Pacific

Journal of

Mathematics

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We prove that every slim double Lie groupoid with proper core action is completely determined by a factorization of a canonically defined "diagonal" Lie groupoid.

Introduction

A double groupoid is a set $\mathcal{B}$ provided with two different but compatible groupoid structures. It is useful to represent the elements of $\mathcal{B}$ as boxes that merge horizontally or vertically according to the groupoid multiplication into consideration. The vertical sides of a box belong to a groupoid $\mathcal{V}$, and the horizontal sides to a groupoid $\mathcal{H}$. A double groupoid is *slim* if any box is determined by its four sides. The notion of double groupoids was introduced by Ehresmann [1963], and later studied in [Brown 2004; Brown and Janelidze 2004; Brown and Mackenzie 1992; Brown and Spencer 1976] and references therein.

The notion of double Lie groupoid was defined and investigated by K. Mackenzie [1992; 2000]; see also [Pradines 1977; Mackenzie 1999; Lu and Weinstein 1989] for applications to differential and Poisson geometry. In particular the question of the classification of double Lie groupoids was raised in [Mackenzie 1992]; see also [Brown and Mackenzie 1992]. In the latter article, a complete answer was given in the restricted case of locally trivial double Lie groupoids. More recently, a description in two stages of discrete double groupoids was given in [Andruskiewitsch and Natale 2009]. To state them, let us recall that a diagram over a pair of groupoids $\mathcal{V}$ and $\mathcal{H}$ is a triple $(\mathcal{D}, j, i)$ where $\mathcal{D}$ is a groupoid and $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$ are morphisms of groupoids (over a fixed set of points). The stages are:

 (a) Any double groupoid is an extension of slim double groupoid (its *frame*) by an abelian group bundle.

(b) The category of slim double groupoids, with fixed vertical and horizontal groupoids $\mathcal{V}$ and $\mathcal{H}$, satisfying the filling condition, is equivalent to the category of diagrams over $\mathcal{V}$ and $\mathcal{H}$.

In this paper, we extend stage (b) to the setting of double Lie groupoids. In this context the top-right map has to be a surjective submersion, a differentiable version of the filling condition [Mackenzie 1992]. As one may expect, there are some topological and geometrical ingredients in our main result, which is this:

Theorem 3.7. *The category of slim double Lie groupoids*, *with fixed vertical and horizontal Lie groupoids $\mathcal{V}$ and $\mathcal{H}$, and proper core action, is equivalent to the category of diagrams of Lie groupoids $(\mathcal{D}, j, i)$ such that the maps j and i are transversal at the identities.*

Our proof of this theorem relies on [Andruskiewitsch and Natale 2009, Theorem 2.8] and some topological and differentiable considerations. We should note that the properness of the core action really boils down to the closedness of the anchor map of the corresponding action groupoid, since the core action is actually free.

We have started to investigate the stage (a) of the above construction in the differentiable setting; we expect to present this analysis in a future paper.

1. Preliminaries on Lie groupoids an double Lie groupoids

We denote a groupoid in the form $\mathcal{G} \underset{e}{\overset{s}{\rightrightarrows}} \mathcal{P}$, where s and e stand for "source" and "end"; and the identity map is $\mathrm{id} : \mathcal{P} \to \mathcal{G}$. Recall that a groupoid $\mathcal{G} \underset{e}{\overset{s}{\rightrightarrows}} \mathcal{P}$ is a *Lie groupoid* [Mackenzie 2005], if $\mathcal{P}$ and $\mathcal{G}$ are smooth manifolds, s and e are surjective submersions and the other structural maps are smooth. The *anchor* of $\mathcal{G}$ is the map $\chi : \mathcal{G} \to \mathcal{P} \times \mathcal{P}$ given by $\chi(g) = (s(g), e(g))$.

Definition 1.1. Let $\mathcal{G} \underset{e}{\overset{s}{\rightrightarrows}} \mathcal{P}$ be a Lie groupoid and let $U \subseteq \mathcal{P}$ be an open subset. A *local bisection* of $\mathcal{G}$ on U is a smooth map $\sigma : U \to \mathcal{G}$ which is a section of e and such that $V := (s \circ \sigma)(U)$ is an open subset of $\mathcal{P}$ and $s \circ \sigma : U \to V$ is a diffeomorphism. Define $\mathcal{G}^U = s^{-1}(U)$ and $\mathcal{G}_U = e^{-1}(U)$; the *local left* and *right translations* induced by σ are the maps

$$(1\text{-}1) \qquad \begin{aligned} L_\sigma : \mathcal{G}^U \to \mathcal{G}^V, & \quad g \mapsto \sigma(s(g))g; \\ R_\sigma : \mathcal{G}_V \to \mathcal{G}_U, & \quad g \mapsto g\sigma((s \circ \sigma)^{-1}(e(g))). \end{aligned}$$

For more on bisections see [Mackenzie 2005, Section 1.4].

Double Lie groupoids.

Definition 1.2 (Ehresmann). A *double groupoid* is a groupoid object internal to the category of groupoids. In other terms, a *double groupoid* consists of a set $\mathcal{B}$ with two groupoid structures with *bases* $\mathcal{H}$ and $\mathcal{V}$, which are themselves groupoids

over a common base $\mathcal{P}$, all subject to the compatibility condition that the structure maps of each structure are morphisms with respect to the other.

It is usual to represent a double groupoid $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ as a diagram of four related groupoids

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow[\;\;r\;\;]{\;\;l\;\;} & \mathcal{V} \\
{\scriptstyle t}\downarrow\downarrow{\scriptstyle b} & & {\scriptstyle t}\downarrow\downarrow{\scriptstyle b} \\
\mathcal{H} & \xrightarrow[\;\;r\;\;]{\;\;l\;\;} & \mathcal{P}
\end{array}
$$

where t, b, l, r mean "top", "bottom", "left" and "right". We sketch the main axioms that these groupoids should satisfy and refer to, for instance, [Andruskiewitsch and Natale 2005, Section 2] and [Andruskiewitsch and Natale 2006, Section 1] for detailed exposition and other conventions.

The elements of $\mathcal{B}$ are called boxes and will be denoted by

$$
A = \;\begin{array}{c} {\scriptstyle t(A)} \\ l(A)\boxed{A}r(A) \\ {\scriptstyle b(A)} \end{array}\; \in \mathcal{B}.
$$

Here $t(A)$, $b(A) \in \mathcal{H}$ and $l(A)$, $r(A) \in \mathcal{V}$. The identity maps will be denoted $\mathbf{id} : \mathcal{V} \to \mathcal{B}$ and $\mathbf{id} : \mathcal{H} \to \mathcal{B}$. The product in the groupoid $\mathcal{B}$ with base $\mathcal{V}$ is called *horizontal* and denoted by AB or $\{AB\}$, for $A, B \in \mathcal{B}$ with $r(A) = l(B)$. The product in the groupoid $\mathcal{B}$ with base $\mathcal{H}$ is called *vertical* and we shall denote it by $\begin{smallmatrix} A \\ B \end{smallmatrix}$ $\left\{ \begin{smallmatrix} A \\ B \end{smallmatrix} \right\}$, for $A, B \in \mathcal{B}$ with $b(A) = t(B)$. This pictorial notation is useful for understanding products in the double structure. For instance, the compatibility axioms between the horizontal and vertical products are described by

$$
l\boxed{\begin{smallmatrix} t \\ A \\ b \end{smallmatrix}}r \; r\boxed{\begin{smallmatrix} t' \\ B \\ b' \end{smallmatrix}}r' = l\boxed{\begin{smallmatrix} tt' \\ \{AB\} \\ bb' \end{smallmatrix}}r' \quad \text{and} \quad \begin{array}{c} l\boxed{\begin{smallmatrix} t \\ A \\ b \end{smallmatrix}}r \\ l'\boxed{\begin{smallmatrix} b \\ B \\ b' \end{smallmatrix}}r' \end{array} = ll'\left\{\begin{smallmatrix} t \\ \left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] \\ b' \end{smallmatrix}\right\}rr'
$$

We omit the letter inside the box if no confusion arises. We also write A^h and A^v to denote the inverse of $A \in \mathcal{B}$ with respect to the horizontal and vertical structures of groupoid over $\mathcal{B}$ respectively. When one of the sides of a box is an identity, we draw this side as a double edge. For example, if $t(A) = \mathrm{id}_p$, we draw $\boxed{}$ and say that $t(A) \in \mathcal{P}$.

Definition 1.3 [Mackenzie 1992]. A double groupoid is a *double Lie groupoid* if all the four groupoids involved are Lie groupoids and the *top-right corner map*

$$(1\text{-}2) \qquad \ulcorner : \mathcal{B} \to \mathcal{H}\,{}_l\!\times_t \mathcal{V}, \qquad A \mapsto \ulcorner(A) = (t(A), r(A)),$$

is a surjective submersion.

We shall say that a double groupoid is *discrete* if no differentiable or topological structure is present. A discrete double groupoid satisfies the *filling condition* if the *top-right corner map* defined in (1-2) is surjective. We refer the reader to [Andruskiewitsch and Natale 2009] for more details on discrete double groupoids and the filling condition.

Definition 1.4 [Brown and Mackenzie 1992; Mackenzie 1992]. Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a double Lie groupoid. The core groupoid $\mathbf{E}(\mathcal{B})$ of $\mathcal{B}$ is the set

$$\mathbf{E}(\mathcal{B}) = \{E \in \mathcal{B} : t(E), \ r(E) \in \mathcal{P}\} = (\ulcorner)^{-1}\{(\mathrm{id}_p, \mathrm{id}_p) : p \in \mathcal{P}\}$$

with source and target projections $s_{\mathbf{E}}, e_{\mathbf{E}} : \mathbf{E}(\mathcal{B}) \to \mathcal{P}$, given by $s_{\mathbf{E}}(E) = bl(E)$ and $e_{\mathbf{E}}(E) = tr(E)$ respectively; identity map given by $\mathrm{id}_p = \Theta_p := \mathbf{id} \circ \mathrm{id}(p)$; multiplication and inverse given by

$$(1\text{-}3) \quad E \circ F := \begin{Bmatrix} \mathbf{id}\,l(F) & F \\ E & \mathbf{id}(b(F)) \end{Bmatrix}, \quad E^{(-1)} := (E\,\mathbf{id}\,b(E)^{-1})^v = \begin{Bmatrix} \mathbf{id}\,l(E)^{-1} \\ E^h \end{Bmatrix},$$

for every compatible $E, F \in \mathbf{E}(\mathcal{B})$.

We observe that the elements of $\mathbf{E}(\mathcal{B})$ are of the form $E = \boxed{}$; the source gives the bottom-left vertex and the target gives the top-right vertex of the box. Since the top-right corner map is a surjective submersion, the core groupoid $\mathbf{E}(\mathcal{B})$ is a closed embedded submanifold of $\mathcal{B}$ and clearly $s_{\mathbf{E}}$ and $e_{\mathbf{E}}$ are surjective submersions. With these structural maps $\mathbf{E}(\mathcal{B})$ becomes a Lie groupoid with base $\mathcal{P}$ (differentiability conditions being easily verified because $\mathbf{E}(\mathcal{B})$ is an embedded submanifold of $\mathcal{B}$).

Coarse double groupoid. Let $\mathcal{P}$ be a set and $\mathcal{V}$, $\mathcal{H}$ be groupoids over $\mathcal{P}$. Let $\square(\mathcal{V}, \mathcal{H})$ be the set

$$(\mathcal{V}\,{}_b\!\times_l \mathcal{H})\,{}_{(t,r)}\!\times_{(l,b)} (\mathcal{H}\,{}_t\!\times_r \mathcal{V});$$

that is, the set of quadruples $\left(f \overset{x}{\underset{y}{}} g \right)$ with $x, y \in \mathcal{H}$, $f, g \in \mathcal{V}$ such that

$$l(x) = t(f), \quad r(x) = t(g), \quad l(y) = b(f), \quad r(y) = b(g).$$

If no confusion arises, we shall denote a quadruple as above by a box as follows:

$$h \boxed{\overset{x}{\underset{y}{}}} g.$$

The collection
$$\begin{array}{ccc} \square(\mathcal{V}, \mathcal{H}) & \rightrightarrows & \mathcal{H} \\ \Downarrow & & \Downarrow \\ \mathcal{V} & \rightrightarrows & \mathcal{P} \end{array}$$
forms a double groupoid in the way, called the

coarse double groupoid with sides in $\mathcal{H}$ and $\mathcal{V}$.

Definition 1.5 [Andruskiewitsch and Natale 2009]. A double groupoid $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ is *slim* if the morphism of (discrete) double groupoids $\Pi : \mathcal{B} \to \square(\mathcal{V}, \mathcal{H})$ given by

$$\Pi\left(f \begin{array}{|c|} \hline x \\ \hline A \\ \hline y \\ \hline \end{array} g \right) = \left(f \begin{array}{c} x \\ \\ y \end{array} g \right), \qquad f \begin{array}{|c|} \hline x \\ \hline A \\ \hline y \\ \hline \end{array} g \in \mathcal{B},$$

is injective. That is, any box in $\mathcal{B}$ is determined by its sides.

We now pass to the context of double Lie groupoids. Let M, N and P be smooth manifolds, let $f : M \to P$ and $g : N \to P$ be smooth maps, we remind that f and g are called *transversal* at $p \in \mathcal{P}$, if for every $m \in M$ and $n \in N$ such that $p = f(m) = g(n)$ the condition $(T_m f)(T_m M) + (T_n g)(T_n N) = T_p P$ holds. We said that f and g are *transversal* if they are transversal at every p as above.

Remark 1.6. Let $\mathcal{V}$ and $\mathcal{H}$ be Lie groupoids over the same manifold $\mathcal{P}$. If the anchors maps $\chi_{\mathcal{V}} : \mathcal{V} \to \mathcal{P} \times \mathcal{P}$ and $\chi_{\mathcal{H}} : \mathcal{H} \to \mathcal{P} \times \mathcal{P}$ are transversal then $\square(\mathcal{V}, \mathcal{H})$ is a double Lie groupoid [Brown and Mackenzie 1992].

2. Diagrams of groupoids

The following notions were introduced in [Andruskiewitsch and Natale 2009], in the context of discrete double groupoids. Let $\mathcal{V}$ and $\mathcal{H}$ be groupoids over the same base $\mathcal{P}$. A *diagram* over $\mathcal{H}$ and $\mathcal{V}$ is a triple $(\mathcal{D}, j, i)$ where $\mathcal{D}$ is a groupoid over $\mathcal{P}$ and $i : \mathcal{H} \to \mathcal{D}$, $j : \mathcal{V} \to \mathcal{D}$ are morphisms of groupoids over $\mathcal{P}$. A diagram of groupoids $(\mathcal{D}, j, i)$ over $\mathcal{V}$ and $\mathcal{H}$ is called a $(\mathcal{V}, \mathcal{H})$-*factorization* of $\mathcal{D}$ if $\mathcal{D} = j(\mathcal{V})i(\mathcal{H})$.

To each diagram of groupoids $(\mathcal{D}, j, i)$ we associate a discrete double groupoid, denoted by $\square(\mathcal{D}, j, i)$ and defined as the double subgroupoid of $\square(\mathcal{V}, \mathcal{H})$ whose boxes are of the form

$$A = h \begin{array}{|c|} \hline x \\ \hline \\ \hline y \\ \hline \end{array} g \in \square(\mathcal{V}, \mathcal{H}),$$

with $x, y \in \mathcal{H}$, $g, h \in \mathcal{V}$, such that $i(x)j(g) = j(h)i(y)$ in $\mathcal{D}$.

Diagrams of Lie groupoids.

Definition 2.1. Let $\mathcal{V}$ and $\mathcal{H}$ be Lie groupoids over $\mathcal{P}$. A *diagram of Lie groupoids* over $\mathcal{H}$ and $\mathcal{V}$ is a diagram of groupoids $(\mathcal{D}, j, i)$ such that $\mathcal{D}$ is a Lie groupoid and the maps i and j are smooth. If this diagram is a factorization in the sense above, then we say that it is a $(\mathcal{V}, \mathcal{H})$-*factorization of the Lie groupoid* $\mathcal{D}$.

Let $(\mathcal{D}, j, i)$ be a diagram of Lie groupoids over $\mathcal{H}$ and $\mathcal{V}$. Our aim in this Subsection is to determine when $\square(\mathcal{D}, j, i)$ is a double Lie groupoid. For this, we shall consider two maps; the first one is the composition

$$\mathcal{H}\,_r{\times}_t\,\mathcal{V} \xrightarrow{i \times j} \mathcal{D}\,_e{\times}_s\,\mathcal{D} \xrightarrow{m} \mathcal{D},$$

that is,

$$\Phi : \mathcal{H}\,_r{\times}_t\,\mathcal{V} \to \mathcal{D}, \quad (x, g) \mapsto i(x)\,j(g),$$

where s, e and m are the source, target and composition maps of $\mathcal{D}$. The second one is

$$\Psi : \mathcal{V}\,_b{\times}_l\,\mathcal{H} \to \mathcal{D}, \quad (f, y) \mapsto j(f)\,i(y).$$

Since t, b, l and r are surjective submersions, we have that the fiber products involved in the above maps, $\mathcal{V}\,_b{\times}_l\,\mathcal{H}$ and $\mathcal{H}\,_r{\times}_t\,\mathcal{V}$, are embedded submanifolds of $\mathcal{V} \times \mathcal{H}$ and $\mathcal{H} \times \mathcal{V}$ respectively, and since i and j are smooth then Φ and Ψ also are. With the above maps

$$\square(\mathcal{D}, j, i) = (\mathcal{V}\,_b{\times}_l\,\mathcal{H})\,_\Psi{\times}_\Phi\,(\mathcal{H}\,_r{\times}_t\,\mathcal{V}),$$

and from general transversality theory [Lang 2002, Proposition 2.5], if Φ and Ψ are transversal then $\square(\mathcal{D}, j, i)$ is an embedded submanifold of $(\mathcal{V}\,_b{\times}_l\,\mathcal{H}) \times (\mathcal{H}\,_r{\times}_t\,\mathcal{V})$.

Lemma 2.2. *Let $(\mathcal{D}, j, i)$ be a diagram of Lie groupoids. If i and j are transversal at the identities, then Ψ and Φ defined above are submersions.*

Proof. We take $((f, y), (x, g)) \in (\mathcal{V}\,_b{\times}_r\,\mathcal{H}) \times (\mathcal{H}\,_r{\times}_t\,\mathcal{V})$ such that $\Psi(f, y) = \Phi(x, g)$ i.e., $j(f)i(y) = i(x)j(g)$. Now, by [Lang 2002, Proposition 2.5], we have

$$T_{(x,g)}(\mathcal{H}\,_r{\times}_t\,\mathcal{V}) = \{(Y, X) \in (T_x\mathcal{H}) \times (T_g\mathcal{V})/(T_x\,r)(Y) = (T_g\,t)(X)\}.$$

Let $W \in T_{(x,g)}(\mathcal{H}\,_r{\times}_t\,\mathcal{V})$. We need to prove that there is $(X_1, Y_1) \in T_{(f,y)}(\mathcal{V}\,_b{\times}_l\,\mathcal{H})$ such that $T_{(f,y)}\Psi(X_1, Y_1) = W$. Note that

$$\begin{aligned} T_{(f,y)}\Psi(X_1, Y_1) &= (T_{(j(f),i(y))}m)(T_{(f,y)}\,j \times i)(X_1, Y_1) \\ &= (T_{(j(f),i(y))}m)((T_f\,j)(X_1), (T_y\,i)(Y_1)). \end{aligned}$$

Now, we know that in $\mathcal{H} \rightrightarrows \mathcal{P}$ there is a local bisection $\tau : U \to \mathcal{H}$ with $r(y) \in U \subseteq \mathcal{P}$ open and $\tau(r(y)) = y$ [Mackenzie 2005, Proposition 1.4.9]. Since τ is a bisection, it induces local left and right translations as in (1-1). Concretely, set $V = (l \circ \tau)(U)$ open in $\mathcal{P}$, $\mathcal{H}^U = l^{-1}(U)$ and $\mathcal{H}_U = r^{-1}(U)$ and define

$$\begin{aligned} L_\tau : \mathcal{H}^U \to \mathcal{H}^V, \quad z \mapsto \tau(l(z))z, \\ R_\tau : \mathcal{H}_V \to \mathcal{H}_U, \quad z \mapsto z\,\tau((l \circ \tau)^{-1}r(z)). \end{aligned}$$

Define the map $\tau_\mathcal{D} : U \to \mathcal{D}$ by $i \circ \tau$; since i is a morphism of groupoids, $\tau_\mathcal{D}$ is a local bisection of $\mathcal{D}$. Also note that $\tau_\mathcal{D}(e(i(y))) = (i \circ \tau)(r(y)) = i(y)$.

Analogously, there is a local bisection $\sigma : U' \to \mathcal{V}$ such that $\sigma_{\mathcal{V}}(b(f)) = f$, with $U' \subseteq \mathcal{D}$ open and $b(f) \in U'$. Again this induces a local bisection $\sigma_{\mathcal{D}} : U' \to \mathcal{D}$ such that $\sigma_{\mathcal{D}}(e(j(f))) = j(f)$. Let $(X_1, Y_1) \in T_{(f,y)}(\mathcal{V}\ _b\times_l\ \mathcal{H})$, then by Xu's formula for product in the tangent groupoid [Mackenzie 2005, Theorem 1.4.14] we obtain:

$$
\begin{aligned}
T_{(f,y)}\Psi(X_1, Y_1) &= (T_{(j(f),i(y))}m)((T_f j)(X_1), (T_y i)(Y_1)) \\
&= (T_{i(y)}L_{\sigma_{\mathcal{D}}})(T_y i)(Y_1) + (T_{j(f)}R_{\tau_{\mathcal{D}}})(T_f j)(X_1) \\
&\quad - (T_{i(y)}L_{\sigma_{\mathcal{D}}})(T_{\mathrm{id}_{\mathcal{D}} l(y)}R_{\tau_{\mathcal{D}}})(T_{l(y)}\,\mathrm{id}_{\mathcal{D}})(z),
\end{aligned}
$$

where we write $z = (T_{j(f)}e)(T_f j)(X_1) = (T_{i(y)}s)(T_y i)(Y_1)$.

Now $(T_{j(f)i(y)}L_{\sigma_{\mathcal{D}}}^{-1})(W) \in T_{i(y)}\mathcal{D}$ because we have

$$
\begin{aligned}
L_{\sigma_{\mathcal{D}}}^{-1}(j(f)i(y)) &= \sigma_{\mathcal{D}}^{-1}(s(j(f)i(y)))\,j(f)i(y) \\
&= \sigma_{\mathcal{D}}((s \circ \sigma_{\mathcal{D}})^{-1}(s(j(f)i(y))))^{-1}j(f)i(y) \\
&= \sigma_{\mathcal{D}}((s \circ \sigma_{\mathcal{D}})^{-1}s(j(f)))^{-1}j(f)i(y) \\
&= \sigma_{\mathcal{D}}(e(j(f)))^{-1}j(f)i(y) \\
&= j(f)^{-1}j(f)i(y) = i(y).
\end{aligned}
$$

In an analogous way, we have $(T_{i(y)}R_{\tau_{\mathcal{D}}}^{-1})(T_{j(f)i(y)}L_{\sigma_{\mathcal{D}}}^{-1})(W) \in T_{\mathrm{Id}_{\mathcal{D}} l(y)}\mathcal{D}$ since

$$
\begin{aligned}
R_{\tau_{\mathcal{D}}}^{-1}(i(y)) &= i(y)\tau_{\mathcal{D}}^{-1}((s \circ \tau_{\mathcal{D}}^{-1})^{-1}e(i(y))) \\
&= i(y)\tau_{\mathcal{D}}((s \circ \tau_{\mathcal{D}})^{-1}((s \circ \tau_{\mathcal{D}}^{-1})^{-1}e(i(y))))^{-1} \\
&= i(y)\tau_{\mathcal{D}}((s \circ \tau_{\mathcal{D}})^{-1}((s \circ \tau_{\mathcal{D}})e(i(y))))^{-1} \\
&= i(y)\tau_{\mathcal{D}}(e(i(y)))^{-1} = i(y)^{-1}i(y) = \mathrm{Id}_{\mathcal{D}}\,s(i(y)) = \mathrm{Id}_{\mathcal{D}}(l(y)).
\end{aligned}
$$

Write $p = l(y)$, since i and j are transversal at $\mathrm{Id}_{\mathcal{D}}(p)$ then

$$
T_{\mathrm{Id}_p}\mathcal{D} = (T_{\mathrm{Id}_p}i)(T_{\mathrm{Id}_p}\mathcal{H}) + (T_{\mathrm{Id}_p}j)(T_{\mathrm{Id}_p}\mathcal{V})
$$

and in consequence we can find $X \in T_{\mathrm{Id}_p}\mathcal{H}$ and $Y \in T_{\mathrm{Id}_p}\mathcal{V}$ such that

$$
(T_{i(y)}R_{\tau_{\mathcal{D}}}^{-1})(T_{j(f)i(y)}L_{\sigma_{\mathcal{D}}}^{-1})(W) = (T_{\mathrm{Id}_p}i)(X) + (T_{\mathrm{Id}_p}j)(Y).
$$

Thus, if we take the vectors

$$
X' = X + (T_p\,\mathrm{Id}_{\mathcal{H}})(T_{\mathrm{Id}_p}t)(Y), \quad Y' = Y + (T_p\,\mathrm{Id}_{\mathcal{V}})(T_{\mathrm{Id}_p}l)(X),
$$

a direct calculation shows that

$$
(2\text{-}1) \quad (T_{i(y)}R_{\tau_{\mathcal{D}}}^{-1})(T_{j(f)i(y)}L_{\sigma_{\mathcal{D}}}^{-1})(W) = (T_{\mathrm{Id}_p}i)(X') + (T_{\mathrm{Id}_p}j)(Y') - (T_p\,\mathrm{Id}_{\mathcal{D}})(Z)
$$

where $Z = (T_{\mathrm{Id}_p}b)(Y) + (T_{\mathrm{Id}_p}l)(X)$.

Since $(L_{\sigma_\mathscr{D}} \circ R_{\tau_\mathscr{D}} \circ i) = (L_{\sigma_\mathscr{D}} \circ i \circ R_\tau)$ and $(L_{\sigma_\mathscr{D}} \circ R_{\tau_\mathscr{D}} \circ j) = (R_{\tau_\mathscr{D}} \circ j \circ L_{\sigma_\mathscr{V}})$, we may apply $(T_{i(y)} L_{\sigma_D})(T_{\mathrm{Id}_p} R_{\tau_\mathscr{D}})$ to both sides of (2-1) and arrive at

$$W =$$

$$(T_{i(y)} L_{\sigma_\mathscr{D}})(T_y\, i)(Y_1) + (T_{j(f)} R_{\tau_\mathscr{D}})(T_f\, j)(X_1) - (T_{i(y)} L_{\sigma_\mathscr{D}})(T_{\mathrm{Id}_p} R_{\tau_\mathscr{D}})(T_p\, \mathrm{Id}_\mathscr{D})(Z),$$

where $X_1 = (T_{\mathrm{Id}_p} L_{\sigma_\mathscr{V}})(Y')$ and $Y_1 = (T_{\mathrm{Id}_p} R_\tau)(X')$. It is clear that $(T_f\, b)(X_1) = Z = (T_y\, l)(Y_1)$, thus $T_{(f,y)} \Psi(X_1, Y_1) = W$.

In the same way we prove that Φ is a submersion. $\qquad\square$

From the above result we obtain the following immediate consequence.

Theorem 2.3. *Let $(\mathscr{D}, j, i)$ be a $(\mathscr{V}, \mathscr{H})$-factorization of the Lie groupoid $\mathscr{D}$. If i and j are transversal at the identities, then $\square(\mathscr{D}, j, i)$ is a slim double Lie groupoid.*

Proof. By Lemma 2.2 we have that Φ and Ψ are transversal, thus $\square(\mathscr{D}, j, i)$ is an embedded submanifold of $(\mathscr{V}\,_b\times_l \mathscr{H}) \times (\mathscr{H}\,_r\times_t \mathscr{V})$. Since Φ and Ψ are surjective submersions, both projections from $\square(\mathscr{D}, j, i) = (\mathscr{V}\,_b\times_l \mathscr{H})\,_\Psi\times_\Phi (\mathscr{H}\,_r\times_t \mathscr{V})$ to $(\mathscr{V}\,_b\times_l \mathscr{H})$ and $\mathscr{H}\,_r\times_t \mathscr{V})$ are surjective submersions and the same is true for the projections from the fiber products $\mathscr{V}\,_b\times_l \mathscr{H}$ and $\mathscr{H}\,_r\times_t \mathscr{V}$ to $\mathscr{V}$ and $\mathscr{H}$; then the top, bottom, left and right maps from $\square(\mathscr{D}, j, i)$ are surjective submersions and the same for the top-right corner map. It is clear that the compositions, the identities maps and the inversions maps are smooth. $\qquad\square$

The diagonal groupoid. In this subsection, all groupoids are discrete. We review from [Andruskiewitsch and Natale 2009] the construction of the diagonal groupoid. Let $\mathscr{B}$ be a double groupoid that satisfies the filling condition and let $\mathscr{V} \circledast \mathscr{H}$ be the free product (over $\mathscr{P}$) of the vertical and horizontal groupoids; see the same reference and [Higgins 2005]. We set

$$[A] := xgy^{-1}h^{-1} \in \mathscr{V} \circledast \mathscr{H} \quad \text{if } {}_h\!\begin{array}{|c|}\hline \overset{x}{} \\ A \\ \underset{y}{} \\\hline \end{array}\!{}^g \in \mathscr{B}.$$

We write $J_\circledast(\mathscr{B})$ for the subgroupoid of $\mathscr{V} \circledast \mathscr{H}$ generated by $\{[A] | A \in \mathscr{B}\}$. As $s_\circledast([A]) = e_\circledast([A]) = tl(A)$ we have that the groupoid $J_\circledast(\mathscr{B}) \rightrightarrows \mathscr{P}$ is in fact a group bundle. We know that the group bundle $J_\circledast(\mathscr{B})$ is a normal subgroupoid of $\mathscr{V} \circledast \mathscr{H}$ [Andruskiewitsch and Natale 2009, Lemma 3.5].

Assume that $(\mathscr{B}; \mathscr{V}, \mathscr{H}; \mathscr{P})$ is slim; then the associated *diagonal groupoid* is the quotient $\mathscr{D}(\mathscr{B}) = \mathscr{V} \circledast \mathscr{H}/J_\circledast(\mathscr{B})$. If we compose the natural inclusions of $\mathscr{V}$ and $\mathscr{H}$ in $\mathscr{V} \circledast \mathscr{H}$ with the projections to $\mathscr{D}(\mathscr{B})$ we get two groupoid morphisms

$$i : \mathscr{H} \to \mathscr{D}(\mathscr{B}) \quad \text{and} \quad j : \mathscr{V} \to \mathscr{D}(\mathscr{B}).$$

Thus we have a $(\mathcal{V}, \mathcal{H})$-factorization $(\mathcal{D}(\mathcal{B}), i, j)$. In this subsection our aim is to give another presentation of the diagonal groupoid as a quotient of $\mathcal{V} \, _b\times_l \, \mathcal{H}$.

Proposition 2.4. *Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double groupoid that satisfies the filling condition. We define on $\mathcal{V} \, _b\times_l \, \mathcal{H}$ the following relation $\sim_{\mathcal{B}}$:*

$$(v_1, h_1) \sim_{\mathcal{B}} (v_2, h_2) \Longleftrightarrow r(h_1) = r(h_2), t(v_1) = t(v_2) \text{ and } v_1 h_1 h_2^{-1} v_2^{-1} \in J_{\circledast}(\mathcal{B}).$$

Then $\sim_{\mathcal{B}}$ is an equivalence relation and the map

$$\phi : \mathcal{V} \, _b\times_l \, \mathcal{H}/\sim_{\mathcal{B}} \; \to \mathcal{D}(\mathcal{B}), \qquad [v, h] \mapsto j(v)i(h)$$

is a well defined isomorphism of quivers over $\mathcal{P}$.

Proof. Clearly $\sim_{\mathcal{B}}$ is an equivalence relation; we write $\mathcal{G} := (\mathcal{V} \, _b\times_l \, \mathcal{H})/\sim_{\mathcal{B}}$ and denote by $[f, x]$ the equivalence class of $(f, x) \in \mathcal{V} \, _b\times_l \, \mathcal{H}$. Let $(f_1, x_1), (f_2, x_2)$ be elements of $\mathcal{V} \, _b\times_l \, \mathcal{H}$ such that $[f_1, x_1] = [f_2, x_2]$, then we have $f_1 x_1 x_2^{-1} f_2^{-1} \in J_{\circledast}(\mathcal{B})$ so that $\overline{f_1} \, \overline{x_1} = \overline{f_2} \, \overline{x_2}$ in $\mathcal{D}(\mathcal{B})$ (where $\bar{w}$ denotes the image of w under i if w belongs to $\mathcal{H}$, or under j if w belongs to $\mathcal{V}$). This proves that ϕ is well defined.

Suppose that $g, g' \in \mathcal{V}$ and $x, x' \in \mathcal{H}$ is any collection satisfying $\bar{g} \, \bar{x} = \overline{g'} \, \overline{x'}$. Then $g' x' x^{-1} g^{-1} \in J_{\circledast}(\mathcal{B})$, hence $[g', x'] = [g, x]$. Therefore ϕ is injective.

To prove that ϕ is surjective, let $d \in \mathcal{D}(\mathcal{B})$. Then $d = \overline{d_1} \, \overline{d_2} \cdots \overline{d_n}$ with d_i an element of $\mathcal{V}$ or $\mathcal{H}$. Let $d_i \in \mathcal{H}$, $d_{i+1} \in \mathcal{V}$ with $r(d_i) = t(d_{i+1})$. Since $\mathcal{B}$ satisfies the filling condition, the corner

$$(2\text{-}2) \qquad\qquad \begin{array}{c} \overline{} \\[-4pt] \overline{d_i} \\[-2pt] \boxed{}\,d_{i+1}, \end{array}$$

can be completed to a box in $\mathcal{B}$; that is, there exists $B \in \mathcal{B}$ such that

$$B = f \begin{array}{c} d_i \\ \boxed{} \\ y \end{array} d_{i+1}.$$

Thus, $d_i d_{i+1} y^{-1} f^{-1} \in J_{\circledast}(\mathcal{B})$ and $\overline{d_i} \, \overline{d_{i+1}} = \bar{f} \, \bar{y}$. So, we can commute the d_i's in $d = \overline{d_1} \, \overline{d_2} \cdots \overline{d_n}$ in such a way that we can obtain $d = \bar{g} \, \bar{x}$ with $g \in \mathcal{V}$, $x \in \mathcal{H}$ and $b(g) = l(x)$. This proves that ϕ is surjective. $\square$

Remark 2.5. The map ϕ from the previous lemma induces a structure of groupoid on $\mathcal{G} = \mathcal{V} \, _b\times_l \, \mathcal{H}/\sim_{\mathcal{B}}$ with the following structural maps:

- The source and the target projections are

$$s : \mathcal{G} \to \mathcal{P}, \quad [v, h] \mapsto t(v); \qquad e : \mathcal{G} \to \mathcal{P}, \quad [v, h] \mapsto r(h).$$

- The identity map is $\mathrm{id} : \mathcal{P} \to \mathcal{G}, \quad p \mapsto \mathrm{id}_p = [\mathrm{id}_p, \mathrm{id}_p]$.

- Multiplication is given by $[v_1, h_1][v_2, h_2] = [v_1 f, z\, h_2]$, where $f\ \boxed{}\!\!{}^{\,h_1}_{\,z}\ v_2 \in \mathcal{B}$, when it is defined;

- The inverse map is given by $[v, h]^{-1} = [f^{-1}, z^{-1}]$, for every $[v, h] \in \mathcal{G}$ and where $v\ {}^{z}\!\!\boxed{}\!\!{}_{h}\ f \in \mathcal{B}$.

Now we recall the following lemma which will be very useful for our purposes.

Lemma 2.6 [Andruskiewitsch and Natale 2009, Lemma 3.8]. *Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double groupoid that satisfies the filling condition. Let $f \in \mathcal{V}$ and $x \in \mathcal{H}$ such that*:

- $l(x) = b(f)$ and $t(f) = r(x)$,

- $f x \in J_{\circledast}(\mathcal{B}) \subset \mathcal{V} \circledast \mathcal{H}$.

Then there exists $E \in \mathbf{E}(\mathcal{B})$ such that $E = {}^{f}\!\boxed{}_{x}$. $\square$

We give an alternative description of $\sim_{\mathcal{B}}$ using this lemma. Indeed, $(f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2)$ if and only if $f_1 x_1 x_2^{-1} f_2^{-1} \in J_{\circledast}(\mathcal{B})$. Since $J_{\circledast}(\mathcal{B})$ is a normal subgroupoid, we have that $(f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2)$ if and only if $f_2^{-1} f_1 x_1 x_2^{-1} \in J_{\circledast}(\mathcal{B})$. Hence

$$(2\text{-}3) \qquad (f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2) \quad \Longleftrightarrow \quad \text{there exists} \quad E = {}^{f_2^{-1} f_1}\!\boxed{}_{x_1 x_2^{-1}} \in \mathcal{B}.$$

Thus, the graph of the relation $\sim_{\mathcal{B}}$ is

$$R = \left\{ (f_1, x_1, f_2, x_2) \in (\mathcal{V}\ {}_b\!\times_l\ \mathcal{H})\ {}_\eta\!\times_\eta\ (\mathcal{V}\ {}_b\!\times_l\ \mathcal{H}) \ \middle|\ \exists E \in \mathbf{E}(\mathcal{B}),\ E = {}^{f_2^{-1} f_1}\!\boxed{}_{x_1 x_2^{-1}} \right\},$$

where

$$\eta : \mathcal{V}\ {}_b\!\times_l\ \mathcal{H} \to \mathcal{P} \times \mathcal{P}, \quad (f, x) \mapsto (t(f), r(x));$$

from this we infer that the relation $\sim_{\mathcal{B}}$ is determined by the core groupoid of $\mathcal{B}$.

Lemma 2.7. *Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double groupoid satisfying the filling condition. If $(f_1, x_1), (f_2, x_2) \in \mathcal{V}\ {}_b\!\times_l\ \mathcal{H}$, then $(f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2)$ if and only if there exist $A, B \in \mathcal{B}$ such that*

$$A = f_1\ {}^{x}\!\boxed{}_{x_1}\ g, \quad and \quad B = f_2\ {}^{x}\!\boxed{}_{x_2}\ g.$$

Proof. In fact, if $A = f_1\ {}^{x}\!\boxed{}_{x_1}\ g$ and $B = f_2\ {}^{x}\!\boxed{}_{x_2}\ g$ belong to $\mathcal{B}$, then $x g x_1^{-1} f_1^{-1} \in J_{\circledast}(\mathcal{B})$ and $x g x_2^{-1} f_2^{-1} \in J_{\circledast}(\mathcal{B})$; taking the inverse of the first one and composing with the second one we arrive at $f_1 x_1 x_2^{-1} f_2^{-1} \in J_{\circledast}(\mathcal{B})$, that is, $(f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2)$.

Conversely, if $(f_1, x_1) \sim_{\mathcal{B}} (f_2, x_2)$ then by (2-3) there is $E \in \mathbf{E}(\mathcal{B})$ such that

$$E = f_2^{-1} f_1 \begin{array}{|c|} \hline \\ \hline \end{array}_{x_1 x_2^{-1}}.$$

The filling condition guarantees that given $x_2 \in \mathcal{H}$ and $f_2 \in \mathcal{V}$ with $l(x_2) = b(f_2)$, there is a box $B' \in \mathcal{B}$ with $t(B') = x_2$ and $l(B') = f_2^{-1}$. Let

$$B' = f_2^{-1} \begin{array}{|c|} \hline x_2 \\ \hline x \end{array} g^{-1} \quad \text{and} \quad A' := \left\{ \begin{matrix} E^v & \mathbf{id}(x_2) \\ \mathbf{id}(f_2) & B' \end{matrix} \right\} = f_1^{-1} \begin{array}{|c|} \hline x_1 \\ \hline x \end{array} g^{-1}.$$

Let A, B be the vertical inverses of A' and B'. Thus $A = f_1 \begin{array}{|c|} \hline x \\ \hline x_1 \end{array} g$ and $B = f_2 \begin{array}{|c|} \hline x \\ \hline x_2 \end{array} g$ are both in $\mathcal{B}$ and we get the result. $\qquad\square$

3. Slim double Lie groupoids

Actions of Lie groupoids.

Definition 3.1. A *left action* of a groupoid $\mathcal{G} \rightrightarrows \mathcal{P}$ *along* a map $\varepsilon : N \to \mathcal{P}$ is given by a map $\mathcal{G} \,_e\times_\varepsilon N \to N$, denoted by $(g, n) \mapsto gn$, such that:

$$\varepsilon(hy) = s(h), \quad \mathrm{id}(\varepsilon(y))\, y = y, \quad (gh)y = g(hy),$$

for all $g, h \in \mathcal{G}$ and $y \in N$ such that $e(g) = s(h)$ and $e(h) = \varepsilon(y)$. We shall simply say that the groupoid $\mathcal{G}$ acts on N.

The *transformation* or *action groupoid* $\mathcal{G} \ltimes N \rightrightarrows N$, associated with such an action, is a groupoid with set of arrows $\mathcal{G} \,_e\times_\varepsilon N$. The source and target maps are

$$s', \, e' : \mathcal{G} \ltimes N \to N, \quad \text{given by} \quad s'(g, n) = gn \quad \text{and} \quad e'(g, n) = n,$$

respectively, and composition $(g, n)(h, m) = (gh, m)$.

Let $\mathcal{G} \rightrightarrows \mathcal{P}$ be a Lie groupoid, N a smooth manifold and $\varepsilon : N \to \mathcal{P}$ a smooth map. A left action is *smooth* provided that the map $\mathcal{G} \,_e\times_\varepsilon N \to N$ is smooth. The resulting action groupoid is again a Lie groupoid.

The *canonical left action* of a groupoid $\mathcal{G} \rightrightarrows \mathcal{P}$ along $\mathrm{id} : \mathcal{P} \to \mathcal{P}$ is given by $gp = s(g)$, where $g \in \mathcal{G}$, $p \in \mathcal{P}$ and $e(g) = p$. Recall that the orbit space $\mathcal{P}/\mathcal{G}$ is the quotient space of $\mathcal{P}$ where two points of $\mathcal{P}$ are identified if there exists an arrow between them. Thus, if we have $\mathcal{G} \rightrightarrows \mathcal{P}$ an action along a map $\varepsilon : N \to \mathcal{P}$, the orbit space $N/\mathcal{G}$ is equal to $N/\mathcal{G} \ltimes N$, the orbit space of the action groupoid. Also, the stabilizer of $n \in N$ is defined in the usual way; it coincides with the isotropy group of n with respect to the action groupoid.

We recall that a continuous map $f : X \to Y$ between two topological spaces X and Y, is said to be *proper* if the inverse image of a compact subset of Y is compact.

Definition 3.2. A Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{P}$ is *proper* if the anchor map is proper. An action of a Lie groupoid $\mathcal{G}$ on a smooth manifold N is *proper* if the action groupoid $\mathcal{G} \ltimes N$ is proper.

The following proposition is useful to decide when an action is proper.

Proposition 3.3 [Tu 2004, 2.14]. *Let $\mathcal{G} \rightrightarrows \mathcal{P}$ be a Lie groupoid and N a smooth manifold endowed with a left action of $\mathcal{G}$. Then $\mathcal{G}$ acts properly on N if and only if the anchor map $(s', e') : \mathcal{G} \ltimes N \to N \times N$ is closed and $\forall n \in N$ the stabilizer of n is compact.* $\square$

A groupoid is free when all the isotropy groups are trivial; thus, an action groupoid is free when all stabilizers are trivial.

Proposition 3.4 [Dragulete 2007, Theorem 3.3.1]. *If $\mathcal{G} \rightrightarrows \mathcal{P}$ is a free and proper groupoid, then the orbit space $\mathcal{P}/\mathcal{G}$ is a smooth manifold and the orbit projection $\mathcal{P} \to \mathcal{P}/\mathcal{G}$ is a submersion.* $\square$

In particular, if $\mathcal{G}$ is a Lie groupoid with a free and proper smooth action on N, then the orbit space $N/\mathcal{G}$ is a smooth manifold and the orbit projection $N \to N/\mathcal{G}$ is a submersion.

The core action.

Proposition 3.5 [Andruskiewitsch and Natale 2009, Proposition 1.1]. *Suppose $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ is a slim double groupoid. Define $\gamma : \mathcal{B} \to \mathcal{P}$, $\gamma(A) = lb(A)$, the bottom-left vertex of A.*

 (a) *There is an action of the core groupoid $\mathbf{E}(\mathcal{B})$ on $\gamma : \mathcal{B} \to \mathcal{P}$ given by*

$$(3\text{-}1) \qquad E {\rightarrow} A := \left\{ \begin{matrix} \mathbf{id}\, l(A) & A \\ E & \mathbf{id}\, b(A) \end{matrix} \right\}, \quad A \in \mathcal{B}, E \in \mathbf{E}.$$

 (b) *Let $B \in \mathcal{B}$. Then the stabilizer $\mathbf{E}(\mathcal{B})^B$ is trivial and the orbit of B is $\mathbb{O}_B = \{A \in \mathcal{B} : t(A) = t(B), r(A) = r(B)\}$.* $\square$

We introduce next the core action of the core groupoid on the fibred product $\mathcal{V} \,_b\times_l \mathcal{H}$, that allows us to present the diagonal groupoid as a quotient of $\mathcal{V} \,_b\times_l \mathcal{H}$.

Lemma 3.6. *Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double groupoid and let $\eta : \mathcal{V} \,_b\times_l \mathcal{H} \to \mathcal{P}$ be given by $\eta(f, x) = b(f) = l(x)$. Then $\mathbf{E}(\mathcal{B})$ acts on η by*

$$(3\text{-}2) \qquad E \triangleright (f, x) = (f\, l(E), b(E)\, x), \quad \text{when } \eta(f, x) = e_{\mathbf{E}(\mathcal{B})}(E).$$

The orbit space $\mathcal{V} \,_b\times_l \mathcal{H}/\mathbf{E}(\mathcal{B})$ coincides with $\mathcal{D}(\mathcal{B})$.

The map (3-2) is the *core action* of $\mathbf{E}(\mathcal{B})$ on $\mathcal{V} \,_b\times_l \mathcal{H}$.

Proof. Since $b(f\,l(E)) = bl(E) = s_{\mathbf{E}(\mathscr{B})}(E)$ and $l(b(E)\,x) = lb(E) = s_{\mathbf{E}(\mathscr{B})}(E)$, the map $\triangleright : \mathbf{E}(\mathscr{B})\,_{e_{\mathbf{E}(\mathscr{B})}}\!\times_\eta (\mathscr{V}\,_b\!\times_l \mathscr{H}) \to \mathscr{V}\,_b\!\times_l \mathscr{H}$ is well defined and $\triangleright$ is an action:

$$(E \circ F) \triangleright (f, x) = (f\,l(E \circ F), b(E \circ F)\,x) = (f\,l(F)l(E), b(E)b(F)\,x)$$
$$= E \triangleright (f\,l(F), t(F)\,x) = E \triangleright (F \triangleright (f, x)).$$

Also, $\eta(E \triangleright (f, x)) = \eta(f\,l(E), b(E)\,x) = l(b(E)x) = lb(E) = s_{\mathbf{E}(\mathscr{B})}(E)$.

For (b), if $(f, x) \sim_\mathscr{B} (g, y)$, there are $A, B \in \mathscr{B}$ such that $f\begin{array}{c} z \\ \boxed{A} \\ x \end{array} h$ and $g\begin{array}{c} z \\ \boxed{B} \\ y \end{array} h$;

see Lemma 2.7. Then, by Proposition 3.5, there is a box $E \in \mathbf{E}(\mathscr{B})$ such that $A = E \rightharpoonup B$, in consequence, $x = b(E)y$ and $f = g\,l(E)$.

Conversely, if $A, B \in \mathscr{B}$ and $E \in \mathbf{E}(\mathscr{B})$ satisfy $b(A) = b(E)b(B)$ and $l(A) = l(B)l(E)$, then the boxes $E \rightharpoonup B$ and B have the same top and right sides. By Lemma 2.7, we have

$$(l(E \rightharpoonup B), b(E \rightharpoonup B)) \sim_\mathscr{B} (l(B), b(B)),$$

that is $(l(A), b(A)) \sim_\mathscr{B} (l(B), b(B))$. From this we conclude that given (f, x) and (g, y) in $\mathscr{V}\,_b\!\times_l \mathscr{H}$, we have $(f, x) \sim_\mathscr{B} (g, y)$ if and only if there exists $E \in \mathbf{E}(\mathscr{B})$ with $f = g\,l(E)$ and $x = b(E)y$. Thus the orbit space coincides with the diagonal groupoid. $\qquad\square$

Let $\pi : \mathscr{V}\,_b\!\times_l \mathscr{H} \to \mathscr{V}\,_b\!\times_l \mathscr{H}/\mathbf{E}(\mathscr{B})$ be the projection determined by (3-2). Let

$$\tilde{i} : \mathscr{H} \to \mathscr{V}\,_b\!\times_l \mathscr{H}, \quad x \mapsto (\mathrm{id}(l(x)), x)$$
$$\tilde{j} : \mathscr{V} \to \mathscr{V}\,_b\!\times_l \mathscr{H}, \quad f \mapsto (f, \mathrm{id}(b(f))),$$

be the canonical inclusions of $\mathscr{V}$ and $\mathscr{H}$, and consider the compositions $i = \pi \circ \tilde{i}$ and $j = \pi \circ \tilde{j}$.

The main theorem.

Theorem 3.7. *Let $\mathscr{V}$ and $\mathscr{H}$ be Lie groupoids. The assignments $\mathscr{B} \mapsto \mathscr{D}(\mathscr{B})$ and $(\mathscr{D}, j, i) \mapsto \square(\mathscr{D}, j, i)$ determine mutual category equivalences between*

(a) *the category of slim double Lie groupoids $(\mathscr{B}; \mathscr{V}, \mathscr{H}; \mathscr{P})$ with proper core action, and*

(b) *the category of $(\mathscr{V}, \mathscr{H})$-factorizations of Lie groupoids $(\mathscr{D}, j, i)$ such that the maps i and j are transversal at the identities.*

Proof. The equivalence of categories at the discrete level was obtained in [Andruskiewitsch and Natale 2009]. We see that this equivalence behaves well in the differentiable context. We start with the direction from (a) to (b).

Step 1. *Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double Lie groupoid. If the core action is proper, then $\mathcal{D}(\mathcal{B})$ is a Lie groupoid over $\mathcal{P}$. Furthermore, the maps i and j defined above are transversal at the identities.*

Proof of the step. Since $\mathcal{B}$ is slim, the action (3-2) is free. Hence, the orbit space $\mathcal{V} \,_b\times_l \mathcal{H}/\mathbf{E}(\mathcal{B})$ has a unique manifold structure such that the canonical projection $\pi : \mathcal{V} \,_b\times_l \mathcal{H} \to \mathcal{V} \,_b\times_l \mathcal{H}/\mathbf{E}(\mathcal{B})$ is a surjective submersion, because the action is proper, see Proposition 3.4. We deduce using the identification in Lemma 3.6 that $\mathcal{D}(\mathcal{B})$ is a Lie groupoid over $\mathcal{P}$. In fact, the structure maps are described in Remark 2.5; using local sections of π, it is clear that the source and target maps are surjective submersions and that the other structural maps are smooth.

Let us now proceed with transversality. Let $p \in \mathcal{P}$ and we take a tangent vector $Z \in T_{[\mathrm{id}_p, \mathrm{id}_p]}\mathcal{D}(\mathcal{B})$, where $[\mathrm{id}_p, \mathrm{id}_p] = \pi(\mathrm{id}_p, \mathrm{id}_p)$. Since π is a surjective submersion, there is $(U, W) \in T_{(\mathrm{id}_p, \mathrm{id}_p)}(\mathcal{V} \,_b\times_l \mathcal{H})$ such that $T_{(\mathrm{id}_p, \mathrm{id}_p)}\pi(U, W) = Z$. Choose

$$Y = U \in T_{\mathrm{id}_p}\mathcal{V}, \qquad X = W - (T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b)(U) \in T_{\mathrm{id}_p}\mathcal{H}.$$

It is clear that

$$(T_{\mathrm{id}_p}\tilde{j}\,)(Y) = (U, (T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b(U))) \text{ and } (T_{\mathrm{id}_p}\tilde{i}\,)(X) = ((T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}l)(X), X).$$

We compute

$$
\begin{aligned}
(T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}l)(X) &= (T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}l)(W) - (T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}l)(T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b)(U) \\
&= (T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}l)(W) - (T_p \,\mathrm{id}_{\mathcal{V}})(T_{\mathrm{id}_p}b)(U) = 0;
\end{aligned}
$$

then $(T_{\mathrm{id}_p}\tilde{i}\,)(X) = (0, X)$. In consequence we have

$$
\begin{aligned}
(T_{\mathrm{id}_p}\tilde{j}\,)(Y) + (T_{\mathrm{id}_p}\tilde{i}\,)(X) &= (U, (T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b)(U)) + (0, X) \\
&= (U, (T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b)(U) + W - (T_p \,\mathrm{id}_{\mathcal{H}})(T_{\mathrm{id}_p}b)(U)) \\
&= (U, W).
\end{aligned}
$$

Then if we apply $T_{(\mathrm{id}_p, \mathrm{id}_p)}\pi$ to both sides of the above equation we arrive to

$$(T_{\mathrm{id}_p}j)(Y) + (T_{\mathrm{id}_p}i)(X) = Z,$$

that is, the maps i and j are transversal at the identities. $\qquad\square$

Step 2. *Let $(\mathcal{D}, j, i)$ be a $(\mathcal{V}, \mathcal{H})$-factorization of Lie groupoids such that the maps i and j are transversal at the identities. Then the associated double groupoid $\square(\mathcal{D}, j, i)$ is a slim double Lie groupoid and its core action is proper.*

Proof of the step. By Theorem 2.3, $\square(\mathcal{D}, j, i)$ is a slim double Lie groupoid. By [Brown and Mackenzie 1992], its core groupoid is a Lie groupoid with underlying

manifold

$$\mathcal{V}^{op} {}_j\times_i \mathcal{H} = \{(h, y) \mid j(h^{-1}) = i(y)\}.$$

Up to this description, the core action on $\mathcal{V} {}_b\times_l \mathcal{H}$ is given by

(3-3) $(h, y) \triangleright (f, x) = (fh, yx)$ when $\eta(f, x) = t(h) = r(y)$.

It remains to prove that the core action (3-3) is proper. But the action (3-3) is free, that is the stabilizers are trivial. Thus, in order to prove that it is proper, we only need to check that the anchor map of the corresponding action groupoid

$$(s, t) : (\mathcal{V}^{op} {}_j\times_i \mathcal{H}) \ltimes (\mathcal{V} {}_b\times_l \mathcal{H}) \to (\mathcal{V} {}_b\times_l \mathcal{H}) \times (\mathcal{V} {}_b\times_l \mathcal{H})$$

is closed; see Proposition 3.3. Let $A \subseteq (\mathcal{V} {}_j\times_i \mathcal{H}) \ltimes (\mathcal{V} {}_b\times_l \mathcal{H})$ be a closed set and consider a sequence $\{(f_n, x_n, g_n, y_n)\}_{n\in\mathbb{N}}$ in A such that

$$\{(s, t)(f_n, x_n, g_n, y_n)\}_{n\in\mathbb{N}} = \{(g_n f_n, x_n y_n, g_n, y_n)\}_{n\in\mathbb{N}}$$

converges to $(a, b, g, y) \in (\mathcal{V} {}_b\times_l \mathcal{H}) \times (\mathcal{V} {}_b\times_l \mathcal{H})$. We need to see that $(a, b, g, y) \in (s, t)(A)$.

Clearly we have $g_n \to g$, $y_n \to y$, $g_n f_n \to a$, and $x_n y_n \to b$ as $n \to \infty$. Hence $(f_n, x_n, g_n, y_n) \to (g^{-1}a, by^{-1}, g, y)$. Since A is a closed set, we conclude that $((g^{-1}a, by^{-1}, g, y) \in A$. But $(a, b, g, y) = (s, t)(g^{-1}a, by^{-1}, g, y) \in (s, t)(A)$, hence $(s, t)(A)$ is closed. This finishes the proof of the step and of the theorem. $\square$

Example 3.8. Let G be a simply connected Poisson-Lie group, $\mathfrak{g}$ its Lie algebra. It is well known that $\mathfrak{g}$ is a Lie bialgebra; let $\mathfrak{g}^*$ be the dual Lie algebra and let $\mathfrak{d}$ be the corresponding Drinfeld double. Let G^* and D be simply connected Lie groups with Lie algebras $\mathfrak{g}^*$ and $\mathfrak{d}$, respectively. Then the maps $G \to D$ taking u to $\bar{u}$ and $G^* \to D$ taking $g \mapsto \bar{g}$ give rise to a symplectic double groupoid [Lu and Weinstein 1989, Theorem 3]:

(3-4)

$$
\begin{array}{ccc}
\Gamma & \overset{l}{\underset{r}{\rightrightarrows}} & G^* \\
t \big\Downarrow b & & t \big\Downarrow b \\
G & \overset{l}{\underset{r}{\rightrightarrows}} & \{*\}
\end{array}
$$

where $\Gamma = \{(g, u, v, h) \mid g, h \in G^*, u, v \in G, \text{ and } \bar{g}\,\bar{u} = \bar{v}\,\bar{h}\}$.

It is clear that $\mathbf{E}(\Gamma) = \{(g, u, e, e) \mid g \in G^*, u \in G, \text{ and } \bar{g}\,\bar{u} = e\}$, that the core action over $G \times G^*$ is given by $(g, v, e, e)(a, b) = (ag, vb)$, and that the anchor of the respective action groupoid is

$$\mathfrak{X} : \mathbf{E}(\Gamma) \times (G \times G^*) \to (G \times G^*) \times (G \times G^*), \quad ((g, v, e, e), (a, b)) = (ag, vb).$$

Arguing as in the in the proof of Step 2 of Theorem 3.7, we see that the core action is proper. Observe that $(\Gamma, G, G^*, *)$ does not satisfy, in general, the filling condition. If $\overline{G}\,\overline{G^*} = \overline{G^*}\,\overline{G}$, then Γ is a slim double Lie groupoid.

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