

*Pacific  
Journal of  
Mathematics*

**SMALL COVERS  
AND THE HALPERIN-CARLSSON CONJECTURE**

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# SMALL COVERS AND THE HALPERIN–CARLSSON CONJECTURE

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**We prove that the Halperin–Carlsson conjecture holds for any free  $(\mathbb{Z}_2)^m$ -action on a compact manifold whose orbit space is a small cover.**

## 1. Introduction

For any prime  $p$ , let  $\mathbb{Z}_p$  denote the quotient group  $\mathbb{Z}/p\mathbb{Z}$ , and  $S^1$  the circle group.

**The Halperin–Carlsson Conjecture.** *If  $G = (\mathbb{Z}_p)^m$  or  $(S^1)^m$  can act freely on a finite CW-complex  $X$ , then, respectively,*

$$\sum_{i=0}^{\infty} \dim_{\mathbb{Z}_p} H^i(X, \mathbb{Z}_p) \geq 2^m \quad \text{or} \quad \sum_{i=0}^{\infty} \dim_{\mathbb{Q}} H^i(X, \mathbb{Q}) \geq 2^m.$$

This was proposed by Halperin [1985] for the torus case and by Carlsson [1986] for the  $\mathbb{Z}_p$ -torus case. It is also called the *toral rank conjecture* in some papers.

At first this conjecture mainly took the form of whether a free  $(\mathbb{Z}_p)^m$ -action on a product of spheres  $S^{n_1} \times \cdots \times S^{n_k}$  implies  $m \leq k$ . Many authors have studied this intriguing conjecture in its various aspects [Conner 1957; Carlsson 1982; Adem 1987; Adem and Browder 1988; Adem and Benson 1998; Hanke 2009]. For a survey of results on the topic, see [Adem 2004; Allday and Puppe 1993]. The general case is still open for any prime  $p$ .

For general finite CW-complexes, the conjecture was proved in [Puppe 2009] for  $m \leq 3$  in the torus and  $\mathbb{Z}_2$ -torus cases and  $m \leq 2$  in the odd  $\mathbb{Z}_p$ -torus case. Also we have the following result, achieved independently, which confirmed the Halperin–Carlsson conjecture for some special  $\mathbb{Z}_2$ -torus actions on real moment-angle complexes:

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This work was partially supported by the Japan Society for the Promotion of Science (JSPS grant no. P10018) and the Natural Science Foundation of China (grant no. 11001120). This work was also funded by the PAPD (priority academic program development) of Jiangsu higher education institutions.

*MSC2010:* 57R22, 57R91, 57S17, 57S25.

*Keywords:* free torus action, Halperin–Carlsson conjecture, small cover, moment-angle manifold, glue-back construction.

**Theorem 1.1** [Cao and Lü 2009; Ustinovskii 2011]. *Let  $K^{n-1}$  be an  $(n - 1)$ -dimensional simplicial complex on the vertex set  $[d]$ . Then the real moment-angle complex  $\mathbb{R}\mathcal{L}_{K^{n-1}}$  over  $K^{n-1}$  must satisfy  $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathbb{R}\mathcal{L}_{K^{n-1}}, \mathbb{Z}_2) \geq 2^{d-n}$ . In particular, if  $P^n$  is an  $n$ -dimensional simple convex polytope with  $d$  facets, then the real moment-angle manifold  $\mathbb{R}\mathcal{L}_{P^n}$  must satisfy  $\sum_i \dim_{\mathbb{Z}_2} H^i(\mathbb{R}\mathcal{L}_{P^n}, \mathbb{Z}_2) \geq 2^{d-n}$ .*

**Remark 1.2.** Stronger results were obtained in [Cao and Lü 2009] and [Ustinovskii 2011]; for example, Theorem 1.1 holds even if the  $\mathbb{Z}_2$ -coefficients are replaced by rational coefficients.

**Remark 1.3.** There is a purely algebraic analogue of the Halperin–Carlsson conjecture, which was proposed in [Carlsson 1986] in the context of commutative algebras. Some related results were obtained in [Carlsson 1987].

Here we only study the conjecture for  $G = (\mathbb{Z}_2)^m$  and  $X$  a closed manifold. We use the following conventions:

- we treat  $(\mathbb{Z}_2)^m$  as an additive group;
- all manifolds and submanifolds are smooth;
- we do not distinguish between an embedded submanifold and its image.

Suppose that  $(\mathbb{Z}_2)^m$  acts freely and smoothly on a closed  $n$ -manifold  $M^n$ . Let  $Q^n = M^n / (\mathbb{Z}_2)^m$  be the orbit space. Then  $Q^n$  is a closed  $n$ -manifold too. Let  $\pi : M^n \rightarrow Q^n$  be the orbit map. We can think of  $M^n$  either as a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$  or as a regular covering over  $Q^n$  whose deck transformation group is  $(\mathbb{Z}_2)^m$ . In algebraic topology, we have a standard way to recover  $M^n$  from  $Q^n$ , using the universal covering space of  $Q^n$  and the monodromy of the covering [Hatcher 2002]. However, it is not so easy for us to visualize the total space of the covering with this approach. In [Yu 2012], a new way of constructing principal  $(\mathbb{Z}_2)^m$ -bundles over closed manifolds is introduced, which allows us to visualize this kind of object more easily.

Indeed, it is shown in [Yu 2012] that  $\pi : M^n \rightarrow Q^n$  determines a  $(\mathbb{Z}_2)^m$ -coloring  $\lambda_\pi$  on a nice manifold with corners  $V^n$  (called a  $\mathbb{Z}_2$ -core of  $Q^n$ ), and up to equivariant homeomorphism, we can recover  $M^n$  by a standard *glue-back construction* from  $V^n$  and  $\lambda_\pi$ . Using this new language, we prove the following theorem, which supports the Halperin–Carlsson conjecture.

**Theorem 1.4.** *Suppose that  $(\mathbb{Z}_2)^m$  acts freely on a closed  $n$ -manifold  $M^n$  whose orbit space is homeomorphic to a small cover; then*

$$(1) \quad \sum_i \dim_{\mathbb{Z}_2} H^i(M^n, \mathbb{Z}_2) \geq 2^m.$$

Recall that an  $n$ -dimensional small cover is a closed  $n$ -manifold with a locally

standard  $(\mathbb{Z}_2)^n$ -action whose orbit space can be identified with an  $n$ -dimensional simple convex polytope [Davis and Januszkiewicz 1991].

Given an arbitrary  $n$ -dimensional simple convex polytope  $P^n$ , there may not exist any small cover over  $P^n$ . But we can always define a closed manifold  $\mathbb{R}\mathcal{L}_{P^n}$  associated to  $P^n$  called a *real moment-angle manifold* [Davis and Januszkiewicz 1991, Construction 4.1]. Let  $\mathcal{F}(P^n) = \{F_1, \dots, F_r\}$  be the set of facets of  $P^n$ , and let  $\{e_1, \dots, e_r\}$  be a basis of  $(\mathbb{Z}_2)^r$ . For  $1 \leq i \leq r$ , we define a function  $\lambda^* : \mathcal{F}(P^n) \rightarrow (\mathbb{Z}_2)^r$  by

$$(2) \quad \lambda^*(F_i) = e_i.$$

For any proper face  $f$  of  $P^n$ , let  $G_f$  denote the subgroup of  $(\mathbb{Z}_2)^r$  generated by the set  $\{\lambda^*(F_i) \mid f \subset F_i\}$ . The real moment-angle manifold  $\mathbb{R}\mathcal{L}_{P^n}$  of  $P^n$  is defined to be the quotient space

$$(3) \quad \mathbb{R}\mathcal{L}_{P^n} := P^n \times (\mathbb{Z}_2)^r / \sim,$$

where  $(p, g) \sim (p', g')$  if and only if  $p = p'$  and  $g^{-1}g' \in G_{f(p)}$ , with  $f(p)$  being the unique face of  $P^n$  that contains  $p$  in its relative interior. Let  $[(p, g)]$  denote the equivalence class of  $(p, g)$  in  $\mathbb{R}\mathcal{L}_{P^n}$ . There is a *canonical action* of  $(\mathbb{Z}_2)^r$  on  $\mathbb{R}\mathcal{L}_{P^n}$  by

$$g' \cdot [(p, g)] = [(p, g' + g)],$$

for all  $p \in P^n$  and  $g, g' \in (\mathbb{Z}_2)^r$ . This  $(\mathbb{Z}_2)^r$ -action on  $\mathbb{R}\mathcal{L}_{P^n}$  is not free. But a subgroup  $N \subset (\mathbb{Z}_2)^r$  might act freely on  $\mathbb{R}\mathcal{L}_{P^n}$  through the canonical action. In that case, the quotient space  $\mathbb{R}\mathcal{L}_{P^n}/N$  is called a *partial quotient* of  $\mathbb{R}\mathcal{L}_{P^n}$  [Buchstaber and Panov 2002, Section 7.5]. Also, if there is another subgroup  $\tilde{N}$  of  $(\mathbb{Z}_2)^r$  with  $\tilde{N} \supset N$ , and  $\tilde{N}$  also acts freely on  $\mathbb{R}\mathcal{L}_{P^n}$  through the canonical action, we get an induced free action of  $\tilde{N}/N$  on  $\mathbb{R}\mathcal{L}_{P^n}/N$  whose orbit space is  $\mathbb{R}\mathcal{L}_{P^n}/\tilde{N}$ . By abuse of terminology, we also call this  $(\tilde{N}/N)$ -action on  $\mathbb{R}\mathcal{L}_{P^n}/N$  *canonical*.

It is known that any small cover over  $P^n$  (if it exists) is a partial quotient of  $\mathbb{R}\mathcal{L}_{P^n}$  by a rank  $(r - n)$  subgroup of  $(\mathbb{Z}_2)^r$  [Buchstaber and Panov 2002, Section 7.5].

**Proposition 1.5.** *Suppose that  $Q^n$  is a small cover over a simple convex polytope  $P^n$  of dimension  $n$ , and that  $M^n$  is a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$ . If  $M^n$  is connected, then there exists a subgroup  $N$  of  $(\mathbb{Z}_2)^r$ , where  $r$  is the number of facets of  $P^n$ , such that  $M^n$  is equivalent to the partial quotient  $\mathbb{R}\mathcal{L}_{P^n}/N$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

Recall that two principal  $(\mathbb{Z}_2)^m$ -bundles  $M_1^n$  and  $M_2^n$  over a space  $Q^n$  are called *equivalent* if there are a homeomorphism  $f : M_1^n \rightarrow M_2^n$  and a group automorphism  $\sigma : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^m$  such that

- $f(g \cdot x) = \sigma(g) \cdot f(x)$  for all  $g \in (\mathbb{Z}_2)^m$  and  $x \in M_1^n$ , and
- $f$  induces the identity map on the orbit space.

Under these conditions, we also say that the *free*  $(\mathbb{Z}_2)^m$ -actions on  $M_1^n$  and  $M_2^n$  are *equivalent*.

This paper is organized as follows. In [Section 2](#), we review how to construct principal  $(\mathbb{Z}_2)^m$ -bundles over a manifold from the classical theory of fiber bundles and from the glue-back construction introduced in [[Yu 2012](#)]. We compare these two constructions, using them to prove several lemmas on principal  $(\mathbb{Z}_2)^m$ -bundles, and then give a proof of [Proposition 1.5](#). In [Section 3](#), we prove [Theorem 1.4](#).

### 2. Glue-back construction

Suppose  $(\mathbb{Z}_2)^m$  acts freely and smoothly on an  $n$ -dimensional closed manifold  $M^n$ . Then the orbit space  $Q^n = M^n / (\mathbb{Z}_2)^m$  is naturally a closed manifold. In this section, we assume that  $Q^n$  is connected and that  $H^1(Q^n, \mathbb{Z}_2) \neq 0$ . Indeed, if  $Q^n$  is not connected, we can just apply our discussion to each connected component of  $Q^n$ . And if  $H^1(Q^n, \mathbb{Z}_2) = 0$ , then  $M^n$  must be homeomorphic to  $Q^n \times (\mathbb{Z}_2)^m$ .

Let  $\pi : M^n \rightarrow Q^n$  be the orbit map of the free  $(\mathbb{Z}_2)^m$ -action. If we think of  $M^n$  as a principal  $(\mathbb{Z}_2)^m$ -bundle over  $Q^n$ , then it determines an element

$$(4) \quad \Lambda_\pi \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m) \cong H^1(Q^n, (\mathbb{Z}_2)^m).$$

If we think of  $M^n$  as a regular covering space over  $Q^n$ , its *monodromy* is a group homomorphism  $\mathcal{H}_\pi : \pi_1(Q^n, q_0) \rightarrow (\mathbb{Z}_2)^m$ , where  $q_0$  is a base point of  $Q^n$ . Then  $\mathcal{H}_\pi$  factors through  $\Lambda_\pi$  via the canonical group homomorphism

$$(5) \quad \pi_1(Q^n, q_0) \rightarrow H_1(Q^n, \mathbb{Z}) \rightarrow H_1(Q^n, \mathbb{Z}_2).$$

Conversely, given any element  $\Lambda \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ , we can obtain a principal  $(\mathbb{Z}_2)^m$ -bundle  $X(Q^n, \Lambda)$  over  $Q^n$  as follows. We compose  $\Lambda$  with the group homeomorphism in (5) and obtain a group homomorphism

$$(6) \quad \Phi_\Lambda : \pi_1(Q^n, q_0) \rightarrow (\mathbb{Z}_2)^m.$$

Then we define a left action of  $\pi_1(Q^n, q_0)$  on  $(\mathbb{Z}_2)^m$  by

$$(7) \quad \gamma \cdot g = \Phi_\Lambda(\gamma) + g,$$

for all  $\gamma \in \pi_1(Q^n, q_0)$  and  $g \in (\mathbb{Z}_2)^m$ . Also, suppose  $p : \tilde{Q}^n \rightarrow Q^n$  is a universal covering of  $Q^n$ , and let  $\pi_1(Q^n, q_0)$  act freely on  $\tilde{Q}^n$  from the right. Then we can define a free action of  $\pi_1(Q^n, q_0)$  on  $\tilde{Q}^n \times (\mathbb{Z}_2)^m$  thus: for any  $\gamma \in \pi_1(Q^n, q_0)$  and  $(x, g) \in \tilde{Q}^n \times (\mathbb{Z}_2)^m$ ,

$$(8) \quad \gamma \cdot (x, g) := (x \cdot \gamma^{-1}, \gamma \cdot g) = (x \cdot \gamma^{-1}, \Phi_\Lambda(\gamma) + g).$$

Let  $X(Q^n, \Lambda)$  be the quotient space of this  $\pi_1(Q^n, q_0)$  action on  $\tilde{Q}^n \times (\mathbb{Z}_2)^m$ , and let  $\Theta_\Lambda : \tilde{Q}^n \times (\mathbb{Z}_2)^m \rightarrow X(Q^n, \Lambda)$  be the corresponding quotient map. So for all

$\gamma \in \pi_1(Q^n, q_0)$  and  $(x, g) \in \tilde{Q}^n \times (\mathbb{Z}_2)^m$ , we have

$$\Theta_\Lambda(x \cdot \gamma, g) = \Theta_\Lambda(x, \gamma \cdot g).$$

Now, for any  $(x, g) \in \tilde{Q}^n \times (\mathbb{Z}_2)^m$ , we define a map

$$(9) \quad \pi_\Lambda : X(Q^n, \Lambda) \rightarrow Q^n, \quad \Theta_\Lambda(x, g) \mapsto p(x).$$

Clearly  $\pi_\Lambda : X(Q^n, \Lambda) \rightarrow Q^n$  is a principal  $(\mathbb{Z}_2)^m$ -bundle with a *canonical free*  $(\mathbb{Z}_2)^m$ -action defined by

$$(10) \quad g' \cdot \Theta_\Lambda(x, g) := \Theta_\Lambda(x, g + g'),$$

for all  $x \in \tilde{Q}^n$  and  $g, g' \in (\mathbb{Z}_2)^m$ . Therefore the monodromy of  $X(Q^n, \Lambda)$  is given by  $\Phi_\Lambda$ . We call  $X(Q^n, \Lambda)$  the bundle associated to  $p : \tilde{Q}^n \rightarrow Q^n$  (thought of as a principal  $\pi_1(Q^n)$ -bundle) and the  $\pi_1(Q)$ -action (7) on  $(\mathbb{Z}_2)^m$ . In the theory of fiber bundles, we may also write

$$X(Q^n, \Lambda) = \tilde{Q}^n \times_\Lambda (\mathbb{Z}_2)^m.$$

Also, any subgroup  $H$  of  $(\mathbb{Z}_2)^m$  acts freely on  $X(Q^n, \Lambda)$  via (10). Then the quotient space  $X(Q^n, \Lambda)/H$  is naturally equipped with a free  $(\mathbb{Z}_2)^m/H$ -action. We call  $X(Q^n, \Lambda)/H$  with this free  $(\mathbb{Z}_2)^m/H$ -action a *partial quotient* of  $X(Q^n, \Lambda)$ .

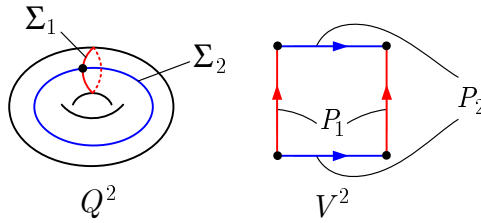
For a principal  $(\mathbb{Z}_2)^m$ -bundle  $\pi : M^n \rightarrow Q^n$ , it is easy to verify that  $X(Q^n, \Lambda_\pi)$  is equivalent to  $M^n$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .

But this way of constructing  $M^n$  from  $Q^n$  and  $\Lambda_\pi$  is not so convenient for the proof of [Theorem 1.4](#), so we use another way of constructing principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ , introduced in [[Yu 2012](#)]. First, we construct a manifold with corners from  $Q^n$  that can carry the information of any element of  $H^1(Q^n, (\mathbb{Z}_2)^m)$ . This is done as follows [[Yu 2012](#)].

By a standard argument of intersection theory in differential topology, we can show that there exists a collection of  $(n-1)$ -dimensional compact embedded submanifolds  $\Sigma_1, \dots, \Sigma_k$  in  $Q^n$  such that their homology classes  $\{[\Sigma_1], \dots, [\Sigma_k]\}$  form a basis of  $H_{n-1}(Q^n, \mathbb{Z}_2) \cong H^1(Q^n, \mathbb{Z}_2) \neq 0$ . Also, we can put  $\Sigma_1, \dots, \Sigma_k$  in *general position* in  $Q^n$ , which means that

- $\Sigma_1, \dots, \Sigma_k$  intersect transversely with each other, and
- if  $\Sigma_{i_1} \cap \dots \cap \Sigma_{i_s}$  is not empty, then it is an embedded submanifold of  $Q^n$  of codimension  $s$ .

Then we cut  $Q^n$  open along  $\Sigma_1, \dots, \Sigma_k$ ; that is, we choose a small tubular neighborhood  $N(\Sigma_i)$  of each  $\Sigma_i$  and remove the interior of each  $N(\Sigma_i)$  from  $Q^n$ . Then we get a nice manifold with corners  $V^n = Q^n - \bigcup_i \text{int}(N(\Sigma_i))$ , which is called a  $\mathbb{Z}_2$ -core of  $Q^n$  from cutting  $Q^n$  open along  $\Sigma_1, \dots, \Sigma_k$  (see [Figure 1](#) for an example). A manifold with corners is called *nice* if each codimension- $l$  face



**Figure 1.** A  $\mathbb{Z}_2$ -core of a torus.

of the manifold belongs to exactly  $l$  facets [Jänich 1968; Davis 1983]. Here, the niceness of  $V^n$  follows from  $\Sigma_1, \dots, \Sigma_k$  being in general position in  $Q^n$ . The boundary of  $N(\Sigma_i)$  is called the *cut section* of  $\Sigma_i$  in  $Q^n$ , and  $\{\Sigma_1, \dots, \Sigma_k\}$  is called a  $\mathbb{Z}_2$ -cut system of  $Q^n$ . We can choose each  $\Sigma_i$  to be connected.

The projection  $\eta_i : \partial N(\Sigma_i) \rightarrow \Sigma_i$  is a double cover, either trivial or nontrivial. Let  $\bar{\tau}_i$  be the generator of the deck transformation of  $\eta_i$ . Then  $\bar{\tau}_i$  is a free involution on  $\partial N(\Sigma_i)$ ; that is,  $\bar{\tau}_i$  is a homeomorphism with no fixed point, and  $\bar{\tau}_i^2 = id$ . By applying some local deformations to these  $\bar{\tau}_i$  if necessary [Yu 2012], we can construct an *involutive panel structure* on  $\partial V^n$ , which means that the boundary of  $V^n$  is the union of some compact subsets  $P_1, \dots, P_k$  (called *panels*) that satisfy the following conditions:

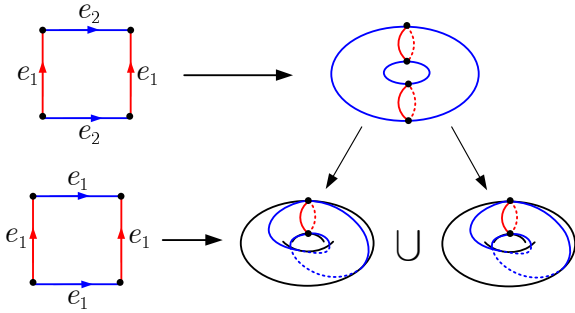
- (a) each panel  $P_i$  is a disjoint union of facets of  $V^n$ , and each facet is contained in exactly one panel;
- (b) there exists a free involution  $\tau_i$  on each  $P_i$  that sends a face  $f \subset P_i$  to a face  $f' \subset P_i$  (it is possible that  $f' = f$ );
- (c) for all  $i \neq j$ , we have  $\tau_i(P_i \cap P_j) \subset P_i \cap P_j$  and  $\tau_i \circ \tau_j = \tau_j \circ \tau_i : P_i \cap P_j \rightarrow P_i \cap P_j$ .

The  $P_i$  above consists of those facets of  $V^n$  that lie in the cut section of  $\Sigma_i$ , and  $\tau_i : P_i \rightarrow P_i$  is the restriction of the modified  $\bar{\tau}_i$  to  $P_i$  (see [Yu 2012] for the details of these constructions).

**Remark 2.1.** A more general notion of involutive panel structure is introduced in [Yu 2012], where the involution  $\tau_i$  in (b) is not required to be free. This general notion is used in [Yu 2012] to unify the construction of all locally standard  $(\mathbb{Z}_2)^m$ -actions on closed manifolds from the orbit spaces.

Let  $\mathcal{P}(V^n) = \{P_1, \dots, P_k\}$  denote the set of all panels in  $V^n$ . Any map  $\lambda : \mathcal{P}(V^n) \rightarrow (\mathbb{Z}_2)^m$  is called a  $(\mathbb{Z}_2)^m$ -coloring on  $V^n$ , and any element in  $(\mathbb{Z}_2)^m$  is called a *color*.

Now, let us see how to recover a principal  $(\mathbb{Z}_2)^m$ -bundle  $\pi : M^n \rightarrow Q^n$  from a  $\mathbb{Z}_2$ -core  $V^n$  of  $Q^n$  and the element  $\Lambda_\pi \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ . By the Poincaré



**Figure 2**

duality for closed manifolds, there is a group isomorphism

$$\kappa : H_{n-1}(Q^n, \mathbb{Z}_2) \rightarrow H_1(Q^n, \mathbb{Z}_2).$$

So we can assign an element of  $(\mathbb{Z}_2)^m$  to each panel  $P_i$  of  $V^n$  by

$$(11) \quad \lambda_\pi(P_i) = \Lambda_\pi(\kappa([\Sigma_i])) \in (\mathbb{Z}_2)^m.$$

We call  $\lambda_\pi$  the *associated  $(\mathbb{Z}_2)^m$ -coloring* of  $\pi : M^n \rightarrow Q^n$  on  $V^n$ .

Generally, for any  $(\mathbb{Z}_2)^m$ -coloring  $\lambda$  on  $V^n$ , we can glue  $2^m$  copies of  $V^n$  by

$$(12) \quad M(V^n, \{P_i, \tau_i\}, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim,$$

where  $(x, g) \sim (x', g')$  whenever  $x' = \tau_i(x)$  for some  $P_i$  and  $g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m$ .

If  $x$  is in the relative interior of  $P_{i_1} \cap \dots \cap P_{i_s}$ , then  $(x, g) \sim (x', g')$  if and only if  $(x', g') = (\tau_{i_s}^{\varepsilon_s} \circ \dots \circ \tau_{i_1}^{\varepsilon_1}(x), g + \varepsilon_1 \lambda(P_{i_1}) + \dots + \varepsilon_s \lambda(P_{i_s}))$ , where  $\varepsilon_j = 0$  or  $1$  for any  $1 \leq j \leq s$  and  $\tau_{i_j}^0 := id$ .

$M(V^n, \{P_i, \tau_i\}, \lambda)$  is called the *glue-back construction* from  $(V^n, \lambda)$ . Also, we use  $M(V^n, \lambda)$  to denote  $M(V^n, \{P_i, \tau_i\}, \lambda)$  in contexts where there is no ambiguity about the involutive panel structure on  $V^n$ .

**Example 2.2.** Figure 2 shows two principal  $(\mathbb{Z}_2)^2$ -bundles over a torus  $T^2$  via glue-back constructions from two different  $(\mathbb{Z}_2)^2$ -colorings on a  $\mathbb{Z}_2$ -core of  $T^2$ . The  $\{e_1, e_2\}$  in the picture is a linear basis of  $(\mathbb{Z}_2)^2$ . The first  $(\mathbb{Z}_2)^2$ -coloring gives a torus, and the second one gives a disjoint union of two tori. Also, we can define a double covering map (as defined later in (16)) from the torus on the top to either one of the tori below it.

**Example 2.3.** Figure 3 shows a  $\mathbb{Z}_2$ -core of the Klein bottle with three different  $\mathbb{Z}_2$ -colorings, where  $\mathbb{Z}_2 = \langle a \rangle$ . So from the glue-back construction, we get three inequivalent double coverings of the Klein bottle. From left to right in Figure 3, the first  $\mathbb{Z}_2$ -coloring gives a torus, while the second and the third both give a Klein bottle.



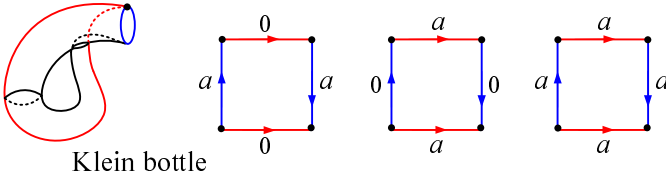


Figure 3

Let  $\theta_\lambda : V^n \times (\mathbb{Z}_2)^m \rightarrow M(V^n, \lambda)$  be the quotient map defined in (12). It is shown in [Yu 2012] that  $M(V^n, \lambda)$  is a closed manifold with a smooth free  $(\mathbb{Z}_2)^m$ -action defined by

$$(13) \quad g' \cdot \theta_\lambda(x, g) := \theta_\lambda(x, g + g'),$$

for all  $x \in V^n$  and  $g, g' \in (\mathbb{Z}_2)^m$ . The orbit space of  $M(V^n, \lambda)$  under this free  $(\mathbb{Z}_2)^m$ -action is homeomorphic to  $Q^n$ . We say that (13) defines the *natural*  $(\mathbb{Z}_2)^m$ -action on  $M(V^n, \lambda)$ . Here, we always associate this natural free  $(\mathbb{Z}_2)^m$ -action to  $M(V^n, \lambda)$ . Any subgroup  $H \subset (\mathbb{Z}_2)^m$  also acts freely on  $M(V^n, \lambda)$  through the natural action. The induced action of  $(\mathbb{Z}_2)^m/H$  on  $M(V^n, \lambda)/H$  is also free, and its orbit space is homeomorphic to  $M(V^n, \lambda)/(\mathbb{Z}_2)^m = Q^n$ . By abuse of terminology, we also call this  $(\mathbb{Z}_2)^m/H$ -action on  $M(V^n, \lambda)/H$  *natural* and call  $M(V^n, \lambda)/H$  with the natural  $(\mathbb{Z}_2)^m/H$ -action a *partial quotient* of  $M(V^n, \lambda)$ .

We have defined “partial quotient” in three different contexts :  $\mathbb{R}\mathcal{L}_{P^n}$ ,  $X(Q^n, \Lambda)$  and  $M(V^n, \lambda)$ . The common property of these notions is that each of them denotes the quotient space of some free  $\mathbb{Z}_2$ -torus action on a space.

**Theorem 2.4** [Yu 2012, Theorem 3.5]. *Let  $\pi : M^n \rightarrow Q^n$  be a principal  $(\mathbb{Z}_2)^m$ -bundle, and let  $\lambda_\pi$  be the associated  $(\mathbb{Z}_2)^m$ -coloring on  $V^n$ . Then  $M(V^n, \lambda_\pi)$  and  $M^n$  are equivalent principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

For any integer  $m \geq 1$ , define

$$\begin{aligned} \text{Col}_m(V^n) &:= \text{the set of all } (\mathbb{Z}_2)^m\text{-colorings on } V^n \\ &= \{\lambda \mid \lambda : \mathcal{P}(V^n) \rightarrow (\mathbb{Z}_2)^m\}, \\ L_\lambda &:= \text{the subgroup of } (\mathbb{Z}_2)^m \text{ generated by } \{\lambda(P_1), \dots, \lambda(P_k)\}, \\ \text{rank}(\lambda) &:= \dim_{\mathbb{Z}_2} L_\lambda, \text{ for all } \lambda \in \text{Col}_m(V^n). \end{aligned}$$

For any  $g \in (\mathbb{Z}_2)^m$ , it is clear from (13) that  $L_\lambda$  acts freely on  $\theta_\lambda(V^n \times (g + L_\lambda))$ , whose orbit space is  $Q^n$ .

**Theorem 2.5** [Yu 2012, Theorem 3.7]. *For any  $(\mathbb{Z}_2)^m$ -coloring  $\lambda$  on  $V^n$ ,  $M(V^n, \lambda)$  has  $2^{m-\text{rank}(\lambda)}$  connected components that are pairwise homeomorphic, and  $L_\lambda \cong (\mathbb{Z}_2)^{\text{rank}(\lambda)}$  acts freely on each connected component of  $M(V^n, \lambda)$  whose orbit space*

is  $Q^n$ . Each connected component of  $M(V^n, \lambda)$  is equivalent to  $\theta_\lambda(V^n \times L_\lambda)$  as principal  $(\mathbb{Z}_2)^{\text{rank}(\lambda)}$ -bundles over  $Q^n$ .

An element  $\lambda \in \text{Col}_m(V^n)$  is called *maximally independent* if  $\text{rank}(\lambda) = k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . If  $\lambda \in \text{Col}_m(V^n)$  is maximally independent, then  $m \geq k$ .

Obviously, the relation in (11) defines a one-to-one correspondence between the elements of  $\text{Col}_m(V^n)$  and  $\text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m) \cong H^1(Q^n, (\mathbb{Z}_2)^m)$ . Suppose  $\Lambda \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$  is the element corresponding to  $\lambda \in \text{Col}_m(V^n)$ ; then  $L_\lambda$  is nothing but the image  $\text{Im}(\Lambda) \subset (\mathbb{Z}_2)^m$  of  $\Lambda$ , and  $\lambda$  is maximally independent if and only if  $\Lambda$  is injective. We define

$$\text{rank}(\Lambda) := \dim_{\mathbb{Z}_2}(\text{Im}(\Lambda)) = \dim_{\mathbb{Z}_2}(L_\lambda) = \text{rank}(\lambda).$$

It is clear that  $X(Q^n, \Lambda)$  and  $M(V^n, \lambda)$  are equivalent principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ , and so are  $\Theta_\Lambda(\tilde{Q}^n \times \text{Im}(\Lambda))$  and  $\theta_\lambda(V^n \times L_\lambda)$ . The canonical free  $(\mathbb{Z}_2)^m$ -action on  $X(Q^n, \Lambda)$  defined by (10) corresponds exactly to the natural  $(\mathbb{Z}_2)^m$ -action on  $M(V^n, \lambda)$  defined by (13). So for any subgroup  $H$  of  $(\mathbb{Z}_2)^m$ , the partial quotients  $X(Q^n, \Lambda)/H$  and  $M(V^n, \lambda)/H$  are equivalent. Then we can write Theorem 2.5 in terms of  $X(Q^n, \Lambda)$  as follows.

**Theorem 2.5\*.** *For any  $\Lambda \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$ ,  $X(Q^n, \Lambda)$  has  $2^{m-\text{rank}(\Lambda)}$  connected components that are pairwise homeomorphic, and  $\text{Im}(\Lambda) \cong (\mathbb{Z}_2)^{\text{rank}(\Lambda)}$  acts freely on each connected component of  $X(Q^n, \Lambda)$  whose orbit space is  $Q^n$ . Each connected component of  $X(Q^n, \Lambda)$  is equivalent to  $\Theta_\Lambda(\tilde{Q}^n \times \text{Im}(\Lambda))$  as principal  $(\mathbb{Z}_2)^{\text{rank}(\Lambda)}$ -bundles over  $Q^n$ .*

We prove several lemmas on principal  $(\mathbb{Z}_2)^m$ -bundles over a closed manifold. The statements of these lemmas are written in the language of glue-back construction. But we use  $X(Q^n, \Lambda)$  and  $M(V^n, \lambda)$  alternatively in the proofs of these lemmas, depending on what is convenient.

**Lemma 2.6.** *For any  $m \geq \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ , if  $\lambda_1, \lambda_2 \in \text{Col}_m(V^n)$  are both maximally independent, then  $M(V^n, \lambda_1)$  must be equivalent to  $M(V^n, \lambda_2)$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ .*

*Proof.* Let  $\Lambda_1$  and  $\Lambda_2$  be the elements of  $\text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^m)$  corresponding to  $\lambda_1$  and  $\lambda_2$ . Then by our assumption,  $\Lambda_1$  and  $\Lambda_2$  are both injective. So there exists a group automorphism  $\sigma$  of  $(\mathbb{Z}_2)^m$  such that  $\sigma \circ \Lambda_1 = \Lambda_2$ . Then we can define a homeomorphism  $\phi : \tilde{Q}^n \times (\mathbb{Z}_2)^m \rightarrow \tilde{Q}^n \times (\mathbb{Z}_2)^m$ , for  $x \in \tilde{Q}^n$  and  $g \in (\mathbb{Z}_2)^m$ , by

$$\phi(x, g) = (x, \sigma(g)).$$

Obviously,  $\Theta_{\Lambda_1}(x, g) = \Theta_{\Lambda_1}(x', g')$  if and only if  $\Theta_{\Lambda_2}(\phi(x, g)) = \Theta_{\Lambda_2}(\phi(x', g'))$ . So  $\phi$  induces an equivalence between the two principal  $(\mathbb{Z}_2)^m$ -bundles  $X(Q^n, \Lambda_1)$  and  $X(Q^n, \Lambda_2)$ . So  $M(V^n, \lambda_1)$  is equivalent to  $M(V^n, \lambda_2)$ .  $\square$

**Lemma 2.7.** *Suppose  $M_1$  and  $M_2$  are two principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ , where  $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . If  $M_1$  and  $M_2$  are both connected, then  $M_1$  must be equivalent to  $M_2$  as principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ .*

*Proof.* Using this notation, for some  $\lambda_i \in \text{Col}_k(V^n)$ ,  $i = 1, 2$ , [Theorem 2.4](#) gives

$$M_i \cong M(V^n, \lambda_i).$$

Also, because  $M_1$  and  $M_2$  are both connected, [Theorem 2.5](#) implies that  $\text{rank}(\lambda_1) = \text{rank}(\lambda_2) = k$ ; that is,  $\lambda_1$  and  $\lambda_2$  are both maximally independent. So by [Lemma 2.6](#),  $M(V^n, \lambda_1)$  and  $M(V^n, \lambda_2)$  are equivalent principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ .  $\square$

We study some relations between  $M(V^n, \lambda)$  for different  $\lambda \in \text{Col}_m(V^n)$ . For conciseness, for any topological space  $B$  and field  $\mathbb{F}$ , we define

$$\text{hrk}(B, \mathbb{F}) := \sum_{i=0}^{\infty} \dim_{\mathbb{F}} H^i(B, \mathbb{F}).$$

**Lemma 2.8.** *For any double covering  $\xi : \tilde{B} \rightarrow B$  and any  $i \geq 0$ ,*

$$\dim_{\mathbb{Z}_2} H^i(\tilde{B}, \mathbb{Z}_2) \leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2).$$

So  $\text{hrk}(\tilde{B}, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(B, \mathbb{Z}_2)$ .

*Proof.* The Gysin sequence of  $\xi : \tilde{B} \rightarrow B$ , in  $\mathbb{Z}_2$ -coefficient, reads:

$$\dots \rightarrow H^{i-1}(B, \mathbb{Z}_2) \xrightarrow{\phi_{i-1}} H^i(B, \mathbb{Z}_2) \xrightarrow{\xi^*} H^i(\tilde{B}, \mathbb{Z}_2) \rightarrow H^i(B, \mathbb{Z}_2) \xrightarrow{\phi_i} \dots,$$

where  $\phi_i(\gamma) = \gamma \cup e$  for all  $\gamma \in H^i(B, \mathbb{Z}_2)$  and  $e \in H^1(B, \mathbb{Z}_2)$  is the first Stiefel–Whitney class (mod 2 Euler class) of  $\tilde{B}$ . Then by the exactness of the Gysin sequence, we have

$$\begin{aligned} \dim_{\mathbb{Z}_2} H^i(\tilde{B}, \mathbb{Z}_2) &= \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) + \dim_{\mathbb{Z}_2} \text{ker}(\phi_i) \\ &= 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_{i-1}) - \dim_{\mathbb{Z}_2} \text{Im}(\phi_i) \\ &\leq 2 \cdot \dim_{\mathbb{Z}_2} H^i(B, \mathbb{Z}_2). \end{aligned} \quad \square$$

**Remark 2.9.** In [Lemma 2.8](#), if we replace the  $\mathbb{Z}_2$ -coefficients by  $\mathbb{Z}_p$  ( $p$  is an odd prime) or  $\mathbb{Q}$  (rational) coefficients, the conclusion in the lemma might fail. For example, let  $B = \mathbb{R}P^2 \vee \mathbb{R}P^2$  be a one-point union of two  $\mathbb{R}P^2$ 's, and let  $\tilde{B}$  be the union of two spheres that intersect at two points (see [Figure 4](#)). It is clear that  $\tilde{B}$  is a double covering of  $B$ . But for any field  $\mathbb{F} = \mathbb{Z}_p$  or  $\mathbb{Q}$ , we have  $\text{hrk}(B, \mathbb{F}) = 1$ , while  $\text{hrk}(\tilde{B}, \mathbb{F}) = 4$ .

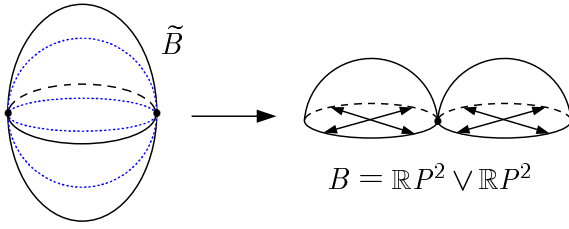


Figure 4

**Lemma 2.10.** *Suppose that  $\lambda_{\max} \in \text{Col}_k(V^n)$  is a maximally independent  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$ , where  $k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2)$ . Then, for any  $\lambda \in \text{Col}_k(V^n)$ ,*

$$\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2).$$

*Proof.* Suppose  $\Lambda$  is the element of  $\text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$  corresponding to  $\lambda$ . Let  $\{\alpha_1, \dots, \alpha_k\}$  be a  $\mathbb{Z}_2$ -linear basis of  $H_1(Q^n, \mathbb{Z}_2)$ . Without loss of generality, we assume that  $\{\Lambda(\alpha_1), \dots, \Lambda(\alpha_s)\}$  is a  $\mathbb{Z}_2$ -linear basis of  $\text{Im}(\Lambda) \subset (\mathbb{Z}_2)^k$ . Then we can choose  $\omega_1, \dots, \omega_{k-s} \in (\mathbb{Z}_2)^k$  such that  $(\mathbb{Z}_2)^k = \text{Im}(\Lambda) \oplus \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-s} \rangle$ .

We define a sequence of elements  $\Lambda_0, \dots, \Lambda_{k-s} \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$  thus: for any  $0 \leq j \leq k-s$ ,

$$(14) \quad \Lambda_j(\alpha_i) := \begin{cases} \Lambda(\alpha_i) & \text{if } 1 \leq i \leq s \text{ or } s+j < i \leq k; \\ \omega_{i-s} & \text{if } s+1 \leq i \leq s+j. \end{cases}$$

Clearly  $\Lambda_0 = \Lambda$  and  $\text{Im}(\Lambda) = \text{Im}(\Lambda_0) \subset \text{Im}(\Lambda_1) \subset \dots \subset \text{Im}(\Lambda_{k-s}) = (\mathbb{Z}_2)^k$ , and for  $1 \leq j \leq k-s$ ,

$$\text{rank}(\Lambda_j) = \text{rank}(\Lambda_{j-1}) + 1.$$

Let  $\lambda_j$  be the elements of  $\text{Col}_k(V^n)$  corresponding to  $\Lambda_j$ , with  $0 \leq j \leq k-s$ . Then  $\lambda_{k-s}$  is maximally independent. So by Lemma 2.6, we have

$$(15) \quad \text{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2) = \text{hrk}(M(V^n, \lambda_{k-s}), \mathbb{Z}_2) = \text{hrk}(X(Q^n, \Lambda_{k-s}), \mathbb{Z}_2).$$

To prove the lemma, it suffices to show that for all  $1 \leq j \leq k-s$ ,

$$\text{hrk}(X(Q^n, \Lambda_{j-1}), \mathbb{Z}_2) \geq \text{hrk}(X(Q^n, \Lambda_j), \mathbb{Z}_2).$$

Notice that  $\text{Im}(\Lambda_j) = \text{Im}(\Lambda_{j-1}) \oplus \langle \omega_j \rangle \subset (\mathbb{Z}_2)^k$ , and the only difference between  $\Lambda_{j-1}$  and  $\Lambda_j$  is that  $\Lambda_{j-1}(\alpha_{s+j}) = \Lambda(\alpha_{s+j})$  while  $\Lambda_j(\alpha_{s+j}) = \omega_j$ . Let

$$K_j = \Theta_{\Lambda_j}(\tilde{Q}^n \times \text{Im}(\Lambda_j))$$

for all  $1 \leq j \leq k-s$ , where  $p: \tilde{Q}^n \rightarrow Q^n$  is a universal covering of  $Q^n$ .

We define a free involution  $t_j$  on  $K_j$ : for any  $(x, g) \in \tilde{Q}^n \times \text{Im}(\Lambda_j)$ ,

$$(16) \quad t_j(\Theta_{\Lambda_j}(x, g)) = \Theta_{\Lambda_j}(x, g + \Lambda(\alpha_{s+j}) + \omega_j).$$

Let  $K_j/t_j$  be the quotient space of  $K_j$  under  $t_j$ , and let  $\overline{\Theta_{\Lambda_j}(x, g)} \in K_j/t_j$  denote the equivalence class of  $\Theta_{\Lambda_j}(x, g)$ . So  $K_j$  is a double covering of  $K_j/t_j$ .

By (9), the bundle map  $\pi_{\Lambda_j} : X(Q^n, \Lambda_j) \rightarrow Q^n$  restricted to  $K_j$  gives a bundle map  $\pi_{\Lambda_j} : K_j \rightarrow Q^n$  that sends any  $\Theta_{\Lambda_j}(x, g)$  to  $p(x)$ , and the monodromy of  $\pi_{\Lambda_j}$  is  $\Phi_{\Lambda_j} : \pi_1(Q^n, q_0) \rightarrow \text{Im}(\Lambda_j) \subset (\mathbb{Z}_2)^k$ ; see (6). So  $\pi_{\Lambda_j}$  induces a map

$$\bar{\pi}_{\Lambda_j} : K_j/t_j \rightarrow Q^n, \quad \overline{\Theta_{\Lambda_j}(x, g)} \mapsto p(x).$$

By the definition (16) of  $t_j$ , we can easily see that  $\bar{\pi}_{\Lambda_j}$  is a fiber bundle whose fiber is  $\text{Im}(\Lambda_j)$  modulo the relation  $\sim$ , where for all  $g \in \text{Im}(\Lambda_j)$ ,

$$g \sim g + \Lambda(\alpha_{s+j}) + \omega_j,$$

or equivalently,  $\omega_j \sim \Lambda(\alpha_{s+j})$ . Now by (14),  $\text{Im}(\Lambda_j)/\sim$  can be identified with  $\text{Im}(\Lambda_{j-1})$ , so the fiber of  $\bar{\pi}_{\Lambda_j} : K_j/t_j \rightarrow Q^n$  is isomorphic to  $\text{Im}(\Lambda_{j-1})$ . Let

$$\varrho : \text{Im}(\Lambda_j) \rightarrow \text{Im}(\Lambda_{j-1}) = \text{Im}(\Lambda_j)/\sim.$$

So the monodromy of  $\bar{\pi}_{\Lambda_j}$  is  $\varrho \circ \Phi_{\Lambda_j} : \pi_1(Q^n, q_0) \rightarrow \text{Im}(\Lambda_{j-1})$ . Also, it is easy to check that  $\varrho \circ \Phi_{\Lambda_j}$  coincides with the monodromy  $\Phi_{\Lambda_{j-1}}$  of the bundle  $\pi_{\Lambda_{j-1}} : K_{j-1} \rightarrow Q^n$ . Therefore, the two bundles  $K_j/t_j$  and  $K_{j-1}$  over  $Q^n$  are actually equivalent. So by Lemma 2.8,

$$\text{hrk}(K_j, \mathbb{Z}_2) \leq 2 \cdot \text{hrk}(K_j/t_j, \mathbb{Z}_2) = 2 \cdot \text{hrk}(K_{j-1}, \mathbb{Z}_2).$$

Also, because by Theorem 2.5\*,  $X(Q^n, \Lambda_j)$  consists of  $2^{k-\text{rank}(\Lambda_j)}$  copies of  $K_j$  for each  $0 \leq j \leq k - s$  and  $\text{rank}(\Lambda_j) = \text{rank}(\Lambda_{j-1}) + 1$ , we get

$$\begin{aligned} \text{hrk}(X(Q^n, \Lambda_{j-1}), \mathbb{Z}_2) &= 2^{k-\text{rank}(\Lambda_{j-1})} \text{hrk}(K_{j-1}, \mathbb{Z}_2) \\ &\geq 2^{k-\text{rank}(\Lambda_j)} \text{hrk}(K_j, \mathbb{Z}_2) = \text{hrk}(X(Q^n, \Lambda_{j+1}), \mathbb{Z}_2). \end{aligned}$$

Therefore,

$$\begin{aligned} \text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) &= \text{hrk}(X(Q^n, \Lambda_0), \mathbb{Z}_2) \geq \text{hrk}(X(Q^n, \Lambda_{k-s}), \mathbb{Z}_2) \\ &= \text{hrk}(M(V^n, \lambda_{\max}), \mathbb{Z}_2), \end{aligned}$$

where we use (15) for the final equality. □

**Lemma 2.11.** *Let  $M^n$  be a connected principal  $(\mathbb{Z}_2)^s$ -bundle over  $Q^n$ . Then there exist a maximally independent coloring  $\tilde{\lambda} \in \text{Col}_k(V^n)$ , where*

$$k = \dim_{\mathbb{Z}_2} H_{n-1}(Q^n, \mathbb{Z}_2),$$

*and a free  $(\mathbb{Z}_2)^{k-s}$ -action on  $M(V^n, \tilde{\lambda})$  whose orbit space is homeomorphic to  $M^n$ . Also,  $M^n$  is equivalent to a partial quotient  $M(V^n, \tilde{\lambda})/H$  for some subgroup  $H$  of  $(\mathbb{Z}_2)^k$  with rank  $k - s$ .*

*Proof.* We use a similar argument to the proof of [Lemma 2.10](#). Because  $M^n$  is connected, [Theorem 2.5](#) implies that  $s \leq k$  and that there is an element  $\lambda \in \text{Col}_k(V^n)$  such that  $\text{rank}(\lambda) = s$  and  $M^n$  is homeomorphic to  $\theta_\lambda(V^n \times L_\lambda) \subset M(V^n, \lambda)$ .

As in the proof of [Lemma 2.10](#), let  $\Lambda$  be the element of  $\text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$  corresponding to  $\lambda$ , and let  $\{\alpha_1, \dots, \alpha_k\}$  be a  $\mathbb{Z}_2$ -linear basis of  $H_1(Q^n, \mathbb{Z}_2)$  such that  $\{\Lambda(\alpha_1), \dots, \Lambda(\alpha_s)\}$  is a  $\mathbb{Z}_2$ -linear basis of  $\text{Im}(\Lambda) \subset (\mathbb{Z}_2)^k$ . Suppose also that  $(\mathbb{Z}_2)^k = \text{Im}(\Lambda) \oplus \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-s} \rangle$ , and define the same sequence of elements  $\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_{k-s} \in \text{Hom}(H_1(Q^n, \mathbb{Z}_2), (\mathbb{Z}_2)^k)$  as in [\(14\)](#) and corresponding elements  $\lambda_0, \lambda_1, \dots, \lambda_{k-s} \in \text{Col}_k(V^n)$ . So  $\lambda_{k-s}$  is maximally independent.

Let  $\widehat{H} = \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-s} \rangle \subset (\mathbb{Z}_2)^k$ . Then  $\widehat{H} \cong (\mathbb{Z}_2)^{k-s}$ , and there exists a free action  $\star$  of  $\widehat{H}$  on  $X(Q^n, \Lambda_{k-s}) \cong M(V^n, \lambda_{k-s})$  defined by

$$\omega_j \star \Theta_{\Lambda_{j-s}}(x, g) := \Theta_{\Lambda_{j-s}}(x, g + \Lambda(\alpha_{s+j}) + \omega_j),$$

for  $1 \leq j \leq k-s$ . As in the proof of [Lemma 2.10](#), we can show that the orbit space of the action of  $\widehat{H}$  is homeomorphic to  $\Theta_\Lambda(\widetilde{Q}^n \times \text{Im}(\Lambda)) \cong \theta_\lambda(V^n \times L_\lambda) \cong M^n$ .

The action of  $\widehat{H}$  on  $X(Q^n, \Lambda_{k-s})$  can be identified with the canonical action [\(10\)](#) of  $H = \langle \Lambda(\alpha_{s+1}) + \omega_1 \rangle \oplus \dots \oplus \langle \Lambda(\alpha_k) + \omega_{k-s} \rangle$  on  $X(Q^n, \Lambda_{k-s})$  via a group isomorphism  $\sigma : \widehat{H} \rightarrow H$ , where for  $1 \leq j \leq k-s$ ,

$$\sigma(\omega_j) = \Lambda(\alpha_{s+j}) + \omega_j.$$

Here  $\sigma$  is an isomorphism because  $(\mathbb{Z}_2)^k = \text{Im}(\Lambda) \oplus \langle \omega_1 \rangle \oplus \dots \oplus \langle \omega_{k-s} \rangle$ . So  $M^n$  is equivalent to the partial quotient  $X(Q^n, \Lambda_{k-s})/H \cong M(V^n, \lambda_{k-s})/H$  as principal  $(\mathbb{Z}_2)^s$ -bundles over  $Q^n$ . This completes the lemma.  $\square$

*Proof of [Proposition 1.5](#).* Suppose the polytope  $P^n$  has  $k+n$  facets. Then

$$H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k.$$

So by [Lemma 2.11](#), there exist a maximally independent coloring  $\widetilde{\lambda} \in \text{Col}_k(V^n)$  and a subgroup  $H \subset (\mathbb{Z}_2)^k$  such that  $M^n$  is equivalent to the partial quotient  $M(V^n, \widetilde{\lambda})/H$  as principal  $(\mathbb{Z}_2)^m$ -bundles over  $Q^n$ . Both  $M(V^n, \widetilde{\lambda})$  and the real moment-angle manifold  $\mathbb{R}\mathcal{L}_{P^n}$  are principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ , and they are both connected. So by [Lemma 2.7](#),  $\mathbb{R}\mathcal{L}_{P^n}$  is equivalent to  $M(V^n, \widetilde{\lambda})$ .

Let  $\widetilde{N} \subset (\mathbb{Z}_2)^{k+n}$  be a subgroup of rank  $k$  such that  $Q^n$  is homeomorphic to the partial quotient  $\mathbb{R}\mathcal{L}_{P^n}/\widetilde{N}$  (such a subgroup  $\widetilde{N}$  is not unique). The equivalence between  $M(V^n, \widetilde{\lambda})$  and  $\mathbb{R}\mathcal{L}_{P^n}$  determines a group isomorphism  $\sigma : (\mathbb{Z}_2)^k \rightarrow \widetilde{N}$ . Then  $M(V^n, \widetilde{\lambda})/H$  is equivalent to the partial quotient  $\mathbb{R}\mathcal{L}_{P^n}/N$  of  $\mathbb{R}\mathcal{L}_{P^n}$ , where  $N = \sigma(H) \subset \widetilde{N} \subset (\mathbb{Z}_2)^{k+n}$ . This proves our proposition.  $\square$

### 3. Proof of [Theorem 1.4](#)

We adapt the following lemma for our proof of [Theorem 1.4](#).

**Lemma 3.1** [Ustinovskii 2011]. *Let  $(X, A)$  be a pair of CW-complexes such that  $A$  has a collar neighborhood  $U(A)$  in  $X$ , that is,*

$$(U(A), A) \cong (A \times [0, 1), A \times 0).$$

*Take a homeomorphism  $\varphi : A \rightarrow A$  that can be extended to a homeomorphism  $\tilde{\varphi} : X \rightarrow X$ . Let  $Y = X_1 \cup_{\varphi} X_2$  be the space obtained by gluing two copies of  $X$  along  $A$  via the map  $\varphi$ . Then for any field  $\mathbb{F}$ , we have  $\text{hrk}(Y, \mathbb{F}) \geq \text{hrk}(A, \mathbb{F})$ .*

*Proof.* The argument is almost the same as in [Ustinovskii 2011]. Let  $U_1(A)$  and  $U_2(A)$  be the collar neighborhoods of  $A$  in  $X_1$  and  $X_2$ . Consider an open cover  $Y = W_1 \cup W_2$ , where  $W_1 = X_1 \cup U_2(A)$  and  $W_2 = X_2 \cup U_1(A)$ . Then the Mayer-Vietoris sequence of cohomology groups for this open cover reads (we omit the coefficients  $\mathbb{F}$ ):

$$\begin{aligned} \dots \rightarrow H^{j-1}(W_1 \cap W_2) \xrightarrow{\delta_{(j)}^*} H^j(Y) \\ \xrightarrow{g_{(j)}^*} H^j(W_1) \oplus H^j(W_2) \xrightarrow{p_{(j)}^*} H^j(W_1 \cap W_2) \rightarrow \dots \end{aligned}$$

Here the map  $p_{(j)}^*$  equals  $i_1^* \oplus -i_2^*$ , where  $i_1$  and  $i_2$  are inclusions of  $W_1 \cap W_2$  into  $W_1$  and  $W_2$ . Because  $W_1$  and  $W_2$  are both homotopy equivalent to  $X$  and  $W_1 \cap W_2 = U_1(A) \cup U_2(A) \cong A \times (-1, 1)$ , we get another, equivalent, long exact sequence

$$\dots \rightarrow H^{j-1}(A) \xrightarrow{\widehat{\delta}_{(j)}^*} H^j(Y) \xrightarrow{\widehat{g}_{(j)}^*} H^j(X_1) \oplus H^j(X_2) \xrightarrow{\widehat{p}_{(j)}^*} H^j(A) \rightarrow \dots$$

Now  $\widehat{p}_{(j)}^* = \iota_1^* \oplus -(\iota_2 \circ \varphi)^*$ , where  $\iota_1$  and  $\iota_2$  are inclusions of  $A$  into  $X_1$  and  $X_2$ . For any  $\gamma \in H^j(X_1)$ , it is easy to see that  $(\gamma, (\tilde{\varphi}^{-1})^*\gamma)$  is in  $\ker(\widehat{p}_{(j)}^*)$ . Thus  $\dim \ker(\widehat{p}_{(j)}^*) \geq \dim H^j(X)$ , and so  $\dim \text{Im}(\widehat{p}_{(j)}^*) \leq \dim H^j(X)$ . Then

$$\begin{aligned} \dim H^j(Y) &= \dim \ker(\widehat{g}_{(j)}^*) + \dim \text{Im}(\widehat{g}_{(j)}^*) = \dim \text{Im}(\widehat{\delta}_{(j)}^*) + \dim \ker(\widehat{p}_{(j)}^*) \\ &\geq \dim H^{j-1}(A) - \dim \text{Im}(\widehat{p}_{(j-1)}^*) + \dim H^j(X) \\ &\geq \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X). \end{aligned}$$

Summing up these inequalities over all indices  $j$ , we get

$$\begin{aligned} \text{hrk}(Y, \mathbb{F}) = \sum_j \dim H^j(Y) &\geq \sum_j \dim H^{j-1}(A) - \dim H^{j-1}(X) + \dim H^j(X) \\ &= \sum_j \dim H^{j-1}(A) = \text{hrk}(A, \mathbb{F}). \quad \square \end{aligned}$$

**Remark 3.2.** In Lemma 3.1, the assumption that  $\varphi : A \rightarrow A$  can be extended to a homeomorphism  $\tilde{\varphi} : X \rightarrow X$  is essential; otherwise the claim may not hold. For example, let  $X$  be a solid torus and  $A \cong T^2$  the boundary of  $X$ . Let  $\varphi : A \rightarrow A$  be the

homeomorphism interchanging the meridian and longitude of  $T^2$ . If we glue two copies of  $X$  along their boundaries via  $\varphi$ , we get a 3-sphere  $S^3$ . But  $\text{hrk}(S^3, \mathbb{Z}_2) = 2$ , while  $\text{hrk}(A, \mathbb{Z}_2) = 4$ . The reason why the conclusion of [Lemma 3.1](#) does not hold in this example is that  $\varphi$  cannot be extended to a homeomorphism on the whole  $X$ .

We introduce an auxiliary notion that plays an important role in our proof of [Theorem 1.4](#). Suppose  $V^n$  is a  $\mathbb{Z}_2$ -core of a closed manifold  $Q^n$  and the involutive panel structure on  $V^n$  is  $\{P_i, \tau_i\}$ . For any panel  $P_j$  of  $V^n$ , we define the space

$$(17) \quad M_{\setminus P_j}(V^n, \lambda) := V^n \times (\mathbb{Z}_2)^m / \sim_{P_j},$$

where  $(x, g) \sim_{P_j} (x', g')$  whenever  $x' = \tau_i(x)$  for some  $P_i \neq P_j$  and

$$g' = g + \lambda(P_i) \in (\mathbb{Z}_2)^m.$$

In other words,  $M_{\setminus P_j}(V^n, \lambda)$  is the quotient space of  $V^n \times (\mathbb{Z}_2)^m$  under the rule in (12), except that we leave the interior of those facets in  $P_j \times (\mathbb{Z}_2)^m$  open. We call  $M_{\setminus P_j}(V^n, \lambda)$  a *partial glue-back* from  $(V^n, \lambda)$ . Let the corresponding quotient map be

$$(18) \quad \theta_\lambda^{\setminus P_j} : V^n \times (\mathbb{Z}_2)^m \rightarrow M_{\setminus P_j}(V^n, \lambda).$$

Then the boundary of  $M_{\setminus P_j}(V^n, \lambda)$  can be written as  $\theta_\lambda^{\setminus P_j}(P_j \times (\mathbb{Z}_2)^m)$ .

*Proof of Theorem 1.4.* The proof is by induction on the dimension of  $M^n$ . When  $n = 1$ , the only small cover is a circle. Because a principal  $(\mathbb{Z}_2)^m$ -bundle over a circle must be a disjoint union of  $2^m$  or  $2^{m-1}$  circles, the theorem holds. Now we assume the theorem holds for manifolds with dimension less than  $n$ .

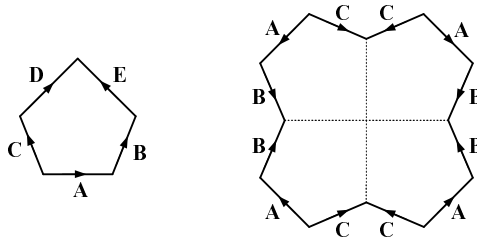
Suppose that  $P^n$  is an  $n$ -dimensional simple convex polytope with  $k + n$  facets  $F_1, \dots, F_{k+n}$ , with  $k \geq 1$ , and that  $\pi_\mu : Q^n \rightarrow P^n$  is a small cover over  $P^n$  with the characteristic function  $\mu$ . For any face  $f = F_{i_1} \cap \dots \cap F_{i_l}$  of  $P^n$ , let  $G_f^\mu$  be the rank- $l$  subgroup of  $(\mathbb{Z}_2)^n$  generated by  $\mu(F_{i_1}), \dots, \mu(F_{i_l})$ . Then by definition,

$$(19) \quad Q^n = P^n \times (\mathbb{Z}_2)^n / \sim, \quad \text{with} \\ (p, w) \sim (p', w') \iff p = p' \text{ and } w - w' \in G_{f(p)}^\mu,$$

where  $f(p)$  is the unique face of  $P^n$  that contains  $p$  in its relative interior. It was shown in [[Davis and Januszkiewicz 1991](#)] that the  $\mathbb{Z}_2$ -Betti numbers of  $Q^n$  can be computed from the  $h$ -vector of  $P^n$ . In particular,  $H_{n-1}(Q^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^k$ .

We choose an arbitrary vertex  $v_0$  of  $P^n$ . By reindexing the facets of  $P^n$ , we can assume that  $F_1, \dots, F_k$  are all the facets of  $P^n$  that are not incident to  $v_0$ . Then according to [[Davis and Januszkiewicz 1991](#)], the homology classes of the embedded submanifolds  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$  (called *facial submanifolds* of  $Q^n$ ) form a  $\mathbb{Z}_2$ -linear basis of  $H_{n-1}(Q^n, \mathbb{Z}_2)$ . Cutting  $Q^n$  open along  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$  gives us a  $\mathbb{Z}_2$ -core of  $Q^n$ , denoted by  $V^n$ . We can think of  $V^n$  as a partial gluing of the  $2^n$





**Figure 5.** A  $\mathbb{Z}_2$ -core of a small cover in dimension 2.

copies of  $P^n$  according to the rule in (19), except that we leave the facets  $F_1, \dots, F_k$  in each copy of  $P^n$  open (see Figure 5 for an example). Let  $\zeta : P^n \times (\mathbb{Z}_2)^n \rightarrow V^n$  denote the quotient map and let  $P_1, \dots, P_k$  be the panels of  $V^n$  corresponding to  $\pi_\mu^{-1}(F_1), \dots, \pi_\mu^{-1}(F_k)$ . Then each  $P_i$  consists of  $2^n$  copies of  $F_i$ , and for all  $p \in F_i$  and  $w \in (\mathbb{Z}_2)^n$ , the involutive panel structure  $\{\tau_i : P_i \rightarrow P_i\}_{1 \leq i \leq k}$  on  $V^n$  can be written

$$(20) \quad \tau_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)).$$

Obviously, each  $\tau_i$  extends to an automorphism  $\tilde{\tau}_i$  of  $V^n$  given by the same form: for all  $p \in P^n$  and  $w \in (\mathbb{Z}_2)^n$ ,

$$(21) \quad \tilde{\tau}_i(\zeta(p, w)) = \zeta(p, w + \mu(F_i)).$$

These  $\tilde{\tau}_i$  commute with each other; that is,  $\tilde{\tau}_i \circ \tilde{\tau}_j = \tilde{\tau}_j \circ \tilde{\tau}_i$ , for  $1 \leq i, j \leq k$ . So each  $\tilde{\tau}_i$  preserves any panel  $P_j$  of  $V^n$ .

To prove Theorem 1.4, it suffices to show that  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$  for any  $\lambda \in \text{Col}_m(V^n)$ , because of Theorem 2.4.

We assume  $m = k$ . Let  $\lambda_0$  be a maximally independent  $(\mathbb{Z}_2)^k$ -coloring of  $V^n$ ; that is,  $\text{rank}(\lambda_0) = k$ . By Lemma 2.10,  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq \text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2)$  for all  $\lambda \in \text{Col}_k(V^n)$ . So it suffices to prove that

$$(22) \quad \text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k.$$

Inequality (22) follows from Theorem 1.1 and Lemma 2.7 (see Remark 3.3 below). But here we give another proof of (22), which only uses Lemma 3.1. This proof takes advantage of the interior symmetries of small covers (see (20) and (21)), and is more natural from the viewpoint of the glue-back construction.

Because  $\lambda_0$  is maximally independent, by Lemma 2.6, we can assume  $\lambda_0(P_i) = e_i$  for  $1 \leq i \leq k$ , where  $\{e_1, \dots, e_k\}$  is a linear basis of  $(\mathbb{Z}_2)^k$ . Let  $\theta_{\lambda_0} : V^n \times (\mathbb{Z}_2)^k \rightarrow M(V^n, \lambda_0)$  be the quotient map defined by (12).

Now take an arbitrary panel of  $V^n$ , say  $P_1$ , and let  $M_{\setminus P_1}(V^n, \lambda_0)$  be a partial glue-back from  $(V^n, \lambda_0)$  defined by (17). Let  $\theta_{\lambda_0}^{\setminus P_1} : V^n \times (\mathbb{Z}_2)^k \rightarrow M_{\setminus P_1}(V^n, \lambda_0)$

be the corresponding quotient map. Suppose  $H$  is the subgroup of  $(\mathbb{Z}_2)^k$  generated by  $\{e_2, \dots, e_k\}$ . Then we define

$$(23) \quad Y_1 = \theta_{\lambda_0}^{\setminus P_1}(V^n \times H), \quad Y_2 = \theta_{\lambda_0}^{\setminus P_1}(V^n \times (e_1 + H)),$$

$$(24) \quad A_1 = \theta_{\lambda_0}^{\setminus P_1}(P_1 \times H), \quad A_2 = \theta_{\lambda_0}^{\setminus P_1}(P_1 \times (e_1 + H)).$$

Obviously,  $A_1 = \partial Y_1$  and  $A_2 = \partial Y_2$ , and there is a homeomorphism  $\Pi : Y_1 \rightarrow Y_2$  with  $\Pi(A_1) = A_2$ . Indeed, for all  $x \in V^n$  and  $h \in H$ ,  $\Pi$  is given by

$$\Pi(\theta_{\lambda_0}^{\setminus P_1}(x, h)) = \theta_{\lambda_0}^{\setminus P_1}(x, h + e_1).$$

It is easy to see that  $M(V^n, \lambda_0)$  is the gluing of  $Y_1$  and  $Y_2$  along their boundary by a homeomorphism  $\varphi : A_1 \rightarrow A_2$  defined by

$$\varphi(\theta_{\lambda_0}^{\setminus P_1}(x_1, h)) = \theta_{\lambda_0}^{\setminus P_1}(\tau_1(x_1), h + e_1),$$

for all  $x_1 \in P_1$  and  $h \in H$ .

Also, because  $\tau_1 : P_1 \rightarrow P_1$  extends to a homeomorphism  $\tilde{\tau}_1 : V^n \rightarrow V^n$  (see (20) and (21)), we can extend  $\varphi$  to a homeomorphism  $\tilde{\varphi} : Y_1 \rightarrow Y_2$  by

$$\tilde{\varphi}(\theta_{\lambda_0}^{\setminus P_1}(x, h)) = \theta_{\lambda_0}^{\setminus P_1}(\tilde{\tau}_1(x), h + e_1),$$

for all  $x \in V^n$  and  $h \in H$ . We know  $\tilde{\varphi}$  is well-defined because  $\tilde{\tau}_1$  commutes with each  $\tau_i$  on  $P_i$  (see (12) and (21)).

Identifying  $(Y_1, A_1)$  with  $(Y_2, A_2)$  via  $\Pi$ , we get a decomposition of  $M(V^n, \lambda_0)$  that satisfies all the conditions in Lemma 3.1. So Lemma 3.1 implies that

$$(25) \quad \text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq \text{hrk}(A_1, \mathbb{Z}_2).$$

Also, let  $q : Y_1 \cup Y_2 \rightarrow M(V^n, \lambda_0)$  be the quotient map and let

$$\xi_{\lambda_0} : M(V^n, \lambda_0) \rightarrow Q^n$$

be the orbit map of the natural  $(\mathbb{Z}_2)^k$ -action on  $M(V^n, \lambda_0)$  (see (13)). It is easy to see that

$$A_1 \cong q(A_1) = \xi_{\lambda_0}^{-1}(\pi_{\mu}^{-1}(F_1)).$$

Because  $\xi_{\lambda_0}^{-1}(\pi_{\mu}^{-1}(F_1))$  is a principal  $(\mathbb{Z}_2)^k$ -bundle over  $\pi_{\mu}^{-1}(F_1)$  and  $\pi_{\mu}^{-1}(F_1)$  is a small cover over  $F_1$  of dimension  $n - 1$ , we have  $\text{hrk}(\xi_{\lambda_0}^{-1}(\pi_{\mu}^{-1}(F_1)), \mathbb{Z}_2) \geq 2^k$ , by the induction hypothesis. Then  $\text{hrk}(A_1, \mathbb{Z}_2) \geq 2^k$  also. So the case  $m = k$  is confirmed, because by (25),  $\text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$ .

Now we assume  $m < k$ . Let  $\iota : (\mathbb{Z}_2)^m \hookrightarrow (\mathbb{Z}_2)^k$  be the standard inclusion, and define  $\hat{\lambda} := \iota \circ \lambda$ . We consider  $\hat{\lambda}$  as a  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$ . So by the above argument,  $\text{hrk}(M(V^n, \hat{\lambda}), \mathbb{Z}_2) \geq 2^k$ . By Theorem 2.5,  $M(V^n, \hat{\lambda})$  consists of  $2^{k-m}$  copies of  $M(V^n, \lambda)$ , so  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ .

Finally, we assume  $m > k$ . Because  $\text{rank}(\lambda) \leq k$ , with a proper change of basis, we can assume  $L_\lambda \subset (\mathbb{Z}_2)^k \subset (\mathbb{Z}_2)^m$ . Let  $\varrho : (\mathbb{Z}_2)^m \rightarrow (\mathbb{Z}_2)^k$  be the standard projection. Define  $\bar{\lambda} := \varrho \circ \lambda$ . Similarly, we consider  $\bar{\lambda}$  as a  $(\mathbb{Z}_2)^k$ -coloring on  $V^n$ , and so we have  $\text{hrk}(M(V^n, \bar{\lambda}), \mathbb{Z}_2) \geq 2^k$ . By [Theorem 2.5](#),  $M(V^n, \lambda)$  consists of  $2^{m-k}$  copies of  $M(V^n, \bar{\lambda})$ , so  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ .

So for any  $m \geq 1$  and  $\lambda \in \text{Col}_m(V^n)$ , we always have  $\text{hrk}(M(V^n, \lambda), \mathbb{Z}_2) \geq 2^m$ . The induction is complete.  $\square$

**Remark 3.3.** Both  $M(V^n, \lambda_0)$  and  $\mathbb{R}\mathcal{E}_{P^n}$  are connected principal  $(\mathbb{Z}_2)^k$ -bundles over  $Q^n$ . Then by [Lemma 2.7](#),  $M(V^n, \lambda_0)$  is homeomorphic to  $\mathbb{R}\mathcal{E}_{P^n}$ . So the conclusion of [Theorem 1.1](#) also tells us that  $\text{hrk}(M(V^n, \lambda_0), \mathbb{Z}_2) \geq 2^k$ .

A crucial step in this proof is that when  $\lambda_0 \in \text{Col}_k(V^n)$  is maximally independent, we can always get the type of decomposition of  $M(V^n, \lambda_0)$  as in [Lemma 3.1](#), which allows us to use the induction hypothesis. However, for an arbitrary  $\lambda \in \text{Col}_k(V^n)$ , this type of decomposition of  $M(V^n, \lambda)$  may not exist (at least not obviously).

For example, in the lower picture in [Figure 2](#), we have a principal  $(\mathbb{Z}_2)^2$ -bundle  $\pi : M^2 \rightarrow T^2$ , where  $M^2$  is a disjoint union of two tori. The union of the two meridians in  $M^2$  is the inverse image of a meridian in  $T^2$  under  $\pi$ . If we cut  $M^2$  open along these two meridians, we get two circular cylinders. But  $M^2$  is not obtained by gluing these two cylinders together, because the colors of the  $(\mathbb{Z}_2)^2$ -coloring on the two panels are not linearly independent. So the construction in [\(23\)](#) for this case fails to give us the type of decomposition of  $M^2$  as in [Lemma 3.1](#).

So when  $\lambda \in \text{Col}_k(V^n)$  is not maximally independent, we may not be able to directly apply the induction hypothesis to  $M(V^n, \lambda)$  as we do to  $M(V^n, \lambda_0)$  above. But these cases are settled by [Lemma 2.10](#).

### Acknowledgement

The author thanks B. Hanke for [Remark 3.2](#).

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Received March 1, 2011. Revised November 15, 2011.

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The subscription price for 2012 is US \$420/year for the electronic version, and \$485/year for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. Prior back issues are obtainable from Periodicals Service Company, 11 Main Street, Germantown, NY 12526-5635. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 969 Evans Hall, Berkeley, CA 94720-3840, is published monthly except July and August. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PJM peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS

at the University of California, Berkeley 94720-3840

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Typeset in L<sup>A</sup>T<sub>E</sub>X

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Volume 256    No. 2    April 2012

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