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We prove that the Halperin–Carlsson conjecture holds for any free $(\mathbb{Z}_2)^m$-action on a compact manifold whose orbit space is a small cover.

1. Introduction

For any prime p, let $\mathbb{Z}_p$ denote the quotient group $\mathbb{Z}/p\mathbb{Z}$, and S^1 the circle group.

The Halperin–Carlsson Conjecture. *If $G = (\mathbb{Z}_p)^m$ or $(S^1)^m$ can act freely on a finite CW-complex X, then, respectively,*

$$\sum_{i=0}^{\infty} \dim_{\mathbb{Z}_p} H^i(X, \mathbb{Z}_p) \geq 2^m \quad \text{or} \quad \sum_{i=0}^{\infty} \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \geq 2^m.$$

This was proposed by Halperin [1985] for the torus case and by Carlsson [1986] for the $\mathbb{Z}_p$-torus case. It is also called the *toral rank conjecture* in some papers.

At first this conjecture mainly took the form of whether a free $(\mathbb{Z}_p)^m$-action on a product of spheres $S^{n_1} \times \cdots \times S^{n_k}$ implies $m \leq k$. Many authors have studied this intriguing conjecture in its various aspects [Conner 1957; Carlsson 1982; Adem 1987; Adem and Browder 1988; Adem and Benson 1998; Hanke 2009]. For a survey of results on the topic, see [Adem 2004; Allday and Puppe 1993]. The general case is still open for any prime p.

For general finite CW-complexes, the conjecture was proved in [Puppe 2009] for $m \leq 3$ in the torus and $\mathbb{Z}_2$-torus cases and $m \leq 2$ in the odd $\mathbb{Z}_p$-torus case. Also we have the following result, achieved independently, which confirmed the Halperin–Carlsson conjecture for some special $\mathbb{Z}_2$-torus actions on real moment-angle complexes:

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Theorem 1.1 [Cao and Lü 2009; Ustinovskii 2011]. *Let K^{n-1} be an $(n-1)$-dimensional simplicial complex on the vertex set $[d]$. Then the real moment-angle complex $\mathbb{R}\mathcal{Z}_{K^{n-1}}$ over K^{n-1} must satisfy $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathbb{R}\mathcal{Z}_{K^{n-1}}, \mathbb{Z}_2) \geq 2^{d-n}$. In particular, if P^n is an n-dimensional simple convex polytope with d facets, then the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ must satisfy $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathbb{R}\mathcal{Z}_{P^n}, \mathbb{Z}_2) \geq 2^{d-n}$.*

Remark 1.2. Stronger results were obtained in [Cao and Lü 2009] and [Ustinovskii 2011]; for example, Theorem 1.1 holds even if the $\mathbb{Z}_2$-coefficients are replaced by rational coefficients.

Remark 1.3. There is a purely algebraic analogue of the Halperin–Carlsson conjecture, which was proposed in [Carlsson 1986] in the context of commutative algebras. Some related results were obtained in [Carlsson 1987].

Here we only study the conjecture for $G = (\mathbb{Z}_2)^m$ and X a closed manifold. We use the following conventions:

- we treat $(\mathbb{Z}_2)^m$ as an additive group;
- all manifolds and submanifolds are smooth;
- we do not distinguish between an embedded submanifold and its image.

Suppose that $(\mathbb{Z}_2)^m$ acts freely and smoothly on a closed n-manifold M^n. Let $Q^n = M^n/(\mathbb{Z}_2)^m$ be the orbit space. Then Q^n is a closed n-manifold too. Let $\pi : M^n \to Q^n$ be the orbit map. We can think of M^n either as a principal $(\mathbb{Z}_2)^m$-bundle over Q^n or as a regular covering over Q^n whose deck transformation group is $(\mathbb{Z}_2)^m$. In algebraic topology, we have a standard way to recover M^n from Q^n, using the universal covering space of Q^n and the monodromy of the covering [Hatcher 2002]. However, it is not so easy for us to visualize the total space of the covering with this approach. In [Yu 2012], a new way of constructing principal $(\mathbb{Z}_2)^m$-bundles over closed manifolds is introduced, which allows us to visualize this kind of object more easily.

Indeed, it is shown in [Yu 2012] that $\pi : M^n \to Q^n$ determines a $(\mathbb{Z}_2)^m$-coloring λ_π on a nice manifold with corners V^n (called a $\mathbb{Z}_2$-*core* of Q^n), and up to equivariant homeomorphism, we can recover M^n by a standard *glue-back construction* from V^n and λ_π. Using this new language, we prove the following theorem, which supports the Halperin–Carlsson conjecture.

Theorem 1.4. *Suppose that $(\mathbb{Z}_2)^m$ acts freely on a closed n-manifold M^n whose orbit space is homeomorphic to a small cover; then*

$$\sum_i \dim_{\mathbb{Z}_2} H^i(M^n, \mathbb{Z}_2) \geq 2^m. \tag{1}$$

Recall that an n-dimensional small cover is a closed n-manifold with a locally standard $(\mathbb{Z}_2)^n$-action whose orbit space can be identified with an n-dimensional simple convex polytope [Davis and Januszkiewicz 1991].

Given an arbitrary n-dimensional simple convex polytope P^n, there may not exist any small cover over P^n. But we can always define a closed manifold $\mathbb{R}\mathcal{Z}_{P^n}$ associated to P^n called a *real moment-angle manifold* [Davis and Januszkiewicz 1991, Construction 4.1]. Let $\mathcal{F}(P^n) = \{F_1, \dots, F_r\}$ be the set of facets of P^n, and let $\{e_1, \dots, e_r\}$ be a basis of $(\mathbb{Z}_2)^r$. For $1 \le i \le r$, we define a function $\lambda^* : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^r$ by

$$\lambda^*(F_i) = e_i. \tag{2}$$

For any proper face f of P^n, let G_f denote the subgroup of $(\mathbb{Z}_2)^r$ generated by the set $\{\lambda^*(F_i) \mid f \subset F_i\}$. The real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ of P^n is defined to be the quotient space

$$\mathbb{R}\mathcal{Z}_{P^n} := P^n \times (\mathbb{Z}_2)^r / \sim, \tag{3}$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_{f(p)}$, with $f(p)$ being the unique face of P^n that contains p in its relative interior. Let $[(p, g)]$ denote the equivalence class of (p, g) in $\mathbb{R}\mathcal{Z}_{P^n}$. There is a *canonical action* of $(\mathbb{Z}_2)^r$ on $\mathbb{R}\mathcal{Z}_{P^n}$ by

$$g' \cdot [(p, g)] = [(p, g' + g)],$$

for all $p \in P^n$ and $g, g' \in (\mathbb{Z}_2)^r$. This $(\mathbb{Z}_2)^r$-action on $\mathbb{R}\mathcal{Z}_{P^n}$ is not free. But a subgroup $N \subset (\mathbb{Z}_2)^r$ might act freely on $\mathbb{R}\mathcal{Z}_{P^n}$ through the canonical action. In that case, the quotient space $\mathbb{R}\mathcal{Z}_{P^n}/N$ is called a *partial quotient* of $\mathbb{R}\mathcal{Z}_{P^n}$ [Buchstaber and Panov 2002, Section 7.5]. Also, if there is another subgroup $\widetilde{N}$ of $(\mathbb{Z}_2)^r$ with $\widetilde{N} \supset N$, and $\widetilde{N}$ also acts freely on $\mathbb{R}\mathcal{Z}_{P^n}$ through the canonical action, we get an induced free action of $\widetilde{N}/N$ on $\mathbb{R}\mathcal{Z}_{P^n}/N$ whose orbit space is $\mathbb{R}\mathcal{Z}_{P^n}/\widetilde{N}$. By abuse of terminology, we also call this $(\widetilde{N}/N)$-action on $\mathbb{R}\mathcal{Z}_{P^n}/N$ *canonical*.

It is known that any small cover over P^n (if it exists) is a partial quotient of $\mathbb{R}\mathcal{Z}_{P^n}$ by a rank $(r - n)$ subgroup of $(\mathbb{Z}_2)^r$ [Buchstaber and Panov 2002, Section 7.5].

Proposition 1.5. *Suppose that Q^n is a small cover over a simple convex polytope P^n of dimension n, and that M^n is a principal $(\mathbb{Z}_2)^m$-bundle over Q^n. If M^n is connected, then there exists a subgroup N of $(\mathbb{Z}_2)^r$, where r is the number of facets of P^n, such that M^n is equivalent to the partial quotient $\mathbb{R}\mathcal{Z}_{P^n}/N$ as principal $(\mathbb{Z}_2)^m$-bundles over Q^n.*

Recall that two principal $(\mathbb{Z}_2)^m$-bundles M_1^n and M_2^n over a space Q^n are called *equivalent* if there are a homeomorphism $f : M_1^n \to M_2^n$ and a group automorphism $\sigma : (\mathbb{Z}_2)^m \to (\mathbb{Z}_2)^m$ such that

- $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M_1^n$, and
- f induces the identity map on the orbit space.

Under these conditions, we also say that the *free* $(\mathbb{Z}_2)^m$*-actions* on M_1^n and M_2^n *are equivalent*.

This paper is organized as follows. In Section 2, we review how to construct principal $(\mathbb{Z}_2)^m$-bundles over a manifold from the classical theory of fiber bundles and from the glue-back construction introduced in [Yu 2012]. We compare these two constructions, using them to prove several lemmas on principal $(\mathbb{Z}_2)^m$-bundles, and then give a proof of Proposition 1.5. In Section 3, we prove Theorem 1.4.

2. Glue-back construction

Suppose $(\mathbb{Z}_2)^m$ acts freely and smoothly on an n-dimensional closed manifold M^n. Then the orbit space $Q^n = M^n/(\mathbb{Z}_2)^m$ is naturally a closed manifold. In this section, we assume that Q^n is connected and that $H^1(Q^n, \mathbb{Z}_2) \neq 0$. Indeed, if Q^n is not connected, we can just apply our discussion to each connected component of Q^n. And if $H^1(Q^n, \mathbb{Z}_2) = 0$, then M^n must be homeomorphic to $Q^n \times (\mathbb{Z}_2)^m$.

Let $\pi : M^n \to Q^n$ be the orbit map of the free $(\mathbb{Z}_2)^m$-action. If we think of M^n as a principal $(\mathbb{Z}_2)^m$-bundle over Q^n, then it determines an element

$$\Lambda_\pi \in \operatorname{Hom}\big(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m\big) \cong H^1(Q^n, (\mathbb{Z}_2)^m). \tag{4}$$

If we think of M^n as a regular covering space over Q^n, its *monodromy* is a group homomorphism $\mathcal{H}_\pi : \pi_1(Q^n, q_0) \to (\mathbb{Z}_2)^m$, where q_0 is a base point of Q^n. Then $\mathcal{H}_\pi$ factors through Λ_π via the canonical group homomorphism

$$\pi_1(Q^n, q_0) \to H_1(Q^n, \mathbb{Z}) \to H_1(Q^n, \mathbb{Z}_2). \tag{5}$$

Conversely, given any element $\Lambda \in \operatorname{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$, we can obtain a principal $(\mathbb{Z}_2)^m$-bundle $X(Q^n, \Lambda)$ over Q^n as follows. We compose Λ with the group homeomorphism in (5) and obtain a group homomorphism

$$\Phi_\Lambda : \pi_1(Q^n, q_0) \to (\mathbb{Z}_2)^m. \tag{6}$$

Then we define a left action of $\pi_1(Q^n, q_0)$ on $(\mathbb{Z}_2)^m$ by

$$\gamma \cdot g = \Phi_\Lambda(\gamma) + g, \tag{7}$$

for all $\gamma \in \pi_1(Q^n, q_0)$ and $g \in (\mathbb{Z}_2)^m$. Also, suppose $p : \widetilde{Q}^n \to Q^n$ is a universal covering of Q^n, and let $\pi_1(Q^n, q_0)$ act freely on $\widetilde{Q}^n$ from the right. Then we can define a free action of $\pi_1(Q^n, q_0)$ on $\widetilde{Q}^n \times (\mathbb{Z}_2)^m$ thus: for any $\gamma \in \pi_1(Q^n, q_0)$ and $(x, g) \in \widetilde{Q}^n \times (\mathbb{Z}_2)^m$,

$$\gamma \cdot (x, g) := (x \cdot \gamma^{-1}, \gamma \cdot g) = (x \cdot \gamma^{-1}, \Phi_\Lambda(\gamma) + g). \tag{8}$$

Let $X(Q^n, \Lambda)$ be the quotient space of this $\pi_1(Q^n, q_0)$ action on $\widetilde{Q}^n \times (\mathbb{Z}_2)^m$, and let $\Theta_\Lambda : \widetilde{Q}^n \times (\mathbb{Z}_2)^m \to X(Q^n, \Lambda)$ be the corresponding quotient map. So for all

$\gamma \in \pi_1(Q^n, q_0)$ and $(x, g) \in \widetilde{Q}^n \times (\mathbb{Z}_2)^m$, we have

$$\Theta_\Lambda(x \cdot \gamma, g) = \Theta_\Lambda(x, \gamma \cdot g).$$

Now, for any $(x, g) \in \widetilde{Q}^n \times (\mathbb{Z}_2)^m$, we define a map

$$\pi_\Lambda : X(Q^n, \Lambda) \to Q^n, \quad \Theta_\Lambda(x, g) \mapsto p(x). \tag{9}$$

Clearly $\pi_\Lambda : X(Q^n, \Lambda) \to Q^n$ is a principal $(\mathbb{Z}_2)^m$-bundle with a *canonical free* $(\mathbb{Z}_2)^m$*-action* defined by

$$g' \cdot \Theta_\Lambda(x, g) := \Theta_\Lambda(x, g + g'), \tag{10}$$

for all $x \in \widetilde{Q}^n$ and $g, g' \in (\mathbb{Z}_2)^m$. Therefore the monodromy of $X(Q^n, \Lambda)$ is given by Φ_Λ. We call $X(Q^n, \Lambda)$ the bundle associated to $p : \widetilde{Q}^n \to Q^n$ (thought of as a principal $\pi_1(Q^n)$-bundle) and the $\pi_1(Q)$-action (7) on $(\mathbb{Z}_2)^m$. In the theory of fiber bundles, we may also write

$$X(Q^n, \Lambda) = \widetilde{Q}^n \times_\Lambda (\mathbb{Z}_2)^m.$$

Also, any subgroup H of $(\mathbb{Z}_2)^m$ acts freely on $X(Q^n, \Lambda)$ via (10). Then the quotient space $X(Q^n, \Lambda)/H$ is naturally equipped with a free $(\mathbb{Z}_2)^m/H$-action. We call $X(Q^n, \Lambda)/H$ with this free $(\mathbb{Z}_2)^m/H$-action a *partial quotient* of $X(Q^n, \Lambda)$.

For a principal $(\mathbb{Z}_2)^m$-bundle $\pi : M^n \to Q^n$, it is easy to verify that $X(Q^n, \Lambda_\pi)$ is equivalent to M^n as principal $(\mathbb{Z}_2)^m$-bundles over Q^n.

But this way of constructing M^n from Q^n and Λ_π is not so convenient for the proof of Theorem 1.4, so we use another way of constructing principal $(\mathbb{Z}_2)^m$-bundles over Q^n, introduced in [Yu 2012]. First, we construct a manifold with corners from Q^n that can carry the information of any element of $H^1(Q^n, (\mathbb{Z}_2)^m)$. This is done as follows [Yu 2012].

By a standard argument of intersection theory in differential topology, we can show that there exists a collection of $(n-1)$-dimensional compact embedded submanifolds $\Sigma_1, \dots, \Sigma_k$ in Q^n such that their homology classes $\{[\Sigma_1], \dots, [\Sigma_k]\}$ form a basis of $H_{n-1}(Q^n, \mathbb{Z}_2) \cong H^1(Q^n, \mathbb{Z}_2) \neq 0$. Also, we can put $\Sigma_1, \dots, \Sigma_k$ in *general position* in Q^n, which means that

- $\Sigma_1, \dots, \Sigma_k$ intersect transversely with each other, and
- if $\Sigma_{i_1} \cap \cdots \cap \Sigma_{i_s}$ is not empty, then it is an embedded submanifold of Q^n of codimension s.

Then we cut Q^n open along $\Sigma_1, \dots, \Sigma_k$; that is, we choose a small tubular neighborhood $N(\Sigma_i)$ of each Σ_i and remove the interior of each $N(\Sigma_i)$ from Q^n. Then we get a nice manifold with corners $V^n = Q^n - \bigcup_i \mathrm{int}(N(\Sigma_i))$, which is called a $\mathbb{Z}_2$*-core* of Q^n from cutting Q^n open along $\Sigma_1, \dots, \Sigma_k$ (see Figure 1 for an example). A manifold with corners is called *nice* if each codimension-l face

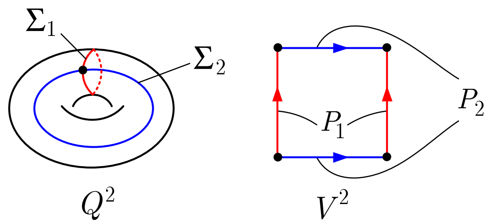

Figure 1. A $\mathbb{Z}_2$-core of a torus.

of the manifold belongs to exactly l facets [Jänich 1968; Davis 1983]. Here, the niceness of V^n follows from $\Sigma_1, \ldots, \Sigma_k$ being in general position in Q^n. The boundary of $N(\Sigma_i)$ is called the *cut section* of Σ_i in Q^n, and $\{\Sigma_1, \ldots, \Sigma_k\}$ is called a $\mathbb{Z}_2$*-cut system* of Q^n. We can choose each Σ_i to be connected.

The projection $\eta_i : \partial N(\Sigma_i) \to \Sigma_i$ is a double cover, either trivial or nontrivial. Let $\bar{\tau}_i$ be the generator of the deck transformation of η_i. Then $\bar{\tau}_i$ is a free involution on $\partial N(\Sigma_i)$; that is, $\bar{\tau}_i$ is a homeomorphism with no fixed point, and $\bar{\tau}_i^2 = id$. By applying some local deformations to these $\bar{\tau}_i$ if necessary [Yu 2012], we can construct an *involutive panel structure* on ∂V^n, which means that the boundary of V^n is the union of some compact subsets $P_1, \ldots, P_k$ (called *panels*) that satisfy the following conditions:

(a) each panel P_i is a disjoint union of facets of V^n, and each facet is contained in exactly one panel;

(b) there exists a free involution τ_i on each P_i that sends a face $f \subset P_i$ to a face $f' \subset P_i$ (it is possible that $f' = f$);

(c) for all $i \neq j$, we have $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$ and $\tau_i \circ \tau_j = \tau_j \circ \tau_i : P_i \cap P_j \to P_i \cap P_j$.

The P_i above consists of those facets of V^n that lie in the cut section of Σ_i, and $\tau_i : P_i \to P_i$ is the restriction of the modified $\bar{\tau}_i$ to P_i (see [Yu 2012] for the details of these constructions).

Remark 2.1. A more general notion of involutive panel structure is introduced in [Yu 2012], where the involution τ_i in (b) is not required to be free. This general notion is used in [Yu 2012] to unify the construction of all locally standard $(\mathbb{Z}_2)^m$-actions on closed manifolds from the orbit spaces.

Let $\mathcal{P}(V^n) = \{P_1, \ldots, P_k\}$ denote the set of all panels in V^n. Any map $\lambda : \mathcal{P}(V^n) \to (\mathbb{Z}_2)^m$ is called a $(\mathbb{Z}_2)^m$*-coloring on* V^n, and any element in $(\mathbb{Z}_2)^m$ is called a *color*.

Now, let us see how to recover a principal $(\mathbb{Z}_2)^m$-bundle $\pi : M^n \to Q^n$ from a $\mathbb{Z}_2$-core V^n of Q^n and the element $\Lambda_\pi \in \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$. By the Poincaré

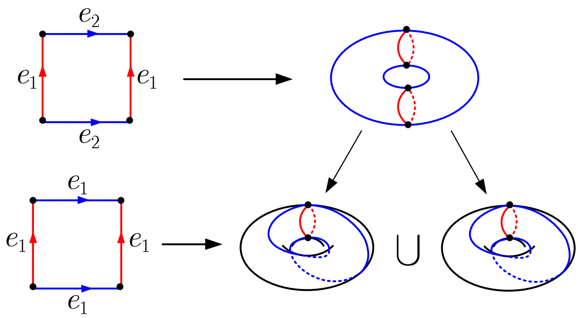

Figure 2

duality for closed manifolds, there is a group isomorphism

$$\kappa : H_{n-1}(Q^n, \mathbb{Z}_2) \to H_1(Q^n, \mathbb{Z}_2).$$

So we can assign an element of $(\mathbb{Z}_2)^m$ to each panel P_i of V^n by

$$\lambda_\pi(P_i) = \Lambda_\pi(\kappa([\Sigma_i])) \in (\mathbb{Z}_2)^m. \tag{11}$$

We call λ_π the *associated* $(\mathbb{Z}_2)^m$*-coloring* of $\pi : M^n \to Q^n$ on V^n.

Generally, for any $(\mathbb{Z}_2)^m$-coloring λ on V^n, we can glue 2^m copies of V^n by

$$M(V^n, \{P_i, \tau_i\}, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim, \tag{12}$$

where $(x, g) \sim (x', g')$ whenever $x' = \tau_i(x)$ for some P_i and $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$.

If x is in the relative interior of $P_{i_1} \cap \cdots \cap P_{i_s}$, then $(x, g) \sim (x', g')$ if and only if $(x', g') = \big(\tau_{i_s}^{\varepsilon_s} \circ \cdots \circ \tau_{i_1}^{\varepsilon_1}(x), g + \varepsilon_1\lambda(P_1) + \cdots + \varepsilon_s\lambda(P_s)\big)$, where $\varepsilon_j = 0$ or 1 for any $1 \le j \le s$ and $\tau_{i_j}^0 := id$.

$M(V^n, \{P_i, \tau_i\}, \lambda)$ is called the *glue-back construction* from (V^n, λ). Also, we use $M(V^n, \lambda)$ to denote $M(V^n, \{P_i, \tau_i\}, \lambda)$ in contexts where there is no ambiguity about the involutive panel structure on V^n.

Example 2.2. Figure 2 shows two principal $(\mathbb{Z}_2)^2$-bundles over a torus T^2 via glue-back constructions from two different $(\mathbb{Z}_2)^2$-colorings on a $\mathbb{Z}_2$-core of T^2. The $\{e_1, e_2\}$ in the picture is a linear basis of $(\mathbb{Z}_2)^2$. The first $(\mathbb{Z}_2)^2$-coloring gives a torus, and the second one gives a disjoint union of two tori. Also, we can define a double covering map (as defined later in (16)) from the torus on the top to either one of the tori below it.

Example 2.3. Figure 3 shows a $\mathbb{Z}_2$-core of the Klein bottle with three different $\mathbb{Z}_2$-colorings, where $\mathbb{Z}_2 = \langle a \rangle$. So from the glue-back construction, we get three inequivalent double coverings of the Klein bottle. From left to right in Figure 3, the first $\mathbb{Z}_2$-coloring gives a torus, while the second and the third both give a Klein bottle.

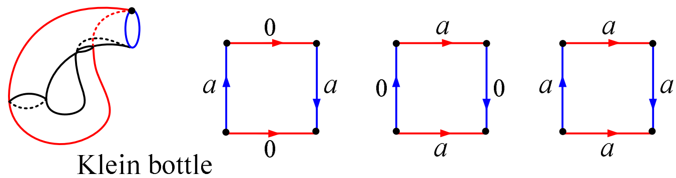

Figure 3

Let $\theta_\lambda : V^n \times (\mathbb{Z}_2)^m \to M(V^n, \lambda)$ be the quotient map defined in (12). It is shown in [Yu 2012] that $M(V^n, \lambda)$ is a closed manifold with a smooth free $(\mathbb{Z}_2)^m$-action defined by

$$g' \cdot \theta_\lambda(x, g) := \theta_\lambda(x, g + g'), \tag{13}$$

for all $x \in V^n$ and $g, g' \in (\mathbb{Z}_2)^m$. The orbit space of $M(V^n, \lambda)$ under this free $(\mathbb{Z}_2)^m$-action is homeomorphic to Q^n. We say that (13) defines the *natural* $(\mathbb{Z}_2)^m$-*action* on $M(V^n, \lambda)$. Here, we always associate this natural free $(\mathbb{Z}_2)^m$-action to $M(V^n, \lambda)$. Any subgroup $H \subset (\mathbb{Z}_2)^m$ also acts freely on $M(V^n, \lambda)$ through the natural action. The induced action of $(\mathbb{Z}_2)^m/H$ on $M(V^n, \lambda)/H$ is also free, and its orbit space is homeomorphic to $M(V^n, \lambda)/(\mathbb{Z}_2)^m = Q^n$. By abuse of terminology, we also call this $(\mathbb{Z}_2)^m/H$-action on $M(V^n, \lambda)/H$ *natural* and call $M(V^n, \lambda)/H$ with the natural $(\mathbb{Z}_2)^m/H$-action a *partial quotient* of $M(V^n, \lambda)$.

We have defined "partial quotient" in three different contexts : $\mathbb{R}\mathcal{Z}_{P^n}$, $X(Q^n, \Lambda)$ and $M(V^n, \lambda)$. The common property of these notions is that each of them denotes the quotient space of some free $\mathbb{Z}_2$-torus action on a space.

Theorem 2.4 [Yu 2012, Theorem 3.5]. *Let $\pi : M^n \to Q^n$ be a principal $(\mathbb{Z}_2)^m$-bundle, and let λ_π be the associated $(\mathbb{Z}_2)^m$-coloring on V^n. Then $M(V^n, \lambda_\pi)$ and M^n are equivalent principal $(\mathbb{Z}_2)^m$-bundles over Q^n.*

For any integer $m \geq 1$, define

$$\begin{aligned} \mathrm{Col}_m(V^n) &:= \text{the set of all } (\mathbb{Z}_2)^m\text{-colorings on } V^n \\ &= \{\lambda \mid \lambda : \mathcal{P}(V^n) \to (\mathbb{Z}_2)^m\}, \\ L_\lambda &:= \text{the subgroup of } (\mathbb{Z}_2)^m \text{ generated by } \{\lambda(P_1), \dots, \lambda(P_k)\}, \\ \mathrm{rank}(\lambda) &:= \dim_{\mathbb{Z}_2} L_\lambda, \quad \text{for all } \lambda \in \mathrm{Col}_m(V^n). \end{aligned}$$

For any $g \in (\mathbb{Z}_2)^m$, it is clear from (13) that L_λ acts freely on $\theta_\lambda(V^n \times (g + L_\lambda))$, whose orbit space is Q^n.

Theorem 2.5 [Yu 2012, Theorem 3.7]. *For any $(\mathbb{Z}_2)^m$-coloring λ on V^n, $M(V^n, \lambda)$ has $2^{m-\mathrm{rank}(\lambda)}$ connected components that are pairwise homeomorphic, and $L_\lambda \cong (\mathbb{Z}_2)^{\mathrm{rank}(\lambda)}$ acts freely on each connected component of $M(V^n, \lambda)$ whose orbit space* *is Q^n. Each connected component of $M(V^n, \lambda)$ is equivalent to $\theta_\lambda(V^n \times L_\lambda)$ as principal $(\mathbb{Z}_2)^{\mathrm{rank}(\lambda)}$-bundles over Q^n.*

An element $\lambda \in \mathrm{Col}_m(V^n)$ is called *maximally independent* if $\mathrm{rank}(\lambda) = k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. If $\lambda \in \mathrm{Col}_m(V^n)$ is maximally independent, then $m \geq k$.

Obviously, the relation in (11) defines a one-to-one correspondence between the elements of $\mathrm{Col}_m(V^n)$ and $\mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m) \cong H^1(Q^n, (\mathbb{Z}_2)^m)$. Suppose $\Lambda \in \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ is the element corresponding to $\lambda \in \mathrm{Col}_m(V^n)$; then L_λ is nothing but the image $\mathrm{Im}(\Lambda) \subset (\mathbb{Z}_2)^m$ of Λ, and λ is maximally independent if and only if Λ is injective. We define

$$\mathrm{rank}(\Lambda) := \dim_{\mathbb{Z}_2}(\mathrm{Im}(\Lambda)) = \dim_{\mathbb{Z}_2}(L_\lambda) = \mathrm{rank}(\lambda).$$

It is clear that $X(Q^n, \Lambda)$ and $M(V^n, \lambda)$ are equivalent principal $(\mathbb{Z}_2)^m$-bundles over Q^n, and so are $\Theta_\Lambda(\widetilde{Q}^n \times \mathrm{Im}(\Lambda))$ and $\theta_\lambda(V^n \times L_\lambda)$. The canonical free $(\mathbb{Z}_2)^m$-action on $X(Q^n, \Lambda)$ defined by (10) corresponds exactly to the natural $(\mathbb{Z}_2)^m$-action on $M(V^n, \lambda)$ defined by (13). So for any subgroup H of $(\mathbb{Z}_2)^m$, the partial quotients $X(Q^n, \Lambda)/H$ and $M(V^n, \lambda)/H$ are equivalent. Then we can write Theorem 2.5 in terms of $X(Q^n, \Lambda)$ as follows.

Theorem 2.5*. *For any $\Lambda \in \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$, $X(Q^n, \Lambda)$ has $2^{m-\mathrm{rank}(\Lambda)}$ connected components that are pairwise homeomorphic, and $\mathrm{Im}(\Lambda) \cong (\mathbb{Z}_2)^{\mathrm{rank}(\Lambda)}$ acts freely on each connected component of $X(Q^n, \Lambda)$ whose orbit space is Q^n. Each connected component of $X(Q^n, \Lambda)$ is equivalent to $\Theta_\Lambda(\widetilde{Q}^n \times \mathrm{Im}(\Lambda))$ as principal $(\mathbb{Z}_2)^{\mathrm{rank}(\Lambda)}$-bundles over Q^n.*

We prove several lemmas on principal $(\mathbb{Z}_2)^m$-bundles over a closed manifold. The statements of these lemmas are written in the language of glue-back construction. But we use $X(Q^n, \Lambda)$ and $M(V^n, \lambda)$ alternatively in the proofs of these lemmas, depending on what is convenient.

Lemma 2.6. *For any $m \geq \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$, if $\lambda_1, \lambda_2 \in \mathrm{Col}_m(V^n)$ are both maximally independent, then $M(V^n, \lambda_1)$ must be equivalent to $M(V^n, \lambda_2)$ as principal $(\mathbb{Z}_2)^m$-bundles over Q^n.*

Proof. Let Λ_1 and Λ_2 be the elements of $\mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ corresponding to λ_1 and λ_2. Then by our assumption, Λ_1 and Λ_2 are both injective. So there exists a group automorphism σ of $(\mathbb{Z}_2)^m$ such that $\sigma \circ \Lambda_1 = \Lambda_2$. Then we can define a homeomorphism $\phi : \widetilde{Q}^n \times (\mathbb{Z}_2)^m \to \widetilde{Q}^n \times (\mathbb{Z}_2)^m$, for $x \in \widetilde{Q}^n$ and $g \in (\mathbb{Z}_2)^m$, by

$$\phi(x, g) = (x, \sigma(g)).$$

Obviously, $\Theta_{\Lambda_1}(x, g) = \Theta_{\Lambda_1}(x', g')$ if and only if $\Theta_{\Lambda_2}(\phi(x, g)) = \Theta_{\Lambda_2}(\phi(x', g'))$. So ϕ induces an equivalence between the two principal $(\mathbb{Z}_2)^m$-bundles $X(Q^n, \Lambda_1)$ and $X(Q^n, \Lambda_2)$. So $M(V^n, \lambda_1)$ is equivalent to $M(V^n, \lambda_2)$. $\square$

Lemma 2.7. *Suppose M_1 and M_2 are two principal $(\mathbb{Z}_2)^k$-bundles over Q^n, where $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. If M_1 and M_2 are both connected, then M_1 must be equivalent to M_2 as principal $(\mathbb{Z}_2)^k$-bundles over Q^n.*

Proof. Using this notation, for some $\lambda_i \in \mathrm{Col}_k(V^n)$, $i = 1, 2$, Theorem 2.4 gives

$$M_i \cong M(V^n, \lambda_i).$$

Also, because M_1 and M_2 are both connected, Theorem 2.5 implies that $\mathrm{rank}(\lambda_1) = \mathrm{rank}(\lambda_2) = k$; that is, λ_1 and λ_2 are both maximally independent. So by Lemma 2.6, $M(V^n, \lambda_1)$ and $M(V^n, \lambda_2)$ are equivalent principal $(\mathbb{Z}_2)^k$-bundles over Q^n. □

We study some relations between $M(V^n, \lambda)$ for different $\lambda \in \mathrm{Col}_m(V^n)$. For conciseness, for any topological space B and field $\mathbb{F}$, we define

$$\mathrm{hrk}(B, \mathbb{F}) := \sum_{i=0}^{\infty} \dim_{\mathbb{F}} H^i(B, \mathbb{F}).$$

Lemma 2.8. *For any double covering $\xi : \widetilde{B} \to B$ and any $i \geq 0$,*

$$\dim_{\mathbb{Z}_2} H^i(\widetilde{B}, \mathbb{Z}_2) \leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2).$$

So $\mathrm{hrk}(\widetilde{B}, \mathbb{Z}_2) \leq 2 \cdot \mathrm{hrk}(B, \mathbb{Z}_2)$.

Proof. The Gysin sequence of $\xi : \widetilde{B} \to B$, in $\mathbb{Z}_2$-coefficient, reads:

$$\cdots \to H^{i-1}(B, \mathbb{Z}_2) \xrightarrow{\phi_{i-1}} H^i(B, \mathbb{Z}_2) \xrightarrow{\xi^*} H^i(\widetilde{B}, \mathbb{Z}_2) \to H^i(B, \mathbb{Z}_2) \xrightarrow{\phi_i} \ldots,$$

where $\phi_i(\gamma) = \gamma \cup e$ for all $\gamma \in H^i(B, \mathbb{Z}_2)$ and $e \in H^1(B, \mathbb{Z}_2)$ is the first Stiefel–Whitney class (mod 2 Euler class) of $\widetilde{B}$. Then by the exactness of the Gysin sequence, we have

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^i(\widetilde{B}, \mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \mathrm{Im}(\phi_{i-1}) + \dim_{\mathbb{Z}_2} \ker(\phi_i) \\ &= 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \mathrm{Im}(\phi_{i-1}) - \dim_{\mathbb{Z}_2} \mathrm{Im}(\phi_i) \\ &\leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2). \end{aligned}$$ □

Remark 2.9. In Lemma 2.8, if we replace the $\mathbb{Z}_2$-coefficients by $\mathbb{Z}_p$ (p is an odd prime) or $\mathbb{Q}$ (rational) coefficients, the conclusion in the lemma might fail. For example, let $B = \mathbb{R}P^2 \vee \mathbb{R}P^2$ be a one-point union of two $\mathbb{R}P^2$'s, and let $\widetilde{B}$ be the union of two spheres that intersect at two points (see Figure 4). It is clear that $\widetilde{B}$ is a double covering of B. But for any field $\mathbb{F} = \mathbb{Z}_p$ or $\mathbb{Q}$, we have $\mathrm{hrk}(B, \mathbb{F}) = 1$, while $\mathrm{hrk}(\widetilde{B}, \mathbb{F}) = 4$.

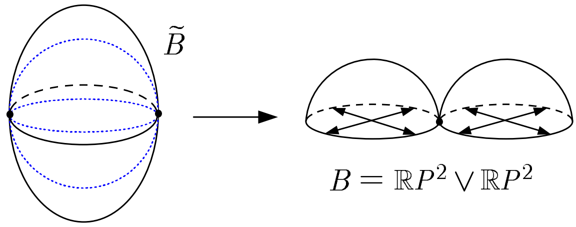

Figure 4

Lemma 2.10. *Suppose that $\lambda_{\max} \in \mathrm{Col}_k(V^n)$ is a maximally independent $(\mathbb{Z}_2)^k$-coloring on V^n, where $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$. Then, for any $\lambda \in \mathrm{Col}_k(V^n)$,*

$$\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq \mathrm{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2).$$

Proof. Suppose Λ is the element of $\mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$ corresponding to λ. Let $\{\alpha_1, \ldots, \alpha_k\}$ be a $\mathbb{Z}_2$-linear basis of $H_1(Q^n, \mathbb{Z}_2)$. Without loss of generality, we assume that $\{\Lambda(\alpha_1), \ldots, \Lambda(\alpha_s)\}$ is a $\mathbb{Z}_2$-linear basis of $\mathrm{Im}(\Lambda) \subset (\mathbb{Z}_2)^k$. Then we can choose $\omega_1, \ldots, \omega_{k-s} \in (\mathbb{Z}_2)^k$ such that $(\mathbb{Z}_2)^k = \mathrm{Im}(\Lambda) \oplus \langle \omega_1 \rangle \oplus \cdots \oplus \langle \omega_{k-s} \rangle$.

We define a sequence of elements $\Lambda_0, \ldots, \Lambda_{k-s} \in \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$ thus: for any $0 \leq j \leq k - s$,

$$\Lambda_j(\alpha_i) := \begin{cases} \Lambda(\alpha_i) & \text{if } 1 \leq i \leq s \text{ or } s + j < i \leq k; \\ \omega_{i-s} & \text{if } s + 1 \leq i \leq s + j. \end{cases} \tag{14}$$

Clearly $\Lambda_0 = \Lambda$ and $\mathrm{Im}(\Lambda) = \mathrm{Im}(\Lambda_0) \subset \mathrm{Im}(\Lambda_1) \subset \cdots \subset \mathrm{Im}(\Lambda_{k-s}) = (\mathbb{Z}_2)^k$, and for $1 \leq j \leq k - s$,

$$\mathrm{rank}(\Lambda_j) = \mathrm{rank}(\Lambda_{j-1}) + 1.$$

Let λ_j be the elements of $\mathrm{Col}_k(V^n)$ corresponding to Λ_j, with $0 \leq j \leq k - s$. Then λ_{k-s} is maximally independent. So by Lemma 2.6, we have

$$\mathrm{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2) = \mathrm{hrk}(M(V^n, \lambda_{k-s}), \mathbb{Z}_2) = \mathrm{hrk}(X(Q^n, \Lambda_{k-s}), \mathbb{Z}_2). \tag{15}$$

To prove the lemma, it suffices to show that for all $1 \leq j \leq k - s$,

$$\mathrm{hrk}(X(Q^n, \Lambda_{j-1}), \mathbb{Z}_2) \geq \mathrm{hrk}(X(Q^n, \Lambda_j), \mathbb{Z}_2).$$

Notice that $\mathrm{Im}(\Lambda_j) = \mathrm{Im}(\Lambda_{j-1}) \oplus \langle \omega_j \rangle \subset (\mathbb{Z}_2)^k$, and the only difference between Λ_{j-1} and Λ_j is that $\Lambda_{j-1}(\alpha_{s+j}) = \Lambda(\alpha_{s+j})$ while $\Lambda_j(\alpha_{s+j}) = \omega_j$. Let

$$K_j = \Theta_{\Lambda_j}(\widetilde{Q}^n \times \mathrm{Im}(\Lambda_j))$$

for all $1 \leq j \leq k - s$, where $p : \widetilde{Q}^n \to Q^n$ is a universal covering of Q^n.

We define a free involution t_j on K_j: for any $(x, g) \in \widetilde{Q}^n \times \mathrm{Im}(\Lambda_j)$,

$$t_j(\Theta_{\Lambda_j}(x, g)) = \Theta_{\Lambda_j}(x, g + \Lambda(\alpha_{s+j}) + \omega_j). \tag{16}$$

Let K_j/t_j be the quotient space of K_j under t_j, and let $\overline{\Theta_{\Lambda_j}(x,g)} \in K_j/t_j$ denote the equivalence class of $\Theta_{\Lambda_j}(x,g)$. So K_j is a double covering of K_j/t_j.

By (9), the bundle map $\pi_{\Lambda_j} : X(Q^n, \Lambda_j) \to Q^n$ restricted to K_j gives a bundle map $\pi_{\Lambda_j} : K_j \to Q^n$ that sends any $\Theta_{\Lambda_j}(x,g)$ to $p(x)$, and the monodromy of π_{Λ_j} is $\Phi_{\Lambda_j} : \pi_1(Q^n, q_0) \to \mathrm{Im}(\Lambda_j) \subset (\mathbb{Z}_2)^k$; see (6). So π_{Λ_j} induces a map

$$\overline{\pi}_{\Lambda_j} : K_j/t_j \to Q^n, \quad \overline{\Theta_{\Lambda_j}(x,g)} \mapsto p(x).$$

By the definition (16) of t_j, we can easily see that $\overline{\pi}_{\Lambda_j}$ is a fiber bundle whose fiber is $\mathrm{Im}(\Lambda_j)$ modulo the relation $\sim$, where for all $g \in \mathrm{Im}(\Lambda_j)$,

$$g \sim g + \Lambda(\alpha_{s+j}) + \omega_j,$$

or equivalently, $\omega_j \sim \Lambda(\alpha_{s+j})$. Now by (14), $\mathrm{Im}(\Lambda_j)/\sim$ can be identified with $\mathrm{Im}(\Lambda_{j-1})$, so the fiber of $\overline{\pi}_{\Lambda_j} : K_j/t_j \to Q^n$ is isomorphic to $\mathrm{Im}(\Lambda_{j-1})$. Let

$$\varrho : \mathrm{Im}(\Lambda_j) \to \mathrm{Im}(\Lambda_{j-1}) = \mathrm{Im}(\Lambda_j)/\sim .$$

So the monodromy of $\overline{\pi}_{\Lambda_j}$ is $\varrho \circ \Phi_{\Lambda_j} : \pi_1(Q^n, q_0) \to \mathrm{Im}(\Lambda_{j-1})$. Also, it is easy to check that $\varrho \circ \Phi_{\Lambda_j}$ coincides with the monodromy $\Phi_{\Lambda_{j-1}}$ of the bundle $\pi_{\Lambda_{j-1}} : K_{j-1} \to Q^n$. Therefore, the two bundles K_j/t_j and K_{j-1} over Q^n are actually equivalent. So by Lemma 2.8,

$$\mathrm{hrk}(K_j, \mathbb{Z}_2) \le 2 \cdot \mathrm{hrk}(K_j/t_j, \mathbb{Z}_2) = 2 \cdot \mathrm{hrk}(K_{j-1}, \mathbb{Z}_2).$$

Also, because by Theorem 2.5*, $X(Q^n, \Lambda_j)$ consists of $2^{k-\mathrm{rank}(\Lambda_j)}$ copies of K_j for each $0 \le j \le k-s$ and $\mathrm{rank}(\Lambda_j) = \mathrm{rank}(\Lambda_{j-1}) + 1$, we get

$$\begin{aligned}\mathrm{hrk}(X(Q^n, \Lambda_{j-1}), \mathbb{Z}_2) &= 2^{k-\mathrm{rank}(\Lambda_{j-1})}\, \mathrm{hrk}(K_{j-1}, \mathbb{Z}_2) \\ &\ge 2^{k-\mathrm{rank}(\Lambda_j)}\, \mathrm{hrk}(K_j, \mathbb{Z}_2) = \mathrm{hrk}(X(Q^n, \Lambda_{j+1}), \mathbb{Z}_2).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) = \mathrm{hrk}(X(Q^n, \Lambda_0), \mathbb{Z}_2) &\ge \mathrm{hrk}(X(Q^n, \Lambda_{k-s}), \mathbb{Z}_2) \\ &= \mathrm{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2),\end{aligned}$$

where we use (15) for the final equality. □

Lemma 2.11. *Let M^n be a connected principal $(\mathbb{Z}_2)^s$-bundle over Q^n. Then there exist a maximally independent coloring $\widetilde{\lambda} \in \mathrm{Col}_k(V^n)$, where*

$$k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2),$$

and a free $(\mathbb{Z}_2)^{k-s}$-action on $M(V^n, \widetilde{\lambda})$ whose orbit space is homeomorphic to M^n. Also, M^n is equivalent to a partial quotient $M(V^n, \widetilde{\lambda})/H$ for some subgroup H of $(\mathbb{Z}_2)^k$ with rank $k-s$.

Proof. We use a similar argument to the proof of Lemma 2.10. Because M^n is connected, Theorem 2.5 implies that $s \le k$ and that there is an element $\lambda \in \mathrm{Col}_k(V^n)$ such that $\mathrm{rank}(\lambda) = s$ and M^n is homeomorphic to $\theta_\lambda(V^n \times L_\lambda) \subset M(V^n, \lambda)$.

As in the proof of Lemma 2.10, let Λ be the element of $\mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$ corresponding to λ, and let $\{\alpha_1, \dots, \alpha_k\}$ be a $\mathbb{Z}_2$-linear basis of $H_1(Q^n, \mathbb{Z}_2)$ such that $\{\Lambda(\alpha_1), \dots, \Lambda(\alpha_s)\}$ is a $\mathbb{Z}_2$-linear basis of $\mathrm{Im}(\Lambda) \subset (\mathbb{Z}_2)^k$. Suppose also that $(\mathbb{Z}_2)^k = \mathrm{Im}(\Lambda) \oplus \langle\omega_1\rangle \oplus \cdots \oplus \langle\omega_{k-s}\rangle$, and define the same sequence of elements $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_{k-s} \in \mathrm{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$ as in (14) and corresponding elements $\lambda_0, \lambda_1, \dots, \lambda_{k-s} \in \mathrm{Col}_k(V^n)$. So λ_{k-s} is maximally independent.

Let $\widehat{H} = \langle\omega_1\rangle \oplus \cdots \oplus \langle\omega_{k-s}\rangle \subset (\mathbb{Z}_2)^k$. Then $\widehat{H} \cong (\mathbb{Z}_2)^{k-s}$, and there exists a free action $\star$ of $\widehat{H}$ on $X(Q^n, \Lambda_{k-s}) \cong M(V^n, \lambda_{k-s})$ defined by

$$\omega_j \star \Theta_{\Lambda_{j-s}}(x, g) := \Theta_{\Lambda_{j-s}}(x, g + \Lambda(\alpha_{s+j}) + \omega_j),$$

for $1 \le j \le k - s$. As in the proof of Lemma 2.10, we can show that the orbit space of the action of $\widehat{H}$ is homeomorphic to $\Theta_\Lambda(\widetilde{Q}^n \times \mathrm{Im}(\Lambda)) \cong \theta_\lambda(V^n \times L_\lambda) \cong M^n$.

The action of $\widehat{H}$ on $X(Q^n, \Lambda_{k-s})$ can be identified with the canonical action (10) of $H = \langle\Lambda(\alpha_{s+1}) + \omega_1\rangle \oplus \cdots \oplus \langle\Lambda(\alpha_k) + \omega_{k-s}\rangle$ on $X(Q^n, \Lambda_{k-s})$ via a group isomorphism $\sigma : \widehat{H} \to H$, where for $1 \le j \le k - s$,

$$\sigma(\omega_j) = \Lambda(\alpha_{s+j}) + \omega_j.$$

Here σ is an isomorphism because $(\mathbb{Z}_2)^k = \mathrm{Im}(\Lambda) \oplus \langle\omega_1\rangle \oplus \cdots \oplus \langle\omega_{k-s}\rangle$. So M^n is equivalent to the partial quotient $X(Q^n, \Lambda_{k-s})/H \cong M(V^n, \lambda_{k-s})/H$ as principal $(\mathbb{Z}_2)^s$-bundles over Q^n. This completes the lemma. □

Proof of Proposition 1.5. Suppose the polytope P^n has $k + n$ facets. Then

$$H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k.$$

So by Lemma 2.11, there exist a maximally independent coloring $\widetilde{\lambda} \in \mathrm{Col}_k(V^n)$ and a subgroup $H \subset (\mathbb{Z}_2)^k$ such that M^n is equivalent to the partial quotient $M(V^n, \widetilde{\lambda})/H$ as principal $(\mathbb{Z}_2)^m$-bundles over Q^n. Both $M(V^n, \widetilde{\lambda})$ and the real moment-angle manifold $\mathbb{R}\mathcal{Z}_{P^n}$ are principal $(\mathbb{Z}_2)^k$-bundles over Q^n, and they are both connected. So by Lemma 2.7, $\mathbb{R}\mathcal{Z}_{P^n}$ is equivalent to $M(V^n, \widetilde{\lambda})$.

Let $\widetilde{N} \subset (\mathbb{Z}_2)^{k+n}$ be a subgroup of rank k such that Q^n is homeomorphic to the partial quotient $\mathbb{R}\mathcal{Z}_{P^n}/\widetilde{N}$ (such a subgroup $\widetilde{N}$ is not unique). The equivalence between $M(V^n, \widetilde{\lambda})$ and $\mathbb{R}\mathcal{Z}_{P^n}$ determines a group isomorphism $\sigma : (\mathbb{Z}_2)^k \to \widetilde{N}$. Then $M(V^n, \widetilde{\lambda})/H$ is equivalent to the partial quotient $\mathbb{R}\mathcal{Z}_{P^n}/N$ of $\mathbb{R}\mathcal{Z}_{P^n}$, where $N = \sigma(H) \subset \widetilde{N} \subset (\mathbb{Z}_2)^{k+n}$. This proves our proposition. □

3. Proof of Theorem 1.4

We adapt the following lemma for our proof of Theorem 1.4.

Lemma 3.1 [Ustinovskii 2011]. *Let (X, A) be a pair of CW-complexes such that A has a collar neighborhood $U(A)$ in X, that is,*

$$(U(A), A) \cong (A \times [0, 1), A \times 0).$$

Take a homeomorphism $\varphi : A \to A$ that can be extended to a homeomorphism $\widetilde{\varphi} : X \to X$. Let $Y = X_1 \cup_\varphi X_2$ be the space obtained by gluing two copies of X along A via the map φ. Then for any field $\mathbb{F}$, we have $\mathrm{hrk}(Y, \mathbb{F}) \geq \mathrm{hrk}(A, \mathbb{F})$.

Proof. The argument is almost the same as in [Ustinovskii 2011]. Let $U_1(A)$ and $U_2(A)$ be the collar neighborhoods of A in X_1 and X_2. Consider an open cover $Y = W_1 \cup W_2$, where $W_1 = X_1 \cup U_2(A)$ and $W_2 = X_2 \cup U_1(A)$. Then the Mayer–Vietoris sequence of cohomology groups for this open cover reads (we omit the coefficients $\mathbb{F}$):

$$\cdots \to H^{j-1}(W_1 \cap W_2) \xrightarrow{\delta^*_{(j)}} H^j(Y) \xrightarrow{g^*_{(j)}} H^j(W_1) \oplus H^j(W_2) \xrightarrow{p^*_{(j)}} H^j(W_1 \cap W_2) \to \cdots.$$

Here the map $p^*_{(j)}$ equals $i_1^* \oplus -i_2^*$, where i_1 and i_2 are inclusions of $W_1 \cap W_2$ into W_1 and W_2. Because W_1 and W_2 are both homotopy equivalent to X and $W_1 \cap W_2 = U_1(A) \cup U_2(A) \cong A \times (-1, 1)$, we get another, equivalent, long exact sequence

$$\cdots \to H^{j-1}(A) \xrightarrow{\widehat{\delta}^*_{(j)}} H^j(Y) \xrightarrow{\widehat{g}^*_{(j)}} H^j(X_1) \oplus H^j(X_2) \xrightarrow{\widehat{p}^*_{(j)}} H^j(A) \to \cdots.$$

Now $\widehat{p}^*_{(j)} = \iota_1^* \oplus -(\iota_2 \circ \varphi)^*$, where ι_1 and ι_2 are inclusions of A into X_1 and X_2. For any $\gamma \in H^j(X_1)$, it is easy to see that $(\gamma, (\widetilde{\varphi}^{-1})^*\gamma)$ is in $\ker(\widehat{p}^*_{(j)})$. Thus $\dim \ker(\widehat{p}^*_{(j)}) \geq \dim H^j(X)$, and so $\dim \mathrm{Im}(\widehat{p}^*_{(j)}) \leq \dim H^j(X)$. Then

$$\begin{aligned}\dim H^j(Y) &= \dim \ker(\widehat{g}^*_{(j)}) + \dim \mathrm{Im}(\widehat{g}^*_{(j)}) = \dim \mathrm{Im}(\widehat{\delta}^*_{(j)}) + \dim \ker(\widehat{p}^*_{(j)}) \\ &\geq \dim H^{j-1}(A) - \dim \mathrm{Im}(\widehat{p}^*_{(j-1)}) + \dim H^j(X) \\ &\geq \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X).\end{aligned}$$

Summing up these inequalities over all indices j, we get

$$\begin{aligned}\mathrm{hrk}(Y, \mathbb{F}) = \sum_j \dim H^j(Y) &\geq \sum_j \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X) \\ &= \sum_j \dim H^{j-1}(A) = \mathrm{hrk}(A, \mathbb{F}).\end{aligned}$$

□

Remark 3.2. In Lemma 3.1, the assumption that $\varphi : A \to A$ can be extended to a homeomorphism $\widetilde{\varphi} : X \to X$ is essential; otherwise the claim may not hold. For example, let X be a solid torus and $A \cong T^2$ the boundary of X. Let $\varphi : A \to A$ be the homeomorphism interchanging the meridian and longitude of T^2. If we glue two copies of X along their boundaries via φ, we get a 3-sphere S^3. But $\mathrm{hrk}(S^3, \mathbb{Z}_2) = 2$, while $\mathrm{hrk}(A, \mathbb{Z}_2) = 4$. The reason why the conclusion of Lemma 3.1 does not hold in this example is that φ cannot be extended to a homeomorphism on the whole X.

We introduce an auxiliary notion that plays an important role in our proof of Theorem 1.4. Suppose V^n is a $\mathbb{Z}_2$-core of a closed manifold Q^n and the involutive panel structure on V^n is $\{P_i, \tau_i\}$. For any panel P_j of V^n, we define the space

$$M_{\backslash P_j}(V^n, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim_{P_j}, \tag{17}$$

where $(x, g) \sim_{P_j} (x', g')$ whenever $x' = \tau_i(x)$ for some $P_i \neq P_j$ and

$$g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m.$$

In other words, $M_{\backslash P_j}(V^n, \lambda)$ is the quotient space of $V^n \times (\mathbb{Z}_2)^m$ under the rule in (12), except that we leave the interior of those facets in $P_j \times (\mathbb{Z}_2)^m$ open. We call $M_{\backslash P_j}(V^n, \lambda)$ a *partial glue-back* from (V^n, λ). Let the corresponding quotient map be

$$\theta_\lambda^{\backslash P_j} : V^n \times (\mathbb{Z}_2)^m \to M_{\backslash P_j}(V^n, \lambda). \tag{18}$$

Then the boundary of $M_{\backslash P_j}(V^n, \lambda)$ can be written as $\theta_\lambda^{\backslash P_j}(P_j \times (\mathbb{Z}_2)^m)$.

Proof of Theorem 1.4. The proof is by induction on the dimension of M^n. When $n = 1$, the only small cover is a circle. Because a principal $(\mathbb{Z}_2)^m$-bundle over a circle must be a disjoint union of 2^m or 2^{m-1} circles, the theorem holds. Now we assume the theorem holds for manifolds with dimension less than n.

Suppose that P^n is an n-dimensional simple convex polytope with $k+n$ facets $F_1, \dots, F_{k+n}$, with $k \geq 1$, and that $\pi_\mu : Q^n \to P^n$ is a small cover over P^n with the characteristic function μ. For any face $f = F_{i_1} \cap \cdots \cap F_{i_l}$ of P^n, let G_f^μ be the rank-l subgroup of $(\mathbb{Z}_2)^n$ generated by $\mu(F_1), \dots, \mu(F_l)$. Then by definition,

$$Q^n = P^n \times (\mathbb{Z}_2)^n / \sim, \quad \text{with} \tag{19}$$

$$(p, w) \sim (p', w') \iff p = p' \text{ and } w - w' \in G^\mu_{f(p)},$$

where $f(p)$ is the unique face of P^n that contains p in its relative interior. It was shown in [Davis and Januszkiewicz 1991] that the $\mathbb{Z}_2$-Betti numbers of Q^n can be computed from the h-vector of P^n. In particular, $H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$.

We choose an arbitrary vertex v_0 of P^n. By reindexing the facets of P^n, we can assume that $F_1, \dots, F_k$ are all the facets of P^n that are not incident to v_0. Then according to [Davis and Januszkiewicz 1991], the homology classes of the embedded submanifolds $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$ (called *facial submanifolds* of Q^n) form a $\mathbb{Z}_2$-linear basis of $H_{n-1}(Q^n, \mathbb{Z}_2)$. Cutting Q^n open along $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$ gives us a $\mathbb{Z}_2$-core of Q^n, denoted by V^n. We can think of V^n as a partial gluing of the 2^n

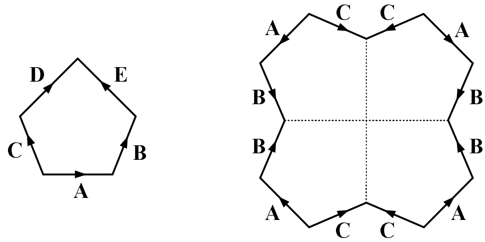

Figure 5. A $\mathbb{Z}_2$-core of a small cover in dimension 2.

copies of P^n according to the rule in (19), except that we leave the facets $F_1, \ldots, F_k$ in each copy of P^n open (see Figure 5 for an example). Let $\zeta : P^n \times (\mathbb{Z}_2)^n \to V^n$ denote the quotient map and let $P_1, \ldots, P_k$ be the panels of V^n corresponding to $\pi_\mu^{-1}(F_1), \ldots, \pi_\mu^{-1}(F_k)$. Then each P_i consists of 2^n copies of F_i, and for all $p \in F_i$ and $w \in (\mathbb{Z}_2)^n$, the involutive panel structure $\{\tau_i : P_i \to P_i\}_{1\le i\le k}$ on V^n can be written

$$\tau_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)). \tag{20}$$

Obviously, each τ_i extends to an automorphism $\widetilde{\tau}_i$ of V^n given by the same form: for all $p \in P^n$ and $w \in (\mathbb{Z}_2)^n$,

$$\widetilde{\tau}_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)). \tag{21}$$

These $\widetilde{\tau}_i$ commute with each other; that is, $\widetilde{\tau}_i \circ \widetilde{\tau}_j = \widetilde{\tau}_j \circ \widetilde{\tau}_i$, for $1 \le i, j \le k$. So each $\widetilde{\tau}_i$ preserves any panel P_j of V^n.

To prove Theorem 1.4, it suffices to show that $\operatorname{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \ge 2^m$ for any $\lambda \in \operatorname{Col}_m(V^n)$, because of Theorem 2.4.

We assume $m = k$. Let λ_0 be a maximally independent $(\mathbb{Z}_2)^k$-coloring of V^n; that is, $\operatorname{rank}(\lambda_0) = k$. By Lemma 2.10, $\operatorname{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \ge \operatorname{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2)$ for all $\lambda \in \operatorname{Col}_k(V^n)$. So it suffices to prove that

$$\operatorname{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \ge 2^k. \tag{22}$$

Inequality (22) follows from Theorem 1.1 and Lemma 2.7 (see Remark 3.3 below). But here we give another proof of (22), which only uses Lemma 3.1. This proof takes advantage of the interior symmetries of small covers (see (20) and (21)), and is more natural from the viewpoint of the glue-back construction.

Because λ_0 is maximally independent, by Lemma 2.6, we can assume $\lambda_0(P_i) = e_i$ for $1 \le i \le k$, where $\{e_1, \ldots, e_k\}$ is a linear basis of $(\mathbb{Z}_2)^k$. Let $\theta_{\lambda_0} : V^n \times (\mathbb{Z}_2)^k \to M(V^n, \lambda_0)$ be the quotient map defined by (12).

Now take an arbitrary panel of V^n, say P_1, and let $M_{\backslash P_1}(V^n, \lambda_0)$ be a partial glue-back from (V^n, λ_0) defined by (17). Let $\theta_{\lambda_0}^{\backslash P_1} : V^n \times (\mathbb{Z}_2)^k \to M_{\backslash P_1}(V^n, \lambda_0)$

be the corresponding quotient map. Suppose H is the subgroup of $(\mathbb{Z}_2)^k$ generated by $\{e_2, \dots, e_k\}$. Then we define

$$Y_1 = \theta_{\lambda_0}^{\backslash P_1}(V^n \times H), \quad Y_2 = \theta_{\lambda_0}^{\backslash P_1}(V^n \times (e_1 + H)), \tag{23}$$

$$A_1 = \theta_{\lambda_0}^{\backslash P_1}(P_1 \times H), \quad A_2 = \theta_{\lambda_0}^{\backslash P_1}(P_1 \times (e_1 + H)). \tag{24}$$

Obviously, $A_1 = \partial Y_1$ and $A_2 = \partial Y_2$, and there is a homeomorphism $\Pi : Y_1 \to Y_2$ with $\Pi(A_1) = A_2$. Indeed, for all $x \in V^n$ and $h \in H$, Π is given by

$$\Pi(\theta_{\lambda_0}^{\backslash P_1}(x, h)) = \theta_{\lambda_0}^{\backslash P_1}(x, h + e_1).$$

It is easy to see that $M(V^n, \lambda_0)$ is the gluing of Y_1 and Y_2 along their boundary by a homeomorphism $\varphi : A_1 \to A_2$ defined by

$$\varphi(\theta_{\lambda_0}^{\backslash P_1}(x_1, h)) = \theta_{\lambda_0}^{\backslash P_1}(\tau_1(x_1), h + e_1),$$

for all $x_1 \in P_1$ and $h \in H$.

Also, because $\tau_1 : P_1 \to P_1$ extends to a homeomorphism $\widetilde{\tau}_1 : V^n \to V^n$ (see (20) and (21)), we can extend φ to a homeomorphism $\widetilde{\varphi} : Y_1 \to Y_2$ by

$$\widetilde{\varphi}(\theta_{\lambda_0}^{\backslash P_1}(x, h)) = \theta_{\lambda_0}^{\backslash P_1}(\widetilde{\tau}_1(x), h + e_1),$$

for all $x \in V^n$ and $h \in H$. We know $\widetilde{\varphi}$ is well-defined because $\widetilde{\tau}_1$ commutes with each τ_i on P_i (see (12) and (21)).

Identifying (Y_1, A_1) with (Y_2, A_2) via Π, we get a decomposition of $M(V^n, \lambda_0)$ that satisfies all the conditions in Lemma 3.1. So Lemma 3.1 implies that

$$\mathrm{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq \mathrm{hrk}(A_1, \mathbb{Z}_2). \tag{25}$$

Also, let $q : Y_1 \cup Y_2 \to M(V^n, \lambda_0)$ be the quotient map and let

$$\xi_{\lambda_0} : M(V^n, \lambda_0) \to Q^n$$

be the orbit map of the natural $(\mathbb{Z}_2)^k$-action on $M(V^n, \lambda_0)$ (see (13)). It is easy to see that

$$A_1 \cong q(A_1) = \xi_{\lambda_0}^{-1}(\pi_\mu^{-1}(F_1)).$$

Because $\xi_{\lambda_0}^{-1}(\pi_\mu^{-1}(F_1))$ is a principal $(\mathbb{Z}_2)^k$-bundle over $\pi_\mu^{-1}(F_1)$ and $\pi_\mu^{-1}(F_1)$ is a small cover over F_1 of dimension $n-1$, we have $\mathrm{hrk}\big(\xi_{\lambda_0}^{-1}(\pi_\mu^{-1}(F_1)), \mathbb{Z}_2\big) \geq 2^k$, by the induction hypothesis. Then $\mathrm{hrk}(A_1, \mathbb{Z}_2) \geq 2^k$ also. So the case $m = k$ is confirmed, because by (25), $\mathrm{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$.

Now we assume $m < k$. Let $\iota : (\mathbb{Z}_2)^m \hookrightarrow (\mathbb{Z}_2)^k$ be the standard inclusion, and define $\widehat{\lambda} := \iota \circ \lambda$. We consider $\widehat{\lambda}$ as a $(\mathbb{Z}_2)^k$-coloring on V^n. So by the above argument, $\mathrm{hrk}(M(V^n, \widehat{\lambda}), \mathbb{Z}_2) \geq 2^k$. By Theorem 2.5, $M(V^n, \widehat{\lambda})$ consists of 2^{k-m} copies of $M(V^n, \lambda)$, so $\mathrm{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$.

Finally, we assume $m > k$. Because $\text{rank}(\lambda) \leq k$, with a proper change of basis, we can assume $L_\lambda \subset (\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^m$. Let $\varrho : (\mathbb{Z}_2)^m \to (\mathbb{Z}_2)^k$ be the standard projection. Define $\bar{\lambda} := \varrho \circ \lambda$. Similarly, we consider $\bar{\lambda}$ as a $(\mathbb{Z}_2)^k$-coloring on V^n, and so we have $\text{hrk}(M(V^n, \bar{\lambda}), \mathbb{Z}_2) \geq 2^k$. By Theorem 2.5, $M(V^n, \lambda)$ consists of 2^{m-k} copies of $M(V^n, \bar{\lambda})$, so $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$.

So for any $m \geq 1$ and $\lambda \in \text{Col}_m(V^n)$, we always have $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$. The induction is complete. □

Remark 3.3. Both $M(V^n, \lambda_0)$ and $\mathbb{R}\mathscr{Z}_{P^n}$ are connected principal $(\mathbb{Z}_2)^k$-bundles over Q^n. Then by Lemma 2.7, $M(V^n, \lambda_0)$ is homeomorphic to $\mathbb{R}\mathscr{Z}_{P^n}$. So the conclusion of Theorem 1.1 also tells us that $\text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$.

A crucial step in this proof is that when $\lambda_0 \in \text{Col}_k(V^n)$ is maximally independent, we can always get the type of decomposition of $M(V^n, \lambda_0)$ as in Lemma 3.1, which allows us to use the induction hypothesis. However, for an arbitrary $\lambda \in \text{Col}_k(V^n)$, this type of decomposition of $M(V^n, \lambda)$ may not exist (at least not obviously).

For example, in the lower picture in Figure 2, we have a principal $(\mathbb{Z}_2)^2$-bundle $\pi : M^2 \to T^2$, where M^2 is a disjoint union of two tori. The union of the two meridians in M^2 is the inverse image of a meridian in T^2 under π. If we cut M^2 open along these two meridians, we get two circular cylinders. But M^2 is not obtained by gluing these two cylinders together, because the colors of the $(\mathbb{Z}_2)^2$-coloring on the two panels are not linearly independent. So the construction in (23) for this case fails to give us the type of decomposition of M^2 as in Lemma 3.1.

So when $\lambda \in \text{Col}_k(V^n)$ is not maximally independent, we may not be able to directly apply the induction hypothesis to $M(V^n, \lambda)$ as we do to $M(V^n, \lambda_0)$ above. But these cases are settled by Lemma 2.10.

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