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ENERGY AND VOLUME OF VECTOR FIELDS ON SPHERICAL DOMAINS

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We present a "boundary version" for theorems about minimality of volume and energy functionals on a spherical domain of an odd-dimensional Euclidean sphere.

1. Introduction

Let (M, g) be a closed, n-dimensional Riemannian manifold and T^1M the unit tangent bundle of M considered as a closed Riemannian manifold with the Sasaki metric. Let $X : M \to T^1M$ be a unit vector field defined on M, regarded as a smooth section of the unit tangent bundle T^1M. The volume of X was defined in [Gluck and Ziller 1986] by $\operatorname{vol} X := \operatorname{vol} X(M)$, where $\operatorname{vol} X(M)$ is the volume of the submanifold $X(M) \subset T^1M$. Using an orthonormal local frame $\{e_1, e_2, \dots, e_{n-1}, e_n = X\}$, the volume of the unit vector field X is given by

$$\operatorname{vol} X = \int_M \Bigg(1 + \sum_{a=1}^{n} \|\nabla_{e_a} X\|^2 + \sum_{a<b} \|\nabla_{e_a} X \wedge \nabla_{e_b} X\|^2 + \cdots + \sum_{a_1<\cdots<a_{n-1}} \|\nabla_{e_{a_1}} X \wedge \cdots \wedge \nabla_{e_{a_{n-1}}} X\|^2\Bigg)^{1/2} \nu_M(g)$$

and the energy of the vector field X is given by

$$\mathscr{E}(X) = \frac{n}{2} \operatorname{vol} M + \frac{1}{2} \int_M \sum_{a=1}^{n} \|\nabla_{e_a} X\|^2 \, \nu_M(g).$$

The Hopf vector fields on $\mathbb{S}^{2k+1}$ are unit vector fields tangent to the classical Hopf fibration $\mathbb{S}^1 \hookrightarrow \mathbb{S}^{2k+1}$. The following theorems gives a characterization of Hopf flows as absolute minima of volume and energy functionals:

Theorem 1 [Gluck and Ziller 1986]. *The unit vector fields of minimum volume on the sphere $\mathbb{S}^3$ are precisely the Hopf vector fields and no others.*

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Theorem 2 [Brito 2000]. *The unit vector fields of minimum energy on the sphere $\mathbb{S}^3$ are precisely the Hopf vector fields and no others.*

We prove in this paper the following boundary version for these theorems:

Theorem 3. *Let U be an open set of the $(2k+1)$-dimensional unit sphere $\mathbb{S}^{2k+1}$ and let $K \subset U$ be a connected $(2k+1)$-submanifold with boundary of the sphere $\mathbb{S}^{2k+1}$. Let $\vec{v}$ be an unit vector field on U which coincides with a Hopf flow H along the boundary of K. Then*

$$\mathscr{E}(\vec{v}) \geq \left(\frac{2k+1}{2} + \frac{k}{2k-1}\right) \operatorname{vol} K \quad \textit{and} \quad \operatorname{vol} \vec{v} \geq \frac{4^k}{\binom{2k}{k}} \operatorname{vol} K.$$

(Other results for higher dimensions may be found in [Brito et al. 2004; Borrelli and Gil-Medrano 2006; Chacón et al. 2001].)

2. Preliminaries

Let $U \subset \mathbb{S}^{2k+1}$ be an open set of the unit sphere and let $K \subset U$ be a connected $(2k+1)$-submanifold with boundary of $\mathbb{S}^{2k+1}$. Let H be a Hopf vector field on $\mathbb{S}^{2k+1}$ and let $\vec{v}$ be an unit vector field defined on U. We also consider the map $\varphi_t^{\vec{v}} : U \to \mathbb{S}^{2k+1}(\sqrt{1+t^2})$ given by $\varphi_t^{\vec{v}}(x) = x + t\vec{v}(x)$. This map was introduced in [Asimov 1978; Brito et al. 1981; Milnor 1978].

Lemma 4. *For $t > 0$ sufficiently small, the map $\varphi_t^{\vec{v}}$ is a diffeomorphism.*

Proof. A simple application of the identity perturbation method. □

From now on, we assume that $t > 0$ is small enough so that the map $\varphi_t^{\vec{v}}$ is a diffeomorphism. In order to find the Jacobian matrix of $\varphi_t^{\vec{v}}$, we define the unit vector field $\vec{u}$ on $\varphi_t^{\vec{v}}(U) \subset \mathbb{S}^{2k+1}(\sqrt{1+t^2})$ by

$$\vec{u}(x) := \frac{1}{\sqrt{1+t^2}}\vec{v}(x) - \frac{t}{\sqrt{1+t^2}}x.$$

Using an adapted orthonormal frame $\{e_1, \ldots, e_{2k}, \vec{v}\}$ on a neighborhood V of U, we obtain an adapted orthonormal frame on $\varphi_t^{\vec{v}}(V)$ given by $\{\bar{e}_1, \ldots, \bar{e}_{2k}, \vec{u}\}$, where $\bar{e}_i = e_i$ for all $i \in \{1, \ldots, 2k\}$.

In this manner, we can write

$$\begin{aligned}
d\varphi_t^{\vec{v}}(e_1) &= \langle d\varphi_t^{\vec{v}}(e_1), e_1\rangle e_1 + \ldots + \langle d\varphi_t^{\vec{v}}(e_1), e_{2k}\rangle e_{2k} + \langle d\varphi_t^{\vec{v}}(e_1), \vec{u}\rangle \vec{u},\\
d\varphi_t^{\vec{v}}(e_2) &= \langle d\varphi_t^{\vec{v}}(e_2), e_1\rangle e_1 + \ldots + \langle d\varphi_t^{\vec{v}}(e_2), e_{2k}\rangle e_{2k} + \langle d\varphi_t^{\vec{v}}(e_2), \vec{u}\rangle \vec{u},\\
&\ \vdots\\
d\varphi_t^{\vec{v}}(e_{2k}) &= \langle d\varphi_t^{\vec{v}}(e_{2k}), e_1\rangle e_1 + \ldots + \langle d\varphi_t^{\vec{v}}(e_{2k}), e_{2k}\rangle e_{2k} + \langle d\varphi_t^{\vec{v}}(e_{2k}), \vec{u}\rangle \vec{u},\\
d\varphi_t^{\vec{v}}(\vec{v}) &= \langle d\varphi_t^{\vec{v}}(\vec{v}), e_1\rangle e_1 + \ldots + \langle d\varphi_t^{\vec{v}}(\vec{v}), e_{2k}\rangle e_{2k} + \langle d\varphi_t^{\vec{v}}(\vec{v}), \vec{u}\rangle \vec{u}.
\end{aligned}$$

Now, by Gauss's equation for the trivial immersion $\mathbb{S}^{2k+1} \hookrightarrow \mathbb{R}^{2k+2}$, we have

$$\tilde{\nabla}_Y \vec{v} = d\vec{v}(Y) = \nabla_Y \vec{v} - \langle \vec{v}, Y \rangle\, x$$

for every vector field Y on $\mathbb{S}^{2k+1}$, and then

$$\big\langle d\varphi_t^{\vec{v}}(e_1), e_1 \big\rangle = \big\langle e_1 + t\, d\vec{v}(e_1), e_1 \big\rangle = 1 + t \big\langle \nabla_{e_1} \vec{v}, e_1 \big\rangle$$

Analogously, we can conclude that

$$\begin{aligned}
\big\langle d\varphi_t^{\vec{v}}(e_i), e_i \big\rangle &= 1 + t\big\langle \nabla_{e_i}\vec{v}, e_i \big\rangle && \text{for } i \in \{1, \dots, 2k\},\\
\big\langle d\varphi_t^{\vec{v}}(e_i), e_j \big\rangle &= t\big\langle \nabla_{e_i}\vec{v}, e_j \big\rangle && \text{for } i, j \in \{1, \dots, 2k\},\ i \neq j,\\
\big\langle d\varphi_t^{\vec{v}}(e_i), \vec{u} \big\rangle &= 0 && \text{for } i \in \{1, \dots, 2k\},\\
\big\langle d\varphi_t^{\vec{v}}(\vec{v}), \vec{u} \big\rangle &= \sqrt{1+t^2}.
\end{aligned}$$

By employing the notation $h_{ij}(\vec{v}) := \langle \nabla_{e_i}\vec{v}, e_j \rangle$ (where $i, j \in \{1, \dots, 2k\}$), we can express the determinant of the Jacobian matrix of $\varphi_t^{\vec{v}}$ in the form

$$\det(d\varphi_t^{\vec{v}}) = \sqrt{1+t^2}\left(1 + \sum_{i=1}^{2k} \sigma_i(\vec{v}) t^2\right),$$

where, by definition, the functions σ_i are the i-symmetric functions of the h_{ij}. For instance, if $k = 1$, we have

$$\begin{aligned}
\sigma_1(\vec{v}) &:= h_{11}(\vec{v}) + h_{22}(\vec{v}),\\
\sigma_2(\vec{v}) &:= h_{11}(\vec{v})h_{22}(\vec{v}) - h_{12}(\vec{v})h_{21}(\vec{v}).
\end{aligned}$$

3. Proof of the Theorem

The energy of the vector field $\vec{v}$ (on K) is given by

$$\mathscr{E}(\vec{v}) := \frac{1}{2}\int_K \|d\vec{v}\|^2 = \frac{2k+1}{2}\operatorname{vol} K + \frac{1}{2}\int_K \|\nabla \vec{v}\|^2$$

Using the notation above, we have

$$\mathscr{E}(\vec{v}) = \frac{2k+1}{2}\operatorname{vol} K + \frac{1}{2}\int_K \left(\sum_{i,j=1}^{2k} (h_{ij}(\vec{v}))^2 + \sum_{i=1}^{2k} \langle \nabla_{\vec{v}}\vec{v}, e_i \rangle^2\right)$$

and then

$$\mathscr{E}(\vec{v}) \geq \frac{2k+1}{2}\operatorname{vol} K + \frac{1}{2}\int_K \sum_{i,j=1}^{2k} (h_{ij}(\vec{v}))^2. \tag{1}$$

Now observe that

$$\sum_{i<j}(h_{ii}-h_{jj})^2=(2k-1)\sum_i h_{ii}^2-2\sum_{i<j}h_{ii}h_{jj}$$

and

$$\sum_{i<j}(h_{ij}+h_{ji})^2=\sum_{i\neq j}h_{ij}^2+2\sum_{i<j}h_{ij}h_{ji}.$$

If we sum these last two equations, we get

$$(2k-1)\sum_i h_{ii}^2+\sum_{i\neq j}h_{ij}^2\geq 2\sigma_2$$

and then

$$\sum_i h_{ii}^2+\frac{1}{2k-1}\sum_{i\neq j}h_{ij}^2\geq\frac{2}{2k-1}\sigma_2. \tag{2}$$

Also, we can write

$$\sum_{i,j=1}^{2k}h_{ij}^2=\sum_{i\neq j}h_{ij}^2+\sum_i h_{ii}^2\geq\sum_i h_{ii}^2+\frac{1}{2k-1}\sum_{i\neq j}h_{ij}^2.$$

From this and (2), we obtain

$$\sum_{i,j=1}^{2k}(h_{ij}(\vec{v}))^2\geq\frac{2}{2k-1}\sigma_2(\vec{v}).$$

But then, using inequality (1), we find that

$$\mathscr{E}(\vec{v})\geq\frac{2k+1}{2}\operatorname{vol}K+\frac{1}{2k-1}\int_K\sigma_2(\vec{v}). \tag{3}$$

On the other hand, by the change of variables theorem, we obtain

$$\operatorname{vol}\varphi_t^H(K)=\int_K\sqrt{1+t^2}\Big(1+\sum_{i=1}^{2k}\sigma_i(H)t^i\Big)$$

By a straightforward computation shown in [Chacón 2000] and [Brito et al. 2004], we have $\sigma_i(H)=\eta_i$ for all $i\in\{1,\dots,2k\}$, where

$$\eta_i=\begin{cases}\binom{k}{i/2} & \text{if } i \text{ is even},\\ 0 & \text{if } i \text{ is odd}.\end{cases}$$

We know that the vector fields $\vec{v}$ and H are the same on ∂K. Thus, $\varphi_t^{\vec{v}}(K)$ and $\varphi_t^H(K)$ are $(2k+1)$-submanifolds of $\mathbb{S}^{2k+1}(\sqrt{1+t^2})$ with the same boundary. We claim that $\varphi_t^{\vec{v}}(K) = \varphi_t^H(K)$ for all t sufficiently small. In fact, if p is an interior point of K,

$$\lim_{t\to 0} \varphi_t^{\vec{v}}(p) = \lim_{t\to 0} \varphi_t^H(p) = p$$

and then we have necessarily

$$\varphi_t^{\vec{v}}(K) = \varphi_t^H(K)$$

for all t sufficiently small; equivalently,

$$\int_K \sqrt{1+t^2}\left(1+\sum_{i=1}^{2k} \sigma_i(\vec{v})t^i\right) = \int_K \sqrt{1+t^2}\left(1+\sum_{i=1}^{2k} \eta_i t^i\right)$$

for all $t > 0$ sufficiently small. Consequently, after canceling the factor $\sqrt{1+t^2}$ and rearranging the terms, we obtain

$$\left(\int_K [\sigma_1(\vec{v}) - \eta_1]\right)t + \left(\int_K [\sigma_2(\vec{v}) - \eta_2]\right)t^2 + \ldots + \left(\int_K [\sigma_{2k}(\vec{v}) - \eta_{2k}]\right)t^{2k} = 0$$

for all sufficiently small t. By identity of polynomials, we conclude

$$\int_K \sigma_i(\vec{v}) = \int_K \eta_i = \eta_i \operatorname{vol} K \quad \text{for } i \in \{1, \ldots, 2k\}.$$

Using this (for $i = 2$) together with (3), we get

$$\mathscr{E}(\vec{v}) \geq \frac{2k+1}{2} \operatorname{vol} K + \frac{\eta_2}{2k-1} \operatorname{vol} K = \left(\frac{2k+1}{2} + \frac{k}{2k-1}\right) \operatorname{vol} K.$$

We can obtain an analogue of this result for volumes using the following inequality (see [Brito et al. 2004] or [Chacón 2000, page 59]):

$$\operatorname{vol} \vec{v} \geq \int_K \left(1 + \sum_{i=1}^{k} \frac{\binom{k}{i}}{\binom{2k}{2i}} \sigma_{2i}(\vec{v})\right).$$

But $\int_K \sigma_{2i} = \int_K \eta_{2i} = \eta_{2i} \operatorname{vol} K$ for all $i \in \{1, \ldots, k\}$. Then, we have

$$\operatorname{vol} \vec{v} \geq \left(1 + \sum_{i=1}^{k} \frac{\binom{k}{i}^2}{\binom{2k}{2i}}\right) \operatorname{vol} K \geq \frac{4^k}{\binom{2k}{k}} \operatorname{vol} K$$

4. Final remarks

(1) If K is a spherical cap (the closure of a connected open set with round boundary of the three unit sphere), the theorem provides a "boundary version" for

the minimalization theorem of energy and volume functionals on [Brito 2000] and [Gluck and Ziller 1986].

(2) The “Hopf boundary” hypothesis is essential. In fact, if there is no constraint for the unit vector field $\vec{v}$ on ∂K, it is possible to construct vector fields on “small caps” such that $\|\nabla\vec{v}\|$ is small on K (exponential maps may be used on that construction). A consequence of this is that $\mathscr{E}(\vec{v})$ and $\operatorname{vol}\vec{v}$ are less than volume and energy of Hopf vector fields respectively.

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