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CLAUS GERHARDT

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A NOTE ON INVERSE CURVATURE FLOWS IN ASYMPTOTICALLY ROBERTSON–WALKER SPACETIMES

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We prove that the leaves of the rescaled curvature flow considered in earlier work converge to the graph of a constant function.

1. Introduction

In [Gerhardt 2004] and [Gerhardt 2006a, Chapter 7] we considered the inverse mean curvature flow in a Lorentzian manifold $N = N^{n+1}$ which we called an asymptotically Robertson–Walker space, and which is defined by the following conditions:

Definition 1.1. A cosmological spacetime N, $\dim N = n+1$, is said to be *asymptotically Robertson–Walker* (ARW) with respect to the future, if a future end of N, N_+, can be written as a product $N_+ = [a, b) \times \mathcal{S}_0$, where $\mathcal{S}_0$ is a compact Riemannian space, and there exists a future directed time function $\tau = x^0$ such that the metric in N_+ can be written as

$$d\breve{s}^2 = e^{2\tilde{\psi}}\{-(dx^0)^2 + \sigma_{ij}(x^0, x)\, dx^i dx^j\}, \tag{1-1}$$

where $\mathcal{S}_0$ corresponds to $x^0 = a$, $\tilde{\psi}$ is of the form

$$\tilde{\psi}(x^0, x) = f(x^0) + \psi(x^0, x), \tag{1-2}$$

and we assume that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on $\mathcal{S}_0$ such that

$$\lim_{\tau\to b} e^{\psi} = c_0 \quad \text{and} \quad \lim_{\tau\to b} \sigma_{ij}(\tau, x) = \bar{\sigma}_{ij}(x), \tag{1-3}$$

and

$$\lim_{\tau\to b} f(\tau) = -\infty. \tag{1-4}$$

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Without loss of generality we shall assume $c_0 = 1$. Then N is ARW with respect to the future if the metric is close to the Robertson–Walker metric

$$d\bar{s}^2 = e^{2f}\{-(dx^0)^2 + \overline{\sigma}_{ij}(x)\,dx^i dx^j\} \tag{1-5}$$

near the singularity $\tau = b$. By *close* we mean that the derivatives of arbitrary order with respect to space and time of the conformal metric $e^{-2f}\breve{g}_{\alpha\beta}$ in (1-1) should converge to the corresponding derivatives of the conformal limit metric in (1-5) when x^0 tends to b. We emphasize that in our terminology Robertson–Walker metric does not imply that $(\overline{\sigma}_{ij})$ is a metric of constant curvature, it is only the spatial metric of a warped product.

We assume, furthermore, that f satisfies the following five conditions:

$$-f' > 0. \tag{1-6}$$

There exists $\omega \in \mathbb{R}$ such that

$$n + \omega - 2 > 0 \quad \text{and} \quad \lim_{\tau \to b} |f'|^2 e^{(n+\omega-2)f} = m > 0. \tag{1-7}$$

Set $\tilde{\gamma} = \frac{1}{2}(n + \omega - 2)$, then the limit

$$\lim_{\tau \to b}\big(f'' + \tilde{\gamma}|f'|^2\big) \tag{1-8}$$

exists and

$$|D^m_\tau(f'' + \tilde{\gamma}|f'|^2)| \le c_m |f'|^m \qquad \text{for all } m \ge 1, \tag{1-9}$$

as well as

$$|D^m_\tau f| \le c_m |f'|^m \qquad \text{for all } m \ge 1. \tag{1-10}$$

We call N a *normalized* ARW spacetime if

$$\int_{\mathcal{S}_0} \sqrt{\det \overline{\sigma}_{ij}} = |S^n|. \tag{1-11}$$

Remark 1.2. (i) If these assumptions are satisfied, then we proved in [Gerhardt 2004] that the range of τ is finite, hence, we shall assume without loss of generality that $b = 0$, that is,

$$a < \tau < 0. \tag{1-12}$$

(ii) Any ARW spacetime can be normalized as one easily checks. For normalized ARW spaces the constant m in (1-7) is defined uniquely and can be identified with the mass of N, see [Gerhardt 2006b].

(iii) In view of the assumptions on f the mean curvature of the coordinate slices $M_\tau = \{x^0 = \tau\}$ tends to ∞ if τ goes to zero.

(iv) ARW spaces satisfy a strong volume decay condition, see [Gerhardt 2008, Definition 0.1].

(v) Similarly one can define N to be ARW with respect to the past. In this case the singularity would lie in the past, correspond to $\tau = 0$, and the mean curvature of the coordinate slices would tend to $-\infty$.

We assume that N satisfies the timelike convergence condition. Consider the future end N_+ of N and let $M_0 \subset N_+$ be a spacelike hypersurface with positive mean curvature $\breve{H}_{|M_0} > 0$ with respect to the past directed normal vector $\breve{\nu}$—we shall explain in Section 2 why we use the symbols $\breve{H}$ and $\breve{\nu}$ and not the usual ones H and ν. Then, as we have proved in [Gerhardt 2008], the inverse mean curvature flow

$$\dot{x} = -\breve{H}^{-1}\,\breve{\nu} \tag{1-13}$$

with initial hypersurface M_0 exists for all time, is smooth, and runs straight into the future singularity.

If we express the flow hypersurfaces $M(t)$ as graphs over $\mathcal{S}_0$

$$M(t) = \operatorname{graph} u(t,\cdot), \tag{1-14}$$

then one of the main results in our former paper was:

Theorem 1.3. (i) *Let N satisfy the above assumptions, then the range of the time function x^0 is finite, that is, we may assume that $b = 0$. Set*

$$\tilde{u} = ue^{\gamma t}, \tag{1-15}$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, then there are positive constants c_1, c_2 such that

$$-c_2 \le \tilde{u} \le -c_1 < 0, \tag{1-16}$$

and $\tilde{u}$ converges in $C^\infty(\mathcal{S}_0)$ to a smooth function, if t goes to infinity. We shall also denote the limit function by $\tilde{u}$.

(ii) *Let $\breve{g}_{ij}$ be the induced metric of the leaves $M(t)$, then the rescaled metric*

$$e^{\frac{2}{n}t}\breve{g}_{ij} \tag{1-17}$$

converges in $C^\infty(\mathcal{S}_0)$ to

$$(\tilde{\gamma}^2 m)^{\frac{1}{\gamma}}(-\tilde{u})^{\frac{2}{\gamma}}\overline{\sigma}_{ij}. \tag{1-18}$$

(iii) *The leaves $M(t)$ get more umbilical if t tends to infinity, namely,*

$$\breve{H}^{-1}\big|\breve{h}_i^j - \tfrac{1}{n}\breve{H}\delta_i^j\big| \le ce^{-2\gamma t}. \tag{1-19}$$

In case $n+\omega-4>0$, *we even get a better estimate*

$$\left|\breve h_i^j-\tfrac{1}{n}\breve H\delta_i^j\right|\le ce^{-\frac{1}{2n}(n+\omega-4)t}. \tag{1-20}$$

The results for the mean curvature flow have recently also been proved for other inverse curvature flows, where the mean curvature is replaced by a curvature function F of class (K^*) homogeneous of degree 1, which includes the n-th root of the Gaussian curvature, see Kröner [2011].

In this note we want to prove that the functions in (1-15) converge to a constant. This result will also be valid when, instead of the mean curvature, other curvature functions F homogeneous of degree one will be considered satisfying

$$F(1,\dots,1)=n \tag{1-21}$$

provided the rescaled functions in (1-15) can be estimated as in (1-16) and converge in $C^3(\mathcal{S}_0)$. For simplicity we shall formulate the result only for the solution in Theorem 1.3, but it will be apparent from the proof that the result is also valid for different curvature functions.

Theorem 1.4. *The functions* $\tilde u$ *in* (1-15) *converge to a constant.*

2. Proof of Theorem 1.4

When we proved the convergence results for the inverse mean curvature flow in [Gerhardt 2004], we considered the flow hypersurfaces to be embedded in N equipped with the conformal metric

$$d\bar s^2=-\big(dx^0\big)^2+\sigma_{ij}(x^0,x)\,dx^i dx^j. \tag{2-1}$$

Though, formally, we have a different ambient space we still denote it by the same symbol N and distinguish only the metrics $\breve g_{\alpha\beta}$ and $\bar g_{\alpha\beta}$

$$\breve g_{\alpha\beta}=e^{2\widetilde\psi}\bar g_{\alpha\beta} \tag{2-2}$$

and the corresponding geometric quantities of the hypersurfaces $\breve h_{ij}$, $\breve g_{ij}$, $\breve\nu$, respectively h_{ij}, g_{ij}, ν, and so on.

The second fundamental forms $\breve h_i^j$ and h_i^j are related by

$$e^{\widetilde\psi}\breve h_i^j=h_i^j+\widetilde\psi_\alpha\nu^\alpha\delta_i^j \tag{2-3}$$

and, if we define F by

$$F=e^{\widetilde\psi}\breve H, \tag{2-4}$$

then

$$F=H-n\tilde v f'+n\psi_\alpha\nu^\alpha, \tag{2-5}$$

where

$$\tilde{v} = v^{-1}, \tag{2-6}$$

and

$$v^2 = 1 - \sigma^{ij} u_u u_j \equiv 1 - |Du|^2. \tag{2-7}$$

The evolution equation can be written as

$$\dot{x} = -F^{-1} \nu, \tag{2-8}$$

since

$$\check{\nu} = e^{-\tilde{\psi}} \nu. \tag{2-9}$$

The flow (2-8) can also be considered to comprise more general curvature functions F by assuming that $F = F(\check{h}^i_j)$, where $\check{h}^i_j$ is an abbreviation for the right-hand side of (2-3). Stipulating that indices of tensors will be raised or lowered with the help of the metric

$$g_{ij} = -u_i u_j + \sigma_{ij}, \tag{2-10}$$

we may also consider F to depend on

$$\check{h}_{ij} = h_{ij} - \tilde{v} f' g_{ij} + \psi_\alpha \nu^\alpha g_{ij} \tag{2-11}$$

and we define accordingly

$$F^{ij} = \frac{\partial F}{\partial \check{h}_{ij}}. \tag{2-12}$$

Now, let us prove Theorem 1.4. We use the relation

$$\tilde{v}^2 = 1 + \|Du\|^2 = 1 + g^{ij} u_i u_j \tag{2-13}$$

and shall prove that

$$\lim_{t\to\infty} \left(\|Du\|^2\right)' e^{2\gamma t} = 2\gamma \Delta\tilde{u}\, \tilde{u}, \tag{2-14}$$

where

$$\tilde{u} = \lim_{t\to\infty} u e^{\gamma t}, \tag{2-15}$$

as well as

$$\lim_{t\to\infty} \left(\tilde{v}^2\right)' e^{2\gamma t} = -2\gamma \|D\tilde{u}\|^2 \tag{2-16}$$

yielding

$$-\Delta\tilde{u}\, \tilde{u} = \|D\tilde{u}\|^2 \tag{2-17}$$

on the compact limit hypersurface M. Since $\tilde u$ is strictly negative we then conclude

$$\int_M \|D\tilde u\|^2 \tilde u^{-1} = 0, \tag{2-18}$$

hence $\|D\tilde u\| = 0$.

Let us first derive (2-14). Using

$$\dot g_{ij} = -2F^{-1}h_{ij}, \tag{2-19}$$

see [Gerhardt 2006a, Lemma 2.3.1], where we write $g_{ij} = g_{ij}(t,\xi)$, $\xi = (\xi^i)$ are local coordinates for $\mathcal{S}_0$, and where

$$\dot g_{ij} = \frac{\partial g_{ij}}{\partial t} = \dot u_i u_j + u_i \dot u_j + \dot\sigma_{ij}\dot u, \tag{2-20}$$

and $\dot\sigma_{ij}$ is defined by

$$\dot\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial u}, \tag{2-21}$$

we deduce

$$\begin{aligned}\big(\|Du\|^2\big)' = \big(g^{ij}u_iu_j\big)' &= 2g^{ij}\dot u_i u_j - \dot g_{ij}u^iu^j\\ &= 2F^{-1}H + g^{ij}\dot\sigma_{ij}\dot u - \dot g_{ij}u^iu^j\\ &= 2F^{-1}H + \tilde v F^{-1} g^{ij}\dot\sigma_{ij} + 2F^{-1}h_{ij}u^iu^j\\ &= 2F^{-1}H + \tilde v F^{-1}\sigma^{ij}\dot\sigma_{ij} + \tilde v^3 F^{-1}\dot\sigma_{ij}\check u^i\check u^j + 2F^{-1}h_{ij}u^iu^j,\end{aligned} \tag{2-22}$$

where we used the relation

$$g^{ij} = \sigma^{ij} + \tilde v^2 \check u^i \check u^j \tag{2-23}$$

and where $\check u^i$ is defined by

$$\check u^i = \sigma^{ij}u_j. \tag{2-24}$$

The last two terms on the right-hand side of (2-22) are an $o\big(e^{-2\gamma t}\big)$, thus we have

$$\big(\|Du\|^2\big)' = 2F^{-1}\big(H + \tilde v \tfrac12 \sigma^{ij}\dot\sigma_{ij}\big) + o\big(e^{-2\gamma t}\big). \tag{2-25}$$

On the other hand,

$$h_{ij}\tilde v = -u_{ij} + \bar h_{ij}, \tag{2-26}$$

where $\bar h_{ij}$ is the second fundamental form of the slices $\{x^0 = \text{const}\}$

$$\bar h_{ij} = -\tfrac12 \dot\sigma_{ij} \tag{2-27}$$

and we infer

$$H\tilde v = -\Delta u + g^{ij}\bar h_{ij} = -\Delta u + \bar H + \tilde v^2 \bar h_{ij}\check u^i \check u^j. \tag{2-28}$$

Combining (2-22), (2-27) and (2-28) we obtain

$$\begin{aligned}(\|Du\|^2)' &= 2F^{-1}(H - \tilde v \bar H) + o(e^{-2\gamma t})\\ &= 2F^{-1}(H - \bar H) + o(e^{-2\gamma t})\\ &= -2F^{-1}\Delta u + o(e^{-2\gamma t}).\end{aligned} \tag{2-29}$$

In view of [Gerhardt 2006a, Lemma 7.3.4], the estimates for h_{ij}, u, and ψ, and the homogeneity of F, we have

$$\lim_{t\to\infty} F(-u) = n\tilde\gamma^{-1} = \gamma^{-1}, \tag{2-30}$$

hence we deduce

$$\lim_{t\to\infty} (\|Du\|^2)' e^{2\gamma t} = 2\gamma\,\Delta\tilde u\,\tilde u. \tag{2-31}$$

Let us now differentiate $\tilde v^2$. From the relation

$$\tilde v = \eta_\alpha \nu^\alpha, \quad (\eta_\alpha) = (-1, 0, \dots, 0), \tag{2-32}$$

we infer

$$\dot{\tilde v} = \eta_{\alpha\beta}\nu^\alpha \dot x^\beta + \eta_\alpha \dot\nu^\alpha = -F^{-1}\eta_{\alpha\beta}\nu^\alpha\nu^\beta + (F^{-1})_k u^k, \tag{2-33}$$

where we used

$$\dot\nu = (-F^{-1})^k x_k, \tag{2-34}$$

see [Gerhardt 2006a, Lemma 2.3.2]. The first term on the right-hand side of (2-33) is an $o(e^{-2\gamma t})$ in view of the asymptotic behavior of an ARW space, see the definition of *close* in Definition 1.1, while

$$\begin{aligned}(F^{-1})_k\\ = -F^{-2}F^{ij}\{h_{ij;k} - \tilde v_k f' g_{ij} - \tilde f'' u_k g_{ij} + \psi_{\alpha\beta}\nu^\alpha x^\beta_k g_{ij} + \psi_\alpha x^\alpha_l h^l_k g_{ij}\},\end{aligned} \tag{2-35}$$

where we applied the Weingarten equation to derive the last term on the right-hand side. Therefore, we infer

$$\lim_{t\to\infty} (F^{-1})_k u^k e^{2\gamma t} = \|D\tilde u\|^2 \tfrac{1}{n} \lim \frac{f''}{|f'|^2} = -\frac{\tilde\gamma}{n}\|D\tilde u\|^2 = -\gamma\|D\tilde u\|^2, \tag{2-36}$$

in view of (1-8) and the definition of γ in Theorem 1.3, and we deduce further

$$\lim_{t\to\infty} (\tilde v^2)' e^{2\gamma t} = -2\gamma\|D\tilde u\|^2, \tag{2-37}$$

hence the limit function $\tilde{u}$ satisfies

$$\|D\tilde{u}\|^2 = -\Delta\tilde{u}\,\tilde{u} \tag{2-38}$$

completing the proof of Theorem 1.4.

Remark 2.1. We believe that this method of proof will also work for other curvature flows driven by extrinsic curvatures, in Riemannian or Lorentzian manifolds, to prove that the leaves of the rescaled curvature flows converge to the graph of a constant function.

Indeed, applying this method we proved in [Gerhardt 2011, Lemma 6.12] that the rescaled curvature flow converges to a sphere.

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CLAUS GERHARDT
INSTITUTE OF APPLIED MATHEMATICS
RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG
IM NEUENHEIMER FELD 294
D-69120 HEIDELBERG
GERMANY

gerhardt@math.uni-heidelberg.de
http://www.math.uni-heidelberg.de/studinfo/gerhardt/

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