Pacific Journal of Mathematics

COMBINATORIAL CONSTRUCTIONS OF THREE-DIMENSIONAL SMALL COVERS

YASUZO NISHIMURA

Volume 256 No. 1 March 2012

PACIFIC JOURNAL OF MATHEMATICS
Vol. 256, No. 1, 2012

COMBINATORIAL CONSTRUCTIONS OF THREE-DIMENSIONAL SMALL COVERS

YASUZO NISHIMURA

We study two operations on 3-dimensional small covers called connected sum and surgery. These operations correspond to combinatorial operations on $(\mathbb{Z}_2)^3$-colored simple convex polytopes. Then we show that each 3-dimensional small cover can be constructed from T^3, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using these operations. This is a generalization of the results of Izmest'ev and Nishimura, and an improvement of the results of Kuroki and Lü and Yu.

1. Introduction

Davis and Januszkiewicz [1991] introduced a small cover as an n-dimensional closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n$-action whose orbit space is a simple convex polytope P, where $\mathbb{Z}_2$ is the quotient additive group $\mathbb{Z}/2\mathbb{Z}$. They showed that there exists a one-to-one correspondence between small covers and $(\mathbb{Z}_2)^n$-colored polytopes [ibid., Proposition 1.8]. Here a pair (P, λ) is called a *$(\mathbb{Z}_2)^n$-colored polytope* when P is an n-dimensional simple convex polytope with the set of facets $\mathcal{F}$ and a function $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ satisfying the condition that

($\star$) if $F_1 \cap \cdots \cap F_n \neq \varnothing$, then $\{\lambda(F_1), \ldots, \lambda(F_n)\}$ is linearly independent.

We say that two $(\mathbb{Z}_2)^n$-colored polytopes (P_1, λ_1) and (P_2, λ_2) are *equivalent* when there exists a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ such that $\lambda_2\phi = \theta\lambda_1$ for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$. The n-dimensional torus T^n and the real projective space $\mathbb{R}P^n$ with the standard $(\mathbb{Z}_2)^n$-actions are examples of small covers over the n-cube I^n and the n-simplex Δ^n, respectively.

In this paper we are interested in constructions of 3-dimensional small covers M^3 from basic small covers by using some operations. Izmest'ev [2001] studied a class of 3-dimensional small covers that are called *linear models* and correspond to 3-colored polytopes. He introduced two operations on linear models called *connected sum* $\sharp$ and *surgery* $\natural$ and proved the following theorem.

MSC2010: primary 57M50, 57M60, 57S17; secondary 52B10.
Keywords: small cover, equivariant surgery, connected sum, 3-polytope.

Theorem 1.1 [Izmest'ev 2001, Theorem 3]. *Each linear model M^3 can be constructed from T^3 using the operations $\sharp$, $\natural$ and $\natural^{-1}$, where $\natural^{-1}$ is the inverse of $\natural$.*

In [Nishimura 2004], we generalized Theorem 1.1 to orientable small covers M^3 that correspond to 4-colored polytopes. We introduced a new operation called the Dehn surgery $\natural^D$, and showed that each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using four operations $\sharp$, $\natural$, $\natural^{-1}$ and $\natural^D$ [ibid., Theorem 1.10]. Later Lü and Yu [2011] considered a construction of general small covers M^3. They introduced new operations $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$ and $\sharp_i^{©}$ for $i \geq 3$ and showed the following theorem.

Theorem 1.2 [Lü and Yu 2011, Theorem 1.2]. *Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with a certain $(\mathbb{Z}_2)^3$-action using the operations $\sharp$, $\natural^{-1}$, $\sharp^e$, $\sharp^{eve}$, $\sharp^{\Delta}$, $\sharp_4^{©}$ and $\sharp_5^{©}$.*

Operations appeared in Theorem 1.2 are all "nondecreasing", that is, they do not decrease the number of faces of an orbit polytope. In other words the surgery $\natural$ is not used in Theorem 1.2, unlike in Theorem 1.1. Kuroki [2010] pointed out that the operations $\natural^D$, $\sharp^e$ and $\sharp^{eve}$ can be obtained as compositions of $\sharp$ and $\natural$ such as $\natural^D = \natural \circ \sharp\mathbb{R}P^3$, $\sharp^e = \natural \circ \sharp$ and $\sharp^{eve} = \natural^2 \circ \sharp$ [ibid., Theorem 4.1]. Therefore our result in [Nishimura 2004] can be improved as follows: Each orientable small cover M^3 can be constructed from $\mathbb{R}P^3$ and T^3 by using three operations $\sharp$, $\natural$ and $\natural^{-1}$; see [Kuroki 2010, Corollary 4.4]. Moreover Theorem 1.2 can be rewritten by using $\natural$ instead of $\sharp^e$ and $\sharp^{eve}$ as follows [Kuroki 2010, Corollary 4.8]: Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with a certain $(\mathbb{Z}_2)^3$-action by using six operations $\sharp$, $\natural$, $\natural^{-1}$, $\sharp^{\Delta}$, $\sharp_4^{©}$ and $\sharp_5^{©}$. Then a problem arises:

Problem 1.3 [Kuroki 2010, Problem 5.2]. What are basic small covers from which we can construct all 3-dimensional small covers using the operations $\sharp$, $\natural$ and $\natural^{-1}$?

We give a solution to this problem, our main result:

Theorem 1.4. *Every small cover M^3 can be constructed from T^3, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using two operations $\sharp$ and $\natural$.*

In this theorem we do not use the inverse surgery $\natural^{-1}$. As a corollary we obtain improvements of Theorem 1.1 and our previous result in [Nishimura 2004].

Corollary 1.5. (1) *Each linear model M^3 can be constructed from T^3 by using two operations $\sharp$ and $\natural$.*

(2) *Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using two operations $\sharp$ and $\natural$.*

These results are equivariant analogues of the well-known result [Kirby 1978] that "each closed 3-manifold can be constructed from the 3-sphere by using Dehn surgeries".

This paper is organized as follows. In Section 2, we briefly recall the definition and basic facts about small covers, and we introduce some basic 3-dimensional small covers. In Section 3, we establish several operations on $(\mathbb{Z}_2)^3$-colored polytopes. In Section 4, we discuss the constructions of $(\mathbb{Z}_2)^3$-colored polytopes, and prove Theorem 1.4. In Section 5, we follow the point of view of Lü and Yu, and discuss a nondecreasing construction of small covers by using the inverse surgery $\natural^{-1}$ instead of the surgery $\natural$. We will point out in Remark 5.5 that there is a gap in the proof of [Lü and Yu 2011, Theorem 1.2] and improve their result as follows.

Theorem 1.6. (1) *Each linear model M^3 can be constructed from T^3 by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.*

(2) *Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.*

(3) *Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using four operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^{©}$.*

In Section 6 we shall make a remark on a 2-*torus manifold*, which is an object of a little wider class than small covers. If objects are expanded to this class, then the argument becomes easier. We prove the following theorem.

Theorem 1.7. (1) *Each linear model of a locally standard 2-torus manifold over D^3 can be constructed from S^3 by using inverse surgery $\natural^{-1}$.*

(2) *Each orientable locally standard 2-torus manifold over D^3 can be constructed from S^3 by using two surgeries $\natural^{-1}$ and $\natural^D$ and the blow up $\sharp\mathbb{R}P^3$.*

(3) *Each locally standard 2-torus manifold over D^3 can be constructed from S^3 by using the inverse surgery $\natural^{-1}$ and connecting $\mathbb{R}P^3$, $S^1 \times_{\mathbb{Z}_2} S^2$, $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions by operations $\sharp$ and $\sharp^e$.*

2. Basics of small covers

Here we recall definitions and basic facts on small covers; for details, see [Davis and Januszkiewicz 1991]. Let P be an n-dimensional simple convex polytope with facets (that is, codimension-one faces) $\mathcal{F} = \{F_1, \dots, F_m\}$. A *small cover M* over P is an n-dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$-action whose orbit space is P. For a facet F of P, we define $\lambda(F)$ to be the generator of the isotropy subgroup at $x \in \pi^{-1}(\operatorname{int} F)$ where $\pi : M \to P$ is the orbit projection. Then a function $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ is called a *characteristic function* of M if it satisfies the condition $(\star)$.

Therefore λ is a kind of face-coloring of P. We call a function $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$ satisfying $(\star)$ a $(\mathbb{Z}_2)^n$-*coloring* of P. We say that two $(\mathbb{Z}_2)^n$-colored polytopes (P_1, λ_1) and (P_2, λ_2) are *equivalent* when there exists a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ such that $\lambda_2\phi = \theta\lambda_1$ for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$. Conversely, given a simple convex polytope P and a $(\mathbb{Z}_2)^n$-coloring $\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$, we can construct a small cover M whose characteristic function is the given λ as

$$M(P, \lambda) := P \times (\mathbb{Z}_2)^n / \sim ,$$

where $(x, t) \sim (y, s)$ is defined as $x = y \in P$ and $s - t$ is contained in the subgroup generated by $\lambda(F_1), \ldots, \lambda(F_k)$ such that $x \in \mathrm{int}(F_1 \cap \cdots \cap F_k)$. We say that two small covers M_i over P_i for $i = 1, 2$ are $\mathrm{GL}(n, \mathbb{Z}_2)$-*equivalent* on a combinatorial equivalence of polytopes $\phi : P_1 \to P_2$ when there exists a θ-equivariant homeomorphism $f : M_1 \to M_2$ such that $\pi_2 \circ f = \phi \circ \pi_1$ and $f(g \cdot x) = \theta(g) \cdot f(x)$ for $g \in (\mathbb{Z}_2)^n$ and $x \in M_1$ and for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$. Moreover we say that two small covers are *equivalent* when they are $\mathrm{GL}(n, \mathbb{Z}_2)$-*equivalent* on some combinatorial equivalence of polytopes $\phi : P_1 \to P_2$. In [Lü and Masuda 2009], this equivalence and a $\mathrm{GL}(n, \mathbb{Z}_2)$-equivalence on the identity are called a *weakly equivariantly homeomorphism* and *D-J equivalence*, respectively. Davis and Januszkiewicz [1991, Proposition 1.8] proved that a small cover M over P with a characteristic function λ is D-J equivalent to $M(P, \lambda)$. Therefore we can identify an equivalence class of a small cover $M(P, \lambda)$ with the equivalence class of a $(\mathbb{Z}_2)^n$-colored polytope (P, λ).

Example 2.1. The real projective space $\mathbb{R}P^n$ and the n-dimensional torus T^n with the standard $(\mathbb{Z}_2)^n$-actions are examples of small covers over the n-simplex Δ^n and the n-cube I^n respectively. Figure 1 shows their characteristic functions on the polytopes (Schlegel diagram) in the case $n = 3$, where $\{\alpha, \beta, \gamma\}$ is a basis of $(\mathbb{Z}_2)^3$. We notice that a $(\mathbb{Z}_2)^n$-coloring on Δ^n is unique up to equivalence. Therefore we denote the colored simplex by Δ^n by omitting coloring.

An n-dimensional small cover $M(P, \lambda)$ with an n-coloring λ (that is, $\lambda(\mathcal{F})$ is a basis of $(\mathbb{Z}_2)^n$) is called a *linear model*. An example of a linear model is the torus T^n shown in Example 2.1. Obviously an n-coloring of P (that is, a linear model) is unique up to equivalence. In case $n = 3$, it is well known that a simple convex polytope is 3-colorable if and only if each face contains an even number of edges.

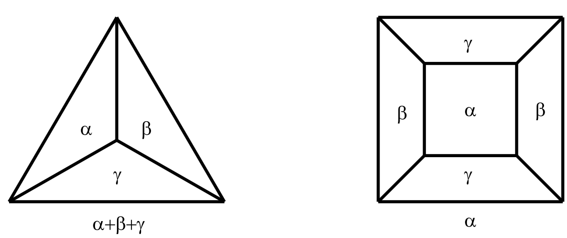

Figure 1. Characteristic functions of $\mathbb{R}P^3$ and T^3.

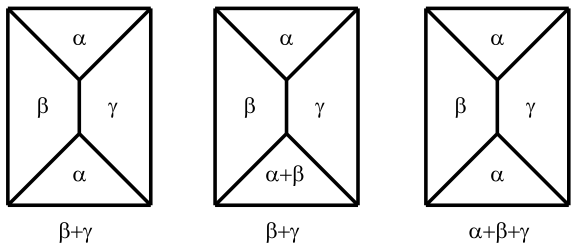

Figure 2. Three types of $(\mathbb{Z}_2)^3$-coloring on the 3-sided prism $P^3(3) = I \times \Delta^2$; λ_1, λ_2 and λ_3, respectively.

In [Nakayama and Nishimura 2005, Theorem 1.7], we gave a criterion for a small cover to be orientable. We recall the criterion in the case $n = 3$.

Theorem 2.2. *A* 3*-dimensional small cover* $M(P, \lambda)$ *is orientable if and only if* $\lambda(\mathcal{F})$ *is contained in* $\{\alpha, \beta, \gamma, \alpha+\beta+\gamma\}$ *for a suitable basis* $\{\alpha, \beta, \gamma\}$ *of* $(\mathbb{Z}_2)^3$.

From this theorem, the small covers $\mathbb{R}P^3$ and T^3 given in Figure 1 are both orientable. We call a $(\mathbb{Z}_2)^3$-coloring satisfying the orientability condition in this theorem an *orientable coloring* of P. Since each triple of $\{\alpha, \beta, \gamma, \alpha+\beta+\gamma\}$ is linearly independent, an orientable coloring is just an ordinary 4-coloring.

Example 2.3. We consider small covers on the 3-sided prism $P^3(3) = I \times \Delta^2$. There exist three types of $(\mathbb{Z}_2)^3$-coloring on $P^3(3)$ shown in Figure 2 up to equivalence. The first example $M(P^3(3), \lambda_1)$ is homeomorphic to $S^1 \times \mathbb{R}P^2$. The second example $M(P^3(3), \lambda_2)$ is also homeomorphic to $S^1 \times \mathbb{R}P^2$ but not equivariantly homeomorphic to $M(P^3(3), \lambda_1)$; see [Lü and Yu 2011, Lemmas 4.2 and 4.3]. The last example $M(P^3(3), \lambda_3)$ is orientable and homeomorphic to $\mathbb{R}P^3 \sharp \mathbb{R}P^3$, where $\sharp$ is the connected sum (see the following section).

Example 2.4. It is easily verified that there exist four types of $(\mathbb{Z}_2)^3$-coloring on the 3-cube $I^3 = P^3(4)$. One of them is the 3-colored cube already seen in Figure 1, and is denoted by (I^3, λ_0). The other three types are shown in Figure 3. The

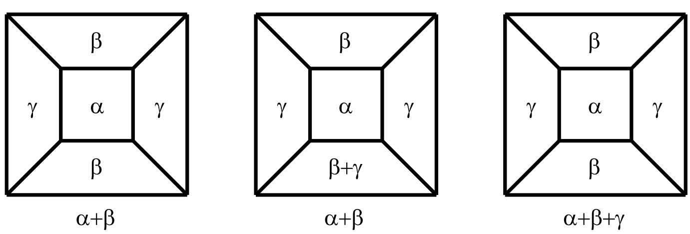

Figure 3. Three types of $(\mathbb{Z}_2)^3$-coloring on the 3-cube I^3; λ_1, λ_2 and λ_3, respectively (except the 3-colored cube of Figure 1).

associated small covers are homeomorphic to $S^1 \times K$, a twisted K-bundle over S^1, and a twisted T^2-bundle over S^1 according to λ_1, λ_2 and λ_3, respectively, where $K = \mathbb{R}P^2 \sharp \mathbb{R}P^2$ is Klein's bottle; for precise statements, see [Lü and Yu 2011, Lemmas 5.3 and 5.4].

Remark 2.5. Lü and Yu [2011] discussed D-J equivalence classes of 3-dimensional small covers. Therefore they wrote that there exist five and seven types of $(\mathbb{Z}_2)^3$-coloring on $P^3(3)$ and I^3, respectively. In this paper we discuss our equivalence (weakly equivariantly homeomorphism) classes instead of D-J equivalence classes in order to argue simply. The difference between the D-J equivalence and our equivalence does not affect the discussion on the following sections.

3. Operations on small covers

From this point on, we assume that $n = 3$ and that (P, λ) is a pair of a 3-dimensional simple convex polytope P and a $(\mathbb{Z}_2)^3$-coloring λ, and $\{\alpha, \beta, \gamma\}$ is a basis of $(\mathbb{Z}_2)^3$. We call a 3-dimensional simple convex polytope a *3-polytope* for simplicity. From Steinitz's theorem (see for example [Grünbaum 2003]), combinatorially equivalent classes of 3-polytopes bijectively correspond to 3-connected 3-valent simple planner graphs, that is, the 1-skeletons of polytopes. Here a graph Γ is called *k-connected*, *l-valent* and *simple* if Γ is connected after cutting any $(k-1)$ edges, the degree of each vertex is l, and there is no loop and no multiedge, respectively. Here, we recall some operations on $(\mathbb{Z}_2)^3$-colored polytopes (or small covers) that were introduced in [Izmest'ev 2001; Lü and Yu 2011; Nishimura 2004].

Definition 3.1 (connected sum $\sharp$). The operation $\sharp$ in Figure 4 (from left to right) is called the *connected sum* (*at vertices*) and its inverse (from right to left) is denoted by $\sharp^{-1}$. These operations also can be defined for noncolored polytopes. Note that $P_1 \sharp P_2$ is also a 3-polytope for any 3-polytopes P_1 and P_2 from Steinitz's theorem. The operation $\sharp$ corresponds to the connected sum $M(P_1, \lambda_1) \sharp M(P_2, \lambda_2)$ around fixed points of them; see [Davis and Januszkiewicz 1991, 1.11] or [Izmest'ev 2001, Definition 3]. We say that (P, λ) is *decomposable* (as a $(\mathbb{Z}_2)^3$-colored polytope) when there exist two $(\mathbb{Z}_2)^3$-colored polytopes (P_i, λ_i) for $i = 1, 2$ such that $(P, \lambda) = (P_1, \lambda_1) \sharp (P_2, \lambda_2)$. Similarly we say that P is *decomposable as a noncolored polytope* when $P = P_1 \sharp P_2$ as noncolored polytopes for some P_1 and P_2.

Specifically the connected sum with Δ^3 on polytopes, denoted by $\sharp\Delta^3$ (and often called the *cutting vertex* or *bistellar* 0*-move*), corresponds to the operation called the *blow up* on small covers; see Figure 5. Its inverse $\sharp^{-1}\Delta^3$ (often called the *bistellar* 2*-move*) is called the *blow down*.

Definition 3.2 (surgery $\natural$). The operation $\natural$ in Figure 6 (from left to right) is called the *surgery* along an edge e and its inverse $\natural^{-1}$ (from right to left) is called the

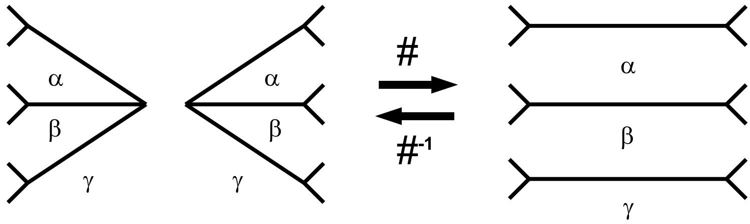

Figure 4. Connected sum $\sharp$ and its inverse $\sharp^{-1}$.

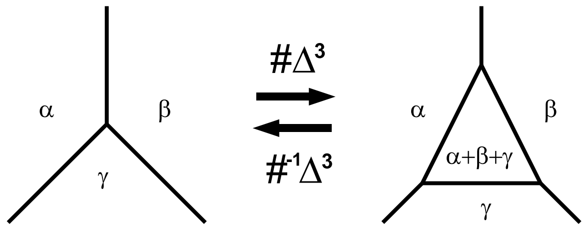

Figure 5. Blow up $\sharp\Delta^3$ and blow down $\sharp^{-1}\Delta^3$.

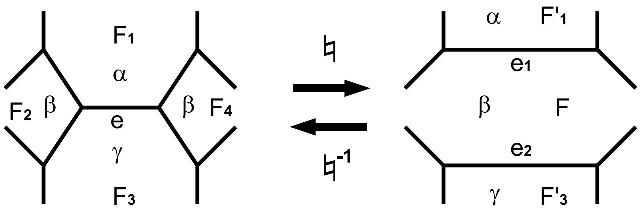

Figure 6. Surgery $\natural$ and its inverse $\natural^{-1}$.

inverse surgery along a pair of edges e_1 and e_2. The operations $\natural$ and $\natural^{-1}$ both correspond to the ordinary surgeries on small covers; see [Izmest'ev 2001]. In the previous papers [Izmest'ev 2001; Kuroki 2010; Lü and Yu 2011; Nishimura 2004], surgeries $\natural$ and $\natural^{-1}$ were not distinguished but instead were denoted by the same symbol $\natural$.

We do not allow the surgeries $\natural$ and $\natural^{-1}$ when the 3-connectedness of the 1-skeleton of P is destroyed after doing it, that is, the following cases respectively:

Case $\natural$. If and only if F_2 and F_4 are adjacent to a same face except F_1 and F_3 (involving the case when F_1 or F_3 is a quadrilateral),

Case $\natural^{-1}$. If and only if F_1' is adjacent to F_3'.

Definition 3.3 (connected sum along edges $\sharp^e$). The operation $\sharp^e$ in Figure 7 (from left to right) is called the *connected sum along edges* and its inverse is denoted

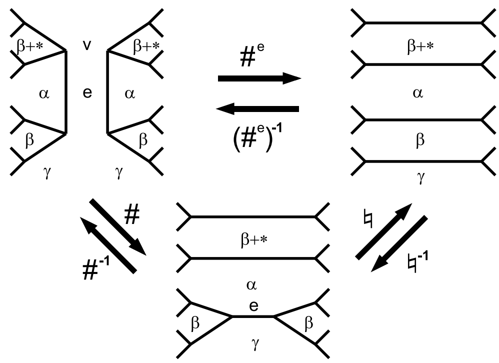

Figure 7. Connected sum along edges $\sharp^e$ and its inverse $(\sharp^e)^{-1}$. The figure also shows that $\sharp^e = \natural \circ \sharp$.

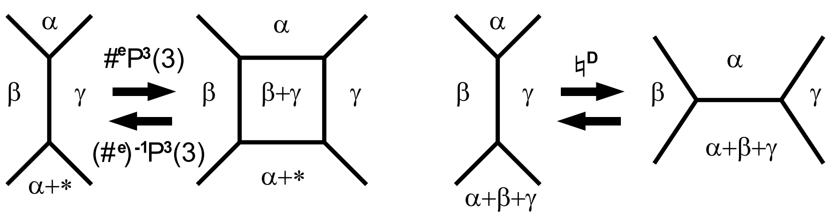

Figure 8. Cutting edge $\sharp^e P^3(3)$ and Dehn surgery $\natural^D = \sharp^e \Delta^3$.

by $(\sharp^e)^{-1}$. We notice that the operation $\sharp^e$ is obtained as the composition $\sharp^e = \natural \circ \sharp$ as shown in the same figure; see [Kuroki 2010, Theorem 4.1(2)]. The operation $\sharp^e$ corresponds to the connected sum around the circle $\pi^{-1}(e)$ of each small cover, where $\pi : M \to P$ is the projection; see [Lü and Yu 2011].

Specifically the operations $\sharp^e P^3(3)$ (along a vertical edge in Figure 2) and $\sharp^e \Delta^3$ are often called the *cutting edge* and the *bistellar* 1*-move*, respectively; see Figure 8. The former (left figure) corresponds to a blow up along the circle $\pi^{-1}(e)$ on a small cover. In this figure we can choose not only $\beta+\gamma$ but also $\alpha+\beta+\gamma$ as a color of the center square when $* = 0$. The latter operation $\sharp^e \Delta^3 = \natural \circ \sharp \Delta^3$ corresponds to the Dehn surgery of type $\frac{2}{1}$ on a small cover; see [Nishimura 2004] or [Kuroki 2010, 3.5]. This operation is denoted by $\natural^D$ and is called the *Dehn surgery*. This operation can be done along an edge e that satisfies the condition

$$\sum_e \lambda(F) := \sum_{\{F \in \mathcal{F} \mid e \cap F \neq \varnothing\}} \lambda(F) = 0.$$

We call such an edge a 0-*sum edge* (or a 4-*colored edge* in orientable case). Note that the Dehn surgery $\natural^D$ does not change the number of faces, and is invertible because $(\natural^D)^{-1} = \natural^D$.

From Steinitz's theorem, a 3-polytope P is decomposable as a noncolored polytope if and only if there exist three edges such that they are not adjacent to each other and the 1-skeleton of P becomes disconnected after cutting them. Obviously if an orientable (4-)colored polytope P is decomposable as a noncolored polytope, then (P, λ) is also decomposable as a $(\mathbb{Z}_2)^3$-colored polytope. However we need to pay a little attention to nonorientable colored polytopes. We say that (P, λ) is *quasidecomposable* when there exist two $(\mathbb{Z}_2)^3$-colored polytopes (P_1, λ_1) and (P_2, λ_2) such that either $(P, \lambda) = (P_1, \lambda_1)\sharp(P_2, \lambda_1)$ or $(P, \lambda) = (P_1, \lambda_1)\sharp^e(P_2, \lambda_2)$, except when $P = P_1 \sharp^e \Delta^3 (= \natural^D P_1)$.

Remark 3.4. If the 1-skeleton of a 3-polytope P becomes disconnected after cutting three edges $\{e', e'', e'''\}$, then these three edges are not adjacent to each other or meet at a vertex. In fact, if a pair $\{e', e''\}$ of these three edges is adjacent to each other and the other edge e''' is not adjacent to $e' \cap e''$, then the 1-skeleton of P becomes disconnected after cutting the edge e''' and the edge that is adjacent to $e' \cap e''$ and different from e' and e''. This contradicts the 3-connectedness of the 1-skeleton of P.

Proposition 3.5. *Let (P, λ) be a $(\mathbb{Z}_2)^3$-colored polytope, but not $P^3(3)$. If P is decomposable as a noncolored polytope, then (P, λ) is quasidecomposable.*

Proof. It is sufficient to treat the case that P is indecomposable as a $(\mathbb{Z}_2)^3$-colored polytope. Since P is decomposable as a noncolored polytope, there exist three nonadjacent edges such that P becomes disconnected after cutting them. Because of the assumption, colors of the three faces adjacent to these edges are not linearly independent as shown in the first figure of Figure 9.

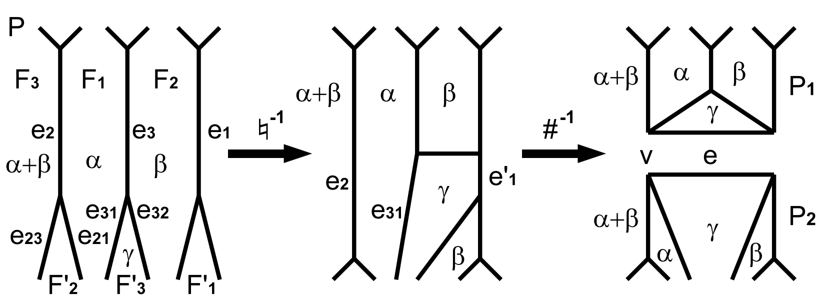

Figure 9. A decomposition of a polytope along a 3-cycle of 2-independent faces.

Since $P \neq P^3(3)$, P has at least six faces so we may assume that there are at least two distinct faces under the pillar (the F_i) in the first figure. We first assume that $F_2' = F_3'$ (equivalently $e_{21} = e_{31}$ because if it is not so, the 1-skeleton of P becomes disconnected after cutting these two edges). Then the 1-skeleton of P becomes disconnected after cutting three edges e_1, e_{23} and e_{32}. Since P is indecomposable as a $(\mathbb{Z}_2)^3$-colored polytope and the color of F_3 is $\alpha + \beta$, these three edges actually meet at a vertex $F_1' \cap F_2 \cap F_3$; see Remark 3.4. It should be $F_1' = F_2' = F_3'$ and it is a triangle. This contradicts the assumption that there are at least two faces under the pillar. Therefore, $F_2' \neq F_3'$. By a similar method, we can prove that F_i' for $i = 1, 2, 3$ are distinct faces. Note that if $F_3 \cap F_3' \neq \varnothing$, it is clear that $F_1 \cap F_1' = F_2 \cap F_2' = \varnothing$. Therefore we can assume that $F_3 \cap F_3' = \varnothing$ by changing the role of the F_i if necessary.

Now we can do the surgery $\natural^{-1}$ along edges e_1 and e_{32} (see the second figure). Moreover $\natural^{-1} P$ can be decomposed into two $(\mathbb{Z}_2)^3$-colored polytopes P_1 and P_2 by cutting three nonadjacent edges e_1', e_2 and e_{31} (see the third figure). Therefore we have $\natural^{-1} P = P_1 \sharp P_2$ or equivalently $P = P_1 \sharp^e P_2$. □

The surgery $\natural$ and the Dehn surgery $\natural^D$ are not allowed along an edge of a quadrilateral and a triangle respectively, and the inverse surgery $\natural^{-1}$ is not allowed along a pair of adjacent edges. The following is a key lemma.

Lemma 3.6. *Suppose (P, λ) is a $(\mathbb{Z}_2)^3$-colored polytope. Suppose that the 3-connectedness of the 1-skeleton of P is destroyed after doing surgeries $\natural^{-1}$ or $\natural^D$, but not the trivial prohibited cases above. Then (P, λ) is quasidecomposable. In particular, when (P, λ) is (orientable) 4-colored, (P, λ) is decomposable as a $(\mathbb{Z}_2)^3$-colored polytope.*

Proof. From Proposition 3.5, it is suffices to prove that (P, λ) is decomposable as a noncolored polytope:

Case $\natural^{-1}$. When the inverse surgery $\natural^{-1}$ is not allowed in the right figure of Figure 6, F_1' is adjacent to F_3'. Then cutting the three nonadjacent edges e_1, e_2 and $F_1' \cap F_3'$ makes the 1-skeleton of P disconnected. Thus, P is decomposable as a noncolored polytope.

Case $\natural^D$. Since $\natural^D = (\sharp^{-1} \Delta^3) \circ \natural^{-1}$ and there is no obstacle for the blow down $\sharp^{-1} \Delta^3$, the allowance of $\natural^D$ depends only on that of $\natural^{-1}$. □

4. Constructions of small covers

In this section we discuss constructions of $(\mathbb{Z}_2)^3$-colored polytopes (that is, small covers) by using two operations $\sharp$ and $\natural$. Henceforth polytopes are considered as $(\mathbb{Z}_2)^3$-colored polytopes. Izmest'ev [2001] proved the following theorem, which is a combinatorial translation of Theorem 1.1.

Theorem 4.1. *Each* 3*-colored polytope* (P^3, λ) *can be constructed from* (I^3, λ_0) *by using three operations* $\sharp$, $\natural$ *and* $\natural^{-1}$.

We start from linear models and consider constructions of orientable small covers (that is, 4-colored polytopes). Let F be an l-gonal face of P. We say that F is *j-independent* (for $j = 2$ or 3) when the rank of $\{\lambda(F_1), \dots, \lambda(F_l)\}$ is j, where $F_1, \dots, F_l$ are faces adjacent to F. In the case of orientable small covers, a j-independent face is a face such that the number of colors of adjacent faces is j (for $j = 2$ or 3). Similarly we say that an edge of P is *j-colored* (for $j = 3$ or 4) when the number of colors of the four faces adjacent to this edge is j.

Proposition 4.2. *Each* 4*-colored polytope* (P^3, λ) *can be constructed from* 3*-colored polytopes and* Δ^3 *by using two operations* $\sharp$ *and* $\natural^D$.

Proof. By induction on the number of faces of P, it is sufficient to prove that

(∗) each 4-colored polytope $P \neq \Delta^3$ can be decomposed into two polytopes after doing the Dehn surgery $\natural^D (= (\natural^D)^{-1})$ finitely many times.

Assume that P is 4-colored and not Δ^3. Then there exists a 3-independent face. Let F be a 3-independent face whose the number of edges is minimum among 3-independent faces of P, and let k be this number. We prove (∗) by induction on k. If $k = 3$ (that is, F is a triangle), then we get a colored decomposition $P = P' \sharp \Delta^3$ immediately. We assume $k \geq 4$. Since F is a 3-independent face, there exists a 4-colored edge e of F; see Figure 10.

We note that there exists no triangular face of P because $k \geq 4$. If the Dehn surgery $\natural^D$ is not allowed along an edge, then P decomposes into two polytopes by $\sharp$ or $\sharp^e$ from Lemma 3.6. Therefore we may assume that the Dehn surgery $\natural^D$ is allowed along every 4-colored edge of F. If the 3-independence of F is preserved under the Dehn surgery $\natural^D$ along some edge of F, then we can reduce P to $\natural^D P$. Because $\natural^D P$ has a $(k-1)$-gonal 3-independent face, the proof ends by induction on k. Therefore it is sufficient to show the existence of such an edge.

In Figure 10 we assume that F becomes 2-independent after doing $\natural^D$ along the edge e. Then an adjacent face of F that is painted as β must be unique, and the other faces are painted by α and γ alternatively such as $* = \gamma, \dots, \star = \alpha$. In

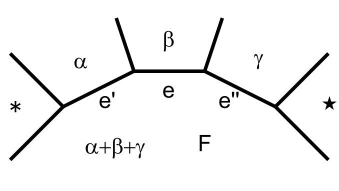

Figure 10. A 4-colored edge e of a 3-independent face F.

particular when $k = 4$ (or even), the contradiction arises because $* = \star$. When $k \geq 5$ and this situation arises, we can do the Dehn surgery $\natural^D$ along the edge e' (or e'') preserving the 3-independence of F. $\square$

Remark 4.3. In the proof of Proposition 4.2 when we ignore the colors of P, the Dehn surgery $\natural^D$ can be continued until a triangle appears for all faces. This leads to a well-known fact that "each 3-polytope is bistellarly equivalent to each other" or equivalently "the *PL*-homeomorphism class of S^2 is unique"; see [Moise 1977].

Combining Proposition 4.2 and Theorem 4.1 and noting that $\natural^D = \natural \circ (\sharp \Delta^3)$, we have the following corollary immediately; see [Nishimura 2004, Theorem 1.10] and [Kuroki 2010, Corollary 4.4].

Corollary 4.4. *Each* 4*-colored polytope* (P^3, λ) *can be constructed from* (I^3, λ_0) *and* Δ^3 *by using three operations* $\sharp$, $\natural$ *and* $\natural^{-1}$.

Next we consider a construction of all $(\mathbb{Z}_2)^3$-colored polytopes. We recall the basic fact that each 3-polytope has a face with less than six edges; see for example [Grünbaum 2003]. Such a face is called a *small face*, and otherwise a *big face*. If each small face can be compressed so that the number of faces of P decreases, then we can reduce all $(\mathbb{Z}_2)^3$-colored polytopes to some basic polytopes by induction on the number of faces. At first we compress 3-independent small faces.

Proposition 4.5. *Let* P *be a* $(\mathbb{Z}_2)^3$*-colored polytope other than* Δ^3 *and* $P^3(3)$ *as noncolored polytopes. If there exists a* 3*-independent small face of* P, *then either* P *or* $\natural^D P$ *is quasidecomposable.*

Proof. If there exists a triangular face of P other than Δ^3 and $P^3(3)$, then P is decomposable as a noncolored polytope and so (P, λ) is quasidecomposable from Proposition 3.5. Therefore we can assume that P has no triangular face. Let F be a 3-independent small face of P.

Case: *F is a quadrilateral.* The situation around F is shown as left of Figure 11 where $a_i, b_j \in \mathbb{Z}_2$ with $b_2 a_3 = 0$ and at least one of a_1 and b_1 is nonzero. By a symmetry we may assume that $a_1 = 1$. Since a triangular face of P does not exist, we can always do $(\sharp^e)^{-1} P^3(3)$ for F along either the horizontal edges (when $a_3 b_1 = 0$) or the vertical edges (when $b_1 = 1, b_2 = 0$), as shown in Figure 8. Therefore $P = P' \sharp^e P^3(3)$, so P is quasidecomposable.

Case: *F is a pentagon.* The situation around F is shown as right of Figure 11, where $a_i, b_j, c_k \in \mathbb{Z}_2$ with $a_2 b_3 + b_2 = 1$, $b_2 c_3 + b_3 = 1$ and at least one of a_1, b_1 and c_1 is nonzero. We prove that there exists a 0-sum edge of F such that F is transformed by $\natural^D P$ into a 3-independent quadrilateral. Then $\natural^D P$ is quasidecomposable from the earlier case. Here if the Dehn surgery $\natural^D$ along this edge is not allowed, then P is quasidecomposable from Lemma 3.6.

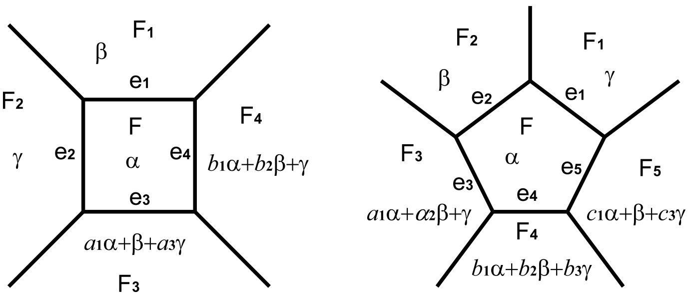

Figure 11. $(\mathbb{Z}_2)^3$-colorings around a quadrilateral and a pentagon.

(i) The case $a_1 = 1$ (the case $c_1 = 1$ can be treated similarly).

If $a_2 = 1$, then e_2 is a 0-sum edge. If $c_1 = 0$ or $b_1 + b_2 = 1$, then the Dehn surgery $\natural^D$ along the edge e_2 preserves the 3-independence of F because the rank of $\{\lambda(F_1), \lambda(F_3), \lambda(F_4), \lambda(F_5)\}$ is three. If $c_1 = 1$ and $b_1 = b_2 = 0$, then we have $b_3 = 1$. Therefore, e_3 is a 0-sum edge and $\{\lambda(F_1), \lambda(F_2), \lambda(F_5)\}$ is linearly independent. If $c_1 = 1$ and $b_1 = b_2 = 1$, then we have $b_3 = 0$ and $c_3 = 1$. Therefore, e_1 is a 0-sum edge and $\{\lambda(F_2), \lambda(F_3), \lambda(F_4)\}$ is linearly independent. In all cases the Dehn surgery $\natural^D$ along a certain edge preserves the 3-independence of F.

If $a_2 = 0$, then we have $b_2 = 1$ and $b_3 + c_3 = 1$. Therefore we obtain $\sum_{e_4} \lambda(F) = (b_1 + c_1)\alpha$ and $\sum_{e_5} \lambda(F) = (b_1 + c_1 + 1)\alpha$, so either e_4 or e_5 is a 0-sum edge. Since $\{\lambda(F_1), \lambda(F_2), \lambda(F_3)\}$ is linearly independent, the Dehn surgery $\natural^D$ along e_4 or e_5 preserves the 3-independence of F. This establishes the statement for the case when $a_1 = 1$ or $c_1 = 1$.

(ii) The case $a_1 = c_1 = 0$.

Because of the assumption, $b_1 = 1$. We have $a_2b_3 + b_2 = 1$, $b_2c_3 + b_3 = 1$ and $\{\lambda(F_1), \lambda(F_2), \lambda(F_4)\}$ is linearly independent. In this case, since $\sum_{e_3} \lambda(F) = (a_2 + b_2 + 1)\beta + (b_3 + 1)\gamma = a_2(1 + b_3)\beta + b_2c_3\gamma$ and $\sum_{e_5} \lambda(F) = (b_2 + 1)\beta + (b_3 + c_3 + 1)\gamma = a_2b_3\beta + c_3(1 + b_2)\gamma$, it is easy to check that either e_3 or e_5 is a 0-sum edge. Then the Dehn surgery $\natural^D$ along e_3 or e_5 preserves the 3-independence of F. $\square$

Remark 4.6. In the proposition above, we get a decomposition $P = P' \sharp \Delta^3$ when P has a 3-independent triangle, $P = P' \sharp^e P^3(3)$ or $P' \sharp P^3(3)$ when P has a 3-independent quadrilateral, $P = \natural^D(P' \sharp^e P^3(3))$ or $\natural^D(P' \sharp P^3(3))$ when P has a 3-independent pentagon and $\natural^D$ is allowed, and otherwise $P = P' \sharp P''$ or $P' \sharp^e P''$. In all cases, P' has fewer faces than P.

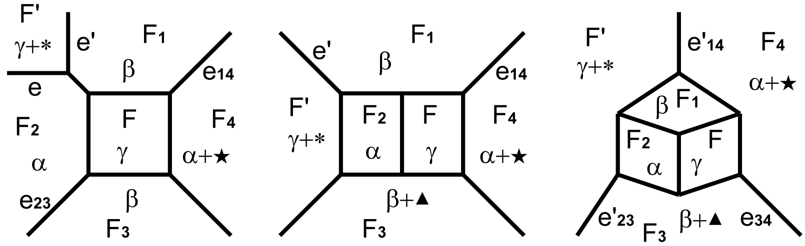

Figure 12. The compression of a 2-independent quadrilateral.

Proposition 4.7. *Let P be a $(\mathbb{Z}_2)^3$-colored polytope other than Δ^3, $P^3(3)$ and I^3 as noncolored polytopes. If there exists a 2-independent small face of P, then either P or $\natural^{-1}P$ is quasidecomposable.*

Proof. Assume that $P \neq \Delta^3$, $P^3(3)$, I^3. If there exists a triangular face of P, then P is decomposable as a noncolored polytope and so (P, λ) is quasidecomposable from Proposition 3.5. More precisely, in this case P is expressed as one of $P' \sharp \Delta^3$, $P' \sharp P^3(3)$ and $P = P' \sharp^e P^3(3)$ (along a horizontal edge in Figure 2). Therefore we can assume that P has no triangular face. Let F be a 2-independent small face of P. We note that (P, λ) is quasidecomposable when the inverse surgery $\natural^{-1}$ is not allowed in the following discussion by Lemma 3.6.

Case: *F is a quadrilateral.* Because $P \neq I^3$, it easily follows from Steinitz's theorem that the number of quadrilaterals adjacent to F is at most two. Then, the situation around F is shown as one of three figures in Figure 12 where $\star = \beta$ or 0 and $\blacktriangle = \alpha$ or 0. If F_1 and F_2 are quadrilateral (see the third figure), then P can be decomposed into the connected sum of a certain polytope P' and I^3 with a certain coloring because the 1-skeleton of P becomes disconnected after cutting three edges e'_{14}, e'_{23} and e_{34} (see the fact mentioned before Remark 3.4). If F_2 is quadrilateral and F_1 and F_3 have both at least five edges (the second figure), then we can do the surgery $\natural^{-1}$ along edges e' and e_{14} because $\lambda(F')$, $\lambda(F_1)$ and $\lambda(F_4)$ are linearly independent. This leads to the third figure, so $\natural^{-1}P$ is decomposable (that is, P is quasidecomposable). More precisely $(P, \lambda) = (P', \lambda') \sharp^e (I^3, \lambda'')$ for some $(\mathbb{Z}_2)^3$-colored polytope (P', λ') and a $(\mathbb{Z}_2)^3$-coloring λ'' on I^3. If F is not adjacent to a quadrilateral (the first figure), then we can do the surgery $\natural^{-1}$ along edges e and e_{23} because $\lambda(F')$, $\lambda(F_2)$ and $\lambda(F_3)$ are linearly independent. This leads to the second figure. Therefore $\natural^{-1}P$ is quasidecomposable.

Case: *F is a pentagon.* The situation around F is shown as the first figure in Figure 13. We can assume that P has no triangle and no quadrilateral from the previous case and the proof of Proposition 4.5. We do the surgery $\natural^{-1}$ along the edges e and e' and divide F into a triangle and a quadrilateral (the second figure).

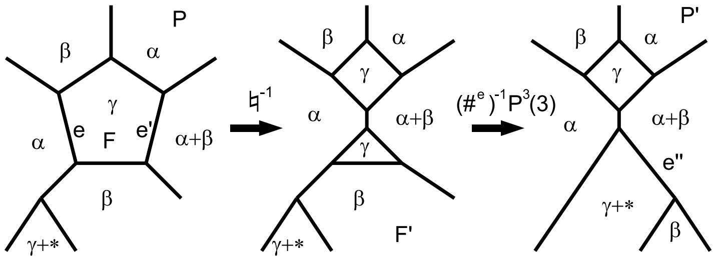

Figure 13. The compression of a 2-independent pentagon.

Since $\natural^{-1}P$ has a triangular face, it is quasidecomposable from Proposition 3.5. More precisely, $\natural^{-1}P = P' \sharp^e P^3(3)$ along the edge e'' (see the third figure). $\square$

Remark 4.8. When F is a pentagon in the proof of Proposition 4.7, although the compression of the triangle of $\natural^{-1}P$ does not change the number of faces compared with that of P, a pentagon F is transformed into a quadrilateral by this step (see the third figure in Figure 13). Then we apply the argument the first case in the proof of Proposition 4.7 to this quadrilateral so that the number of faces in the resulting polytope is one less than the number of faces in P.

In consequence of Propositions 4.5 and 4.7, we can reduce any $(\mathbb{Z}_2)^3$-colored polytope to Δ^3, I^3 and $P^3(3)$ with a certain coloring by using the surgeries $\natural^{-1}$ and $\natural^D = (\sharp^{-1}\Delta^3)\circ\natural^{-1}$ (without $\natural$) and the inverses of connected sums $\sharp$ and $\sharp^e(=\natural\circ\sharp)$. From Examples 2.3 and 2.4 the possible colorings on $P^3(3)$ and I^3 are only three and four types, respectively. We notice that $(I^3, \lambda_i) = (P^3(3), \lambda_i) \sharp^e (P^3(3), \lambda_i)$ for $i = 1, 2, 3$ along vertical edges and $(P^3(3), \lambda_3) = \Delta^3 \sharp \Delta^3$. Therefore there exist four basic $(\mathbb{Z}_2)^3$-colored polytopes: (I^3, λ_0) (3-colored), Δ^3 (orientable 4-colored), $(P^3(3), \lambda_1)$ (nonorientable 4-colored) and $(P^3(3), \lambda_2)$ (nonorientable 5-colored). Since the surgeries $\natural$ and $\natural^{-1}$ preserves the number of colors of faces, and the connected sum $\sharp$ increases the number of faces, it is clear that these four polytopes can not be constructed from others by using only $\sharp$, $\natural$ and $\natural^{-1}$. Therefore:

Theorem 4.9. *Each $(\mathbb{Z}_2)^3$-colored polytope (P^3, λ) can be constructed from Δ^3, (I^3, λ_0), $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using two operations $\sharp$ and $\natural$.*

The topological translation of this theorem is Theorem 1.4 shown in the introduction. We restrict the theorem above to 3- and 4-colored polytopes, and obtain improvements of Theorem 4.1 and Corollary 4.4, respectively:

Corollary 4.10. (1) *Each 3-colored polytope (P^3, λ) can be constructed from (I^3, λ_0) by using two operations $\sharp$ and $\natural$.*

(2) *Each 4-colored polytope (P^3, λ) can be constructed from Δ^3 and (I^3, λ_0) by using two operations $\sharp$ and $\natural$.*

5. Nondecreasing constructions of small covers

Since the operations $\natural$ and its inverse $\natural^{-1}$ both correspond to surgeries on small covers, from Izmest'ev's point of view in [2001], we used the surgeries $\natural$ and $\natural^{-1}$ in the previous section. However Lü and Yu [2011] considered a "nondecreasing" construction by only operations that do not decrease the number of faces. Therefore they did not use $\natural$ in [2011]. To overcome some obstacles, they introduced new operations $\sharp^{eve}$, $\sharp^{\Delta}$ and $\sharp_i^{©}$, with the following result:

Theorem 5.1 [Lü and Yu 2011, Theorem 1.1]. *Each $(\mathbb{Z}_2)^3$-colored polytope (P^3, λ) can be constructed from Δ^3 and $(P^3(3), \lambda_2)$ by using seven operations $\sharp$, $\sharp^e$, $\sharp^{eve}$, $\natural^{-1}$, $\sharp^{\Delta}$, $\sharp_4^{©}$ and $\sharp_5^{©}$.*

However there is a gap in the proof of their paper, which we will point out. In this section we also consider a nondecreasing construction of small covers in their point of view. At first we start with 3-colored polytopes (that is, linear models). Izmest'ev [2001] claimed that each 3-colored polytope can be constructed from 3-colored prisms $P^3(2l)$ by using $\sharp$ and $\natural^{-1}$ in the proof of Theorem 4.1. From the relation $P^3(2l) = I^3 \sharp^e \cdots \sharp^e I^3$, we can obtain a construction of 3-colored polytopes as follows.

Proposition 5.2. *Each 3-colored polytope (P^3, λ) can be constructed from (I^3, λ_0) by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.*

Above, we use the operation $\sharp^e$ instead of $\natural$ used in Theorem 4.1. Then we can also use the Dehn surgery $\natural^D$ and its inverse because of the relations $\natural^D = \sharp^e \Delta^3$ and $(\natural^D)^{-1} = \natural^D$. Applying Proposition 4.2 to the proposition above, we have this:

Proposition 5.3. *Each 4-colored polytope (P^3, λ) can be constructed from Δ^3 and (I^3, λ_0) by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.*

However, the operations $\sharp$, $\sharp^e$ and $\natural^{-1}$ are not enough to construct all $(\mathbb{Z}_2)^3$-colored polytopes. To analyze the "nondecreasing" construction of general $(\mathbb{Z}_2)^3$-colored polytopes, we first prove the following lemma.

Lemma 5.4. *Let (P, λ) be a $(\mathbb{Z}_2)^3$-colored polytope and e be an edge of P but not an edge of a quadrilateral. Suppose that the 3-connectedness of the 1-skeleton of P is destroyed after doing surgery $\natural$ along the edge e. Then (P, λ) is quasi-decomposable.*

Proof. In Figure 6, we assume that the surgery $\natural$ destroys the 3-connectedness of the 1-skeleton of P. Then there exists a face F such that $F \cap F_2 \neq \varnothing$ and $F \cap F_4 \neq \varnothing$; see Figure 14. Since neither F_1 nor F_3 is a quadrilateral, $R = Q$ and $R' = Q'$ cannot both hold simultaneously (in particular $P \neq P^3(3)$). By a symmetry we can assume that $R \neq Q$, that is, e_1 is not adjacent to e_2. When e_1' is adjacent to e_4 (that is, when $R' = Q'$), the 1-skeleton of P becomes disconnected

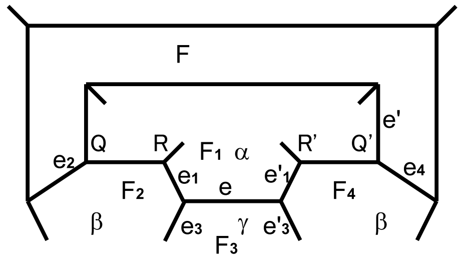

Figure 14. Obstacle to surgery ♮ (correctly, F is a convex polygon).

after cutting the three nonadjacent edges e_1, e_2, e'. Therefore P is decomposable as a noncolored polytope, so P is quasidecomposable from Proposition 3.5. Hence we assume that e'_1 is not adjacent to e_4 (that is, $R' \neq Q'$). We do the inverse surgery $\natural^{-1}$ along the pair of edges $\{e'_i, e_4\}$ where $i = 3$ when $\lambda(F)$ is either α or $\alpha + \beta$, and $i = 1$ when it is not so. If the inverse surgery $\natural^{-1}$ is not allowed, then (P, λ) is quasidecomposable from Lemma 3.6. Otherwise, the graph of $\natural^{-1}P$ becomes disconnected after cutting the three nonadjacent edges e_2, e_i (where $i = 1, 3$) and the edge constructed by gluing e'_i and e_4 by $\natural^{-1}$, and $\{\lambda(F), \lambda(F_2), \lambda(F_i)\}$ is linearly independent. Therefore $\natural^{-1}P$ is decomposable as a $(\mathbb{Z}_2)^3$-colored polytope such as $\natural^{-1}P = P_1 \sharp P_2$, or equivalently $P = P_1 \sharp^e P_2$. Thus, P is quasidecomposable. □

Remark 5.5. Izmest'ev [2001] used the lemma above only when F_4 in Figure 14 is a quadrilateral. In this case, P is always decomposable as a noncolored polytope. Lü and Yu [2011] claimed without proof that this argument can be generalized to every case under the hypothesis of Lemma 5.4 (see [ibid., Proposition 2.5]), and proved Theorem 5.1 using this claim when F_4 is also a pentagon. However their claim is incorrect; see Figure 15. This gap in their proof of Theorem 5.1 is filled by using Lemma 5.4 instead of using [Proposition 2.5]. Furthermore, Theorem 5.1 is improved by replacing $\sharp^{\Delta}$ with $\sharp_3^{©}$ as follows: Each $(\mathbb{Z}_2)^3$-colored polytope (P^3, λ) can be constructed from Δ^3 and $(P^3(3), \lambda_2)$ by using seven operations $\sharp$, $\sharp^e$, $\sharp^{eve}$, $\natural^{-1}$ and $\sharp_i^{©}$ for $i = 3, 4, 5$.

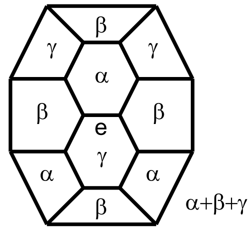

Figure 15. A counterexample of [Lü and Yu 2011, Proposition 2.5].

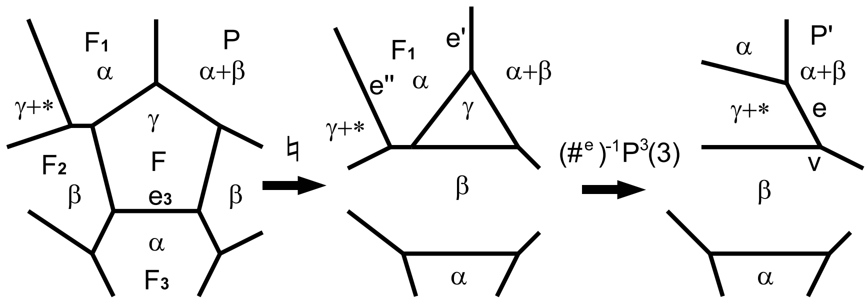

Figure 16. Another compression of a 2-independent pentagon. In the first figure we may assume that F_3 is not a quadrilateral by replacing it by F_2 if necessary because there is no pair of quadrilaterals adjacent to each other in P. Then we can do the surgery $\natural$ along the edge e_3 and transform F into a triangle (the second figure). Here when the surgery $\natural$ is not allowed, P is quasidecomposable from Lemma 5.4. Then the triangle can be compressed by $(\sharp^e)^{-1}P^3(3)$ along the edge e (which is also the composition of $\natural^{-1}$ along e', e'' and $\sharp^{-1}P^3(3)$ at v) and we have $P = \natural^{-1}(P' \sharp^e P^3(3))$ in the third figure.

From the discussion of Propositions 4.5 and 4.7, the number of faces of $(\mathbb{Z}_2)^3$-colored polytopes can be reduced by using the inverses of $\sharp$ and $\sharp^e$ when P has a 3-independent small face (see Remark 4.6 and $\natural^D = (\sharp^e \Delta^3)^{-1}$), or a 2-independent triangle ($P = P' \sharp P^3(3)$ or $P' \sharp^e P^3(3)$ along a horizontal edge), or a pair of 2-independent quadrilaterals adjacent to each other ($P = P' \sharp I^3$ or $P' \sharp^e I^3$). Moreover, each 2-independent pentagon can be compressed by using the surgery $\natural$ as shown in Figure 16.

In general when colors of two faces on ends of a common edge of big faces coincide, we can do the surgery $\natural$ along this edge and decrease the number of faces. In such a way, we reduce the number of faces of P by using $\natural$, $\sharp^{-1}$ and $(\sharp^e)^{-1}$. Here, we denote an ultimate polytope obtained by these operations by $\tilde{P}$. By the argument as above, $\tilde{P}$ satisfies the conditions that

(1) $\tilde{P}$ is not quasidecomposable,

(2) each small face of $\tilde{P}$ is an isolated 2-independent quadrilateral, and

(3) the colors of any two faces on the end of every edge that is adjacent to big faces do not coincide.

There are many polytopes satisfying this condition; see Figure 17. Such a polytope $\tilde{P}$ is called *irreducible*. To reduce $\tilde{P}$ we need a coloring change operation $\sharp_4^{©}$ introduced in [Lü and Yu 2011].

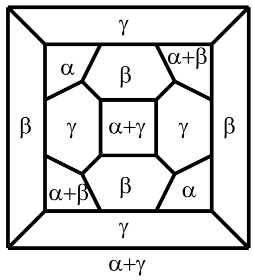

Figure 17. Example of an irreducible polytope; truncated octahedron with a $(\mathbb{Z}_2)^3$-coloring; see [Lü and Yu 2011, Example 2.1].

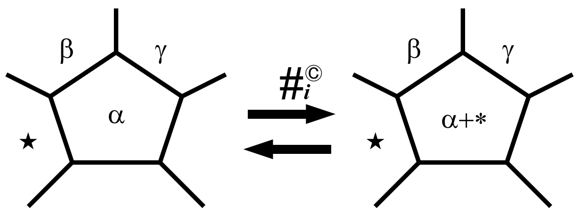

Figure 18. Coloring change $\sharp_i^{©}$ for 2-independent i-gon.

Definition 5.6. The operation in Figure 18 is called the *coloring change* $\sharp_i^{©}$ for a 2-independent i-gon. This operation is defined as the connected sum along faces to an i-gonal prism $P^3(i)$. In particular, $\sharp_3^{©} = \sharp^{\Delta}(P^3(3), \lambda_2)$; see [Lü and Yu 2011]. It is clear that $\sharp_i^{©}$ is invertible because $(\sharp_i^{©})^{-1} = \sharp_i^{©}$.

By using the operation $\sharp_4^{©}$, we can change a color of each 2-independent quadrilateral F of an irreducible polytope $\tilde{P}$, and compress it by the surgery $\natural$ as the following way. The situation around F is shown as Figure 19. Here F_i for $1 \leq i \leq 4$ are all big faces, and $F_5 \neq F_6$ because F_1 is not a quadrilateral. Moreover F_5 is not adjacent to F_6 because $\tilde{P}$ is not decomposable as a noncolored polytope. After changing color of F as $\lambda(F_5)$ by the operation $\sharp_4^{©}$, if the surgery $\natural$ along the edge e_5 is allowed, then we can do it and reduce the number of faces of $\tilde{P}$. If this surgery is not allowed, then F_5 is adjacent to F_2 or F_3. In this case F_6 is not adjacent to F_3 and F_4. Therefore we can do the surgery $\natural$ along e_6 after changing color of F as $\lambda(F_6)$ by the operation $\sharp_4^{©}$, and reduce the number of faces of $\tilde{P}$.

Moreover the 3-colored cube (I^3, λ_0) is obtained by this operation from other basic polytopes such as $\sharp_4^{©}(I^3, \lambda_i)$ for $i = 1$ or 3. Therefore we have an improvement of Theorem 5.1 as follows.

Theorem 5.7. *Each $(\mathbb{Z}_2)^3$-colored polytope (P^3, λ) can be constructed from Δ^3, $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using four operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^{©}$.*

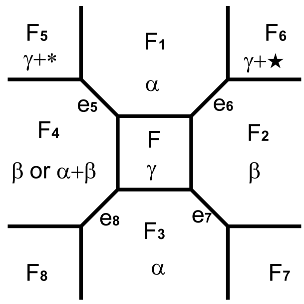

Figure 19. Compression of an isolated 2-independent quadrilateral.

The topological translations of Propositions 5.2 and 5.3 and Theorem 5.7 are stated in Theorem 1.6.

6. Locally standard 2-torus manifolds over D^3

A 2-*torus manifold* M^n is an n-dimensional closed smooth manifold with an effective action of $(\mathbb{Z}_2)^n$; see [Lü 2009; Lü and Masuda 2009] for details. If the action is locally standard, then the orbit space Q is a nice manifold with corners. When Q is a simple convex polytope, M is a small cover.

We consider the case that Q is a 3-dimensional disc D^3 with a simple cell decomposition of the boundary ∂D^3, that is, *a locally standard* 2-*torus manifold over* D^3. This class is a little wider than 3-dimensional small covers. In fact the 1-skeleton of Q is a 2-connected 3-valent planar graph. This graph is simple and 3-connected if and only if Q is a simple convex polytope. In this category there is no obstacle to surgeries. Therefore the argument in the previous section becomes easy.

Example 6.1. In Figure 20 we show the characteristic functions of S^3 with the standard $(\mathbb{Z}_2)^3$-action and three different $(\mathbb{Z}_2)^3$-colorings of the 2-sided prism $P^3(2)$, respectively. The associated 2-torus manifolds $M(P^3(2), \lambda_i)$ are homeomorphic to $S^1 \times S^2$, the S^2-bundle over S^1 characterized by the conjugation $z \mapsto \bar{z}$ on $S^2 = \mathbb{C}P^1$

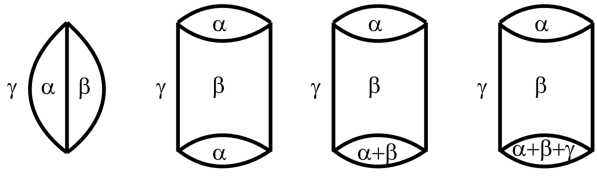

Figure 20. The $(\mathbb{Z}_2)^3$-colored simple cell decompositions of D^3; $\emptyset$, $(P^3(2), \lambda_0)$, $(P^3(2), \lambda_1)$ and $(P^3(2), \lambda_2)$.

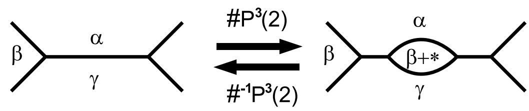

Figure 21. Blow up $\sharp P^3(2)$ and its inverse. In particular $\sharp(P^3(2), \lambda_0)$ (when $* = 0$) is identified with the inverse surgery $\natural^{-1}$ along a pair of adjacent edges. In [Kuroki 2010] the blow down $\sharp^{-1} P^3(2)$ is written by $\natural^0$.

and $S^1 \times S^2$ respectively as $i = 0, 1, 2$. We denote $M(P^3(2), \lambda_1)$ by $S^1 \times_{\mathbb{Z}_2} S^2$, where the $\mathbb{Z}_2$-action on $S^1 \times S^2$ is given as $t \cdot (s, z) = (-s, \bar{z})$.

Remark 6.2. We can easily verify the following relations:

(1) $\sharp\oslash$ is trivial and $\sharp^e\oslash = \natural$.

(2) $\sharp P^3(2)$ (or $\sharp^e P^3(2)$ along the horizontal edge) is a blow up shown in Figure 21 and $\sharp^e P^3(2)$ (along the vertical edge) is trivial.

(3) $\natural^2(I^3, \lambda_0) = (P^3(2), \lambda_0)$ and $\natural(P^3(2), \lambda_0) = \oslash$.

(4) $\natural(P^3(3), \lambda_1) = (P^3(2), \lambda_1)$ and $\natural(P^3(3), \lambda_3) = (P^3(2), \lambda_2)$.

(5) $\natural^D \Delta^3 = (P^3(2), \lambda_2)$.

Assume that $Q \neq P^3(2)$ and the 1-skeleton of Q is 2-connected but not 3-connected. There exist two edges e_1 and e_2 such they are not adjacent to each other and the 1-skeleton of Q becomes disconnected after cutting them. Then, the 1-skeleton of Q becomes disconnected after cutting e_1 and other two edges that are adjacent to a vertex of e_2; see Remark 3.4. Here we can choose these three edges such that they do not adjoin one vertex because Q is not $P^3(2)$. Therefore we have a decomposition $Q = Q' \sharp Q''$ for some $Q', Q'' \neq \oslash$, that is, Q is decomposable as a $(\mathbb{Z}_2)^3$-colored cell decomposition of D^3. Applying (3), (4) and (5) of Remark 6.2 to Theorem 4.9, we obtain the following corollary immediately.

Corollary 6.3. *Each $(\mathbb{Z}_2)^3$-colored cell decomposition of D^3 can be constructed from Δ^3, (I^3, λ_0), $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using two operations $\sharp$ and $\natural$.*

In the category of 2-torus manifolds, there is no obstacle to surgeries and blow downs. Therefore we need not consider the case that surgeries are not allowed (for example, Lemmas 3.6 and 5.4), and obtain the following theorem.

Theorem 6.4. (1) *Each 3-colored cell decomposition of D^3 can be constructed from $\oslash$ by using the inverse surgery $\natural^{-1}$.*

(2) *Each 4-colored cell decomposition of D^3 can be constructed from $\oslash$ by using the inverse surgery $\natural^{-1}$, the Dehn surgery $\natural^D (= \sharp^e \Delta^3)$ and the blow up $\sharp\Delta^3$.*

(3) *Each* $(\mathbb{Z}_2)^3$*-colored cell decomposition of* D^3 *can be constructed from* $\oslash$ *by using the inverse surgery* $\natural^{-1}$ *and connecting* Δ^3, $(P^3(2), \lambda_1)$ *and* $(P^3(3), \lambda_2)$ *by the operations* $\sharp$ *and* $\sharp^e$.

Proof. Let (Q, λ) be a $(\mathbb{Z}_2)^3$-colored cell decomposition of D^3 but not $\oslash$. If a 2-gonal face appears in the following discussion, then $(P^3(2), \lambda_1)$ is separated from Q or we do the surgery $\natural$ and this 2-gon is compressed such as Figure 21 immediately.

First, each 3-colored cell decomposition except $\oslash$ can be done by the surgery $\natural$ along some edge. Clearly, this operation decreases the number of faces.

Second, in the proof of Proposition 4.2, the Dehn surgery $\natural^D$ can be continued until a triangle appears because there is no obstacle to $\natural^D$. Therefore each 4-colored cell decomposition of D^3 can be reduced to a 3-colored cell decomposition by using $\natural^D$ and the blow down $\sharp^{-1}\Delta^3$.

Third, since there is no obstacle to the surgeries $\natural$ and $\natural^D$ in this category, in the proofs of Propositions 4.5 and 4.7, we need not consider a quasidecomposition by prohibition of surgeries. By Proposition 4.5, when Q has a 3-independent small face, Q can be reduced by one of the blow downs $\sharp^{-1}\Delta^3$, $\sharp^{-1}P^3(3)$ and $(\sharp^e)^{-1}P^3(3)$ and the Dehn surgery $\natural^D$; see Remark 4.6. By Proposition 4.7, when Q has a 2-independent triangle, the number of faces of Q can be reduced by the blow downs $\sharp^{-1}P^3(3)$ or $(\sharp^e)^{-1}P^3(3)$ (along a horizontal edge in Figure 2). Since each 2-independent quadrilateral (or pentagon) has a 3-colored edge, we can do the surgery $\natural$ along this edge in this category and decrease the number of faces. Therefore either Q or $\natural^D Q$ can be expressed as one of $\natural^{-1}Q'$, $\Delta^3 \sharp Q'$, $P^3(2) \sharp Q'$, $P^3(3) \sharp Q'$ or $P^3(3) \sharp^e Q'$ for some Q' such that the number of faces of Q' is less than that of Q.

From the relations (3), (4) and (5) in Remark 6.2, $(P^3(2), \lambda_i)$ for $i = 0, 2$ and $(P^3(3), \lambda_j)$ for $j = 1, 3$ can be constructed from $\oslash$, Δ^3, $(P^3(2), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using $\sharp$, $\sharp^e$ and $\natural^{-1}$. Here $\sharp$ (or $\sharp^e$) and $\natural^{-1}$ (or $\natural^D$) are commutative in this category such as $\sharp(P^3(2), \lambda_2) = \natural^D \circ \sharp\Delta^3$ or $\sharp(P^3(3), \lambda_1) = \natural^{-1} \circ \sharp(P^3(2), \lambda_1)$ and so on. Therefore Q can be constructed from Q' by using operations $\natural^{-1}$, $\natural^D = \sharp^e\Delta^3$, $\sharp\Delta^3$, $\sharp(P^3(2), \lambda_1)$, $\sharp(P^3(3), \lambda_2)$ and $\sharp^e(P^3(3), \lambda_2)$. For example, if $\natural^D Q$ can be expressed as $(P^3(3), \lambda_1) \sharp Q'$, then $Q = \natural^D((P^3(3), \lambda_1) \sharp Q') = \Delta^3 \sharp^e (\natural^{-1} \circ (P^3(2), \lambda_1) \sharp Q')$. By induction on the number of faces of Q, the proof is complete. □

The topological translation of Theorem 6.4 is stated in Theorem 1.7.

Acknowledgment

I thank Professor M. Masuda for his advice and stimulating discussions.

References

[Davis and Januszkiewicz 1991] M. W. Davis and T. Januszkiewicz, "Convex polytopes, Coxeter orbifolds and torus actions", *Duke Math. J.* **62**:2 (1991), 417–451. MR 92i:52012 Zbl 0733.52006

[Grünbaum 2003] B. Grünbaum, *Convex polytopes*, 2nd ed., Graduate Texts in Mathematics **221**, Springer, New York, 2003. MR 2004b:52001 Zbl 1024.52001

[Izmest'ev 2001] I. V. Izmest'ev, "Three-dimensional manifolds defined by a coloring of the faces of a simple polytope", *Mat. Zametki* **69**:3 (2001), 375–382. In Russian; translated in *Math. Notes* **69**:3 (2001), 340–346. MR 2002g:57005 Zbl 0991.57016

[Kirby 1978] R. Kirby, "A calculus for framed links in S^3", *Invent. Math.* **45**:1 (1978), 35–56. MR 57 #7605 Zbl 0377.55001

[Kuroki 2010] S. Kuroki, "Operations on 3-dimensional small covers", *Chin. Ann. Math. Ser. B* **31**:3 (2010), 393–410. MR 2011f:57029 Zbl 1217.57010

[Lü 2009] Z. Lü, "2-torus manifolds, cobordism and small covers", *Pacific J. Math.* **241**:2 (2009), 285–308. MR 2010k:55008 Zbl 1181.57036

[Lü and Masuda 2009] Z. Lü and M. Masuda, "Equivariant classification of 2-torus manifolds", *Colloq. Math.* **115**:2 (2009), 171–188. MR 2010k:52023 Zbl 1165.57023

[Lü and Yu 2011] Z. Lü and L. Yu, "Topological types of 3-dimensional small covers", *Forum Math.* **23**:2 (2011), 245–284. MR 2787623 Zbl 1222.52015

[Moise 1977] E. E. Moise, *Geometric topology in dimensions* 2 *and* 3, Graduate Texts in Mathematics **47**, Springer, New York, 1977. MR 58 #7631 Zbl 0349.57001

[Nakayama and Nishimura 2005] H. Nakayama and Y. Nishimura, "The orientability of small covers and coloring simple polytopes", *Osaka J. Math.* **42**:1 (2005), 243–256. MR 2006a:57023 Zbl 1065.05041

[Nishimura 2004] Y. Nishimura, "Equivariant surgeries of small covers", *RIMS Kôkyûroku* **1393** (2004), 44–47. In Japanese.

Received April 5, 2011. Revised March 28, 2012.

YASUZO NISHIMURA
FACULTY OF EDCATION AND REGIONAL STUDIES
UNIVERSITY OF FUKUI
3-9-1 BUNKYO
FUKUI 910-8507
JAPAN
y-nishi@u-fukui.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

http://pacificmath.org

Founded in 1951 by
E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or pacificmath.org for submission instructions.

The subscription price for 2012 is US $420/year for the electronic version, and $485/year for print and electronic.
Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and the Science Citation Index.

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS
at the University of California, Berkeley 94720-3840
A NON-PROFIT CORPORATION
Typeset in LaTeX

PACIFIC JOURNAL OF MATHEMATICS

Volume 256 No. 1 March 2012

0030-8730(201203)256:1;1-A