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ENERGY IDENTITY AND REMOVABLE SINGULARITIES OF MAPS FROM A RIEMANN SURFACE WITH TENSION FIELD UNBOUNDED IN L^2

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We prove removable singularity results for maps with bounded energy from the unit disk B of $\mathbb{R}^2$ centered at the origin to a closed Riemannian manifold whose tension field is unbounded in $L^2(B)$ but satisfies the following condition:

$$\left(\int_{B_t\setminus B_{t/2}} |\tau(u)|^2\right)^{\frac{1}{2}} \le C_1\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ and any $0 < t < 1$, where C_1 is a constant independent of t.

We will also prove that if a sequence $\{u_n\}$ has uniformly bounded energy and satisfies

$$\left(\int_{B_t\setminus B_{t/2}} |\tau(u_n)|^2\right)^{\frac{1}{2}} \le C_2\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ and any $0 < t < 1$, where C_2 is a constant independent of n and t, then the energy identity holds for this sequence and there will be no neck formation during the blow up process.

1. Introduction

Let (M, g) be a Riemannian manifold and (N, h) a Riemannian manifold without boundary. For a $W^{1,2}(M, N)$ map u, the energy density of u is defined by

$$e(u) = \tfrac{1}{2}|\nabla u|^2 = \mathrm{Tr}_g(u^*h),$$

where u^*h is the pullback of the metric tensor h.

The energy functional of the mapping u is defined as

$$E(u) = \int_M e(u)\, dV.$$

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A map $u \in C^1(M, N)$ is called a harmonic map if it is a critical point of the energy.

By the Nash embedding theorem, N can be isometrically embedded into a Euclidean space $\mathbb{R}^K$ for some positive integer K. Then (N, h) can be viewed as a submanifold of $\mathbb{R}^K$, and a map $u \in W^{1,2}(M, N)$ is a map in $W^{1,2}(M, \mathbb{R}^K)$ whose image lies on N. The space $C^1(M, N)$ should be understood in the same way. In this sense we have the following Euler–Lagrangian equation for harmonic maps.

$$\Delta u = A(u)(\nabla u, \nabla u).$$

The tension field of a map u, $\tau(u)$, is defined by

$$\tau(u) = \Delta u - A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of N in $\mathbb{R}^K$. So u is a harmonic map if and only if $\tau(u) = 0$.

Notice that, when M is a Riemann surface, the functional $E(u)$ is conformal invariant. Harmonic maps are of special interest in this case. Consider a harmonic map u from a Riemann surface M to N. Recall that Sacks and Uhlenbeck, in a fundamental paper [1981], established the well-known removable singularity theorem by using a class of piecewise smooth harmonic functions to approximate the weak harmonic map. Li and Wang [2010] gave a slightly different proof of the following removable singularity theorem.

Theorem 1.1 [Li and Wang 2010]. *Let B be the unit disk in $\mathbb{R}^2$ centered at the origin. If $u : B \setminus \{0\} \to N$ is a $W^{2,2}_{\text{loc}}(B \setminus \{0\}, N) \cap W^{1,2}(B, N)$ map and u satisfies*

$$\tau(u) = g \in L^2(B, \mathbb{R}^K),$$

then u can be extended to a map belonging to $W^{2,2}(B, N)$.

In this direction we will prove the following result:

Proposition 1.2. *Let B be the unit disk in $\mathbb{R}^2$ centered at the origin. If*

$$u : B \setminus \{0\} \to N$$

is a $W^{2,2}_{\text{loc}}(B \setminus \{0\}, N) \cap W^{1,2}(B, N)$ map and u satisfies

$$\left(\int_{B_t \setminus B_{t/2}} |\tau(u)|^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ and any $0 < t < 1$, where C is a constant independent of t, then there exists some $s > 1$ such that

$$\nabla u \in L^{2s}(B).$$

A direct corollary of this result is the following removable singularity theorem:

Theorem 1.3. *Assume that* $u \in W^{2,2}_{\mathrm{loc}}(B \setminus \{0\}, N) \cap W^{1,2}(B, N)$ *and* u *satisfies*

$$\left(\int_{B_t \setminus B_{t/2}} |\tau(u)|^2\right)^{\frac{1}{2}} \le C\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ *and any* $0 < t < 1$, *where* C *is a constant independent of* t. *Then we have*

$$u \in \bigcap_{1 < p < \frac{2}{1+a}} W^{2,p}(B, N).$$

Consider a sequence of maps $\{u_n\}$ from a Riemann surface M to N with uniformly bounded energy. Clearly $\{u_n\}$ converges to u weakly in $W^{1,2}(M, N)$ for some $u \in W^{1,2}(M, N)$, but in general it may not converge strongly in $W^{1,2}(M, N)$ to u, and the falling of the strong convergence is due to the energy concentration at finite points. Jost [1987] and Parker [1996] independently proved that, when $\tau(u_n) = 0$, that is, u_n are harmonic maps, the lost energy is exactly the sum of the energy of the bubbles. Recall that Sacks and Uhlenbeck [1981] proved that the bubbles for such a sequence are harmonic spheres defined as nontrivial harmonic maps from S^2 to N. This result is called energy identity. Furthermore they proved that there is no neck formation during the blow up process, that is, the bubble tree convergence holds true.

For the case when $\tau(u_n)$ is bounded in L^2, that is, $\{u_n\}$ is an approximated harmonic map sequence, the energy identity was proved for N is a sphere by Qing [1995], and for the general target manifold N by Ding and Tian [1995] and, independently, by Wang [1996]. Qing and Tian [1997] proved that there is no neck formation during the blow up process; see also [Lin and Wang 1998]. For the heat flow of harmonic maps, related results can also be found in [Topping 2004a; 2004b]. For the case where the target manifold is a sphere, the energy identity and bubble tree convergence were proved by Lin and Wang [2002] for sequences with tension fields uniform bounded in L^p, for any $p > 1$. In fact, they proved this result under a scaling invariant condition which can be deduced from the uniform boundness of the tension field in L^p.

By virtue of Fanghua Lin and Changyou Wang's result, it is natural to ask the following question.

Question. Let $\{u_n\}$ be a sequence from a closed Riemann surface to a closed Riemannian manifold with tension field uniformly bounded in L^p for some $p > 1$. Do energy identity and bubble tree convergence results hold true during blowing up for such a sequence?

Remark 1.4. Parker [1996] constructed a sequence from a Riemann surface whose tension field is uniformly bounded in L^1, in which the energy identity fails.

Theorem 1.5 [Li and Zhu 2010]. *Let $\{u_n\}$ be a sequence of maps from B to N in $W^{1,2}(B, N)$ with tension field $\tau(u_n)$, where B is the unit disk of $\mathbb{R}^2$ centered at the origin. If*

(I) $\|u_n\|_{W^{1,2}(B)}+\|\tau(u_n)\|_{W^{1,p}(B)} \leq \Lambda$ *for some* $p \geq \frac{6}{5}$, *and*

(II) $u_n \to u$ *strongly in* $W^{1,2}_{\mathrm{loc}}(B \setminus \{0\}, N)$ *as* $n \to \infty$,

there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and some nonnegative integer k such that, for any $i = 1, \ldots, k$, there are some points x^i_n, positive numbers r^i_n, and a nonconstant harmonic sphere ω^i (viewed as a map from $\mathbb{R}^2 \cup \{\infty\} \to N$) such that:

(1) $x^i_n \to 0$ *and* $r^i_n \to 0$ *as* $n \to \infty$;

(2) $\lim_{n\to\infty}\left(\frac{r^i_n}{r^j_n} + \frac{r^j_n}{r^i_n} + \frac{|x^i_n - x^j_n|}{r^i_n + r^j_n}\right) = \infty$ *for any* $i \neq j$;

(3) ω^i *is the weak limit or strong limit of* $u_n(x^i_n + r^i_n x)$ *in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, N)$;

(4) *Energy identity*:

$$\lim_{n\to\infty} E(u_n, B) = E(u, B) + \sum_{i=1}^{k} E(\omega^i, \mathbb{R}^2);$$

(5) *Necklessness*: *the image* $u(B)\bigcup_{i=1}^{k} \omega^i(\mathbb{R}^2)$ *is a connected set.*

Lemma 1.6. *Suppose $\tau(u)$ satisfies*

$$\left(\int_{B_t\setminus B_{t/2}} |\tau(u)|^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{t}\right)^a,$$

for some $0 < a < \frac{2}{3}$ and any $0 < t < 1$. Then $\tau(u)$ is bounded in $L^p(B)$ for some $p \geq \frac{6}{5}$.

Proof. We have

$$\begin{aligned}\int_{B_{2^{-k+1}}\setminus B_{2^{-k}}} |\tau(u)|^p &\leq C(2^{-k})^{2-p}\|\tau(u)\|^p_{L^2(B_{2^{-k+1}}\setminus B_{2^{-k}})}\\ &\leq C(2^{-k})^{2-p-ap}.\end{aligned}$$

Hence

$$\int_B |\tau(u)|^p \leq C\sum_{k=1}^{\infty}(2^{-k})^{2-p-ap}.$$

When $0 < a < \frac{2}{3}$, we can choose some $p \geq \frac{6}{5}$ such that $2 - p - ap > 0$, and so

$$\sum_{k=1}^{\infty} (2^{-k})^{2-p-ap} \leq C,$$

which implies that $\tau(u)$ is bounded in $L^p(B)$ for some $p \geq \frac{6}{5}$. □

Thus Theorem 1.5 holds for sequences $\{u_n\}$ satisfying the following conditions.

(I) $\|u_n\|_{W^{1,2}(B)} \leq \Lambda$ and $(\int_{B_t \setminus B_{t/2}} |\tau(u_n)|^2)^{\frac{1}{2}} \leq C(\frac{1}{t})^a$ for some $0 < a < \frac{2}{3}$ and any $0 < t < 1$, where C is independent of n and t, and

(II) $u_n \to u$ strongly in $W^{1,2}_{\mathrm{loc}}(B \setminus \{0\}, N)$ as $n \to \infty$.

With the help of this observation, we find the following theorem.

Theorem 1.7. *Let $\{u_n\}$ be a sequence of maps from B to N in $W^{1,2}(B, N)$ with tension field $\tau(u_n)$, where B is the unit disk of $\mathbb{R}^2$ centered at the origin. If*

(I) $\|u_n\|_{W^{1,2}(B)} \leq \Lambda$ *and*

$$\left(\int_{B_t \setminus B_{t/2}} |\tau(u_n)|^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ and any $0 < t < 1$, where C is independent of n and t, and

(II) $u_n \to u$ *strongly in* $W^{1,2}_{\mathrm{loc}}(B \setminus \{0\}, N)$ *as* $n \to \infty$,

then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and some nonnegative integer k such that, for any $i = 1, \ldots, k$, there are some points x_n^i, positive numbers r_n^i, and a nonconstant harmonic sphere ω^i (which is viewed as a map from $\mathbb{R}^2 \cup \{\infty\} \to N$), such that:

(1) $x_n^i \to 0$, $r_n^i \to 0$ *as* $n \to \infty$;

(2) $$\lim_{n\to\infty}\left(\frac{r_n^i}{r_n^j} + \frac{r_n^j}{r_n^i} + \frac{|x_n^i - x_n^j|}{r_n^i + r_n^j}\right) = \infty$$ *for any* $i \neq j$;

(3) ω^i *is the weak limit or strong limit of* $u_n(x_n^i + r_n^i x)$ *in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, N)$;

(4) *Energy identity*: $\lim_{n\to\infty} E(u_n, B) = E(u, B) + \sum_{i=1}^k E(\omega^i, \mathbb{R}^2)$;

(5) *Neckless*: *the image* $u(B) \bigcup_{i=1}^k \omega^i(\mathbb{R}^2)$ *is a connected set.*

Remark 1.8. When

$$\left(\int_{B_t \setminus B_{t/2}} |\tau(u_n)|^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{t}\right)^a$$

for some $0 < a < 1$ and any $0 < t < 1$, where C is independent of n and t, we can deduce that $\tau(u_n)$ is uniformly bounded in $L^p(B)$ for any $p < 2/(1+a)$, and when $a \to 1$, $p \to 1$. Hence our condition is stronger than the condition that the tension field is bounded in L^p for some $p > 1$, and this result suggests that we probably have a positive answer to the Question on page 367.

Organization of the paper. In Section 2 we quote and prove several important results. In Section 3 we prove the removable singularity result. Theorem 1.7 is proved in Section 4. Throughout the paper, the letter C is used to denote positive constants which vary from line to line. We do not always distinguish between sequences and their subsequences.

2. The ϵ-regularity lemma and the Pohozaev inequality

This section contains a well-known small energy regularity lemma for approximated harmonic maps and a version of the Pohozaev inequality, which will be important later. We assume that the disk $B \subseteq \mathbb{R}^2$ is the unit disk centered at the origin, which has the standard flat metric.

Lemma 2.1. *Suppose that $u \in W^{2,2}(B, N)$ and $\tau(u) = g \in L^2(B, \mathbb{R}^K)$. Then there exists an $\varepsilon_0 > 0$ such that if $\int_B |\nabla u|^2 \le \varepsilon_0^2$, we have*

$$\|u - \bar{u}\|_{W^{2,2}(B_{1/2})} \le C(\|\nabla u\|_{L^2(B)} + \|g\|_{L^2(B)}). \tag{2-1}$$

Here $\bar{u}$ is the mean value of u over $B_{1/2}$.

Proof. We can find a complete proof of this lemma in [Ding and Tian 1995]. □

Using the standard elliptic estimates and the embedding theorems, we can derive from the above lemma that

Corollary 2.2. *Under the assumptions of Proposition 1.2, we have*

$$\begin{aligned} \operatorname{Osc}_{B_{2r}\setminus B_r} u &\le C(\|\nabla u\|_{L^2(B_{4r}\setminus B_{r/2})} + r\|g\|_{L^2(B_{4r}\setminus B_{r/2})}) \\ &\le C(\|\nabla u\|_{L^2(B_{4r}\setminus B_{r/2})} + r^{1-a}). \end{aligned} \tag{2-2}$$

Lemma 2.3 (Pohozaev inequality). *Under the assumptions of Proposition 1.2, for $0 < t_2 < t_1 < 1$,*

$$\int_{\partial(B_{t_1}\setminus B_{t_2})} r\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2\Big) \le t_1\|\nabla u\|_{L^2(B_{t_1}\setminus B_{t_2})}\|g\|_{L^2(B_{t_1}\setminus B_{t_2})}. \tag{2-3}$$

Proof. Multiplying both sides of the equation $\tau(u) = g$ by $r(\partial u/\partial r)$, we get

$$\int_{B_{t_1}\setminus B_{t_2}} r\frac{\partial u}{\partial r} \triangle u = \int_{B_{t_1}\setminus B_{t_2}} r\frac{\partial u}{\partial r} g.$$

Integrating by parts, we get

$$\int_{B_{t_1}\setminus B_{t_2}} r\frac{\partial u}{\partial r} \triangle u\, dx = \int_{\partial(B_{t_1}\setminus B_{t_2})} r\Big|\frac{\partial u}{\partial r}\Big|^2 - \int_{B_{t_1}\setminus B_{t_2}} \nabla\Big(r\frac{\partial u}{\partial r}\Big)\nabla u\, dx$$

and

$$\begin{aligned}\int_{B_{t_1}\setminus B_{t_2}} \nabla\Big(r\frac{\partial u}{\partial r}\Big)\nabla u\,dx &= \int_{B_{t_1}\setminus B_{t_2}} \nabla\Big(x^k\frac{\partial u}{\partial x^k}\Big)\nabla u\,dx\\ &= \int_{B_{t_1}\setminus B_{t_2}} |\nabla u|^2 + \int_{t_2}^{t_1}\int_0^{2\pi} \frac{r}{2}\frac{\partial}{\partial r}|\nabla u|^2 r\,d\theta\,dr\\ &= \int_{B_{t_1}\setminus B_{t_2}} |\nabla u|^2 + \tfrac{1}{2}\int_{\partial(B_{t_1}\setminus B_{t_2})} |\nabla u|^2 r - \int_{B_{t_1}\setminus B_{t_2}} |\nabla u|^2\\ &= \tfrac{1}{2}\int_{\partial(B_{t_1}\setminus B_{t_2})} |\nabla u|^2 r.\end{aligned}$$

This implies the conclusion of the lemma. □

Corollary 2.4. *Under the assumptions of Proposition 1.2, we have*

$$\frac{\partial}{\partial t}\int_{B_t\setminus B_{t/2}} \Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2 \le C\|\nabla u\|_{L^2(B_t\setminus B_{t/2})} t^{-a}. \tag{2-4}$$

Proof. In the previous lemma, let $t_1 = t$ and $t_2 = t/2$. Then

$$\begin{aligned}\frac{\partial}{\partial t}\int_{B_t\setminus B_{t/2}} \Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2 &= \int_{\partial B_t}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2\Big) - \tfrac{1}{2}\int_{\partial B_{t/2}}\Big(\Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2\Big)\\ &\le \|g\|_{L^2(B_t\setminus B_{t/2})}\|\nabla u\|_{L^2(B_t\setminus B_{t/2})}\\ &\le C\|\nabla u\|_{L^2(B_t\setminus B_{t/2})} t^{-a}.\end{aligned}$$ □

Corollary 2.5. *Under the assumptions of Proposition 1.2,*

$$\int_{B_t\setminus B_{t/2}} \Big|\frac{\partial u}{\partial r}\Big|^2 - \tfrac{1}{2}|\nabla u|^2 \le C\|\nabla u\|_{L^2(B_t)} t^{1-a}. \tag{2-5}$$

Proof. Integrating both sides of the inequality (2-4) from 0 to t and noting that $\|\nabla u\|_{L^2(B_s\setminus B_{s/2})} \le \|\nabla u\|_{L^2(B_t)}$ for any $s \le t$, we get (2-5). □

3. Removal of singularities

We now discuss the removal of singularities of a class of approximated harmonic maps from the unit disk of $\mathbb{R}^2$ to a closed Riemannian manifold N.

Lemma 3.1. *Assume that u satisfies the assumptions of Proposition 1.2. Then there are constants $\lambda > 0$ and $C > 0$ such that*

$$\int_{B_r} |\nabla u|^2 \le C r^{\lambda} \tag{3-1}$$

for r small enough.

Proof. Because we only need to prove the lemma for r small, we can assume that $E(u,B)<\varepsilon_0$. Let $u^*(r):(0,1)\to\mathbb{R}^K$ be a curve defined by

$$u^*(r)=\frac{1}{2\pi}\int_0^{2\pi}u(r,\theta)\,d\theta.$$

Then

$$\frac{\partial u^*}{\partial r}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial r}\,d\theta.$$

On the one hand, we have

$$\begin{aligned}\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}\nabla u\nabla(u-u^*)&\ge\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}\Bigl(|\nabla u|^2-\Bigl|\frac{\partial u}{\partial r}\Bigr|^2\Bigr)\\&\ge\tfrac12\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}|\nabla u|^2-C(2^{-k}t)^{1-a},\end{aligned}$$

where the second inequality makes use of (2-5).

Summing k from 0 to infinity, we get

$$\int_{B_t}\nabla u\nabla(u-u^*)\ge\tfrac12\int_{B_t}|\nabla u|^2-Ct^{1-a}.$$

On the other hand,

$$\begin{aligned}&\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}\nabla u\nabla(u-u^*)\\&\quad=-\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}(u-u^*)\Delta u+\int_{\partial(B_{2^{-k}t}\backslash B_{2^{-k-1}t})}\frac{\partial u}{\partial r}(u-u^*)\\&\quad=-\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}(u-u^*)(\tau(u)-A(u)(\nabla u,\nabla u))+\int_{\partial(B_{2^{-k}t}\backslash B_{2^{-k-1}t})}\frac{\partial u}{\partial r}(u-u^*).\end{aligned}$$

Hence, by summing k from 0 to infinity, we get

$$\begin{aligned}&\int_{B_t}\nabla u\nabla(u-u^*)\\&\quad\le\sum_{k=0}^{\infty}\|u-u^*\|_{L^\infty(B_{2^{-k}t}\backslash B_{2^{-k-1}t})}\Bigl(\|A\|_{L^\infty}\int_{B_{2^{-k}t}\backslash B_{2^{-k-1}t}}|\nabla u|^2+C(2^{-k}t)^{1-a}\Bigr)\\&\qquad\qquad+\int_{\partial B_t}\frac{\partial u}{\partial r}(u-u^*)\\&\quad\le\epsilon\int_{B_t}|\nabla u|^2+Ct^{1-a}+\int_{\partial B_t}\frac{\partial u}{\partial r}(u-u^*).\end{aligned}$$

Note that we used Corollary 2.2 and ensured that ϵ is small by letting t be small.

Note that

$$\left|\int_{\partial B_t}\frac{\partial u}{\partial r}(u-u^*)\right| \le \left(\int_{\partial B_t}\left|\frac{\partial u}{\partial r}\right|^2\right)^{\frac{1}{2}}\left(\int_{\partial B_t}|u-u^*|^2\right)^{\frac{1}{2}}$$
$$\le \left(\int_0^{2\pi} t^2\left|\frac{\partial u}{\partial r}\right|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^{2\pi}\left|\frac{\partial u}{\partial \theta}\right|^2 d\theta\right)^{\frac{1}{2}}$$
$$\le \tfrac{1}{2}\int_0^{2\pi}\left(\left|\frac{\partial u}{\partial \theta}\right|^2+t^2\left|\frac{\partial u}{\partial r}\right|^2\right)d\theta = \frac{t}{2}\int_{\partial B_t}|\nabla u|^2.$$

Combining the two sides of the inequalities and letting ϵ be small (we can do this by letting t be small), we conclude that there is a constant $\lambda \in (0,1)$ such that

$$\lambda\int_{B_t}|\nabla u|^2 \le t\int_{\partial B_t}|\nabla u|^2 + Ct^{1-a}.$$

Set $f(t)=\int_{B_t}|\nabla u|^2$. Then we get the ordinary differential inequality

$$\left(\frac{f(t)}{t^\lambda}\right)' \ge -Ct^{-\lambda-a}.$$

Letting λ be small enough that $\lambda+a<1$, we get

$$f(t)=\int_{B_t}|\nabla u|^2 \le Ct^\lambda$$

for t small enough. □

Proof of Proposition 1.2. Let $r_k=2^{-k}$ and $v_k(x)=u(r_kx)$. Then

$$\left(\int_{B_2\setminus B_1}|\nabla v_k|^{2s}\right)^{\frac{1}{2s}} \le C\|v_k-\bar{v_k}\|_{W^{2,2}(B_2\setminus B_1)}$$
$$\le \left(\int_{B_4\setminus B_{1/2}}|\nabla v_k|^2\right)^{\frac{1}{2}} + C\left(\int_{B_{4r_k}\setminus B_{r_k/2}} r_k^2|\tau|^2\right)^{\frac{1}{2}}.$$

Therefore we deduce that

$$\int_{B_2\setminus B_1}|\nabla v_k|^{2s} \le C\left(\int_{B_4\setminus B_{1/2}}|\nabla v_k|^2\right)^s + C\left(\int_{B_{4r_k}\setminus B_{r_k/2}} r_k^2|\tau|^2\right)^s$$
$$\le C\left(\int_{B_4\setminus B_{1/2}}|\nabla v_k|^2\right)^s + Cr_k^{2s(1-a)}.$$

Note that when k is large enough,

$$\int_{B_{4r_k}\setminus B_{r_k/2}}|\nabla u|^2 \le 1.$$

Hence

$$r_k^{2s-2}\int_{B_{2r_k}\setminus B_{r_k}}|\nabla u|^{2s}\le C\Big(\int_{B_{4r_k}\setminus B_{r_k/2}}|\nabla u|^2\Big)^s+Cr_k^{2s(1-a)}$$
$$\le C\int_{B_{4r_k}\setminus B_{r_k/2}}|\nabla u|^2+Cr_k^{2s(1-a)}.$$

This implies that

$$\int_{B_{2r_k}\setminus B_{r_k}}|\nabla u|^{2s}\le Cr_k^{2-2s}r_k^{\lambda}+Cr_k^{2-2sa}.$$

Now choose $s>1$ such that $2s-2<\lambda/2$ and $2-2sa>0$. There exists a positive integer k_0 such that when $k\ge k_0$,

$$\int_{B_{2^{-k+1}}\setminus B_{2^{-k}}}|\nabla u|^{2s}\le C(2^{(-\lambda/2)k}+2^{-k(2-2sa)}).$$

Therefore $\int_{B_r}|\nabla u|^{2s}\le C\sum_{k=k_0}^{\infty}(2^{(-\lambda/2)k}+2^{-k(2-2sa)})\le C$ for any $r\le 2^{-k_0+1}$, which completes the proof. □

Proof of Theorem 1.3. Note that

$$\int_{B_{2^{-k}}\setminus B_{2^{-k-1}}}|\tau(u)|^p\le C(2^{-k})^{2-p}\Big(\int_{B_{2^{-k}}\setminus B_{2^{-k-1}}}|\tau(u)|^2\Big)^{p/2}\le C(2^{-k})^{2-p-pa}.$$

Summing over k from 0 to infinity, we deduce that $\int_B|\tau(u)|^p\le C$ for $p<2/(1+a)$.

Recall that we have proved that $\nabla u\in L^{2s}(B)$ for some $s>1$. Hence, by standard elliptic estimates and the bootstrap argument, we can deduce that

$$u\in\bigcap_{1<p<\frac{2}{1+a}}W^{2,p}(B,N).$$ □

4. The bubble tree structure

Energy identity. Assume that $\{u_n\}$ is a uniformly bounded sequence in $W^{1,2}(B,N)$ and that there exists a constant C, independent of n and t, such that

$$\Big(\int_{B_t\setminus B_{t/2}}|\tau(u_n)|^2\Big)^{\frac12}\le C\Big(\frac{1}{t}\Big)^a$$

for some $0<a<1$ and any $0<t<1$. In this section, we will prove the energy identity for this sequence. For convenience, we will assume that there is only one bubble ω, which is the strong limit of $u_n(r_n.)$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2,N)$. Under this assumption we can deduce the following by a standard blowup argument.

Lemma 4.1. *For any $\epsilon > 0$, there exist R and δ such that*

$$\int_{B_{2\lambda}\setminus B_\lambda} |\nabla u_n|^2 \le \epsilon^2 \quad \textit{for any} \quad \lambda \in \left(\frac{Rr_n}{2}, 2\delta\right). \tag{4-1}$$

Proof of the energy identity. For a given $R > 0$, we have

$$\lim_{n\to\infty}\int_B |\nabla u_n|^2 = \lim_{n\to\infty}\int_{B\setminus B_\delta} |\nabla u_n|^2 + \lim_{n\to\infty}\int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 + \lim_{n\to\infty}\int_{B_{Rr_n}} |\nabla u_n|^2,$$

$$\lim_{\delta\to 0}\lim_{n\to\infty}\int_{B\setminus B_\delta} |\nabla u_n|^2 = \int_B |\nabla u|^2, \quad \text{and} \quad \lim_{R\to\infty}\lim_{n\to\infty}\int_{B_{Rr_n}} |\nabla u_n|^2 = \int_{\mathbb{R}^2} |\nabla \omega|^2,$$

Hence, to prove the energy identity, we only need to prove that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty}\int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 = 0. \tag{4-2}$$

The proof is a little similar to the proof in the previous section. We assume that $\delta = 2^{m_n} Rr_n$, where m_n is a positive integer.

On the one hand, we have

$$\begin{aligned}\int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} \nabla u_n \nabla(u_n - u_n^*) &\ge \int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} \left(|\nabla u_n|^2 - \left|\frac{\partial u_n}{\partial r}\right|^2\right)\\ &\ge \tfrac{1}{2}\int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} |\nabla u_n|^2 - C(2^k Rr_n)^{1-a}.\end{aligned}$$

This implies that

$$\int_{B_\delta\setminus B_{Rr_n}} \nabla u_n \nabla(u_n - u_n^*) \ge \tfrac{1}{2}\int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 - C\delta^{1-a}.$$

On the other hand, we have

$$\begin{aligned}&\int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} \nabla u_n \nabla(u_n - u_n^*)\\ &\quad= -\int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} (u_n - u_n^*)\Delta u_n + \int_{\partial(B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n})} \frac{\partial u_n}{\partial r}(u_n - u_n^*)\\ &\quad= -\int_{B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n}} (u_n - u_n^*)(\tau(u_n) - A(u_n)(\nabla u_n, \nabla u_n))\\ &\qquad\qquad + \int_{\partial(B_{2^k Rr_n}\setminus B_{2^{k-1}Rr_n})} \frac{\partial u_n}{\partial r}(u_n - u_n^*).\end{aligned}$$

Summing from 1 to m_n, we deduce that

$$\begin{aligned}
&\int_{B_\delta\setminus B_{Rr_n}} \nabla u_n \nabla(u_n-u_n^*)\\
&\le \sum_{k=1}^{m_n} \|u_n-u_n^*\|_{L^\infty(B_{2^kRr_n}\setminus B_{2^{k-1}Rr_n})}\Big(\|A\|_{L^\infty}\int_{B_{2^kRr_n}\setminus B_{2^{k-1}Rr_n}} |\nabla u_n|^2 + C(2^kRr_n)^{1-a}\Big)\\
&\qquad\qquad + \int_{\partial(B_\delta\setminus B_{Rr_n})} \frac{\partial u_n}{\partial r}(u_n-u_n^*)\\
&\le \epsilon \int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 + C\delta^{1-a} + \int_{\partial(B_\delta\setminus B_{Rr_n})} \frac{\partial u_n}{\partial r}(u_n-u_n^*).
\end{aligned}$$

Comparing the two sides, we get

$$(1-2\epsilon)\int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 \le C\delta^{1-a} + 2\int_{\partial(B_\delta\setminus B_{Rr_n})} \frac{\partial u_n}{\partial r}(u_n-u_n^*).$$

As for the boundary terms, we have

$$\begin{aligned}
\int_{\partial B_\delta} \frac{\partial u_n}{\partial r}(u_n-u_n^*) &\le \Big(\int_{\partial B_\delta}\Big|\frac{\partial u_n}{\partial r}\Big|^2\Big)^{\frac12}\Big(\int_{\partial B_\delta} |u_n-u_n^*|^2\Big)^{\frac12}\\
&\le \Big(\int_0^{2\pi} \delta^2\Big|\frac{\partial u_n}{\partial r}\,d\theta\Big|^2\Big)^{\frac12}\Big(\int_0^{2\pi}\Big|\frac{\partial u_n}{\partial \theta}\Big|^2 d\theta\Big)^{\frac12}\\
&\le \tfrac12\int_0^{2\pi} \delta^2\Big|\frac{\partial u_n}{\partial r}d\theta\Big|^2 + \Big|\frac{\partial u_n}{\partial \theta}\Big|^2 d\theta = \frac{\delta^2}{2}\int_0^{2\pi} |\nabla u_n|^2 d\theta.
\end{aligned}$$

Now, by the trace embedding theorem, we have

$$\begin{aligned}
\int_0^{2\pi} |\nabla u_n(\,\cdot\,,\delta)|^2\delta d\theta &= \int_{\partial B_\delta} |\nabla u_n(\,\cdot\,,\delta)|^2 dS_\delta\\
&\le C\delta\|\nabla u_n\|^2_{W^{1,2}(B_{3\delta/2}\setminus B_{\delta/2})}\\
&\le C\delta\|u_n-\bar u_n\|^2_{W^{2,2}(B_{3\delta/2}\setminus B_{\delta/2})}\\
&\le C\delta\Big(\frac{1}{\delta}\|\nabla u_n\|^2_{L^2(B_{2\delta})} + \|\tau(u_n)\|^2_{L^2(B_{2\delta}\setminus B_{\delta/4})}\Big)\\
&\le C\delta^{1-2a},
\end{aligned}$$

for δ small. From this we deduce that

$$\int_{\partial B_\delta} \frac{\partial u_n}{\partial r}(u_n-u_n^*) \le C\delta^{2(1-a)}.$$

Similarly we get

$$\int_{\partial B_{Rr_n}} \frac{\partial u_n}{\partial r}(u_n-u_n^*) \le C(Rr_n)^{2(1-a)},$$

for n big enough. Therefore

$$(1-2\epsilon)\int_{B_\delta\setminus B_{Rr_n}} |\nabla u_n|^2 \le C\delta^{1-a} + C\delta^{2(1-a)} + C(Rr_n)^{2(1-a)},$$

which clearly implies (4-2), and we are done. □

Necklessness. In this part we prove that there is no neck between the base map u and the bubble ω, that is, the C^0 compactness of the sequence modulo bubbles.

Proof. We only need to prove that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} \operatorname{Osc}_{B_\delta\setminus B_{Rr_n}} u_n = 0. \tag{4-3}$$

Again we assume that $\delta = 2^{m_n}Rr_n$ and let $Q(t) = B_{2^{t+t_0}Rr_n}\setminus B_{2^{t_0-t}Rr_n}$. Similarly to the proof of the previous part, we can get

$$\begin{aligned}(1-2\epsilon)&\int_{Q(k)} |\nabla u_n|^2 \\ &\le 2^{k+t_0}Rr_n \int_{\partial B_{2^{k+t_0}Rr_n}} |\nabla u_n|^2 + 2^{t_0-k}Rr_n\int_{\partial B_{2^{t_0-k}Rr_n}} |\nabla u_n|^2 + C(2^{k+t_0}Rr_n)^{1-a}.\end{aligned}$$

Set $f(t) = \int_{Q(t)} |\nabla u_n|^2$. Then we have

$$(1-2\epsilon)f(t) \le (1-2\epsilon)f(k+1) \le \frac{1}{\log 2} f'(k+1) + C(2^{k+t_0}Rr_n)^{1-a}$$

for $k \le t \le k+1$.

Note that

$$\begin{aligned}f'(k+1) - f'(t) &= \int_{\partial(B_{2^{k+1+t_0}Rr_n}\setminus B_{2^{t+t_0}Rr_n})} \frac{\partial u_n}{\partial r}(u_n - u_n^*) + \int_{\partial(B_{2^{t_0-t}Rr_n}\setminus B_{2^{t_0-k-1}Rr_n})} \frac{\partial u_n}{\partial r}(u_n - u_n^*) \\ &\le C(2^{t+t_0}Rr_n)^{2(1-a)}.\end{aligned}$$

Therefore

$$(1-2\epsilon)f(t) \le \frac{1}{\log 2} f'(t) + C(2^{t+t_0}Rr_n)^{1-a}. \tag{4-4}$$

It follows that

$$\begin{aligned}(2^{-(1-2\epsilon)t}f(t))' &= 2^{-(1-2\epsilon)t}f'(t) - (1-2\epsilon)2^{-(1-2\epsilon)t}f(t)\log 2 \\ &\ge -C2^{(1-a-(1-2\epsilon))t}(2^{t_0}Rr_n)^{1-a}.\end{aligned}$$

Integrating from 1 to L, we get

$$\begin{aligned} 2^{-(1-2\epsilon)L} f(L) - 2^{-(1-2\epsilon)} f(1) &\geq -C \int_1^L 2^{(1-a-(1-2\epsilon))t} (2^{t_0} R r_n)^{1-a} \\ &= -C \frac{2^{(1-a-(1-2\epsilon))t}}{\log 2(1-a-(1-2\epsilon))}\Big|_1^L (2^{t_0} R r_n)^{1-a} \\ &\geq -C (2^{t_0} R r_n)^{1-a}. \end{aligned}$$

Therefore we have

$$f(1) \leq f(L) 2^{-(1-2\epsilon)(L-1)} + C(2^{t_0} R r_n)^{1-a}. \tag{4-5}$$

Now let $t_0 = i$ and $D_i = B_{2^{i+1} R r_n} \setminus B_{2^i R r_n}$. Then we have $f(1) = \int_{D_i \cup D_{i-1}} |\nabla u_n|^2$, and the inequality holds true for L satisfying

$$Q(L) \subseteq B_\delta \setminus B_{R r_n} = B_{2^{m_n} R r_n} \setminus B_{R r_n}.$$

In other words, L should satisfy $i - L \geq 0$ and $i + L \leq M_n$.

(I) If $i \leq \frac{1}{2} m_n$, let $L = i$. Then

$$f(1) = \int_{D_i \cup D_{i-1}} |\nabla u_n|^2 \leq C E^2(u_n, B_\delta \setminus B_{R r_n}) 2^{-(1-2\epsilon)i} + C(2^i R r_n)^{1-a}.$$

(II) If $i > \frac{1}{2} m_n$, let $L = m_n - i$. Then

$$f(1) = \int_{D_i \cup D_{i-1}} |\nabla u_n|^2 \leq C E^2(u_n, B_\delta \setminus B_{R r_n}) 2^{-(1-2\epsilon)(m_n - i)} + C(2^i R r_n)^{1-a}.$$

Hence we have

$$\begin{aligned} \sum_{i=1}^{m_n} E(u_n, D_i) \leq C E(u_n, B_\delta \setminus B_{R r_n}) \Big(\sum_{i \leq \frac{1}{2} m_n} 2^{-i(1-2\epsilon)/2} + \sum_{i > \frac{1}{2} m_n} 2^{-(m_n - i)1-2\epsilon/(2)} \Big) \\ + C \sum_{i=1}^{m_n} (2^i R r_n)^{(1-a)/2} \\ \leq C E(u_n, B_\delta \setminus B_{R r_n}) + C \delta^{(1-a)/2}. \end{aligned}$$

Thus we get

$$\begin{aligned} \mathrm{Osc}_{B_\delta \setminus B_{R r_n}} u_n &\leq C \sum_{i=1}^{m_n} (E(u_n, D_i) + (2^i R r_n)^{1-a}) \\ &\leq C E(u_n, B_\delta \setminus B_{R r_n}) + C \delta^{(1-a)/2}. \end{aligned}$$

Clearly this implies (4-3), as needed. □

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