|}[Y, X], \quad [X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]].$$

(2) An element $\gamma \in \mathscr{L}^1$ satisfying $[\gamma, \gamma] = 0$ is called a *Maurer–Cartan element*.

(3) A *filtration* on a GLA is a decreasing sequence of homogeneous subspaces

$$\mathscr{L}^\bullet \supset \mathscr{F}_0\mathscr{L}^\bullet \supset \cdots \supset \mathscr{F}_n\mathscr{L}^\bullet \supset \mathscr{F}_{n+1}\mathscr{L}^\bullet \supset \cdots$$

satisfying

$$[\mathscr{F}_n\mathscr{L}, \mathscr{F}_m\mathscr{L}] \subset \mathscr{F}_{n+m}\mathscr{L}, \quad [\mathscr{L}, \mathscr{F}_n\mathscr{L}] \subset \mathscr{F}_n\mathscr{L}.$$

(4) A filtration $\mathscr{F}\mathscr{L}$ is called *complete* if $\mathscr{L}$ is isomorphic to the projective limit $\varprojlim \mathscr{L}/\mathscr{F}_n\mathscr{L}$.

An example of a GLA with a complete filtration appeared in Section 3: starting from a manifold M and a closed embedded submanifold $P \subset M$, we constructed $(\hat{\mathfrak{X}}_P^{\bullet+1}(M), [\cdot,\cdot])$, the algebra of formal vector fields along P, with filtration given by the powers of the vanishing ideal of P. So, the index of the filtration plays the role of the order to which elements vanish along P.

For a general GLA with a complete filtration $\mathscr{F}\mathscr{L}$, define the *order* of an element as follows:

$$\mathbb{O} : \mathscr{L} \to \{0, 1, \dots, \infty\},$$

$$\mathbb{O}(X) = \begin{cases} 0 & \text{if } X \in \mathscr{L}\backslash\mathscr{F}_1\mathscr{L}, \\ n & \text{if } X \in \mathscr{F}_n\mathscr{L}\backslash\mathscr{F}_{n+1}\mathscr{L}, \\ \infty & \text{if } X = 0. \end{cases}$$

The order has the following properties, which follow from those of the filtration:

- $\mathbb{O}(X) = \infty$ if and only if $X = 0$,
- $\mathbb{O}(X + Y) \geq \mathbb{O}(X) \wedge \mathbb{O}(Y)$[2],
- $\mathbb{O}(\alpha X) \geq \mathbb{O}(X)$ for all $\alpha \in \mathbb{R}$,
- $\mathbb{O}([X, Y]) \geq \mathbb{O}(X) + \mathbb{O}(Y)$.

[2] $u \wedge v$ denotes $\min\{u, v\}$.

The completeness assumption on the filtration implies the following property:

Lemma A.2. *Let $\{X_n\}_{n\geq 0} \in \mathcal{L}$ be a sequence of elements such that*

$$\lim_{n\to\infty} \mathbb{O}(X_n) = \infty.$$

Then there exists a unique element $X \in \mathcal{L}$, denoted $X := \sum_{n\geq 0} X_n$, such that

$$X - \sum_{k=0}^{n} X_k \in \mathcal{F}_m\mathcal{L},$$

for all n big enough.

Note that $\mathfrak{g}(\mathcal{L}) := \mathcal{F}_1\mathcal{L}^0$ forms a Lie subalgebra of $\mathcal{L}^0$. Elements $X \in \mathfrak{g}(\mathcal{L})$ satisfy $\mathbb{O}(\mathrm{ad}_X(Y)) \geq \mathbb{O}(Y) + 1$ for all $Y \in \mathcal{L}$, and therefore, by Lemma A.2, the exponential of ad_X is well defined, and it is a GLA-automorphism of $\mathcal{L}^\bullet$, written

$$\mathrm{Ad}(e^X) : \mathcal{L}^\bullet \to \mathcal{L}^\bullet, \quad \mathrm{Ad}(e^X)Y := e^{\mathrm{ad}_X}(Y) = \sum_{n\geq 0} \frac{\mathrm{ad}_X^n}{n!}(Y).$$

By Lemma A.2, the Campbell–Hausdorff formula converges for all $X, Y \in \mathfrak{g}(\mathcal{L})$:

$$X * Y = X + Y + \sum_{k\geq 1} \frac{(-1)^k}{k+1} D_k(X, Y), \tag{10}$$

where

$$D_k(X, Y) = \sum_{l_i+m_i>0} \frac{\mathrm{ad}_X^{l_1}}{l_1!} \circ \frac{\mathrm{ad}_Y^{m_1}}{m_1!} \circ \dots \circ \frac{\mathrm{ad}_X^{l_k}}{l_k!} \circ \frac{\mathrm{ad}_Y^{m_k}}{m_k!}(X).$$

We use the notation $\mathcal{G}(\mathcal{L}) = \{e^X \mid X \in \mathfrak{g}(\mathcal{L})\}$; that is, $\mathcal{G}(\mathcal{L})$ is the same space as $\mathfrak{g}(\mathcal{L})$, but we denote its elements by e^X. The universal properties of the Campbell–Hausdorff formula (10) imply that $\mathcal{G}(\mathcal{L})$ endowed with the product $e^X e^Y = e^{X*Y}$ forms a group. Also, *Ad* gives an action of $\mathcal{G}(\mathcal{L})$ on $\mathcal{L}$ by graded Lie algebra automorphisms, which preserves the order:

- $\mathrm{Ad}(e^{X*Y}) = \mathrm{Ad}(e^X e^Y) = \mathrm{Ad}(e^X) \circ \mathrm{Ad}(e^Y)$,
- $\mathrm{Ad}(e^X)([U, V]) = [\mathrm{Ad}(e^X)U, \mathrm{Ad}(e^X)V]$,
- $\mathbb{O}(\mathrm{Ad}(e^X)(U)) = \mathbb{O}(U)$,

for all $X, Y \in \mathfrak{g}(\mathcal{L})$ and all $U, V \in \mathcal{L}$.

For later use, we give the following straightforward estimates:

Lemma A.3. *For all $X, Y, X', Y' \in \mathfrak{g}(\mathcal{L})$ and $U \in \mathcal{L}$, we have*

(1) $\mathbb{O}(X * Y - X' * Y') \geq \mathbb{O}(X - X') \wedge \mathbb{O}(Y - Y')$ *and*

(2) $\mathbb{O}(\mathrm{Ad}(e^X)U - \mathrm{Ad}(e^Y)U) \geq \mathbb{O}(X - Y)$.

Let γ be an MC-element. Notice that $[\gamma, \gamma] = 0$ implies that $d_\gamma := \mathrm{ad}_\gamma$ is a differential on $\mathscr{L}^\bullet$. The fact that $\mathscr{F}_k\mathscr{L}$ are ideals implies that $(\mathscr{F}_k\mathscr{L}^\bullet, d_\gamma)$ are subcomplexes of $(\mathscr{L}^\bullet, d_\gamma)$. The induced differential on the consecutive complexes depends only on γ modulo $\mathscr{F}_1$, and their cohomology groups are denoted

$$H^n_\gamma(\mathscr{F}_k\mathscr{L}^\bullet/\mathscr{F}_{k+1}\mathscr{L}^\bullet).$$

$\mathrm{Ad}(e^X)\gamma$ is again an MC-element for $e^X \in \mathscr{G}(\mathscr{L})$, and we call γ and $\mathrm{Ad}(e^X)\gamma$ *gauge equivalent*. The next Lemma gives a linear approximation of the action $\mathscr{G}(\mathscr{L})$ on MC-elements.

Lemma A.4. *For γ an MC-element and $e^X \in \mathscr{G}(\mathscr{L})$, we have*

$$\mathbb{O}(\mathrm{Ad}(e^X)\gamma - \gamma + d_\gamma X) \geq 2\mathbb{O}(X).$$

We have the following criterion for gauge equivalence.

Theorem A.5. *Let $(\mathscr{L}^\bullet, [\cdot, \cdot])$ be a GLA with a complete filtration $\mathscr{F}_n\mathscr{L}$, and let γ, γ' be two Maurer–Cartan elements. If $\mathbb{O}(\gamma - \gamma') \geq 1$, and if for all $q \geq \mathbb{O}(\gamma - \gamma')$ we have*

$$H^1_\gamma(\mathscr{F}_q\mathscr{L}^\bullet/\mathscr{F}_{q+1}\mathscr{L}^\bullet) = 0,$$

then γ and γ' are gauge equivalent; that is, there exists an element $e^X \in \mathscr{G}(\mathscr{L})$ such that $\gamma = \mathrm{Ad}(e^X)\gamma'$.

Proof. Define $p := \mathbb{O}(\gamma - \gamma')$. By hypothesis, for $q \geq p$, we can find homotopy operators

$$h^q_1 : \mathscr{F}_q\mathscr{L}^1 \to \mathscr{F}_q\mathscr{L}^0 \quad \text{and} \quad h^q_2 : \mathscr{F}_q\mathscr{L}^2 \to \mathscr{F}_q\mathscr{L}^1$$

such that $h^q_1(\mathscr{F}_{q+1}\mathscr{L}^1) \subset \mathscr{F}_{q+1}\mathscr{L}^0$, $h^q_2(\mathscr{F}_{q+1}\mathscr{L}^2) \subset \mathscr{F}_{q+1}\mathscr{L}^1$ and

$$(d_\gamma h^q_1 + h^q_2 d_\gamma - Id)(\mathscr{F}_q\mathscr{L}^1) \subset \mathscr{F}_{q+1}\mathscr{L}^1.$$

We first prove an estimate. Let $q \geq p$, and let $\tilde{\gamma}$ be an MC-element such that $\mathbb{O}(\tilde{\gamma} - \gamma) \geq q$. Then for $\widetilde{X} := h^q_1(\tilde{\gamma} - \gamma)$, we claim that the following estimates hold:

$$\mathbb{O}(\widetilde{X}) \geq q, \quad \mathbb{O}(\mathrm{Ad}(e^{\widetilde{X}})\tilde{\gamma} - \gamma) \geq q + 1. \tag{11}$$

The first follows by the properties of h^q_1, and to prove the second we compute:

$$\begin{aligned}
\mathbb{O}(\mathrm{Ad}(e^{\widetilde{X}})\tilde{\gamma} - \gamma) &\geq \mathbb{O}(\mathrm{Ad}(e^{\widetilde{X}})\tilde{\gamma} - \tilde{\gamma} + d_{\tilde{\gamma}}(\widetilde{X})) \wedge \mathbb{O}(\tilde{\gamma} - d_{\tilde{\gamma}}(\widetilde{X}) - \gamma) \\
&\geq 2\mathbb{O}(\widetilde{X}) \wedge \mathbb{O}([\gamma - \tilde{\gamma}, \widetilde{X}]) \wedge \mathbb{O}(\tilde{\gamma} - \gamma - d_\gamma(\widetilde{X})) \\
&\geq 2q \wedge (\mathbb{O}(\gamma - \tilde{\gamma}) + \mathbb{O}(\widetilde{X})) \wedge \mathbb{O}(\tilde{\gamma} - \gamma - d_\gamma(\widetilde{X})) \\
&\geq 2q \wedge \mathbb{O}((Id - d_\gamma h^q_1)(\tilde{\gamma} - \gamma)),
\end{aligned}$$

where for the second inequality we use Lemma A.4. The last term can be evaluated as follows:

$$\begin{aligned}\mathbb{O}((Id - d_\gamma h_1^q)(\tilde{\gamma}-\gamma)) &\geq \mathbb{O}((Id - d_\gamma h_1^q - h_2^q d_\gamma)(\tilde{\gamma}-\gamma)) \wedge \mathbb{O}(h_2^q(d_\gamma(\tilde{\gamma}-\gamma))) \\ &\geq (q+1) \wedge \mathbb{O}(h_2^q(d_\gamma(\tilde{\gamma}-\gamma))).\end{aligned}$$

Since $d_\gamma(\tilde{\gamma}-\gamma) = -\frac{1}{2}[\tilde{\gamma}-\gamma, \tilde{\gamma}-\gamma]$, we have $\mathbb{O}(d_\gamma(\tilde{\gamma}-\gamma)) \geq 2q \geq q+1$, so

$$\mathbb{O}((Id - d_\gamma h_1^q)(\tilde{\gamma}-\gamma)) \geq q+1,$$

and this proves (11).

We construct a sequence of MC-elements $\{\gamma_k\}_{k\geq 0}$ and a sequence of group elements $\{e^{X_k}\}_{k\geq 1} \in \mathcal{G}(\mathcal{L})$ by the following recursive formulas:

$$\begin{aligned}&\gamma_0 := \gamma', \\ &X_k := h_1^{p+k-1}(\gamma_{k-1}-\gamma) \quad \text{for } k \geq 1, \\ &\gamma_k := \mathrm{Ad}(e^{X_k})\gamma_{k-1} \quad \text{for } k \geq 1.\end{aligned}$$

To show that these formulas do indeed give well-defined sequences, we have to check that $\gamma_{k-1}-\gamma \in \mathcal{F}_{p+k-1}\mathcal{L}^1$. This holds for $k=1$, and in general it follows by applying the estimate (11) inductively at each step $k \geq 1$ to $\widetilde{\gamma} = \gamma_{k-1}$ and $q = p+k-1$, to obtain

$$\mathbb{O}(X_k) \geq p+k-1, \quad \mathbb{O}(\gamma_k - \gamma) \geq p+k.$$

Using Lemma A.3(1), we obtain

$$\mathbb{O}(X_k * X_{k-1} \cdots * X_1 - X_{k-1} \cdots * X_1) \geq \mathbb{O}(X_k) \geq p+k-1,$$

and therefore by Lemma A.2, the product $X_k * X_{k-1} * \cdots * X_1$ converges to some element X. Applying Lemma A.3(1) k times, we obtain

$$\mathbb{O}(X_k * X_{k-1} \cdots * X_1) \geq \mathbb{O}(X_k) \wedge \mathbb{O}(X_{k-1}) \wedge \cdots \wedge \mathbb{O}(X_1) \geq 1,$$

and thus $X \in \mathfrak{g}(\mathcal{L})$. On the other hand, we have

$$\begin{aligned}\mathbb{O}(\mathrm{Ad}(e^X)\gamma' - \gamma) &\geq \mathbb{O}(\mathrm{Ad}(e^X)\gamma' - \gamma_k) \wedge \mathbb{O}(\gamma_k - \gamma) \\ &\geq \mathbb{O}(\mathrm{Ad}(e^X)\gamma' - \mathrm{Ad}(e^{X_k * \cdots * X_1})\gamma') \wedge (p+k) \\ &\geq \mathbb{O}(X - X_k * \cdots * X_1) \wedge (p+k),\end{aligned}$$

where for the last estimate we used Lemma A.3(2). If we let $k \to \infty$, we obtain the conclusion: $\mathrm{Ad}(e^X)\gamma' = \gamma$. □

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